

Non-chiral current algebras for deformed supergroup WZW models

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Abstract

We study deformed WZW models on supergroups with vanishing Killing form. The deformation is generated by the isotropic current-current perturbation which is exactly marginal under these assumptions. It breaks half of the global isometries of the original supergroup. The current corresponding to the remaining symmetry is conserved but its components are neither holomorphic nor anti-holomorphic. We obtain the exact two- and three-point functions of this current and a four-point function in the first two leading orders of a $1/k$ expansion but to all orders in the deformation parameter. We further study the operator product algebra of the currents, the equal time commutators and the quantum equations of motion. The form of the equations of motion suggests the existence of non-local charges which generate a Yangian. Possible applications to string theory on Anti-de Sitter spaces and to condensed matter problems are briefly discussed.

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1 Introduction

Conformally invariant σ -models with superspace target play a prominent role in various branches of mathematical physics. In the context of condensed matter theory they arise as means for an effective description of disordered systems [1]. They are also an essential ingredient in the covariant quantisation of superstrings, especially in backgrounds involving Ramond-Ramond fields [2]. In this context one of the main applications concerns the AdS/CFT correspondence where σ -models on supersymmetric versions of $AdS_d \times S^d$ and related spaces are used to describe the string theory side of the duality [3, 4, 5, 6, 7, 8, 9].

The string background $AdS_3 \times S^3$ with pure Neveu-Schwarz flux can be formulated in terms of a WZW model on the supergroup $PSU(1,1|2)$ [4, 10]. In such a formulation one can describe deformations corresponding to switching on a mixture of Neveu-Schwarz and Ramond-Ramond fluxes which preserves the full isometry of $PSU(1,1|2)$. Using G as an abbreviation for the supergroup $PSU(1,1|2)$, the isometry is $G \times G$. In the Lagrangian description the term describing such a deformation is the kinetic term of the WZW theory: $\text{str}(Jg\bar{J}g^{-1})$ [4, 10]. This corresponds to an operator in the WZW theory which can be written as $:J^a\phi_{ab}\bar{J}^b:(z,\bar{z})$. Here $J^a(z)$ and $\bar{J}^b(\bar{z})$ are the left and right WZW currents and $\phi_{ab}(z,\bar{z})$ is the full primary field corresponding to the adjoint representation of the supergroup G . The presence of the field ϕ_{ab} ensures that the perturbation is invariant under the full isometry $G \times G$. It compensates the non-trivial transformation behaviour of J under left multiplication and of \bar{J} under right multiplication by elements from G . The perturbing field is exactly marginal due to the remarkable fact that the supergroup $PSU(1,1|2)$ has a vanishing Killing form. The vanishing of the Killing form in particular implies that the Casimir element is trivial in the adjoint representation and hence that the field ϕ_{ab} has zero conformal dimension.

The deformation above is described geometrically as a principal chiral model on the supergroup G with a Wess-Zumino term. It was first argued in [11] that such principal chiral models (in the absence of a Wess-Zumino term) are conformal when the supergroup has a vanishing Killing form. The focus of that paper was on the particular series of supergroups: $PSL(n|n)$ which includes the $PSU(1,1|2)$ case. The considerations of [11] were further extended to general supergroups with vanishing Killing form and to their cosets in [12, 13] (see also [3, 4, 5, 6, 7, 8, 9] for more specific models).

Another interesting deformation of the $PSU(1,1|2)$ WZW theory is realised by the perturbing operator $\text{str}(J\bar{J})$. Such deformations are exactly marginal again due to the vanishing of the Killing form. However, they preserve only the diagonal part of the global symmetry group. Their global symmetry is thus isomorphic to one copy of G . As in the previous kind of deformation we expect a string theory interpretation in terms of a non-trivial background with mixtures of RR and NS flux. Although it is straightforward to compute the metric and B -field in the deformed sigma model, the extraction of physical background fields requires more work and the corresponding calculations have not been carried out so far.

Both of the deformations above easily generalise to WZW theories on arbitrary supergroups with vanishing Killing form. These theories are typically logarithmic, admitting the existence of fields with zero conformal dimension besides the identity. For the supergroups with trivial Killing form one of these operators is the adjoint representation primary field $\phi_{ab}(z,\bar{z})$. Supergroup WZW theories are well understood, at least at the conceptual level [14, 15, 16, 10, 17, 18]. Even though the logarithmic structure leads to many complications, the models can be solved due to the existence of two copies of an affine Kac-Moody superalgebra symmetry. The latter are realised in terms of two current algebras, one being holomorphic, the other anti-holomorphic.

For supergroups with vanishing Killing form, deformed by one of the two types of deformations described above, the global symmetry also implies the existence of conserved currents, however now with a much more complicated operator product expansion. In particular, the conserved local currents no longer split into a holomorphic and an antiholomorphic component which are separately conserved. These statements are well-known for conserved currents belonging to global symmetries in massive

theories. The general structure of the operator product algebra generated by conserved currents in massive 2D theories was first considered in [19]. It was demonstrated in [19] that for the massive $O(n)$ σ -models the OPE algebra of conserved currents allows one to construct an infinite tower of non-local conserved charges. These results were later generalised and reinterpreted in terms of Yangian symmetries in [20]. In [19] the OPE algebras generated by currents were called “massive current algebras”. As we will work with similar algebras in the context of conformal theories we prefer to call them “non-chiral current algebras”.

It is well-known that the $G \times G$ -preserving deformations discussed above are integrable, at least on a classical level, see [21, 22] and references therein. Our deformations are thus also very likely to be both conformal and integrable. It should be noted that the integrability is associated with the global symmetry. More precisely, it can be understood as being a consequence of current conservation and the existence of a Maurer-Cartan equation. While a priori there is no geometric reason to expect that the second, G -preserving, deformation considered in this note leads to an integrable theory, we will show that the algebraic structure of the theory allows one to use the construction of [19, 20] to obtain natural candidates for Yangian charges directly at the quantum level. More technically, we argue that the quantum equation of motion for the G -preserving deformation can be rephrased as a Maurer-Cartan equation for the conserved current associated with the global G -symmetry.

To summarise our considerations so far, we see that there are at least two good reasons to study the non-chiral current algebras generated by the above two deformations. Firstly one can use the methods of [19, 20] to construct an infinite tower of non-local conserved charges and to prove the integrability of such models. And secondly one may hope that such algebras will be useful for organising the spectrum of such conformal models. Besides potential applications such algebras are also interesting in their own right. In particular it would be interesting to understand in detail how the conformal symmetry is interrelated with integrability.

The non-chiral current algebra for the $G \times G$ -preserving deformation was recently studied in [23, 24, 25]. In those papers the operator product algebra of currents was investigated using perturbation theory both around the WZW point and in the classical (large level) limit. As a starting point, the authors postulated a quantum version of the Maurer-Cartan equation. Furthermore an interesting bootstrap approach using the Maurer-Cartan equation was put forward to obtain the OPE algebra of the non-chiral currents and primary fields. While very inspiring, that approach however hinges on the crucial assumption that the OPE algebra of the deformed currents closes on itself. This assumption is quite strong in view of the fact that the dimension zero field $\phi_{ab}(z, \bar{z})$ is involved in the deformation which could appear in various combinations in the OPE of the currents. We provide a quantitative discussion of this question by deriving two relations involving the operator $\phi_{ab}(z, \bar{z})$ that are necessary (but maybe not sufficient) for the closure of the current-current OPEs. Unfortunately further analysis of this issue is stalled due to the lack of knowledge of the OPE of the operator $\phi_{ab}(z, \bar{z})$ with itself. As this question lies outside the main scope of this paper the details of that computation are relegated to appendix E.

In the present paper we study the G -preserving current-current deformation. In this case the technical complications related to the field $\phi_{ab}(z, \bar{z})$ are absent and the situation is under very good control. The OPE closure of the deformed currents is easily established. Furthermore, despite the fact that for these deformations there is no Maurer-Cartan equation of geometric origin, the quantum equation of motion does take up essentially the same form (see section 7). We believe that this property allows one to prove the integrability of the model similarly to how it was done in [20] for the massive models.

Besides having potential applications in the context of the AdS/CFT correspondence the G -preserving deformation is also relevant for the study of Gross-Neveu models with supergroup symmetries. In the case of $OSP(2S+2|2S)$ the latter have been argued to be dual to $S^{2S+1|2S}$ supersphere σ -models [26, 27]. This correspondence is very interesting from a conceptual point of view, since superspheres belong to the class of (one-sided) supercosets for which a genuine CFT description is currently

beyond reach. One would hence hope that the same type of correspondence can be established for other supercoset spaces, in particular for those appearing in the AdS/CFT correspondence.

The main technical tool to be employed in the present paper is abelian conformal perturbation theory, used in conjunction with certain representation-theoretic assumptions valid for the supergroups of interest. The basic ideas of this method first appeared in [11] where it was shown that two-point functions of closed string vertex operators and three-point functions of currents can be determined exactly by extrapolation from the semi-classical (flat) limit. The vanishing of the Killing form and some classification of the low rank invariant tensors on the supergroups $PSU(N|N)$ were used to argue for the absence of corrections involving the structure constants for certain correlators. One can then use the abelian conformal perturbation theory (metric and B -field perturbations in toroidal theories) to obtain those correlators.¹ Using essentially the same method the exact open string spectra for certain D-branes in deformed WZW theories on supergroups were obtained in [28, 27] (see also [29] for a similar calculation in a condensed matter context).

In the present paper we apply the same method to bulk correlation functions of currents in the current-current deformed model. We are able to determine all the two- and three-point functions of currents exactly to all orders in the deformation parameter and we calculate their OPE at leading order in perturbation theory. Furthermore we compute the first non-trivial terms in the inverse level expansion for a four-point function of the currents, again to all orders in the coupling constant. Both the OPE algebra and the four-point function of currents exhibit logarithms, just as expected for this type of theories. Using the exact two- and three-point functions of the currents we compute the quantum equation of motion and the equal time commutators to all orders in the coupling constant. The equation of motion takes up the form of the Maurer-Cartan equation, giving strong indications for the integrability of the model following the arguments of [20]. The main body of the paper is organised as follows. In section 2 we introduce supergroup WZW models and discuss the available conformal deformations in some detail. In section 3 we first calculate the exact two- and three-point functions of the deformed theory. Our calculation is valid to all orders in the deformation parameter and shows a non-trivial coupling between holomorphic and anti-holomorphic currents away from the WZW point. In section 4 we elaborate on the precise form of the full OPE between the currents to lowest order in perturbation theory. We find again that the holomorphicity is spoiled and that there are logarithmic contributions in the OPE between currents of opposite chirality. Section 5 contains a calculation of the leading terms of the current four-point function in an expansion in $1/k$ but to all orders in the deformation parameter. In that result the logarithmic nature of the CFT is clearly visible. In section 6 we use the exact knowledge of the singular terms in the OPE of currents to compute their equal time commutators. Remarkably the commutator algebra is isomorphic to the direct sum of two copies of the current algebra. This means that the phase space of the model is isomorphic to two copies of an affine Kac-Moody superalgebra. The equation of motion, however, exhibits significant differences from that of WZW models. It is derived in section 7 to all orders in the coupling constant.

The appendices A, B, C, and D contain our conventions for Lie superalgebras, a thorough review of the abelian conformal perturbation theory and some details of calculations which are particularly cumbersome. Some technical details pertaining to our discussion of the $G \times G$ -preserving deformation have been put in appendix E.

2 Supergroup WZW models and their deformations

Supergroup WZW models exhibit a number of peculiar features that their bosonic cousins are lacking. One of them is the occurrence of logarithmic correlation functions which is intimately connected to the supergeometry and the non-factorisation of the state space into left and right movers [16, 10, 17, 18].

¹The abelian perturbation series did not explicitly appear in [11]. Presumably, in the context of that paper such series appeared only as overall factors in the correlators and were absorbed in the normalisation conventions.

While the presence of the logarithms is a common feature,² special phenomena arise if the Killing form of the underlying supergroup is vanishing [11]. In that case the supergroup WZW model admits marginal perturbations of current-current type which would otherwise break conformal invariance. In this section we review the construction and the symmetries of WZW models and discuss different types of marginal deformations and their implications.

2.1 Supergroup WZW models

Let us fix a supergroup G and a non-degenerate invariant form $\langle \cdot, \cdot \rangle$. We assume the supergroup to be simple and simply-connected and the invariant form to be normalised in the standard way (see below). The supergroup WZW model is a two-dimensional σ -model describing the propagation of strings on G . The action functional is given by

$$\mathcal{S}^{\text{WZW}}[g] = -\frac{ik}{4\pi} \int_{\Sigma} \langle g^{-1} \partial g, g^{-1} \bar{\partial} g \rangle dz \wedge d\bar{z} - \frac{ik}{24\pi} \int_B \langle g^{-1} dg, [g^{-1} dg, g^{-1} dg] \rangle, \quad (2.1)$$

where Σ is a closed Riemann surface and B is a three-dimensional extension of this surface such that $\partial B = \Sigma$. The form $\langle \cdot, \cdot \rangle$ is supposed to be normalised such that the topological Wess-Zumino term is well-defined up to multiples of $2\pi i$ as long as k is an integer. The level k is thus the only parameter of the model.

By construction, every WZW model has a global symmetry $G \times G$ corresponding to multiplying the field $g(z, \bar{z})$ by arbitrary group elements from the left and from the right. In fact, this symmetry is elevated to an affine Kac-Moody algebra symmetry

$$J^a(z) J^b(w) = \frac{k \kappa^{ab}}{(z-w)^2} + \frac{if^ab_c J^c(w)}{z-w}, \quad \bar{J}^a(\bar{z}) \bar{J}^b(\bar{w}) = \frac{k \kappa^{ab}}{(\bar{z}-\bar{w})^2} + \frac{if^ab_c \bar{J}^c(\bar{w})}{\bar{z}-\bar{w}}, \quad (2.2)$$

if one allows these group elements to depend holomorphically and antiholomorphically on z , respectively. In the last formula, the currents are defined by

$$J = -k \partial g g^{-1}, \quad \bar{J} = k g^{-1} \bar{\partial} g, \quad (2.3)$$

and the equations of motion guarantee that they are holomorphic and antiholomorphic, respectively. The tensor κ^{ab} refers to the non-degenerate invariant form, see appendix A for the details of our Lie superalgebra conventions.

Supergroup WZW models are a very exciting subject by themselves but for the purpose of this paper it is not necessary to introduce further details. The interested reader is referred to [18] where a more comprehensive discussion can be found.

2.2 Marginal deformations

WZW models allow for a number of local deformations which preserve conformal invariance. On an abstract level, such deformations have to be of the form

$$\mathcal{S}_{\lambda}[g] = \mathcal{S}^{\text{WZW}}[g] + \mathcal{S}_{\text{def}}[g] \quad \text{with} \quad \mathcal{S}_{\text{def}} = \lambda \int d^2z \mathcal{O}_{\text{def}}(z, \bar{z}), \quad (2.4)$$

where the perturbing field $\mathcal{O}_{\text{def}}(z, \bar{z})$ has conformal weights $(h, \bar{h}) = (1, 1)$ in order to render the coupling λ dimensionless. Consequently, a canonical candidate for the perturbing field is an arbitrary bilinear in the currents J^a and \bar{J}^a , e.g.

$$\mathcal{O}_{\text{def}}(z, \bar{z}) = m_{ab} : J^a \bar{J}^b : (z, \bar{z}). \quad (2.5)$$

²Possible exceptions are WZW models at low levels which admit a free field description (see e.g. [17, 27]). In that case realisations of logarithmic and non-logarithmic theories both exist.

A simple calculation, however, implies that perturbations of this form are generically marginally relevant, i.e. that conformal invariance is spoiled at higher orders in perturbation theory. Roughly speaking, marginality requires that the currents of one specific chirality which enter eq. (2.5) mutually commute (up to central terms). The only exception are supergroups with vanishing Killing form for which also deformations induced by the perturbing field

$$\mathcal{O}_{\text{def}}(z, \bar{z}) = : \langle J(z), \bar{J}(\bar{z}) \rangle : = \kappa_{ba} : J^a \bar{J}^b : (z, \bar{z}) \quad (2.6)$$

are marginal to all orders in perturbation theory. Here κ_{ba} denotes the inverse of the invariant form κ^{ab} . The argument for the marginality will be reviewed below. The deformation just discussed has obvious generalisations such as

$$\mathcal{O}_{\text{def}}(z, \bar{z}) = : \langle J(z), \Omega(\bar{J}(\bar{z})) \rangle : , \quad (2.7)$$

where Ω is an arbitrary (but constant) automorphism of G . The insertion of the automorphism merely corresponds to a reinterpretation of the current \bar{J} and will hence not be considered in this note.³

Let us now analyse what kind of symmetries are preserved by current-current perturbations. Under the isometry $g \mapsto lgr^{-1}$ the currents transform as

$$J \mapsto lJl^{-1} , \quad \bar{J} \mapsto r\bar{J}r^{-1} . \quad (2.8)$$

This in particular implies that none of the deforming fields (2.5) is invariant under the full $G \times G$ symmetry. For a general matrix m_{ab} hardly any of the global symmetries will remain, even if the deformation preserves conformal invariance. The special choice (2.6) for instance preserves one copy of G and that is the maximal possible global symmetry for an ansatz of the form (2.5).

The full symmetry can only be preserved if one of the currents is conjugated by group elements as in $g\bar{J}g^{-1}$.⁴ In operator language this requires the use of a non-chiral primary field $\phi_{ab}(z, \bar{z})$ that transforms in the representation $\text{ad} \otimes \text{ad}$ with respect to the global $G \times G$ symmetry [10]. It is obvious from the construction that the resulting perturbing field

$$\mathcal{O}_{\text{def}}(z, \bar{z}) = : \langle J, g\bar{J}g^{-1} \rangle : = : J^a \phi_{ab} \bar{J}^b : (z, \bar{z}) \quad (2.9)$$

can only be marginal if $\phi_{ab}(z, \bar{z})$ is a primary field with conformal dimensions $h = \bar{h} = 0$. Since the conformal dimension of $\phi_{ab}(z, \bar{z})$ is proportional to the quadratic Casimir element C_{ad} evaluated in the adjoint representation, this is possible precisely when the underlying supergroup has a vanishing Killing form.

It should be noted that a composite, normal ordered operator as the one in (2.9) has complicated properties and leads to a variety of subtleties in perturbation theory, even more so since it belongs to a non-unitary representation and has the same conformal dimension as the identity operator.⁵ For this reason, we restrict ourselves to pure current-current perturbations which preserve the diagonal G symmetry. As explained above, this is the maximal symmetry which can be preserved under such circumstances.

Throughout the rest of the paper we will only treat the deformation by the perturbing field (2.6).⁶ More concretely, the Lagrangian we are considering corresponds to the deformation

$$\mathcal{S}_{\text{def}}[g] = \frac{\lambda}{\pi k} \int_{\Sigma} d^2 z \langle J, \bar{J} \rangle = -\frac{k\lambda}{\pi} \int_{\Sigma} d^2 z \langle \partial g g^{-1}, g^{-1} \bar{\partial} g \rangle , \quad (2.10)$$

³In other words, the usual right action of G on itself is replaced by an Ω -twisted right action of G on itself.

⁴Or a twisted version of this if a twisted $G \times G$ symmetry is to be preserved.

⁵One should also bear in mind that for non-compact models the representation $\text{ad} \otimes \text{ad}$ is not part of the spectrum. A priori this makes it difficult to make sense out of correlation functions involving the field $\phi_{ab}(z, \bar{z})$.

⁶Note that $\kappa_{ba} : J^a \bar{J}^b :$ and $\kappa_{ab} : J^a \bar{J}^b :$ are basically the same perturbing fields since they result from each other by application of the automorphism $\Omega(T^a) = (-1)^a T^a$ to one of the two currents.

where we have included a convenient normalisation factor $1/k\pi$. From the geometrical construction it is evident that this deformation preserves the diagonal G symmetry. Conformal invariance, however, can only be checked on the level of operators. As is well known, a necessary condition for the marginality of the deformation is that the OPE $\mathcal{O}_{\text{def}}(z, \bar{z})\mathcal{O}_{\text{def}}(w, \bar{w})$ does not contain a copy of $\mathcal{O}_{\text{def}}(w, \bar{w})$ again [30]. This condition arises at first order in perturbation theory. A quick calculation keeping only the relevant terms yields

$$\mathcal{O}_{\text{def}}(z, \bar{z}) \mathcal{O}_{\text{def}}(w, \bar{w}) = \kappa_{ba} \kappa_{dc} (-1)^{bc} J^a(z) J^c(w) \bar{J}^b(\bar{z}) \bar{J}^d(\bar{w}) \quad (2.11)$$

$$= \dots - \kappa_{ba} \kappa_{dc} (-1)^{bc} f_{\quad e}^{ac} f_{\quad f}^{bd} \frac{:J^e \bar{J}^f: (w, \bar{w})}{|z-w|^2} + \dots \quad (2.12)$$

The coefficient of the current bilinear $J^e \bar{J}^f$ can easily be shown to be proportional to $C_{\text{ad}} \kappa_{fe}$. It hence vanishes for supergroups with vanishing Killing form, thus proving our assertion.

Conformal invariance can also be shown at higher orders in perturbation theory. For that purpose we review an argument of Bershadsky et al [11] that in this or a similar form will be used frequently throughout the text. The basic idea is the following: We know that the perturbing field (and hence also the corresponding coupling) transforms trivially under a certain action of the supergroup G . As a consequence, the associated β -function will also be invariant with respect to G . Let us then look at all possible Feynman diagrams that can contribute to the perturbative expansion of the β -function. Since the deformation can be fully expressed in terms of currents whose OPE is given in (2.2), the β -function will be a sum over diagrams made up from trivalent vertices (corresponding to the structure constants). These diagrams all cannot have external legs since otherwise the β -function would not transform trivially under G .

Let us first consider an arbitrary diagram which contains at least one trivalent vertex and let us separate it from the rest of the diagram. We now have a vertex connected by three lines to a blob containing the rest of the diagram. Assuming further that the structure constants are the unique invariant rank three tensor of G (this is known to be true in all cases of interest [11, 13, 28]), we conclude that the diagram is proportional to $(-1)^{da} \kappa_{ba} f_{\quad d}^{ac} f_{\quad c}^{bd}$. This tensor is identically zero due to the vanishing of the dual Coxeter number. Hence only diagrams without trivalent vertices remain. Under these circumstances, however, the perturbative series reduces to that of a multi-component free boson which is known to possess marginal current-current perturbations changing the respective radii. In our context this also proves conformal invariance for our deformed non-abelian model.

One may wonder whether the moduli space is one-dimensional or whether one can combine two marginal deformations in order to explore additional directions. Let us analyse this question by switching on an additional perturbing field of the form (2.7) with automorphism $\Omega(T^a) = (-1)^a T^a$. We focus on this choice since Ω is the only automorphism available for all supergroups. Denoting the second deforming operator by $\tilde{\mathcal{O}}_{\text{def}}(z, \bar{z}) = \kappa_{ab} J^a \bar{J}^b$ one finds

$$\begin{aligned} \mathcal{O}_{\text{def}}(z, \bar{z}) \tilde{\mathcal{O}}_{\text{def}}(w, \bar{w}) &= \kappa_{ba} \kappa_{cd} (-1)^{bc} J^a(z) J^c(w) \bar{J}^b(\bar{z}) \bar{J}^d(\bar{w}) \\ &= \dots - \kappa_{ba} \kappa_{cd} (-1)^{bc} f_{\quad e}^{ac} f_{\quad f}^{bd} \frac{:J^e \bar{J}^f: (w, \bar{w})}{|z-w|^2} + \dots \end{aligned} \quad (2.13)$$

A closer investigation shows that there is no reason for the coefficient of $J^e \bar{J}^f$ to vanish. As a result we conclude that the two deformations \mathcal{O}_{def} and $\tilde{\mathcal{O}}_{\text{def}}$ are incompatible in the sense that conformal invariance is broken as soon as both fields are switched on, even though each of them preserves conformal invariance separately. This is not too surprising since the two deformations preserve different G -symmetries, hence destroying supersymmetry (but not the bosonic symmetry) when both couplings are non-zero.

The operator formalism also allows us to verify our earlier statement that the diagonal G -symmetry is preserved. This is the case precisely when the current $\mathcal{O}_{\text{def}}(z, \bar{z})$ is invariant with respect to the

diagonal action of G . In algebraic terms this amounts to the statement

$$\left[\oint_w \frac{dz}{2\pi i} J^a(z) + \oint_{\bar{w}} \frac{d\bar{z}}{2\pi i} \bar{J}^a(\bar{z}) \right] \mathcal{O}_{\text{def}}(w, \bar{w}) = 0 \quad (2.14)$$

in the undeformed theory. The validity of this equation follows immediately from the OPEs

$$J^a(z) \mathcal{O}_{\text{def}}(w, \bar{w}) = \frac{k\bar{J}^a(\bar{w})}{(z-w)^2} + \frac{if^a_{dc} : J^c \bar{J}^d : (w, \bar{w})}{z-w} \quad (2.15)$$

$$\bar{J}^a(\bar{z}) \mathcal{O}_{\text{def}}(w, \bar{w}) = \frac{kJ^a(w)}{(\bar{z}-\bar{w})^2} - \frac{if^a_{dc} : J^c \bar{J}^d : (w, \bar{w})}{\bar{z}-\bar{w}}. \quad (2.16)$$

An argument similar to the previous one for the β -function, but now involving diagrams with precisely one external leg, can be used to show that a quantity with a single G -index does not receive corrections at higher orders in perturbation theory. Hence we conclude that $\mathcal{O}_{\text{def}}(z, \bar{z})$ is G -invariant to all orders in perturbation theory.

3 Exact two- and three-point functions of currents

In this section we will compute the deformed two- and three-point functions of the currents to all orders in λ . To this end we use certain algebraic properties of the supergroups at hand (similar to the ones employed in the previous section) combined with the abelian conformal perturbation theory of [31].

3.1 Two-point functions

Following [20] we will assume that the ultraviolet limit in the deformed theory is smooth and the perturbed operators are in one-to-one correspondence with the operators at the WZW point. In particular this concerns the currents which for the deformed theory will be denoted by the same symbol but now including both holomorphic and antiholomorphic arguments: $J^a(z, \bar{z})$ and $\bar{J}^a(z, \bar{z})$. On general grounds we expect the deformed currents J^a, \bar{J}^a to remain Virasoro primary fields of conformal weights $(1, 0)$ and $(0, 1)$ respectively. This is because the perturbing operator preserves the spin, the currents are conserved (and thus their scaling dimension is one) and there are no fields that could form a Jordan block with the currents. The last statement follows from the fact that the perturbing fields are built from the currents only and thus cannot generate anything but the fields in the vacuum representation of the underlying affine Lie superalgebra.

A general remark must be made regarding correlation functions. In a WZW theory for a supergroup the expectation value of the identity operator is typically zero: $\langle \mathbf{1} \rangle = 0$. This is because the identity field in such models belongs to the bottom of a Jordan block (the socle of a projective cover). However, this effect is absent for the free field realisations of the WZW models encountered in [27] (giving rise to Gross-Neveu models) which allow to establish non-logarithmic theories. Thus, at least in that case, the computations of correlation functions which are done below with the convention $\langle \mathbf{1} \rangle = 1$ are fully justified. For other WZW theories on supergroups the computations below should be understood more formally as means of obtaining OPE coefficients (see section 4).

The two-point functions of currents in the deformed theory can be obtained by summing up the perturbation theory expansion:

$$\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) \rangle_\lambda = \left\langle J^a(z_1) J^b(z_2) \exp\left(-\frac{\lambda}{k\pi} \int d^2w : J^e \bar{J}^f : \kappa_{re}\right) \right\rangle_0 \quad (3.1)$$

and analogously for the $\langle J^a \bar{J}^b \rangle_\lambda$ correlator. Here and elsewhere $\langle \dots \rangle_0$ stands for a WZW theory correlator.

Each term in the perturbation series is given in terms of integrals of correlators evaluated at the WZW point. To compute such integrals we need to have these correlators defined in the distributional sense (and not merely as functions defined for finite separation of variables as is customarily done in CFT). Such distributional correlators in general contain contact term ambiguities related by reparameterisations of the coupling constant λ . A particular choice of these contact terms should be considered as part of the definition of the composite operator $:J^e \bar{J}^r:$ coupling to λ .

We will fix the distributional correlators of the currents extending the prescription of [31]. In appendix D of that paper, G. Moore gave a prescription for partial integrals of distributions arising in conformal perturbation theory of free bosons. That prescription is easily adopted for integrals of correlators containing currents in WZW theory. Moreover in Appendix B we show how one can define the correlation functions at hand as distributions so that the prescription of [31] holds and we justify, based on that definition, various manipulations with such integrals.

Specialising to a particular class of supergroups crucially simplifies summing up the perturbation series (3.1). In addition to the vanishing of the adjoint Casimir element we will assume that the structure constants f^{abc} are the only invariant 3-tensor. Under these assumptions the terms containing the structure constants in the perturbative series (3.1) drop out [11]. For completeness we repeat here the argument. Since a simple Lie superalgebra has a unique non-degenerate invariant bilinear form, the general form of the deformed correlator is

$$\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) \rangle_\lambda = \kappa^{ab} g(\lambda; z_{12}, \bar{z}_{12}) \quad (3.2)$$

where g transforms trivially under the global symmetry group. Thus if g contains terms dependent on f^{abc} those terms must be of the form $f^{abc} C_{abc}$ where C_{abc} is an invariant tensor. But since the only invariant 3-tensor is given by the structure constants, and since $C_{ad} = 0$ for the quadratic Casimir of the adjoint representation, such contributions vanish. The same argument goes through for the $\langle J^a \bar{J}^b \rangle_\lambda$ correlator. The remaining perturbation series is effectively that of the free boson theory. Using the results of [31] (see Appendix B for details) we obtain

$$\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) \rangle_\lambda = \frac{k\kappa^{ab}}{(1-\lambda^2)z_{12}^2}, \quad \langle \bar{J}^a(z_1, \bar{z}_1) \bar{J}^b(z_2, \bar{z}_2) \rangle_\lambda = \frac{k\kappa^{ab}}{(1-\lambda^2)\bar{z}_{12}^2}, \quad (3.3)$$

$$\langle J^a(z_1, \bar{z}_1) \bar{J}^b(z_2, \bar{z}_2) \rangle_\lambda = 0, \quad (3.4)$$

where the correlators are taken at finite separation.

3.2 Three-point functions

Consider next the deformed three-point functions

$$\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) \rangle_\lambda = \left\langle J^a(z_1) J^b(z_2) J^c(z_3) \exp\left(-\frac{\lambda}{k\pi} \int d^2w J^e \bar{J}^r \kappa_{re}\right) \right\rangle_0. \quad (3.5)$$

It was argued in [11] that the terms in the perturbation series (3.5) containing two or more factors of the structure constants vanish. For the argument of [11] to work one needs to assume that there are only three traceless invariant tensors of rank 4:

$$f_{ab}^e f_{cde}, \quad f_{ac}^e f_{bde}, \quad \kappa_{ab} \kappa_{cd} + (-1)^{bc} \kappa_{ac} \kappa_{bd} + \kappa_{ad} \kappa_{bc} \quad (3.6)$$

Any 3-tensor resulting from a contraction of more than two structure constants can be represented diagrammatically as in figure 1 below. Every structure constant corresponds to a three-vertex in the diagram and any contraction to a link. The blob containing four external lines must correspond to a traceless tensor as there are no corrections to the invariant metric. The desired result follows now

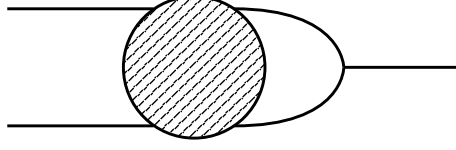


Figure 1: Contraction of tensors for a three-point function

from the fact that every traceless four-tensor listed in (3.6) vanishes upon contracting any two indices with the structure constants.

We therefore only need to extract all terms containing a single factor of the structure constants in (3.5). To this end we first extract terms that are singular as J^a approaches the other insertions. Singularities with one of the external fields give contributions proportional to the deformed two-point functions (3.3) while contractions with the perturbing fields can be rearranged again into correlators of the perturbed theory. We obtain

$$\begin{aligned}
\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) \rangle_\lambda &= \frac{1}{1-\lambda^2} \langle J^a(z_1) J^b(z_2) J^c(z_3) \rangle_0 \\
-(-1)^{a(b+c)} \left(\frac{\lambda}{k\pi} \right) i f^a_{rs} \int \frac{d^2 w}{z_1 - w} \langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^s(w, \bar{w}) \bar{J}^r(w, \bar{w}) \rangle_\lambda \\
-(-1)^{a(b+c)} \left(\frac{\lambda}{\pi} \right) \int \frac{d^2 w_1}{(z_1 - w_1)^2} \langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) \bar{J}^a(w_1, \bar{w}_1) \rangle_\lambda.
\end{aligned} \tag{3.7}$$

The correlators in the second line can be evaluated in the abelian theory that is dropping the structure constants. Using the technique of abelian conformal perturbation explained in Appendix B we obtain

$$-(-1)^{a(b+c)} \left(\frac{\lambda}{k\pi} \right) i f^a_{rs} \int \frac{d^2 w}{z_1 - w} \langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^s(w, \bar{w}) \bar{J}^r(w, \bar{w}) \rangle_\lambda \tag{3.8}$$

$$= \frac{\lambda^2}{(1-\lambda^2)^2} \langle J^a(z_1) J^b(z_2) J^c(z_3) \rangle_0. \tag{3.9}$$

We next take up the correlators in the third line of (3.7). Extracting the singularities of $\bar{J}^a(\bar{w}_1)$ with other fields we obtain

$$\begin{aligned}
&-(-1)^{a(b+c)} \left(\frac{\lambda}{\pi} \right) \int \frac{d^2 w_1}{(z_1 - w_1)^2} \langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) \bar{J}^a(w_1, \bar{w}_1) \rangle_\lambda \\
&= -\frac{\lambda^3}{(1-\lambda^2)^2} \langle J^a(z_1) J^b(z_2) J^c(z_3) \rangle_0 \\
&+ (-1)^{a(b+c)} \left(\frac{\lambda}{\pi} \right)^2 \iint \frac{d^2 w_1 d^2 w_2}{(z_1 - w_1)^2 (\bar{w}_1 - \bar{w}_2)^2} \langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^a(w_2, \bar{w}_2) \rangle_\lambda.
\end{aligned} \tag{3.10}$$

Using the integral (B.12) the integral in (3.10) yields back the original three-point function $\langle J^a J^b J^c \rangle_\lambda$. Collecting together (3.7), (3.8) and (3.10) we finally obtain

$$\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) \rangle_\lambda = \frac{1-\lambda^3}{(1-\lambda^2)^3} \left[\frac{-ik f^{abc}}{z_{12} z_{23} z_{31}} \right]. \tag{3.11}$$

Consider next the mixed three-point function

$$\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) \bar{J}^c(z_3, \bar{z}_3) \rangle_\lambda = \langle J^a(z_1) J^b(z_2) \bar{J}^c(\bar{z}_3) \exp\left(-\frac{\lambda}{k\pi} \int d^2 w J^e \bar{J}^r \kappa_{re}\right) \rangle_0. \tag{3.12}$$

Extracting the singularities of \bar{J}^c with the perturbing fields we obtain

$$\begin{aligned}
& \left\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) \bar{J}^c(z_3, \bar{z}_3) \right\rangle_\lambda \\
&= - \left(\frac{\lambda}{k\pi} \right) i f_{es}^c (-1)^{ce+e} \int \frac{d^2 w}{\bar{z}_3 - \bar{w}} \left\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) : J^e \bar{J}^s : (w, \bar{w}) \right\rangle_\lambda \\
&\quad - \left(\frac{\lambda}{\pi} \right) \int \frac{d^2 w}{(\bar{z}_3 - \bar{w})^2} \left\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) J^c(w, \bar{w}) \right\rangle_\lambda. \tag{3.13}
\end{aligned}$$

The first term in the above equation is easily evaluated using (B.4) while for the second term we use (3.11). Altogether we find

$$\left\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) \bar{J}^c(z_3, \bar{z}_3) \right\rangle_\lambda = \frac{\lambda(1-\lambda)}{(1-\lambda^2)^3} \left[\frac{-ik f^{abc} \bar{z}_{12}}{z_{12}^2 \bar{z}_{23} \bar{z}_{31}} \right]. \tag{3.14}$$

We see that, as was expected on general grounds, the position dependence of the three-point functions (3.11) and (3.14) matches with the fields $J^a(z, \bar{z})$ and $\bar{J}^a(z, \bar{z})$ being primaries of conformal weights (1, 0) and (0, 1) respectively.

4 The OPE algebra of currents

In this section we compute the first order contributions to the current-current OPEs in the deformed theory. Already at this order we find several characteristic features expected from such OPEs [19, 20, 23, 25]. The OPE of J^a with J^b acquires non-holomorphic contributions and the OPE of J^a with \bar{J}^b ceases to be vanishing. In fact, it the latter even receives a logarithmic correction. While we restrict our attention to first order perturbation theory, some of the structure constants can be determined to all orders using the exact knowledge about the two- and three-point functions obtained in section 3. These coefficients are written out in subsection 4.4.

4.1 The method

Our calculation of the OPE between two currents will be based on the method developed in [32]. The starting point is an abstract CFT which we assume to be under complete control, at least conceptually if not computationally. This CFT will then be perturbed by a deformation term $\mathcal{S}_{\text{def}} = \lambda \int d^2 z \mathcal{O}_{\text{def}}(z, \bar{z})$ which could, in principle, be either marginal or relevant. Below, we will focus on one of the marginal current-current deformations considered in section 2. Let us use the symbols Φ^a to denote a basis of operators in the undeformed theory.

The deformed theory has an OPE of the form

$$\Phi^a(z_1, \bar{z}_1) \Phi^b(z_2, \bar{z}_2) = \sum_c C_c^{ab}(z_{12}, \bar{z}_{12}|\lambda) \Phi^c(z_2, \bar{z}_2). \tag{4.1}$$

In this equation, the OPE coefficients $C_c^{ab}(z_1 - z_2|\lambda)$ depend on the deformation parameter λ . Up to potential logarithmic contributions, the explicit coordinate dependence on the right hand side of the equation is completely determined by the conformal dimensions of the operators Φ^c .

The fundamental ingredient in our perturbative evaluation of the OPE coefficients $C_c^{ab}(z_1 - z_2|\lambda)$ is the action principle (see e.g. references in [32]). At the leading order in the deformation parameter λ the action principle reduces to

$$\begin{aligned}
& \left\langle \left[\Phi^a(z_1, \bar{z}_1) \Phi^b(z_2, \bar{z}_2) - \sum_c C_c^{ab}(z_{12}, \bar{z}_{12}|0) \Phi^c(z_2, \bar{z}_2) \right] X(z_3, \bar{z}_3, \dots) \int d^2 z \mathcal{O}_{\text{def}}(z, \bar{z}) \right\rangle_0 \\
&= - \sum_i \partial_\lambda C_c^{ab}(z_{12}, \bar{z}_{12}|0) \left\langle \Phi^c(z_2, \bar{z}_2) X(z_3, \bar{z}_3, \dots) \right\rangle_0, \tag{4.2}
\end{aligned}$$

where all correlators are evaluated in the unperturbed theory. Using a suitable sequence of choices for the multi-local operator $X(z_3, \bar{z}_3, \dots)$ we can therefore determine the derivatives $\partial_\lambda C_c^{ab}(z_{12}, \bar{z}_{12}|0)$ one after another. Together with the knowledge of the unperturbed theory we can then easily reconstruct the OPE coefficients to first order in the deformation parameter λ . It is worth noting that in general infrared divergences are present in perturbation expansion around massless theories. Under certain regularity assumptions the action principle allows one to confine all such divergences to one-point functions so that the deformed OPE coefficients are infrared finite [32].

Let us now specialise our considerations to the current-current perturbation of WZW models that have been introduced in section 2. The perturbation operator in (4.2) now becomes $\mathcal{O}_{\text{def}} = \frac{1}{\pi k} \kappa_{ab} : J^b \bar{J}^a :$. The perturbation theory of such current-current perturbations is infrared finite since the currents fall off as $1/z^2$ and $1/\bar{z}^2$ at infinity, respectively, leaving an integral over $1/|z|^4$ which is integrable. For this reason we will omit all infrared regulators in what follows. However, in order to apply the general formalism presented above we need to discuss the basis of operators we are using. As was already mentioned in section 3 we assume that a basis of operators in the deformed theory can be labelled by the elements of a basis in the undeformed theory and that the limit $\lambda \rightarrow 0$ is smooth. We will therefore denote the deformed operators by the corresponding bare operators. To insert the undeformed operator into the perturbation series one needs to fix the correlation functions as distributions. The distributional correlators are defined up to contact terms. Any particular choice of such contact terms is part of the definition of the composite operator. We will stick to the definition of distributional correlators of currents generalising the prescription of [31] as described in Appendix B.

In the case of the deformed currents we explicitly put both holomorphic and antiholomorphic coordinates in the notation $J^a(z, \bar{z})$, $\bar{J}^b(w, \bar{w})$ to emphasise that these are the deformed currents. Since we are only interested in the OPEs between the currents and since the deformation term itself only contains currents we will only encounter composite operators made up from normal ordered products of currents and their derivatives. Thus we can take a basis labelled by operators in the vacuum sector of the WZW theory. A basis of such operators is built from the composites of currents and their derivatives. Let us express a word of caution regarding the meaning of such composite operators in the deformed theory. The operator $:J^a \bar{J}^b:(z, \bar{z})$ denotes an operator of the deformed theory whose correlators are obtained by inserting the bare composite $:J^a \bar{J}^b:$ in the perturbation theory series. Such an operator in general will not coincide with the normal ordered product of the deformed currents $J^a(z, \bar{z})$ and $\bar{J}^b(w, \bar{w})$, i.e. with an operator of the deformed theory obtained by taking the limit $z \rightarrow w$ in the deformed OPE of the currents $J^a(z, \bar{z})$ and $\bar{J}^b(w, \bar{w})$ and subtracting the singular terms.

In the undeformed theory the OPE of currents will only contain composites which are built of no more than two currents. This will no longer be the case for the deformed OPEs which will contain on the right hand side composites containing an arbitrarily large number of currents. There is a simple rule to be noted for the appearance of such composites: the composite built of N currents may appear at the orders λ^M with $M \geq N - 1$. Thus at the order λ we do not need to include the composite operators beyond bilinears in the currents.

4.2 The OPE of J with itself

We first discuss the deformed OPE of J^a and J^b . Our calculations show that at the leading order in λ it gets deformed as

$$\begin{aligned}
J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) &= \frac{k \kappa^{ab}}{(z_1 - z_2)^2} + \frac{i f^{ab}{}_c J^c(z_2, \bar{z}_2)}{z_1 - z_2} + \frac{i}{2} f^{ab}{}_c \partial J^c(z_2, \bar{z}_2) + \frac{1}{2} : (J^a J^b + J^b J^a) : (z_2, \bar{z}_2) \\
&\quad - \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \frac{\lambda}{k} (-1)^{bd} f^a{}_{dg} f^{gb}{}_c : J^c \bar{J}^d : (z_2, \bar{z}_2) - \frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} i \lambda f^{ab}{}_c \bar{J}^c(z_2, \bar{z}_2) \\
&\quad - \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \frac{i \lambda}{2} f^{ab}{}_c \bar{\partial} \bar{J}^c(z_2, \bar{z}_2) + \mathcal{O}(\lambda^2) + \dots .
\end{aligned} \tag{4.3}$$

The only terms of order λ in this OPE which are not explicitly written on the right hand side are the terms vanishing as $z_1 \rightarrow z_2$.

4.2.1 General ansatz and initial conditions

Since we are only interested in terms which are singular or constant as $z_1 \rightarrow z_2$ we can pick a basis of bare operators from the set containing 1 , J^a , \bar{J}^b , ∂J^a , $\bar{\partial} \bar{J}^b$, $:J^a J^b:$, $:\bar{J}^a \bar{J}^b:$, $:J^a \bar{J}^b:$. In choosing a linear independent set among these operators one needs to take into account the following relation between operators in the WZW model

$$: [J^a, J^b] : = i f^{ab}{}_c \partial J^c \tag{4.4}$$

and a similar one for the antiholomorphic currents. Usually one can choose to form a basis using either the derivatives or the antisymmetric bilinears, but for supergroups with a vanishing dual Coxeter number g^\vee the usual inversion of equation (4.4),

$$\partial J^a = \frac{i}{2g^\vee} f^a{}_{cb} : [J^b, J^c] :, \tag{4.5}$$

does not work. Thus for the subspace at hand we pick the basis containing the operators 1 , J^a , \bar{J}^b , ∂J^a , $\bar{\partial} \bar{J}^b$, $:J^a \bar{J}^b:$ and the symmetric bilinears $: (J^a J^b + J^b J^a) :$, $: (\bar{J}^a \bar{J}^b + \bar{J}^b \bar{J}^a) :$.

We thus write the following ansatz for the deformed OPE

$$\begin{aligned}
J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) &= \frac{k \kappa^{ab}(\lambda)}{(z_1 - z_2)^2} + \frac{i f^{ab}{}_c(\lambda) J^c(z_2, \bar{z}_2)}{z_1 - z_2} + g_{cd}^{ab}(\lambda) : J^c J^d : (z_2, \bar{z}_2) + h_c^{ab}(\lambda) \partial J^c(z_2, \bar{z}_2) \\
&\quad + \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} t_{cd}^{ab}(\lambda) : J^c \bar{J}^d : (z_2, \bar{z}_2) + \frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} u_c^{ab}(\lambda) \bar{J}^c(z_2, \bar{z}_2) \\
&\quad + \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} v_c^{ab}(\lambda) \bar{\partial} \bar{J}^c(z_2, \bar{z}_2) + \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} w_{cd}^{ab}(\lambda) : \bar{J}^c \bar{J}^d : (z_2, \bar{z}_2),
\end{aligned} \tag{4.6}$$

where the coefficients g and w are symmetric in the lower indices:

$$g_{cd}^{ab}(\lambda) = (-1)^{cd} g_{dc}^{ab}(\lambda), \quad w_{cd}^{ab}(\lambda) = (-1)^{cd} w_{dc}^{ab}(\lambda). \tag{4.7}$$

The explicit form of the coordinate dependence could be in principle modified by logarithms but explicit computations below show that this does not happen to first order in λ .

Further constraints on the OPE coefficients are obtained by exchanging the order of the two currents in the OPE and re-expanding around $z_1 = z_2$. Working out the details we find

$$\kappa^{ab}(\lambda) = (-1)^{ab} \kappa^{ba}(\lambda) \qquad f^{ab}_c(\lambda) = -(-1)^{ab} f^{ba}_c(\lambda) \qquad (4.8)$$

$$g_{cd}^{ab}(\lambda) = (-1)^{ab} g_{cd}^{ba}(\lambda) \qquad h_c^{ab}(\lambda) = (-1)^{ab} h_c^{ba}(\lambda) + i f^{ab}_c(\lambda) \qquad (4.9)$$

$$u_c^{ab}(\lambda) = -(-1)^{ab} u_c^{ba}(\lambda) \qquad v_c^{ab}(\lambda) = (-1)^{ab} v_c^{ba}(\lambda) \qquad (4.10)$$

$$w_{cd}^{ab}(\lambda) = (-1)^{ab} w_{cd}^{ba}. \qquad (4.11)$$

More interestingly, we also obtain the equation

$$\left[t_{cd}^{ab}(\lambda) - (-1)^{ab} t_{cd}^{ba}(\lambda) \right] : J^c \bar{J}^d : = u_c^{ab}(\lambda) \partial \bar{J}^c. \qquad (4.12)$$

When considering the OPE $\bar{J}^a(z_1, \bar{z}_1) \bar{J}^b(z_2, \bar{z}_2)$, a similar equation is obtained for $\bar{\partial} J^c$. Additional constraints may arise from the associativity of the OPE. Since we will determine the coefficients perturbatively using an underlying Lagrangian description, associativity should automatically be satisfied.

It remains to write down the undeformed values for the structure constants as they appear in the WZW model. In the conventions chosen for the ansatz (4.6), the non-trivial values are

$$\kappa^{ab}(0) = k \kappa^{ab}, \quad f^{ab}_c(0) = f^{ab}_c, \quad g_{cd}^{ab}(0) = \frac{1}{2} \left[\delta_c^a \delta_d^b + \delta_d^a \delta_c^b \right], \quad h_c^{ab}(0) = \frac{i}{2} f^{ab}_c. \qquad (4.13)$$

All these relations are straightforward to see except for the last two, which follow from

$$: J^a J^b : = \frac{1}{2} : [J^a J^b + J^b J^a] : + \frac{1}{2} : [J^a J^b - J^b J^a] : = \frac{1}{2} \left[\delta_c^a \delta_d^b + \delta_d^a \delta_c^b \right] : J^c J^d : + \frac{i}{2} f^{ab}_c \partial J^c. \qquad (4.14)$$

4.2.2 Further details

Starting from the ansatz (4.6) for the deformed OPE we will now successively determine the individual coefficients using the action principle (4.2). For later convenience and in order to enable a systematic evaluation of the individual contributions we introduce a number of abbreviations. For the first term on the left hand side of eq. (4.2) we use the symbol $A(X)$,

$$A(X) = \kappa_{fe} \left\langle J^a(z_1) J^b(z_2) X(\dots) \frac{1}{\pi k} \int d^2 z : J^e \bar{J}^f : (z, \bar{z}) \right\rangle_0. \qquad (4.15)$$

For the terms originating from the unperturbed OPE we use the symbols $B_i(X)$

$$B_1(X) = \kappa_{fe} \left\langle \frac{k \kappa^{ab}}{(z_1 - z_2)^2} X(\dots) \frac{1}{\pi k} \int d^2 z : J^e \bar{J}^f : (z, \bar{z}) \right\rangle_0, \qquad (4.16)$$

$$B_2(X) = \frac{i f^{ab}_d}{z_1 - z_2} \kappa_{fe} \left\langle J^d(z_2) X(\dots) \frac{1}{\pi k} \int d^2 z : J^e \bar{J}^f : (z, \bar{z}) \right\rangle_0, \qquad (4.17)$$

$$B_3(X) = \kappa_{fe} \left\langle : J^a J^b : (z_2) X(\dots) \frac{1}{\pi k} \int d^2 z : J^e \bar{J}^f : (z, \bar{z}) \right\rangle_0. \qquad (4.18)$$

Here we assume $X(\dots)$ to be a fixed multi-local field.

Determination of κ , f , h and g . We start by checking that $\kappa^{ab}(\lambda)$ does not receive any corrections at first order in perturbation theory (it does so at higher orders, cf. our exact result in eq. (3.3)). Indeed, one can easily check that $A(X) = B_i(X) = 0$ if one chooses $X = 1$. Consistency then requires $\partial_\lambda k^{ab}(0) = 0$. Similar remarks apply to the coefficients $f_c^{ab}(\lambda)$ and $h_c^{ab}(\lambda)$ which both remain undeformed up to $\mathcal{O}(\lambda^2)$. In this case one has to choose $X(\xi) = J^c(\xi)$.

It is also easy to argue that the coefficient g_{cd}^{ab} must not receive any corrections at this order. For that purpose we only note that the operator content of

$$J^a(z_1)J^b(z_2) \int d^2z :J^e \bar{J}^f : (z, \bar{z}) \kappa_{fe} \quad (4.19)$$

must contain operators with a non-trivial \bar{J} component.⁷

A more formal proof of this statement would require more information on the supergroup. Here we briefly point out the problem. In order to determine g it is natural to choose $X(\xi) =: J^c J^d : (\xi)$. The resulting correlators for $A(X)$ and $B_i(X)$ contain precisely one insertion of \bar{J} , implying that $A(X) = B_i(X) = 0$. This implies

$$\partial_\lambda g_{rs}^{ab}(0) \left\langle :J^r J^s : (z_2) :J^c J^d : (\xi) \right\rangle = \frac{\partial_\lambda g_{rs}^{ab}(0)}{(z_2 - \xi)^4} \left\{ k^2 (-1)^{rs} \kappa^{rc} \kappa^{sd} + k^2 \kappa^{rd} \kappa^{sc} \right. \quad (4.20)$$

$$\left. - 2k (-1)^{rs} f^{rc} f^{sed} - k (-1)^{r(s+c)} f^{rd} f^{sce} \right\} = 0. \quad (4.21)$$

One may worry that there exists a non-vanishing tensor such that the contraction above yields zero. However, even if this does happen this merely means that the X we chose was not a good choice to determine $g_{rs}^{ab}(0)$ and we have to consider other X 's. The presence of a non-trivial kernel for the above 4-tensor depends on a particular group. As long as the states $(J^r J^s + J^s J^r) :$ are not of zero norm there will be another X with a nonzero overlap which by the above simple argument must detect $g_{rs}^{ab}(0) = 0$. We will not pursue this more formal line of argument any further.

Determination of u and v . In the next step we select $X(\bar{\xi}) = \bar{J}^c(\bar{\xi})$. This will allow us to determine the coefficients u and v . A straightforward calculation using the explicit form of the undeformed two- and three-point functions as well as a decomposition into partial fractions yields

$$A(X) = \frac{1}{\pi k} \kappa_{fe} (-1)^{ce} \int d^2z \frac{-ik f^{abe}}{(z_1 - z_2)(z_2 - z)(z - z_1)} \frac{k \kappa^{cf}}{(\bar{\xi} - \bar{z})^2} \quad (4.22)$$

$$= -\frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} ik f^{abc} \frac{1}{(\bar{\xi} - \bar{z}_1)(\bar{\xi} - \bar{z}_2)}. \quad (4.23)$$

Here the integral was evaluated using formulas (B.12)-(B.14). Later we will also need the expansion of this expression in terms of inverse powers of $\bar{\xi}$,

$$A(X) = -\frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} ik f^{abc} \frac{1}{\bar{\xi}^2} \left[1 + \frac{\bar{z}_1 + \bar{z}_2}{\bar{\xi}} + \dots \right]. \quad (4.24)$$

One also finds $B_i(X) = 0$ here, even though this time it is a result of integration. The total contribution that has to be matched is thus given by $A(X)$ itself.

⁷It may look like a contact term between J^a and \bar{J}^b proportional to a delta function can spoil this argument, but as formula (B.4) shows such contact terms are of order λ and can thus be discarded at the leading order.

According to eq. (4.2) this result should be compared to the unperturbed correlation functions. With the present choice $X = \bar{J}^c$, the most singular contribution arises from

$$\frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} \partial_\lambda u_d^{ab}(0) \left\langle \bar{J}^d(\bar{z}_2) \bar{J}^c(\bar{\xi}) \right\rangle = \frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} \partial_\lambda u_d^{ab}(0) \frac{k\kappa^{dc}}{(\bar{z}_2 - \bar{\xi})^2} \quad (4.25)$$

$$= \frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} k\kappa^{dc} \partial_\lambda u_d^{ab}(0) \frac{1}{\xi^2} \left[1 + \frac{2\bar{z}_2}{\xi} + \dots \right]. \quad (4.26)$$

Comparison of the leading terms yields

$$u_c^{ab}(\lambda) = -i\lambda f_c^{ab} + \mathcal{O}(\lambda^2). \quad (4.27)$$

While the leading contribution allows to determine u , the subleading contribution provides enough information to calculate v . Plugging our finding for u back into eq. (4.2) leads to the expression

$$A(X) + \frac{\bar{z}_1 - \bar{z}_2}{(z_1 - z_2)^2} \partial_\lambda u_d^{ab}(0) \left\langle \bar{J}^d(\bar{z}_2) \bar{J}^c(\bar{\xi}) \right\rangle = -\frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \frac{1}{\xi^3} ikf^{abc} + \dots, \quad (4.28)$$

which has to be matched by a linear combination of the following two terms:

$$-\frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \partial_\lambda v_d^{ab}(0) \left\langle \bar{\partial} \bar{J}^d(\bar{z}_2) \bar{J}^c(\bar{\xi}) \right\rangle = \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \partial_\lambda v_d^{ab}(0) \frac{2k\kappa^{dc}}{(\bar{z}_2 - \bar{\xi})^3} \quad (4.29)$$

$$= -\frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \frac{1}{\xi^3} 2k\kappa^{dc} \partial_\lambda v_d^{ab}(0) \left[1 + \dots \right], \quad (4.30)$$

$$-\frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \partial_\lambda w_{de}^{ab}(0) \left\langle : \bar{J}^d \bar{J}^e : (\bar{z}_2) \bar{J}^c(\bar{\xi}) \right\rangle = \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \partial_\lambda w_{de}^{ab}(0) \frac{ikf^{dec}}{(\bar{z}_2 - \bar{\xi})^3} = 0. \quad (4.31)$$

Again the last term vanishes since we assumed w to be antisymmetric in the lower indices. We can therefore easily solve for $v(\lambda)$, obtaining

$$v_c^{ab}(\lambda) = -\frac{i\lambda}{2} f_c^{ab} + \mathcal{O}(\lambda^2). \quad (4.32)$$

Determination of t and w . These coefficients are determined in a similar fashion. In order to streamline the presentation in the main text, these calculations have been moved to appendix C.

4.3 The OPE of J with \bar{J}

The next goal is to determine the mixed OPE between J and \bar{J} . In the undeformed theory this OPE vanishes identically. However, the calculations below imply that the deformed OPE acquires a correction that is given by

$$J^a(z_1, \bar{z}_1) \bar{J}^b(z_2, \bar{z}_2) = \frac{i\lambda f_c^{ab}}{z_1 - z_2} \bar{J}^c(z_2, \bar{z}_2) + \frac{i\lambda f_c^{ab}}{\bar{z}_1 - \bar{z}_2} J^c(z_2, \bar{z}_2) \quad (4.33)$$

$$- \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} i\lambda f_c^{ab} \partial J^c(z_2, \bar{z}_2) \quad (4.34)$$

$$- (-1)^{fs} \frac{\lambda}{k} \kappa_{ef} f^{ae} f^{bf} \ln \frac{|z_1 - z_2|^2}{\epsilon^2} : J^c \bar{J}^d : (z_2, \bar{z}_2) + \mathcal{O}(\lambda^2). \quad (4.35)$$

at first order in the coupling constant. Here, the constant ϵ in the logarithmic contribution is a UV regulator which can be thought of as a normal ordering ambiguity. We start with discussing the general ansatz for the OPE $J^a(z_1, \bar{z}_1) \bar{J}^b(z_2, \bar{z}_2)$. Instead of showing all the individual steps leading to our result, we then present the calculation of three coefficients in some detail. The determination of the remaining ones can be found in appendix C.

4.3.1 General ansatz and initial conditions

As in the previous section we start with a general ansatz which in this case reduces to

$$J^a(z_1, \bar{z}_1) \bar{J}^b(z_2, \bar{z}_2) = \frac{A^{ab}(\lambda)}{|z_1 - z_2|^2} + \frac{1}{z_1 - z_2} B_c^{ab}(\lambda) \bar{J}^c(z_2, \bar{z}_2) + \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \bar{B}_c^{ab}(\lambda) \bar{\partial} \bar{J}^c(z_2, \bar{z}_2) \quad (4.36)$$

$$+ \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \hat{B}_{cd}^{ab}(\lambda) : \bar{J}^c \bar{J}^d : (z_2, \bar{z}_2) + \frac{1}{\bar{z}_1 - \bar{z}_2} C_c^{ab}(\lambda) J^c(z_2, \bar{z}_2) \quad (4.37)$$

$$+ \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \tilde{C}_c^{ab}(\lambda) \partial J^c(z_2, \bar{z}_2) + \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \hat{C}_{cd}^{ab}(\lambda) : J^c J^d : (z_2, \bar{z}_2) \quad (4.38)$$

$$+ D_{cd}^{ab}(\lambda) : J^c \bar{J}^d : (z_2, \bar{z}_2). \quad (4.39)$$

Without loss of generality we assume that the coefficients $\hat{B}_{cd}^{ab}(\lambda)$ and $\hat{C}_{cd}^{ab}(\lambda)$ are symmetric in the lower indices, i.e.

$$\hat{B}_{cd}^{ab}(\lambda) = (-1)^{cd} \hat{B}_{dc}^{ab}(\lambda), \quad \hat{C}_{cd}^{ab}(\lambda) = (-1)^{cd} \hat{C}_{dc}^{ab}(\lambda). \quad (4.40)$$

It should also be noted that the explicit coordinate dependence only reflects the contributions needed to account for the proper scaling dimension of the terms on the right hand side. The coefficients may carry an additional implicit logarithmic (and hence dimensionless) coordinate dependence. Indeed, we will soon recognise that such logarithmic contributions arise in the coefficient $D_{cd}^{ab}(\lambda)$.

In the current setting, the initial conditions at $\lambda = 0$ are almost trivial since $J(z)$ and $\bar{J}(w)$ commute in the undeformed theory. We only need to keep track of the non-singular term

$$J^a(z_1) \bar{J}^b(\bar{z}_2) = : J^a \bar{J}^b : (z_2, \bar{z}_2) + \dots \quad (4.41)$$

Consequently we have $D_{cd}^{ab}(0) = \delta_c^a \delta_d^b$, with all other coefficients vanishing.

4.3.2 Further details

The general procedure outlined above instructs us to determine and to compare the following two quantities for suitable choices of the field $X(\dots)$. The first one is

$$A(X) = \kappa_{fe} \left\langle J^a(z_1) \bar{J}^b(z_2) X(\dots) \frac{1}{k\pi} \int d^2 z J^e \bar{J}^f(z, \bar{z}) \right\rangle_0 \quad (4.42)$$

and the second one is

$$B(X) = \lim_{:z_1 \rightarrow z_2:} A(X) = \text{Res}_{z_1 \rightarrow z_2} \frac{A(X)}{z_1 - z_2}, \quad (4.43)$$

that corresponds to the non-singular part of $A(X)$ as z_1 approaches z_2 . Practically, the limit extracts the constant part if $A(X)$ is considered as a Laurent series in $z_1 - z_2$.

Determination of A^{ab} . The result $A^{ab}(\lambda) = 0$ for the central term follows immediately – and even to all orders – from our knowledge of the exact two-point function (3.3). It can also be understood using the general framework using $X = 1$.

Determination of D_{cd}^{ab} . From a physical perspective, the most interesting coefficient in the ansatz (4.36) is certainly $D_{cd}^{ab}(\lambda)$ since it turns out to contain logarithmic contributions. This coefficient can be determined by setting $X(\xi, \bar{\xi}) = : J^c \bar{J}^d : (\xi, \bar{\xi})$. We first evaluate

$$A(X) = (-1)^{bc} \frac{k \kappa_{ef} f^{ace} f^{bdf}}{(z_1 - \xi)^2 (\bar{z}_2 - \bar{\xi})^2} \ln \frac{\epsilon^2 |z_1 - z_2|^2}{|\xi - z_2|^2 |z_1 - \xi|^2}. \quad (4.44)$$

During the calculation we followed the standard recipe of replacing $(\xi - \bar{\xi})^2$ by a regulator ϵ^2 . In a similar fashion we then evaluate

$$B(X) = \lim_{:z_1 \rightarrow z_2:} A(X) = (-1)^{bc} \frac{k\kappa_{ef} f^{ace} f^{bdf}}{|z_2 - \xi|^4} \ln \frac{\epsilon^4}{|z_2 - \xi|^4}. \quad (4.45)$$

For large values of the variable ξ one obtains

$$A(X) - B(X) = (-1)^{bc} \frac{k\kappa_{ef} f^{ace} f^{bdf}}{|\xi|^4} \ln \frac{|z_1 - z_2|^2 |z_2 - \xi|^4}{\epsilon^2 |\xi - z_2|^2 |z_1 - \xi|^2} + \dots \quad (4.46)$$

$$= (-1)^{bc} \frac{k\kappa_{ef} f^{ace} f^{bdf}}{|\xi|^4} \ln \frac{|z_1 - z_2|^2}{\epsilon^2} + \dots. \quad (4.47)$$

This difference has to be compared to

$$-\partial_\lambda D_{rs}^{ab}(0) \left\langle J^r(z_2) \bar{J}^s(\bar{z}_2) : J^c \bar{J}^d : (\xi, \bar{\xi}) \right\rangle = -\frac{k^2 (-1)^{cs} \kappa^{rc} \kappa^{sd}}{(z_2 - \xi)^2 (\bar{z}_2 - \bar{\xi})^2} \partial_\lambda D_{rs}^{ab}(0) \quad (4.48)$$

$$= -k^2 (-1)^{cs} \kappa^{rc} \kappa^{sd} \partial_\lambda D_{rs}^{ab}(0) \frac{1}{|\xi|^4} \left[1 + \dots \right]. \quad (4.49)$$

The comparison yields a logarithmic dependence of the structure constants on the difference $z_1 - z_2$,

$$D_{rs}^{ab}(\lambda) = -(-1)^{fs} \frac{\lambda}{k} \kappa_{ef} f^{ae} f^{bf} f_s \ln \frac{|z_1 - z_2|^2}{\epsilon^2} + \mathcal{O}(\lambda^2). \quad (4.50)$$

The presence of the regulator ϵ can be interpreted as a normal ordering ambiguity ⁸.

Determination of C_c^{ab} and \tilde{C}_c^{ab} . We next explain how to determine the coefficients C and \tilde{C} . Both can be obtained from one single calculation using the choice $X = J^c$. Like before we first evaluate

$$A(X) = -\frac{ikf^{abc}}{(z_1 - \xi)^2} \left[\frac{1}{\bar{z}_1 - \bar{z}_2} - \frac{1}{\bar{\xi} - \bar{z}_2} \right] = -\frac{ikf^{abc}}{\xi^2} \left[\frac{1}{\bar{z}_1 - \bar{z}_2} + \frac{2z_1}{(\bar{z}_1 - \bar{z}_2)\xi} - \frac{1}{\bar{\xi}} + \dots \right]. \quad (4.51)$$

Taking the limit $:z_1 \rightarrow z_2:$ leads to the expression

$$B(X) = \frac{ikf^{abc}}{(z_2 - \xi)^2 (\bar{\xi} - \bar{z}_2)} = \frac{ikf^{abc}}{\xi^2 \bar{\xi}} \left[1 + \frac{2z_2}{\xi} + \frac{\bar{z}_2}{\bar{\xi}} + \dots \right]. \quad (4.52)$$

Putting these together we obtain

$$A(X) - B(X) = -\frac{ikf^{abc}}{(\bar{z}_1 - \bar{z}_2)\xi^2} \left[1 + \frac{2z_1}{\xi} + \dots \right]. \quad (4.53)$$

The leading term in this expression can be accounted for by the term

$$-\frac{1}{\bar{z}_1 - \bar{z}_2} \partial_\lambda C_d^{ab}(0) \left\langle J^d(z_2) J^c(\xi) \right\rangle = -\frac{1}{\bar{z}_1 - \bar{z}_2} \frac{k\kappa^{dc}}{(z_2 - \xi)^2} \partial_\lambda C_d^{ab}(0) \quad (4.54)$$

$$= -\frac{k\kappa^{dc} \partial_\lambda C_d^{ab}(0)}{\bar{z}_1 - \bar{z}_2} \frac{1}{\xi^2} \left[1 + \frac{2z_2}{\xi} + \dots \right]. \quad (4.55)$$

⁸At the level of correlators in the bare theory this is an ambiguity in defining distributional three-point functions $\langle : J^a \bar{J}^b : : J^c \bar{J}^d : : J^e \bar{J}^f : \rangle_{0\kappa_{fe}}$. Such distributions are not considered in appendix B. This can be considered as a further ambiguity in defining the normal ordering in the deforming operator.

A comparison of the most singular terms yields

$$C_c^{ab}(\lambda) = i\lambda f_c^{ab}. \quad (4.56)$$

In order to fix the subleading contributions we analyse

$$A(X) - B(X) + \frac{\partial_\lambda C_d^{ab}(0)}{\bar{z}_1 - \bar{z}_2} \langle J^d(z_2) J^c(\xi) \rangle = -\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \frac{1}{\xi^3} 2ik f^{abc} [1 + \dots]. \quad (4.57)$$

There are in principle two different correlation functions which could give rise to such a contribution,

$$-\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \partial_\lambda \tilde{C}_d^{ab}(0) \langle \partial J^d(z_2) J^c(\xi) \rangle = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \partial_\lambda \tilde{C}_d^{ab}(0) \frac{2k\kappa^{dc}}{(z_2 - \xi)^3} \quad (4.58)$$

$$-\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \partial_\lambda \hat{C}_{de}^{ab}(0) \langle :J^d J^e: (z_2) J^c(\xi) \rangle = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \partial_\lambda \hat{C}_{de}^{ab}(0) \frac{ik f^{dec}}{(z_2 - \xi)^3} = 0. \quad (4.59)$$

The latter vanishes due to our symmetry assumptions. Hence we find

$$\tilde{C}_c^{ab}(\lambda) = -i\lambda f_c^{ab} + \mathcal{O}(\lambda^2). \quad (4.60)$$

Determination of the other coefficients. The computational details related to the remaining OPE coefficients are put into appendix C.

4.4 Exact OPE coefficients

The exact two- and three-point functions of the currents obtained in section 3 allow us to compute the OPE coefficients $\kappa^{ab}(\lambda)$, $f_c^{ab}(\lambda)$, $u_c^{ab}(\lambda)$ from (4.6) and $A^{ab}(\lambda)$, $B_c^{ab}(\lambda)$, $C_c^{ab}(\lambda)$ from (4.36) to all orders in λ . We obtain

$$\kappa^{ab}(\lambda) = \frac{\kappa^{ab}}{1 - \lambda^2}, \quad A^{ab}(\lambda) = 0, \quad f_c^{ab}(\lambda) = \frac{1 - \lambda^3}{(1 - \lambda^2)^2} f_c^{ab}, \quad (4.61)$$

$$u_c^{ab}(\lambda) = C_c^{ab}(\lambda) = B_c^{ab}(\lambda) = i f_c^{ab} \frac{\lambda(1 - \lambda)}{(1 - \lambda^2)^2}. \quad (4.62)$$

5 Four-point function of currents in a $1/k$ expansion

In this section we will use the methods of section 3 to obtain an approximation to a four-point function of the currents. Consider the perturbation series expansion

$$\begin{aligned} A_4(\lambda, k) &= \left\langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) \right\rangle_\lambda \\ &= \left\langle J^a(z_1) J^b(z_2) J^c(z_3) J^d(z_4) \exp\left(-\frac{\lambda}{k\pi} \int d^2w J^e \bar{J}^r \kappa_{re}\right) \right\rangle_0. \end{aligned} \quad (5.1)$$

Every contraction of the bare currents in the perturbative integrals either comes with the metric multiplied by the level k or with the structure constants. Thus the number of the structure constants appearing in the perturbative expansion effectively measures the power of k . For fixed λ we have an expansion in powers of $1/k$ of the form

$$A_4(\lambda, k) = k^2 \left[A_4^{(0)}(\lambda) + \frac{1}{k} A_4^{(1)}(\lambda) + \dots + \frac{1}{k^p} A_4^{(p)}(\lambda) + \dots \right], \quad (5.2)$$

where the term $A^{(p)}(\lambda)$ comes from all terms in the perturbation series with $2p$ structure constant contractions. We compute the functions $A_4^{(0)}(\lambda)$ and $A_4^{(1)}(\lambda)$ to all orders in λ . In this section we sketch the main steps in the computation relegating more details to Appendix D.

We begin by extracting all singularities of the bare current $J^a(z_1)$ in the perturbative integrands. As in section 3 contractions with the perturbing operators are rearranged into integrated correlators in the deformed theory. Using (3.3), (3.11), (3.14) we obtain

$$A_4 = a_4 - \left(\frac{\lambda}{k\pi}\right) \int \frac{d^2w}{z_1 - w} \left\langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) i f^a_{re} : J^e \bar{J}^r : (w, \bar{w}) \right\rangle_{\lambda} (-1)^{a(b+c+d)} \\ - \left(\frac{\lambda}{\pi}\right) \int \frac{d^2w}{(z_1 - w)^2} \left\langle J^b(z_2) J^c(z_3) J^d(z_4) \bar{J}^a(\bar{w}) \right\rangle_{\lambda} (-1)^{a(b+c+d)} \quad (5.3)$$

where

$$a_4 = \frac{k^2}{1 - \lambda^2} \left[\frac{\kappa^{ab} \kappa^{cd}}{z_{12}^2 z_{34}^2} + \frac{(-1)^{ab} \kappa^{ac} \kappa^{bd}}{z_{13}^2 z_{24}^2} + \frac{\kappa^{ad} \kappa^{bc}}{z_{14}^2 z_{23}^2} \right] \\ + \frac{1 - \lambda^3}{(1 - \lambda^2)^3} \left[\frac{f^{ab} f^{scd}}{z_{12}} + \frac{(-1)^{ab} f^{ac} f^{bsd}}{z_{13}} + \frac{(-1)^{a(b+c)} f^{ad} f^{bcs}}{z_{14}} \right] \frac{k}{z_{23} z_{34} z_{42}}. \quad (5.4)$$

We next extract the singularities of $\bar{J}^a(\bar{w})$ in the second term on the right hand side of (5.3). This way we obtain

$$(1 - \lambda^2) A_4 = a_4 + (1 - \lambda) \left(\frac{-\mu}{2\pi}\right) (-1)^{a(b+c+d)} \\ \times \int \frac{d^2w}{z_1 - w} i f^a_{re} \left\langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^e \bar{J}^r : (w, \bar{w}) \right\rangle_{\lambda}. \quad (5.5)$$

Notice that a factor of the structure constants stands in front of the second term on the right hand side of the above expression. Thus, in our approximation, we need to compute all perturbative terms with a single contraction $JJ \rightarrow J$ in the correlation function inside the integral. Extracting singularities of $J^b(z_2)$ in that correlator we obtain terms all containing a single factor of the structure constant except for the contribution

$$- (-1)^{b(c+d+e+r)} \left(\frac{\lambda}{\pi}\right) \int \frac{d^2w_2}{(z_2 - w_2)^2} \left\langle J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^e \bar{J}^r : (w, \bar{w}) \bar{J}^b(w_2, \bar{w}_2) \right\rangle_{\lambda} \quad (5.6)$$

that comes from the abelian contraction of $J^b(z_2)$ with perturbing operators. Extracting the singularities of $\bar{J}^b(\bar{w}_2)$ in the above correlator gives terms which are explicitly evaluated by the methods of Appendix B plus a term proportional to

$$\lambda^2 \left\langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^e \bar{J}^r : (w, \bar{w}) \right\rangle_{\lambda} \quad (5.7)$$

which closes the system of equations. Collecting all contributions we obtain

$$(1 - \lambda^2) \left\langle J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^e \bar{J}^r : (w, \bar{w}) \right\rangle_{\lambda} = \sum_{i=1}^7 a^{(i)} \quad (5.8)$$

where

$$a^{(1)} = (-1)^{b(c+d)} \frac{k\kappa^{be}}{(z_2 - w)^2} \langle J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) \bar{J}^r(w, \bar{w}) \rangle_\lambda, \quad (5.9)$$

$$a^{(2)} = (-1)^{b(c+d+e+r)} k\pi(-\lambda) \kappa^{rb} \delta(z_2 - w) \langle J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) J^e(w, \bar{w}) \rangle_\lambda, \quad (5.10)$$

$$a^{(3)} = (-1)^{b(c+d)} \frac{i f^{be}}{z_2 - w} \langle J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^s \bar{J}^r : (w, \bar{w}) \rangle_\lambda, \quad (5.11)$$

$$a^{(4)} = \lambda(-1)^{b(c+d+e)} \frac{i f^{br}}{z_2 - w} \langle J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^e \bar{J}^s : (w, \bar{w}) \rangle_\lambda, \quad (5.12)$$

$$a^{(5)} = (1 - \lambda) \left(\frac{-\mu}{2\pi} \right) (-1)^{b(c+d+e+r)} i f_{qs}^{bs} \int \frac{d^2 w_2}{z_2 - w_2} \langle J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^e \bar{J}^r : (w, \bar{w}) : J^s \bar{J}^q : (w_2, \bar{w}_2) \rangle_\lambda, \quad (5.13)$$

$$a^{(6)} = \frac{i f^{bc}}{z_{23}} \langle J^s(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) : J^e \bar{J}^r : (w, \bar{w}) \rangle_\lambda, \quad (5.14)$$

$$a^{(7)} = \frac{i(-1)^{bc} f^{bd}}{z_{24}} \langle J^c(z_3, \bar{z}_3) J^s(z_4, \bar{z}_4) : J^e \bar{J}^r : (w, \bar{w}) \rangle_\lambda. \quad (5.15)$$

The first two terms on the right hand side of (5.8) can be calculated using the exact three-point functions. The remaining terms all contain a factor of the structure constants. In our approximation they can be computed using abelian perturbation theory. Substituting the result into (5.5) we finally obtain

$$\begin{aligned} \langle J^a(z_1, \bar{z}_1) J^b(z_2, \bar{z}_2) J^c(z_3, \bar{z}_3) J^d(z_4, \bar{z}_4) \rangle_\lambda &= \frac{k^2}{(1 - \lambda^2)^2} \left[\frac{\kappa^{ab} \kappa^{cd}}{z_{12}^2 z_{34}^2} + \frac{(-1)^{ab} \kappa^{ac} \kappa^{bd}}{z_{13}^2 z_{24}^2} + \frac{\kappa^{ad} \kappa^{bc}}{z_{14}^2 z_{23}^2} \right] \\ &+ \frac{1 - \lambda^4}{(1 - \lambda^2)^4} \left[\frac{f_{s1}^{ab} f_{s2}^{cd}}{z_{12}} + \frac{(-1)^{ab} f_{s1}^{ac} f_{s2}^{bd}}{z_{13}} + \frac{(-1)^{a(b+c)} f_{s1}^{ad} f_{s2}^{bc}}{z_{14}} \right] \frac{k}{z_{23} z_{34} z_{42}} \\ &+ k \frac{\lambda^2 (1 - \lambda)^2}{(1 - \lambda^2)^5} \left[-f_{r1}^{ab} f_{r2}^{cd} \frac{1}{z_{34}^2 z_{12}^2} \ln \left| \frac{z_{13} z_{24}}{z_{23} z_{14}} \right|^2 + (-1)^{a(b+c)} f_{r1}^{ad} f_{r2}^{bc} \frac{1}{z_{14}^2 z_{23}^2} \ln \left| \frac{z_{24} z_{13}}{z_{12} z_{34}} \right|^2 \right. \\ &\left. - (-1)^{ab} f_{r1}^{ac} f_{r2}^{bd} \frac{1}{z_{13}^2 z_{24}^2} \ln \left| \frac{z_{23} z_{14}}{z_{12} z_{34}} \right|^2 \right] + \mathcal{O}(k^0) \end{aligned} \quad (5.16)$$

The above result is crossing symmetric and exhibits logarithms as expected for this type of theories.

Note also that the form of the terms that we obtained does not depend on the particular supergroup we have. This will not be so for further terms in the expansion (5.2) in which we would need to use specific group theoretic identities reducing combinations of four structure constants to those built from two structure constants and the metric.

It is possible to extend the above computation to obtaining higher order corrections $A_4^{(p)}$ as well as to obtaining approximations to four-point functions of the form $\langle J^a J^b J^c \bar{J}^d \rangle_\lambda$ and $\langle J^a J^b \bar{J}^c \bar{J}^d \rangle_\lambda$. However, with the above computation being already quite laborious, it is clear that the method we use quickly stops being efficient. One has to search for more sophisticated methods, perhaps using integrability techniques.

6 Equal time commutators

The OPE for the currents which we analysed in the previous sections appears to have quite a complex structure. One may hope to reveal a simpler structure looking at the equal time commutators of the

currents. This indeed turns out to be the case as we show below.

The equal time commutator algebra of the currents can be obtained from the most singular terms in the current's OPEs via the Bjorken-Johnson-Low limit

$$[J_\mu^a(\sigma_1, 0), J_\nu^b(\sigma_2, 0)] = \lim_{\epsilon \rightarrow 0} \left(J_\mu^a(\sigma_1, i\epsilon) J_\nu^b(\sigma_2, 0) - J_\nu^b(\sigma_2, i\epsilon) J_\mu^a(\sigma_1, 0) \right) \quad (6.1)$$

Using the exact OPE coefficients (4.61) and (4.62) we obtain

$$\begin{aligned} [J^a(\sigma_1, 0), J^b(\sigma_2, 0)] &= 2\pi i \left\{ -\frac{k\kappa^{ab}}{1-\lambda^2} \delta'(\sigma_1 - \sigma_2) + i f^{ab}{}_c \delta(\sigma_1 - \sigma_2) (F_1(\lambda) J^c(\sigma_2, 0) + F_2(\lambda) \bar{J}^c(\sigma_2, 0)) \right\} \\ [J^a(\sigma_1, 0), \bar{J}^b(\sigma_2, 0)] &= 2\pi f^{ab}{}_c \delta(\sigma_1 - \sigma_2) F_2(\lambda) [J^c(\sigma_2, 0) - \bar{J}^c(\sigma_2, 0)] \end{aligned} \quad (6.2)$$

where

$$F_1(\lambda) = \frac{1-\lambda^3}{(1-\lambda^2)^2}, \quad F_2(\lambda) = \frac{\lambda}{(1+\lambda)(1-\lambda^2)}. \quad (6.3)$$

Compactifying the spatial direction on a circle: $\sigma \sim \sigma + 2\pi$ we introduce the Fourier modes

$$J^a(\sigma, \tau) = i \sum_{n \in \mathbb{Z}} e^{-in\sigma} J_n^a(\tau), \quad \bar{J}^a(\sigma, \tau) = -i \sum_{n \in \mathbb{Z}} e^{-in\sigma} \bar{J}_n^a(\tau). \quad (6.4)$$

For these modes we obtain the equal time commutation relations (ETC)

$$\begin{aligned} [J_n^a(\tau), J_m^b(\tau)] &= \frac{k}{1-\lambda^2} \kappa^{ab} n \delta_{n,-m} + i f^{ab}{}_c [F_1(\lambda) J_{n+m}^c(\tau) - F_2(\lambda) \bar{J}_{n+m}^c(\tau)], \\ [\bar{J}_n^a(\tau), \bar{J}_m^b(\tau)] &= -\frac{k}{1-\lambda^2} \kappa^{ab} n \delta_{n,-m} + i f^{ab}{}_c [F_1(\lambda) \bar{J}_{n+m}^c(\tau) - F_2(\lambda) J_{n+m}^c(\tau)], \\ [J_n^a(\tau), \bar{J}_m^b(\tau)] &= i f^{ab}{}_c F_2(\lambda) [\bar{J}_{n+m}^c(\tau) + J_{n+m}^c(\tau)]. \end{aligned} \quad (6.5)$$

In terms of the modes

$$l_n^a(\tau) = J_n^a(\tau) - \lambda \bar{J}_n^a(\tau), \quad r_n^a(\tau) = \bar{J}_n^a(\tau) - \lambda J_n^a(\tau) \quad (6.6)$$

the ETC algebra takes the form

$$[l_n^a(\tau), l_m^b(\tau)] = k \kappa^{ab} n \delta_{n,-m} + i f^{ab}{}_c l_{n+m}^c(\tau), \quad (6.7)$$

$$[r_n^a(\tau), r_m^b(\tau)] = -k \kappa^{ab} n \delta_{n,-m} + i f^{ab}{}_c r_{n+m}^c(\tau), \quad (6.8)$$

$$[r_n^a(\tau), l_m^b(\tau)] = 0. \quad (6.9)$$

We see that the phase space of our model is isomorphic to two commuting copies of the affine current algebra with opposite central extensions. We hope that this simple result will be useful in the further analysis of the model. It is interesting to notice that the phase space of a principal chiral model has exactly the same description (see [33]). The Hamiltonian governing the τ -evolution of the modes is however different. It is derived in the next section.

7 Equations of motion

At the WZW point the equations of motion are the conditions for the (anti-)holomorphicity of the current components: $\partial \bar{J}^a = \bar{\partial} J^a = 0$. In the perturbed theory these equations get deformed. Since the perturbing operator is made only of the currents and the vacuum sector closes on itself via OPEs we

expect the additional term in the deformed equations of motion to be built from the currents. Based on the current conservation, spin conservation and for dimensional reasons the deformed equations must be of the form

$$\bar{\partial}J^a(z, \bar{z}) = -\partial\bar{J}^a(z, \bar{z}) = iG(\lambda)t^a_{bc} : J^c \bar{J}^b : (z, \bar{z}) \quad (7.1)$$

where t^a_{bc} is some invariant group tensor and $G(\lambda)$ is some function of the coupling constant. For the supergroups at hand the structure constants give a unique invariant three tensor so that we can set $t^a_{bc} = f^a_{bc}$. The function $G(\lambda)$ in general depends on the particular definition of the normal ordering in $: J^c \bar{J}^b :$. We defined such an operator following Moore's assignment of contact terms in the abelian conformal perturbation theory. A different choice of the composite operator would in general result in a different function $G(\lambda)$. In our prescription the function $G(\lambda)$ can be computed by matching the leading singular terms in the OPE of (7.1) with the currents. For computing the OPE of $f^a_{bc} : J^c \bar{J}^b : (z, \bar{z})$ with one of the currents the abelian conformal perturbation theory can be used. Using the exact OPE coefficients (4.61) and (4.62) we obtain

$$G(\lambda)t^a_{bc} = -\frac{\lambda}{(1+\lambda)k} f^a_{bc}. \quad (7.2)$$

The equation of motion thus reads

$$\bar{\partial}J^a(z, \bar{z}) = -\partial\bar{J}^a(z, \bar{z}) = -i\frac{\lambda}{(1+\lambda)k} f^a_{bc} : J^c \bar{J}^b : (z, \bar{z}). \quad (7.3)$$

As a consistency check we can compare the quantum equation of motion (7.3) with the classical one that follows from the Lagrangian (2.4), (2.10). Both equations should match in form at the leading order in perturbation. The classical equation can be written as

$$\partial\bar{J} = -\bar{\partial}J = \lambda(\bar{\partial}J_0 - \partial\bar{J}_0) + \frac{\lambda}{k} [\bar{J}_0, J_0] \quad (7.4)$$

where J and \bar{J} are the components of the conserved Noether current

$$J = (1-\lambda)J_0 + \lambda\text{Ad}_g(J_0), \quad (7.5)$$

$$\bar{J} = (1-\lambda)\bar{J}_0 + \lambda\text{Ad}_{g^{-1}}(\bar{J}_0). \quad (7.6)$$

and

$$J_0 = -k\partial g g^{-1}, \quad \bar{J}_0 = k g^{-1}\bar{\partial}g. \quad (7.7)$$

At the leading order in λ the classical equation of motion (7.4) takes the form

$$\partial\bar{J} = -\bar{\partial}J \approx \frac{\lambda}{k} [\bar{J}, J] \quad (7.8)$$

that matches with the leading order term in the quantum equation of motion (7.3).

Another consistency check concerns the time evolution. It is easily verified that the ETC algebra (6.5) is preserved by the time evolution. Moreover, one can check that the Hamiltonian densities giving the equation of motion (7.3) are

$$T(\sigma, \tau) = \left(\frac{1-\lambda^2}{2k}\right) \kappa_{dc} : J^c J^d : (\sigma, \tau), \quad \bar{T}(\sigma, \tau) = \left(\frac{1-\lambda^2}{2k}\right) \kappa_{dc} : \bar{J}^c \bar{J}^d : (\sigma, \tau) \quad (7.9)$$

so that

$$\bar{\partial}J^a(z, \bar{z}) = \frac{i}{2\pi} \left[\int d\sigma \bar{T}(\sigma, \tau), J^a(z, \bar{z}) \right], \quad \partial\bar{J}^a(z, \bar{z}) = \frac{i}{2\pi} \left[\int d\sigma T(\sigma, \tau), \bar{J}^a(z, \bar{z}) \right]. \quad (7.10)$$

In verifying these relations we used⁹

$$f^c{}_{ba} : J^a J^b : = \frac{i}{2} f^c{}_{ba} f^{ab}{}_d \partial J^d = 0, \quad : J^a \bar{J}^b : = (-1)^{ab} : \bar{J}^b J^a : . \quad (7.11)$$

The operators (7.9) are the Virasoro generators of the deformed CFT. Unlike the deformed currents $J^a(z, \bar{z})$, $\bar{J}^b(z, \bar{z})$ these generators remain holomorphic and antiholomorphic. On general grounds this follows from the vanishing of the beta function. However it is instructive to show this more directly using the equation of motion (7.3). To that end we write $T(z, \bar{z})$ as

$$T(z, \bar{z}) = C \left(\frac{1 - \lambda^2}{2k} \right) \kappa_{dc} \lim_{:z \rightarrow w:} J^c(z, \bar{z}) J^d(w, \bar{w}) \quad (7.12)$$

where $\lim_{:z \rightarrow w:}$ stands for taking the limit and subtracting the singular terms in the OPE. In (7.12) C is a constant of proportionality which may depend on λ and, if logarithms are present in the OPE of the deformed currents, on the subtraction scale. The proportionality of the two definitions of the composite field follows from the fact that in the WZW theory $\kappa_{dc} : J^c J^d :$ is the unique group-invariant operator of its conformal weights. Using (7.3) and (7.12) we obtain

$$\bar{\partial} T(w, \bar{w}) = -i \frac{C \lambda (1 - \lambda)}{2k^2} f_{dca} \lim_{:z \rightarrow w:} J^a(z, \bar{z}) : J^c \bar{J}^d : (w, \bar{w}). \quad (7.13)$$

Based on the spin and scaling dimension of the operator on the left hand side and the global symmetry conservation the limit on the right hand side must be a linear combination of the operators $f_{cba} : J^a J^b \bar{J}^c :$ and $\kappa_{ab} : \partial J^b \bar{J}^a :$. However since the right hand side of (7.13) already contains a factor of the structure constants and the metric tensor κ_{ab} does not receive any corrections the second operator cannot appear. Therefore (7.13) can then be rewritten as

$$\bar{\partial} T(w, \bar{w}) = i \tilde{C} f_{dca} : J^a J^c \bar{J}^d : (w, \bar{w}) \quad (7.14)$$

that vanishes due to our definition of the operators $: J^a J^c \bar{J}^d :$ (see formulae (7.11)).

We conclude this section by noting that the form of equation (7.3) is, up to rescaling of the currents, the same as that discussed in [20]. We thus expect that our model is integrable and possesses the same Yangian symmetries as defined in [20]. We leave the detailed investigation of these symmetries to future work.

8 Conclusions

In this paper we have considered current-current perturbations of WZW models on supergroups G . These perturbations break the global symmetry $G \times G$ down to the diagonal action of G but preserve conformal invariance if G has vanishing Killing form. Perturbative calculations provided a number of explicit results regarding the OPEs and correlation functions of currents as well as the quantum equation of motion.

More specifically we were able to determine the most singular terms in the deformed OPE of WZW currents exactly to all orders in the coupling. In turn, this allowed us to obtain the exact quantum equations of motion (7.3) and the equal time commutators of currents (6.5). These exact results provide a non-perturbative Hamiltonian reformulation of the model. In view of the simple form of (6.5) and (7.3) (see also (6.7)) we expect this reformulation to be useful in understanding the structure of the operator product expansion. One of the consequences of our results that could already be seen at first order in perturbation theory is the occurrence of logarithmic contributions in the mixed OPE between the two components of the conserved current.

⁹These relations are true for every term in perturbation series and thus hold in the deformed theory.

The full operator product algebra of currents contains an infinite tower of operators which are composites built of arbitrarily many currents. Our results in section 4 on the first order OPEs and on the four-point function of currents (see section 5) only give very limited information about these terms. Some organising principle is needed to understand the full OPE algebra, perhaps related to the conjectured Yangian structure. We plan to return to this question in the future.

In this paper we have focused on the deformed current algebra. Our main method – quasi-abelian conformal perturbation theory – can also be applied to obtain precise analytical information about the deformed spectrum of conformal dimensions. In [27] this was achieved for the boundary spectrum on symmetry preserving D-branes. As will be reported in [34], similar considerations apply to the deformed bulk spectrum. Our findings will enable additional checks on the conjectured equivalence between supersphere σ -models and $OSP(2S + 2|2S)$ -symmetric Gross-Neveu models [26, 27]. Our results may also shed light onto open questions related to the parabolic paradigm for multifractality spectra in quantum Hall systems [35, 29].

Finally we comment on some potential applications of our results in string theory. First of all, referring to the example mentioned in the previous paragraph one might hope for further examples of dualities between conformal supercoset σ -models and deformed WZW models. It would be particularly interesting to investigate the deformations of the $PSU(2, 2|4)_1$ WZW model and to see whether it can be related to $AdS_5 \times S^5$ string theory which – in the Green-Schwarz formalism – is known to be described by a σ -model on the coset superspace $PSU(2, 2|4)/SO(1, 4) \times SO(5)$ [3]. A more obvious connection of our deformations to string theory exists in the case of the supergroup $PSU(1, 1|2)$ which is known to describe the string background $AdS_3 \times S^3$ with mixtures of Neveu-Schwarz and Ramond-Ramond fluxes [4]. Since the metric and the fluxes preserve the full isometry of $AdS_3 \times S^3$, the $G \times G$ -preserving deformations of the $PSU(1, 1|2)$ WZW model have to be used for their description. On the other hand, the G -preserving deformations discussed here should also correspond to some string background, possibly to some squashed version of $AdS_3 \times S^3$ with fluxes. Extracting the precise form of the metric and the fluxes from the deformed WZW Lagrangian is left to future work.

Note added: While this paper was nearing completion a new preprint has appeared [36] in which the integrability of the $G \times G$ -preserving deformations is discussed.

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A Lie superalgebra conventions

A Lie superalgebra is a graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$. The bracket is required to be bilinear, grade-preserving and graded antisymmetric. Moreover it has to satisfy a graded version of the Jacobi identity. For the definition of the physical action functional it is essential to have a non-degenerate, grade-preserving and graded symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ which plays the role of a metric. For a comprehensive introduction into Lie superalgebras we refer the reader to [37, 38].

For concrete calculations it is convenient to fix a basis T^a of homogeneous generators, with the generator T^a having degree d_a , i.e. $T^a \in \mathfrak{g}_{d_a}$. In most of the paper we will actually use the abbreviation $d_a \equiv a$, hoping that no misunderstandings arise. The structure constants f^{ab}_c are defined in terms of the commutation relation

$$[T^a, T^b] = if^{ab}_c T^c. \tag{A.1}$$

They inherit a number of properties from the requirement that the bracket so defined gives rise to a Lie superalgebra. In particular, in terms of the structure constants, the graded antisymmetry and the Jacobi identity can be written as

$$f^{ba}{}_c = -(-1)^{ab} f^{ab}{}_c \quad , \quad f^{ab}{}_d f^{dc}{}_e + (-1)^{c(a+b)} f^{ca}{}_d f^{db}{}_e + (-1)^{a(b+c)} f^{bc}{}_d f^{da}{}_e = 0. \quad (\text{A.2})$$

For a simple Lie superalgebra the desired non-degenerate metric can be obtained through the definition¹⁰

$$\kappa^{ab} = \langle T^a, T^b \rangle = \text{str}_R(T^a T^b) \quad (\text{A.3})$$

by evaluating the supertrace in a suitable representation R .¹¹ Its inverse is defined by

$$\kappa_{ba} \kappa^{bc} = \delta_a^c. \quad (\text{A.4})$$

Indices are raised and lowered according to the rule

$$A^a = A_b \kappa^{ba} \quad A_a = \kappa_{ab} A^b. \quad (\text{A.5})$$

The last convention in particular implies

$$\kappa^b{}_a = \delta_a^b \quad \kappa_a{}^b = (-1)^a \delta_a^b. \quad (\text{A.6})$$

One also obtains

$$A^a B_a = A_b \kappa^{ba} B_a = (-1)^b A_b \kappa^{ab} B_a = (-1)^a A_a B^a. \quad (\text{A.7})$$

In the main part of the paper, a crucial role will be played by the quadratic Casimir element, defined by

$$C_2 = \kappa_{ba} T^a T^b = (-1)^a \kappa_{ab} T^a T^b. \quad (\text{A.8})$$

If evaluated in the adjoint representation, the quadratic Casimir element provides a simple way of determining whether the Killing form vanishes or not. Indeed, in the former case one finds

$$C_{\text{ad}} \kappa^{ab} = -(-1)^d f^{ac}{}_d f^{bd}{}_c = \text{str}_{\text{ad}}(T^a, T^b) = 0. \quad (\text{A.9})$$

This relation will be at the heart of many of the special features that WZW models exhibit for supergroups with vanishing Killing form.

B Abelian conformal perturbation theory

In Appendix A of [31] a prescription was given for computing the integrals arising in conformal perturbation theory of free bosons when changing the metric and B -field. We refer to these perturbations as abelian conformal perturbations. They are equivalent to current-current perturbations in toroidal WZW theories. The perturbation of the Euclidean action is

$$\Delta S = \frac{\lambda}{k\pi} \int d^2 w : J^e \bar{J}^r : \kappa_{re} \quad (\text{B.1})$$

¹⁰We restrict our attention to simple Lie superalgebras which admit an even non-degenerate form. The other Lie superalgebras are not relevant in our context.

¹¹For a simple Lie superalgebra all metrics are identical up to a scalar factor. Hence the concrete choice of representation R is not essential. It is however important to note that for certain Lie superalgebras the supertrace vanishes identically, for instance in the adjoint representation.

where the OPEs of the currents are

$$J^a(z)J^b(w) \sim \frac{k\kappa^{ab}}{(z-w)^2} + \text{non-sing.} , \quad \bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\kappa^{ab}}{(\bar{z}-\bar{w})^2} + \text{non-sing.} . \quad (\text{B.2})$$

One is interested in computing perturbation theory integrals

$$\int \dots \int d^2w_1 \dots d^2w_k \langle V_1(z_1, \bar{z}_1) \dots V_n(z_n, \bar{z}_n) O(w_1, \bar{w}_1) \dots O(w_k, \bar{w}_k) \rangle \quad (\text{B.3})$$

where $O = :J^e \bar{J}^r : \kappa_{re}$ and V_1, \dots, V_n stand for external insertions. Without loss of generality such insertions can be taken to be affine primaries and their descendants. The correlator entering (B.3) taken at finite separations can be computed using Wick's theorem and the contractions (B.2). Each correlator is a sum over all contraction schemes. Each contraction scheme can be represented as a collection of chains of pairwise contractions. In general a chain of contractions that starts and ends on the same external operator contributes only to the renormalisation of the corresponding operator (change of normal ordering prescription). Also chains that start and end on one of the operators $O(w_i, \bar{w}_i)$ contribute only to the overall normalisation factor. For all computations done in the present paper such contributions are not needed and thus such contraction schemes are assumed to be dropped everywhere where we use abelian conformal perturbation. The remaining contractions each produce functions with non-integrable singularities. A consistent prescription is needed for integrals of such functions.

For the case when the external operators V_i are currents and their composites the prescription of [31] for the integrals defining the deformed correlation functions can be succinctly summarised by specifying dressed contractions of currents

$$\begin{aligned} J^a(z_1)J^b(z_2) &\sim \frac{k\kappa^{ab}}{1-\lambda^2} \frac{1}{(z_1-z_2)^2}, & \bar{J}^a(\bar{z}_1)\bar{J}^b(\bar{z}_2) &\sim \frac{k\kappa^{ab}}{1-\lambda^2} \frac{1}{(\bar{z}_1-\bar{z}_2)^2}, \\ J^a(z_1)\bar{J}^b(\bar{z}_2) &\sim -\frac{\pi\lambda k\kappa^{ab}}{1-\lambda^2} \delta(z_1-z_2). \end{aligned} \quad (\text{B.4})$$

The deformed correlators of currents and their composites are then obtained using Wick's theorem with the above contractions. Strictly speaking this prescription works for the external fields taken at finite separation: $|z_i - z_j| > 0$ for $i \neq j$. This avoids the appearance of meaningless expressions such as squares of delta functions, etc. However for some correlators the distributional answers obtained using (B.4) are correct. In particular this holds for correlators with a single external field being the composite $:J^a \bar{J}^b:$ and the rest of the external fields being currents. For example

$$\begin{aligned} \langle J^c(z_1)J^d(z_2) :J^e \bar{J}^f : (w, \bar{w}) \rangle_\lambda &= \left(-\frac{\pi\lambda k\kappa^{cf}}{1-\lambda^2} \delta(z_1-w) \right) \left(\frac{k\kappa^{de}}{1-\lambda^2} \frac{1}{(z_2-w)^2} \right) \\ &+ (-1)^{cd} \left(\frac{k\kappa^{ce}}{1-\lambda^2} \frac{1}{(z_1-w)^2} \right) \left(-\frac{\pi\lambda k\kappa^{df}}{1-\lambda^2} \delta(z_2-w) \right). \end{aligned} \quad (\text{B.5})$$

In our computations such distributional correlators appear inside integrals and may give non-vanishing contributions at finite separation upon integration. In each case one needs to be careful applying (B.4) to obtain well-defined distributions.

For any prescription of the kind introduced in [31] to be consistent it must come from some *distributional* correlation functions defined in the undeformed theory. Also, besides taking the integrals one sometimes is interested in doing other manipulations with the correlators arising in perturbation series (e.g. of the type we do in sections 3 and 5). To justify such manipulations one needs a rigorous definition of the arising distributions. Below we give such a definition and discuss some properties of these distributions.

There are two basic classes of functions arising from abelian chains of contractions of currents

$$\frac{1}{(z_1 - w_1)^2} \frac{1}{(\bar{w}_1 - \bar{w}_2)^2} \frac{1}{(w_2 - w_3)^2} \cdots \frac{1}{(w_{2k} - z_2)^2}, \quad (\text{B.6})$$

$$\frac{1}{(z_1 - w_1)^2} \frac{1}{(\bar{w}_1 - \bar{w}_2)^2} \frac{1}{(w_2 - w_3)^2} \cdots \frac{1}{(\bar{w}_{2k+1} - \bar{z}_2)^2}. \quad (\text{B.7})$$

These functions correspond to a chain of contractions running from an insertion in z_1 to an insertion in z_2 . We want to promote the first function to a distribution on $\mathbb{R}^{2(2k+2)}$ and the second one to a distribution on $\mathbb{R}^{2(2k+3)}$ so that z_1 and z_2 are distribution variables as well.

Before regularising the above functions we make a couple of general remarks. Note that if $D(z_2 - z_3, z_3 - z_4, \dots, z_{n-1} - z_n)$ is a translation invariant distribution on $(\mathbb{R}^2)^{n-1}$ then there is a natural definition of the product

$$d\delta(z_1 - z_2)D \quad (\text{B.8})$$

where d is a differential operator with constant coefficients. This product is a distribution on $(\mathbb{R}^2)^n$ that acts on a test function $\phi(z_1, z_2, \dots, z_n)$ as

$$\langle d\delta(z_1 - z_2)D, \phi \rangle = \langle d_1^* D, \phi(z_2, z_2, z_3, \dots, z_n) \rangle + \langle D, d_1^* \phi(z_2, z_3, \dots, z_n) \rangle \quad (\text{B.9})$$

where d_1^* is the adjoint differential operator acting on the first variable.

Note also that given a distribution in n variables one can define its partial integral in any variable as a distribution in $n - 1$ variables by taking a test function which does not depend on the given variable on an interval of radius R and then taking R to infinity. If the limit exists it is a distribution in $n - 1$ variables.

We will put square brackets around the regularised functions to denote the corresponding distributions. Using (B.9) we define

$$\left[\frac{1}{(z_1 - w)^2 (\bar{w} - \bar{z}_2)^2} \right] = \partial_w \bar{\partial}_w \partial_{z_1} \bar{\partial}_{z_2} \ln |z_1 - w|^2 \ln |z_2 - w|^2 + \frac{\pi}{\bar{z}_2 - \bar{w}} \partial_w \delta(z_1 - w) + \frac{\pi}{z_1 - w} \bar{\partial}_w \delta(z_2 - w) - \pi^2 \delta(z_1 - w) \delta(z_2 - w) \quad (\text{B.10})$$

Here the first term on the right hand side is defined as a distributional derivative of $\ln |z_1 - w|^2 \ln |z_2 - w|^2$ which is a locally integrable function; the next two terms are well defined by virtue of the above general remark because $\frac{\pi}{\bar{z}_2 - \bar{w}}$ and $\frac{\pi}{z_1 - w}$ are locally integrable. Analogously we define

$$\left[\frac{1}{(z_1 - w)^2 (\bar{z}_2 - \bar{w})} \right] = \partial_w \partial_{z_1} \bar{\partial}_{z_2} \ln |z_1 - w|^2 \ln |z_2 - w|^2 + \frac{\pi}{z_1 - w} \delta(z_2 - w),$$

$$\left[\frac{1}{(z_1 - w)(\bar{z}_2 - \bar{w})^2} \right] = \bar{\partial}_w \partial_{z_1} \bar{\partial}_{z_2} \ln |z_1 - w|^2 \ln |z_2 - w|^2 + \frac{\pi}{\bar{z}_2 - \bar{w}} \delta(z_1 - w). \quad (\text{B.11})$$

We can easily take the integrals of the distributions defined in (B.10), (B.11) to obtain

$$\int d^2 w \left[\frac{1}{(z_1 - w)^2 (\bar{w} - \bar{z}_2)^2} \right] = \pi^2 \delta(z_1 - z_2), \quad (\text{B.12})$$

$$\int d^2 w \left[\frac{1}{(z_1 - w)^2 (\bar{z}_2 - \bar{w})} \right] = \frac{\pi}{z_1 - z_2}, \quad (\text{B.13})$$

$$\int d^2 w \left[\frac{1}{(z_1 - w)(\bar{z}_2 - \bar{w})^2} \right] = \frac{\pi}{\bar{z}_2 - \bar{z}_1}. \quad (\text{B.14})$$

For reference we also include here another useful integral

$$\int_{|w| \leq R} d^2w \frac{1}{(z_1 - w)(\bar{z}_2 - \bar{w})} = -\pi \ln |z_1 - z_2|^2 + \pi \ln(R^2 - z_1 \bar{z}_2). \quad (\text{B.15})$$

We can now define the regularised rational functions (B.6), (B.7) recursively. Take for definiteness (B.6). We can define the corresponding distribution as

$$\begin{aligned} D_{2k} &\equiv \left[\frac{1}{(z_1 - w_1)^2} \frac{1}{(\bar{w}_1 - \bar{w}_2)^2} \frac{1}{(w_2 - w_3)^2} \cdots \frac{1}{(w_{2k} - z_2)^2} \right] \\ &= \partial_{z_1} \partial_{z_2} (\partial_1 \bar{\partial}_1) \cdots (\partial_{2k} \bar{\partial}_{2k}) \ln |z_1 - w_1|^2 \ln |w_2 - w_3|^2 \cdots \ln |w_{2k} - z_2|^2 - C_{2k} \end{aligned} \quad (\text{B.16})$$

where C_{2k} are terms each of the form $d\delta(w_{i_1} - w_{i_2})D$ where D is a distributional regularisation of the rational functions in a smaller number of variables, all containing first or second powers in the denominator, and d is some differential operator with constant coefficients. The precise form of C_{2k} is worked out by differentiating the product of logarithms and using

$$\partial \bar{\partial} \ln |z|^2 = -\pi \delta(z). \quad (\text{B.17})$$

Thus starting with (B.10), (B.11) we can build all the required distributions recursively. It is also clear that this definition also defines regularisations of a more general class of functions of the form

$$\frac{1}{(z_1 - w_1)^{p_1}} \frac{1}{(\bar{w}_1 - \bar{w}_2)^{p_2}} \frac{1}{(w_2 - w_3)^{p_3}} \cdots \frac{1}{(w_{2k} - z_2)^{p_{2k+1}}}, \quad (\text{B.18})$$

$$\frac{1}{(z_1 - w_1)^{q_1}} \frac{1}{(\bar{w}_1 - \bar{w}_2)^{q_2}} \frac{1}{(w_2 - w_3)^{q_3}} \cdots \frac{1}{(\bar{w}_{2k+1} - \bar{z}_2)^{q_{2k+2}}}. \quad (\text{B.19})$$

where $p_i = 1, 2$ and $q_j = 1, 2$.

Next we would like to prove that the distributions D_{2k} satisfy the property

$$\int dw_n D_{2k} = \pi^2 \delta(w_{n-1} - w_{n-2}) D_{2k-1}. \quad (\text{B.20})$$

This property in particular accounts for the values of integrals 3a, 4a in [31]. Naively (B.20) is obtained by using integral (B.12) for a partial integral of D_{2k} . It is not clear however that this is consistent with the recursive definition of D_{2k} . We now explain how one proves this rigorously. Denote for brevity

$$\Delta_i \equiv \partial_i \bar{\partial}_i \quad (\text{B.21})$$

then the total derivative used to define D_{2k} can be written as

$$\Delta_n \Delta_{n-1} \Delta_{n+1} \left[R_1 \ln |w_{n-1} - w_n|^2 \ln |w_n - w_{n+1}|^2 R_2 \right] \quad (\text{B.22})$$

where

$$R_1 = \partial_1 \Delta_1 \cdots \Delta_{n-2} \ln |z_1 - w_1|^2 \ln \cdots \ln |w_{n-2} - w_{n-1}|^2 \quad (\text{B.23})$$

and

$$R_2 = \partial_2 \Delta_{n+2} \cdots \Delta_{2k} \ln |w_{n+1} - w_{n+2}|^2 \cdots \ln |w_{2k} - z_2|^2. \quad (\text{B.24})$$

Using the representation (B.22) we can recast (B.16) into the following form

$$\begin{aligned} D_{2k} &= \left[\text{Sm}(\bar{\partial}_{n-1} R_1) \left(\frac{\pi}{\bar{w}_{n+1} - \bar{w}_n} \partial_n \delta(w_{n-1} - w_n) + \frac{\pi}{w_{n-1} - w_n} \bar{\partial}_n \delta(w_{n+1} - w_n) \right. \right. \\ &\quad \left. \left. - \pi^2 \delta(w_{n-1} - w_n) \delta(w_n - w_{n+1}) \right) \text{Sm}(\partial_{n+1} R_2) \right] + \Delta_n(\dots). \end{aligned} \quad (\text{B.25})$$

Here the square brackets make all rational functions appearing in this expression into a distribution as recursively defined above. $\text{Sm}(\bar{\partial}_{n-1}R_1)$ and $\text{Sm}(\partial_{n+1}R_2)$ stand for the rational functions obtained by taking the derivatives of the products of logarithms at finite separation. The total derivatives $\Delta_n(\dots)$ drop out when taking the integral over w_n . This is ensured by the absence of IR divergences.

We can now take the integral over w_n and we are left with the desired result (B.20) by means of the following lemma

Lemma

$$\partial_n \left[\text{Sm}(\bar{\partial}_{n-1}R_1) \frac{\pi}{\bar{w}_{n+1} - \bar{w}_n} \text{Sm}(\partial_{n+1}R_2) \right] = -\pi^2 \delta(w_{n+1} - w_n) \left[\text{Sm}(\bar{\partial}_{n-1}R_1) \text{Sm}(\partial_{n+1}R_2) \right] \quad (\text{B.26})$$

This lemma is proven by induction in k - the length of the chain. Using the definition we write the left hand side as

$$\begin{aligned} & \partial_n \left(-\partial_{z_1} \partial_{z_2} \Delta_1 \dots \Delta_{n-2} \bar{\partial}_{n-1} \bar{\partial}_n \partial_{n+1} \Delta_{n+2} \dots \Delta_{2k} \right. \\ & \left. \ln |z_1 - w_1|^2 \dots \ln |w_{n-2} - w_{n-1}|^2 \ln |w_{n+1} - w_n|^2 \dots \ln |w_{2k} - z_2|^2 \right. \\ & \left. + \pi \delta(w_n - w_{n+1})(\dots) - C_{k,n} \right) \end{aligned} \quad (\text{B.27})$$

where $C_{k,n}$ are contact terms all containing a factor of $1/(\bar{w}_{n+1} - \bar{w}_n)$. By induction the analogue of formula (B.26) holds for the distributions in $C_{k,n}$ because those terms contain a shorter ‘smooth’ part. For the first two terms the desired identity holds by definition of the distributional derivative.

Using (B.20) repeatedly we obtain integrals 3a and 4a in [31] used in summing up the perturbative series to obtain the dressed contractions given in (B.4). Formula (B.20) easily extends to more general distributions regularising (B.18), (B.19).

C First order calculations

Section 4 was concerned with the determination of the OPE between the currents $J^a(z, \bar{z})$ and $\bar{J}^a(z, \bar{z})$ in deformed WZW models. For the sake of clarity, some of the more technical calculations have been omitted in the main text. For completeness they are summarised in this appendix.

Determination of t . In order to determine this coefficient we will put $X =: J^c \bar{J}^d$. The calculation of the coefficients $A(X)$ and $B_i(X)$ is straightforward but lengthy in this case, and we only report the main steps. In the calculation of $A(X)$ we will encounter the four-point function of J . The quadratic singularities drop out after integration. The simple poles can be determined by moving J^e to the left and performing the contraction with J^a , J^b and J^c . The last contribution drops out after integration while the rest yields

$$A(X) = \frac{k(-1)^{e(a+b+c+d)} \kappa_{fe} \kappa^{df}}{(z_1 - z_2)(z_2 - \xi)(\xi - z_1)} \left[\frac{f^{ea}{}_g f^{gbc}}{\bar{z}_1 - \bar{\xi}} + \frac{f^{eb}{}_g f^{agc}(-1)^{ae}}{\bar{z}_2 - \bar{\xi}} \right]. \quad (\text{C.1})$$

We then split the bracket into its symmetric and its antisymmetric part. After applying the Jacobi identity we find

$$A(X) = \frac{k(-1)^{d(a+b+c)}}{(z_1 - z_2)(z_2 - \xi)(\xi - z_1)} \left\{ \frac{1}{2} [f^{da}_g f^{gbc} + f^{db}_g f^{agc} (-1)^{ad}] \left[\frac{1}{\bar{z}_1 - \bar{\xi}} + \frac{1}{\bar{z}_2 - \bar{\xi}} \right] \right. \quad (\text{C.2})$$

$$\left. + \frac{1}{2} [f^{da}_g f^{gbc} - f^{db}_g f^{agc} (-1)^{ad}] \left[\frac{1}{\bar{z}_1 - \bar{\xi}} - \frac{1}{\bar{z}_2 - \bar{\xi}} \right] \right\} \quad (\text{C.3})$$

$$= \frac{k(-1)^{d(a+b+c)}}{(z_1 - z_2)(z_2 - \xi)(\xi - z_1)} \left\{ -\frac{1}{2} (-1)^{d(a+b)} f^{ab}_g f^{gdc} \left[\frac{1}{\bar{z}_1 - \bar{\xi}} + \frac{1}{\bar{z}_2 - \bar{\xi}} \right] \right. \quad (\text{C.4})$$

$$\left. - \frac{1}{2} [f^{da}_g f^{gbc} - f^{db}_g f^{agc} (-1)^{ad}] \frac{\bar{z}_1 - \bar{z}_2}{(\bar{\xi} - \bar{z}_1)(\bar{\xi} - \bar{z}_2)} \right\}. \quad (\text{C.5})$$

Finally, we expand this expression up to terms involving ξ and $\bar{\xi}$ to the fourth inverse power. After some elementary algebra one obtains

$$A(X) = \frac{1}{2} \frac{k f^{ab}_g f^{gcd}}{z_1 - z_2} \frac{1}{\xi^2 \bar{\xi}} \left[2 + \frac{2(z_1 + z_2)}{\xi} + \frac{\bar{z}_1 + \bar{z}_2}{\bar{\xi}} + \dots \right] \quad (\text{C.6})$$

$$- \frac{k}{2} (-1)^{d(b+c)} \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \frac{1}{\xi^2 \bar{\xi}^2} [f^{ad}_g f^{gbc} + f^{db}_g f^{agc}] [1 + \dots]. \quad (\text{C.7})$$

Fortunately, the remaining terms are easier to determine. The first coefficient $B_1(X)$ vanishes due to the integration. For the second coefficient a simple calculation yields

$$B_2(X) = -\frac{i f^{ab}_g}{z_1 - z_2} (-1)^{de} \kappa_{fe} \frac{1}{\pi k} \int d^2 z \frac{i k f^{gce}}{(z_2 - \xi)(\xi - z)(z - z_2)} \frac{k \kappa^{df}}{(\bar{\xi} - \bar{z})^2} \quad (\text{C.8})$$

$$= \frac{i f^{ab}_g}{z_1 - z_2} \frac{i k f^{gcd}}{(z_2 - \xi)^2 (\bar{z}_2 - \bar{\xi})}. \quad (\text{C.9})$$

Upon expansion we immediately find

$$B_2 = \frac{k f^{ab}_g f^{gcd}}{z_1 - z_2} \frac{1}{\xi^2 \bar{\xi}} \left[1 + \frac{2z_2}{\xi} + \frac{3z_2^2}{\xi^2} + \dots \right] \left[1 + \frac{\bar{z}_2}{\bar{\xi}} + \frac{\bar{z}_2^2}{\bar{\xi}^2} + \dots \right]. \quad (\text{C.10})$$

Finally, we can recycle the knowledge previously obtained about A in order to determine the last coefficient $B_3(X) = \lim_{z_1 \rightarrow z_2} A$. We only need to expand the term $1/(\xi - z_1)$ in a geometric series in $z_1 - z_2$ in order to find

$$B_3(X) = \frac{k}{(\xi - z_2)^3} f^{ab}_g f^{gcd} \frac{1}{\bar{\xi} - \bar{z}_1} = \frac{k}{\xi^3 \bar{\xi}} f^{ab}_g f^{gcd} [1 + \dots]. \quad (\text{C.11})$$

When adding up these contributions, the terms at orders $1/\xi^2 \bar{\xi}$ and $1/\xi^3 \bar{\xi}$ drop out. The remaining term can be simplified using the Jacobi identity. In the end we obtain

$$A(X) - B_1(X) - B_2(X) - B_3(X) = \frac{1}{\xi^2 \bar{\xi}^2} \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} k \left\{ -(-1)^{d(b+c)} f^{ad}_g f^{gbc} \right\} + \dots \quad (\text{C.12})$$

This result should be compared to the unperturbed correlation functions. With the present choice $X = :J^c \bar{J}^d$: the most important contribution arises from

$$\begin{aligned}
& - \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \partial_\lambda t_{rs}^{ab}(0) \left\langle :J^r \bar{J}^s : (z_2, \bar{z}_2) J^c(\xi) \bar{J}^d(\bar{\xi}) \right\rangle \\
& = - \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} (-1)^{cs} \partial_\lambda t_{rs}^{ab}(0) \frac{k \kappa^{rc}}{(z_2 - \xi)^2} \frac{k \kappa^{sd}}{(\bar{z}_2 - \bar{\xi})^2} \\
& = - \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} (-1)^{cs} \partial_\lambda t_{rs}^{ab}(0) \frac{k^2 \kappa^{rc} \kappa^{sd}}{\xi^2 \bar{\xi}^2} [1 + \dots].
\end{aligned} \tag{C.13}$$

Comparing the two expressions and solving for t we find

$$t_{cd}^{ab} = \frac{\lambda}{k} (-1)^{bd} f^a{}_{dg} f^g{}_c. \tag{C.14}$$

Determination of w . The determination of w mimics the calculation for g above. We choose $X = : \bar{J} \bar{J}(\bar{\xi}) :$. The coefficients $B_i(X)$ all vanish. Even though $A(X)$ is non-zero, it is obviously antisymmetric in (ab) and in (cd) . On the other hand such contributions can never arise from w , which is symmetric in the lower indices. Instead they are accounted for by the coefficients u and v that have already been determined above. As a consequence we find

$$0 = \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \partial_\lambda w_{ef}^{ab}(0) \left\langle : \bar{J}^e \bar{J}^f : (\bar{z}_2) : \bar{J}^c \bar{J}^d : (\bar{\xi}) \right\rangle \tag{C.15}$$

$$= \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \partial_\lambda w_{ef}^{ab}(0) \left\{ \frac{k^2 \kappa^{ec} \kappa^{fd} (-1)^{ef}}{(\bar{z}_2 - \bar{\xi})^4} + \frac{k^2 \kappa^{ed} \kappa^{fc}}{(\bar{z}_2 - \bar{\xi})^4} \right. \tag{C.16}$$

$$\left. - \frac{2k f^{ec}{}_g f^g{}_{fd} (-1)^{ef}}{(\bar{z}_2 - \bar{\xi})^4} - \frac{k f^{ed}{}_g f^g{}_{fc} (-1)^{e(f+c)}}{(\bar{z}_2 - \bar{\xi})^4} \right\} \tag{C.17}$$

$$= \frac{(\bar{z}_1 - \bar{z}_2)^2}{(z_1 - z_2)^2} \frac{1}{\xi^4} \partial_\lambda w_{ef}^{ab}(0) \left\{ k^2 \kappa^{ec} \kappa^{fd} (-1)^{ef} + k^2 \kappa^{ed} \kappa^{fc} \right. \tag{C.18}$$

$$\left. - 2k f^{ec}{}_g f^g{}_{fd} (-1)^{ef} - k f^{ed}{}_g f^g{}_{fc} (-1)^{e(f+c)} \right\} + \dots \tag{C.19}$$

Just as for $\partial_\lambda g(0)$, we postulate that the solution for this coefficient is given by $\partial_\lambda w(0) = 0$. In other words, also the coefficient w does not receive any correction at first order in perturbation theory. This concludes our calculations with regard to the OPE of the current J with itself.

Determination of B_c^{ab} and \tilde{B}_c^{ab} . For this case we pick $X = \bar{J}^c$. Following the standard prescription we first evaluate

$$A(X) = - \frac{ik f^{abc}}{(\bar{z}_2 - \bar{\xi})^2} \left[\frac{1}{z_1 - z_2} - \frac{1}{z_1 - \xi} \right] = -ik f^{abc} \left[\frac{1}{(z_1 - z_2)\bar{\xi}^2} + \frac{2\bar{z}_2}{(z_1 - z_2)\bar{\xi}^3} + \frac{1}{\xi \bar{\xi}^2} + \dots \right]. \tag{C.20}$$

In the next step we take the non-singular limit

$$B(X) = \lim_{:z_1 \rightarrow z_2:} A = \frac{ik f^{abc}}{(\bar{z}_2 - \bar{\xi})^2} \frac{1}{z_2 - \xi} = - \frac{ik f^{abc}}{\xi \bar{\xi}^2} [1 + \dots]. \tag{C.21}$$

The total contribution is hence given by

$$A(X) - B(X) = - \frac{ik f^{abc}}{(z_1 - z_2)\bar{\xi}^2} \left[1 + \frac{2\bar{z}_2}{\bar{\xi}} + \dots \right]. \tag{C.22}$$

The result above has to be compared to

$$-\frac{1}{z_1 - z_2} \partial_\lambda B_d^{ab}(0) \left\langle \bar{J}^d(\bar{z}_2) \bar{J}^c(\xi) \right\rangle = -\frac{1}{z_1 - z_2} \frac{k\kappa^{dc}}{(\bar{z}_2 - \xi)^2} \partial_\lambda B_d^{ab}(0) \quad (\text{C.23})$$

$$= -\frac{k\kappa^{dc} \partial_\lambda B_d^{ab}(0)}{z_1 - z_2} \frac{1}{\xi^2} \left[1 + \frac{2\bar{z}_2}{\xi} + \dots \right]. \quad (\text{C.24})$$

A comparison of the leading terms yields

$$B_c^{ab}(\lambda) = i\lambda f_c^{ab}. \quad (\text{C.25})$$

It is obvious that this term already accounts even for subleading contributions up to the order considered. In other words, we have

$$A(X) - B(X) + \frac{\partial_\lambda B_d^{ab}(0)}{z_1 - z_2} \left\langle \bar{J}^d(\bar{z}_2) \bar{J}^c(\xi) \right\rangle = 0 + \dots. \quad (\text{C.26})$$

On the other hand the same result should be obtained when adding up the leading contributions of the following two expressions,

$$-\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \partial_\lambda \tilde{B}_d^{ab}(0) \left\langle \bar{\partial} \bar{J}^d(\bar{z}_2) \bar{J}^c(\xi) \right\rangle = \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \frac{2k\kappa^{dc}}{(\bar{z}_2 - \xi)^3} \partial_\lambda \tilde{B}_d^{ab}(0) \quad (\text{C.27})$$

$$= -\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \frac{1}{\xi^3} 2k\kappa^{dc} \partial_\lambda \tilde{B}_d^{ab}(0) \left[1 + \frac{3\bar{z}_2}{\xi} + \dots \right] \quad (\text{C.28})$$

$$-\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \partial_\lambda \hat{B}_{de}^{ab}(0) \left\langle \bar{J}^d \bar{J}^e(\bar{z}_2) \bar{J}^c(\xi) \right\rangle = \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \frac{ikf^{dec}}{(\bar{z}_2 - \xi)^3} \partial_\lambda \hat{B}_{de}^{ab}(0) = 0. \quad (\text{C.29})$$

The comparison yields

$$\tilde{B}_d^{ab}(\lambda) = 0 + \mathcal{O}(\lambda^2). \quad (\text{C.30})$$

Hence $\tilde{B}_d^{ab}(\lambda)$ remains zero (at least to this order), even after the deformation is switched on.

Determination of \hat{C}_c^{ab} . This case may be treated using $X =: J^c J^d :$. Our first focus rests on

$$A(X) = \frac{1}{\pi} (-1)^{b(c+d)} \int d^2z \left\langle J^a(z_1) : J^c J^d : (\xi) J^b(z) \right\rangle \frac{1}{(\bar{z}_2 - \bar{z})^2}. \quad (\text{C.31})$$

Due to the integration we only need the simple pole contributions from the four-point correlator. The latter is a bit difficult to deal with since two of the currents have a coinciding argument. We can calculate it by expressing the correlator as the non-singular limit $w \rightarrow \xi$ of the correlator

$\langle J^a(z_1)J^c(w)J^d(\xi)J^b(z) \rangle$. A straightforward but lengthy calculation yields

$$\langle J^a(z_1) : J^c J^d : (\xi) J^b(z) \rangle = (-1)^{b(a+c+d)} \lim_{:w \rightarrow \xi:} \frac{k}{(z_1 - w)(w - \xi)(\xi - z_1)} \quad (\text{C.32})$$

$$\left\{ \frac{f^{ba}{}_g f^{gcd}}{z - z_1} + \frac{(-1)^{ab} f^{bc}{}_g f^{agd}}{z - w} + \frac{(-1)^{b(a+c)} f^{bd}{}_g f^{acg}}{z - \xi} \right\} + \dots \quad (\text{C.33})$$

$$= \frac{k(-1)^{b(a+c+d)}}{(\xi - z_1)^3} \quad (\text{C.34})$$

$$\left\{ \frac{f^{ba}{}_g f^{gcd}}{z - z_1} + \frac{(-1)^{ab} f^{bc}{}_g f^{agd}}{z - \xi} + \frac{(-1)^{b(a+c)} f^{bd}{}_g f^{acg}}{z - \xi} \right\} + \dots \quad (\text{C.35})$$

$$- \frac{k(-1)^{b(a+c+d)} (-1)^{ab} f^{bc}{}_g f^{agd}}{(\xi - z_1)^2 (z - \xi)^2}. \quad (\text{C.36})$$

Upon integration the last term drops out, leaving us with

$$A(X) = -\frac{k(-1)^{ab}}{(\xi - z_1)^3} \left\{ \frac{f^{ba}{}_g f^{gcd}}{\bar{z}_2 - \bar{z}_1} + \frac{(-1)^{ab} f^{bc}{}_g f^{agd}}{\bar{z}_2 - \bar{\xi}} + \frac{(-1)^{b(a+c)} f^{bd}{}_g f^{acg}}{\bar{z}_2 - \bar{\xi}} \right\} + \dots \quad (\text{C.37})$$

$$= \frac{k(-1)^{ab} f^{ba}{}_g f^{gcd}}{(\bar{z}_1 - \bar{z}_2)(\xi - z_1)^3} + \frac{k}{(\xi - z_1)^3 (\bar{\xi} - \bar{z}_2)} \left\{ f^{bc}{}_g f^{agd} + (-1)^{bc} f^{bd}{}_g f^{acg} \right\} + \dots \quad (\text{C.38})$$

Expanding $A(X)$ in inverse powers of ξ yields

$$A(X) = -\frac{k f^{ab}{}_g f^{gcd}}{\bar{z}_1 - \bar{z}_2} \frac{1}{\xi^3} \left[1 + \frac{3z_1}{\xi} + \frac{6z_1^2}{\xi^2} + \dots \right] \quad (\text{C.39})$$

$$+ \frac{k}{\xi^3 \bar{\xi}} \left[1 + \frac{3z_1}{\xi} + \frac{\bar{z}_2}{\bar{\xi}} + \dots \right] \left\{ f^{bc}{}_g f^{agd} + (-1)^{bc} f^{bd}{}_g f^{acg} \right\} + \dots \quad (\text{C.40})$$

We then evaluate

$$B(X) = \lim_{:z_1 \rightarrow z_2:} A = \frac{k}{\xi^3 \bar{\xi}} \left[1 + \frac{3z_2}{\xi} + \frac{\bar{z}_2}{\bar{\xi}} + \dots \right] \left\{ f^{bc}{}_g f^{agd} + (-1)^{bc} f^{bd}{}_g f^{acg} \right\} + \dots \quad (\text{C.41})$$

Eventually we arrive at

$$A(X) - B(X) = -\frac{k f^{ab}{}_g f^{gcd}}{\bar{z}_1 - \bar{z}_2} \frac{1}{\xi^3} \left[1 + \frac{3z_1}{\xi} + \frac{6z_1^2}{\xi^2} + \dots \right] \quad (\text{C.42})$$

$$+ \frac{k}{\xi^4 \bar{\xi}} \left[3(z_1 - z_2) + \dots \right] \left\{ f^{bc}{}_g f^{agd} + (-1)^{bc} f^{bd}{}_g f^{acg} \right\} + \dots \quad (\text{C.43})$$

The leading term of this contribution can be attributed to the coefficient C_c^{ab} . Indeed, using the previously obtained result for C_c^{ab} we find

$$-\frac{1}{\bar{z}_1 - \bar{z}_2} \partial_\lambda C_e^{ab}(0) \langle J^e(z_2) : J^c J^d : (\xi) \rangle = -\frac{\partial_\lambda C_e^{ab}(0)}{\bar{z}_1 - \bar{z}_2} \frac{ik f^{ecd}}{(z_2 - \xi)^3} \quad (\text{C.44})$$

$$= \frac{1}{\bar{z}_1 - \bar{z}_2} ik f^{ecd} \partial_\lambda C_e^{ab}(0) \frac{1}{\xi^3} \left[1 + \frac{3z_2}{\xi} + \frac{6z_2^2}{\xi^2} + \dots \right] \quad (\text{C.45})$$

$$= \frac{k f^{ab}{}_e f^{ecd}}{\bar{z}_1 - \bar{z}_2} \frac{1}{\xi^3} \left[1 + \frac{3z_2}{\xi} + \frac{6z_2^2}{\xi^2} + \dots \right]. \quad (\text{C.46})$$

Omitting the terms which are non-singular in $z_1 - z_2$ one is left with

$$A(X) - B(X) + \frac{1}{\bar{z}_1 - \bar{z}_2} \partial_\lambda C_e^{ab}(0) \left\langle J^e(z_2) : J^c J^d : (\xi) \right\rangle = -\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \frac{3k f^{ab}{}_g f^{gcd}}{\xi^4} \left[1 + \frac{2(z_1 + z_2)}{\xi} + \dots \right]. \quad (\text{C.47})$$

The most singular contribution here can now be cancelled by

$$-\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \partial_\lambda \tilde{C}_e^{ab}(0) \left\langle \partial J^e(z_2) : J^c J^d : (\xi) \right\rangle = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} 3\partial_\lambda \tilde{C}_e^{ab}(0) \frac{ik f^{ecd}}{(z_2 - \xi)^4} \quad (\text{C.48})$$

$$= \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} 3k f^{ab}{}_e f^{ecd} \frac{1}{\xi^4} \left[1 + \frac{4z_2}{\xi} + \dots \right], \quad (\text{C.49})$$

where the coefficient \tilde{C}_c^{ab} again has been determined previously. Since the contribution

$$-\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \partial_\lambda \hat{C}_{ef}^{ab}(0) \left\langle : J^e J^f : (z_2) : J^c J^d : (\xi) \right\rangle = -\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \frac{\partial_\lambda \hat{C}_{ef}^{ab}(0)}{(z_2 - \xi)^4} \left\{ \text{some tensor structure} \right\} \quad (\text{C.50})$$

is expected to arise at the same order, we conclude that

$$\hat{C}_{cd}^{ab}(\lambda) = 0 + \mathcal{O}(\lambda^2). \quad (\text{C.51})$$

The reasoning is identical to the reasoning leading to the vanishing of $\partial_\lambda g_{cd}^{ab}(0)$ and $\partial_\lambda w_{cd}^{ab}(0)$.

Determination of \hat{B}_c^{ab} . This coefficient may be determined by putting $X =: \bar{J}^c \bar{J}^d :$. We first evaluate

$$A(X) = \frac{1}{\pi} (-1)^{a(b+c+d)} \int d^2 z \frac{1}{(z_1 - z)^2} \left\langle \bar{J}^b(\bar{z}_2) : \bar{J}^c \bar{J}^d : (\bar{\xi}) \bar{J}^a(\bar{z}) \right\rangle_0. \quad (\text{C.52})$$

As above only the simple poles in the four-point function will contribute after integration. Using the standard procedure we can hence determine

$$\left\langle \bar{J}^b(\bar{z}_2) : \bar{J}^c \bar{J}^d : (\bar{\xi}) \bar{J}^a(\bar{z}) \right\rangle = \lim_{\bar{w} \rightarrow \bar{\xi}} \left\langle \bar{J}^b(\bar{z}_2) \bar{J}^c(\bar{w}) \bar{J}^d(\bar{\xi}) \bar{J}^a(\bar{z}) \right\rangle \quad (\text{C.53})$$

$$= \frac{k(-1)^{a(b+c+d)}}{(\bar{\xi} - \bar{z}_2)^3} \quad (\text{C.54})$$

$$\left\{ \frac{f^{ab}{}_g f^{gcd}}{\bar{z} - \bar{z}_2} + \frac{(-1)^{ba} f^{ac}{}_g f^{bgd}}{\bar{z} - \bar{\xi}} + \frac{(-1)^{a(b+c)} f^{ad}{}_g f^{bcg}}{\bar{z} - \bar{\xi}} \right\} \quad (\text{C.55})$$

$$- \frac{k(-1)^{a(b+c+d)}}{(\bar{\xi} - \bar{z}_2)^2} \left\{ \frac{(-1)^{ba} f^{ac}{}_g f^{bgd}}{(\bar{z} - \bar{\xi})^2} \right\} + \dots \quad (\text{C.56})$$

Upon integration the last term drops out, resulting in

$$A(X) = -\frac{k}{(\bar{\xi} - \bar{z}_2)^3} \frac{f^{ab}{}_g f^{gcd}}{z_1 - z_2} + \frac{k}{(\bar{\xi} - \bar{z}_2)^3 (\xi - z_1)} \left\{ (-1)^{ba} f^{ac}{}_g f^{bgd} + (-1)^{a(b+c)} f^{ad}{}_g f^{bcg} \right\} \quad (\text{C.57})$$

$$= -\frac{k f^{ab}{}_g f^{gcd}}{z_1 - z_2} \left[\frac{1}{\xi^3} + \frac{3\bar{z}_2}{\xi^4} + \frac{6\bar{z}_2^2}{\xi^5} + \dots \right] \quad (\text{C.58})$$

$$+ \pi k^2 \left[\frac{1}{\xi \bar{\xi}^3} + \frac{3\bar{z}_2}{\xi \bar{\xi}^4} + \frac{z_1}{\xi^2 \bar{\xi}^3} + \dots \right] \left\{ (-1)^{ba} f^{ac}{}_g f^{bgd} + (-1)^{a(b+c)} f^{ad}{}_g f^{bcg} \right\}. \quad (\text{C.59})$$

We then evaluate

$$B(X) = \lim_{z_1 \rightarrow z_2} A = \frac{k}{(\bar{\xi} - \bar{z}_2)^3 (\xi - z_2)} \left\{ (-1)^{ba} f^{ac} f^{bgd} + (-1)^{a(b+c)} f^{ad} f^{bcg} \right\} \quad (\text{C.60})$$

$$= \frac{k}{\xi \bar{\xi}^3} \left[1 + \frac{z_2}{\xi} + \frac{3\bar{z}_2}{\bar{\xi}} + \dots \right] \left\{ (-1)^{ba} f^{ac} f^{bgd} + (-1)^{a(b+c)} f^{ad} f^{bcg} \right\}. \quad (\text{C.61})$$

Hence the complete contribution is

$$A(X) - B(X) = -\frac{k f^{ab} f^{gcd}}{z_1 - z_2} \frac{1}{\bar{\xi}^3} \left[1 + \frac{3\bar{z}_2}{\bar{\xi}} + \frac{6\bar{z}_2^2}{\bar{\xi}^2} + \dots \right] \quad (\text{C.62})$$

$$+ \frac{k}{\xi^2 \bar{\xi}^3} \left[z_1 - z_2 + \dots \right] \left\{ (-1)^{ba} f^{ac} f^{bgd} + (-1)^{a(b+c)} f^{ad} f^{bcg} \right\}. \quad (\text{C.63})$$

This expression has to be compared to

$$-\frac{1}{z_1 - z_2} \partial_\lambda B_e^{ab}(0) \left\langle \bar{J}^e(\bar{z}_2) : \bar{J}^c \bar{J}^d : (\bar{\xi}) \right\rangle = -\frac{1}{z_1 - z_2} \partial_\lambda B_e^{ab}(0) \frac{ik f^{ecd}}{(\bar{z}_2 - \bar{\xi})^3} \quad (\text{C.64})$$

$$= \frac{1}{z_1 - z_2} k f^{ab} f^{ecd} \frac{1}{\bar{\xi}^3} \left[1 + \frac{3\bar{z}_2}{\bar{\xi}} + \frac{6\bar{z}_2^2}{\bar{\xi}^2} + \dots \right]. \quad (\text{C.65})$$

Consequently we find

$$A(X) - B(X) + \frac{\partial_\lambda B_e^{ab}(0)}{z_1 - z_2} \left\langle \bar{J}^e(\bar{z}_2) : \bar{J}^c \bar{J}^d : (\bar{\xi}) \right\rangle = \frac{k(-1)^{ab}}{\xi^2 \bar{\xi}^3} \left[z_1 - z_2 + \dots \right] \left\{ f^{ac} f^{bgd} + (-1)^{ac} f^{ad} f^{bcg} \right\}. \quad (\text{C.66})$$

We recognise that the quartic contribution expected to arise in

$$-\frac{1}{z_1 - z_2} \partial_\lambda \tilde{B}_e^{ab}(0) \left\langle \bar{\partial} \bar{J}^e(\bar{z}_2) : \bar{J}^c \bar{J}^d : (\bar{\xi}) \right\rangle = \frac{1}{z_1 - z_2} \partial_\lambda \tilde{B}_e^{ab}(0) \frac{3ik f^{ecd}}{(\bar{z}_2 - \bar{\xi})^4} = 0 \quad (\text{C.67})$$

$$-\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \partial_\lambda \hat{B}_{ef}^{ab}(0) \left\langle \bar{J}^e \bar{J}^f(\bar{z}_2) : \bar{J}^c \bar{J}^d : (\bar{\xi}) \right\rangle = -\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \frac{\partial_\lambda \hat{B}_{ef}^{ab}(0)}{(\bar{z}_2 - \bar{\xi})^4} \left\{ \text{some tensor structure} \right\} \quad (\text{C.68})$$

is absent. Consequently we find that

$$\hat{B}(\lambda) = 0 + \mathcal{O}(\lambda^2). \quad (\text{C.69})$$

This result concludes our calculation of the deformed mixed OPE between the currents J and \bar{J} at the leading order in λ .

D Computation of four-point functions

Here we give the missing details of the computation leading to (5.16). We start with formula (5.8). Each term $a^{(i)}$ contributes a term denoted $A_4^{(i)}$ upon substitution into (5.5) so that we have

$$(1 - \lambda^2) A_4 = A_4^{(0)} + A_4^{(1)} + A_4^{(2)} + A_4^{(3)} + A_4^{(4)} + A_4^{(5)} + A_4^{(6)} + A_4^{(7)}. \quad (\text{D.1})$$

To compute $A_4^{(1)}$ we need the integral

$$I_1 = \int \frac{d^2 w}{z_1 - w} \frac{1}{(z_2 - w)^2} \frac{\bar{z}_{34}}{(z_{34})^2 (\bar{z}_3 - \bar{w})(\bar{z}_4 - \bar{w})} = \frac{\pi}{z_{34}^2 z_{12}^2} \ln \left| \frac{z_{13} z_{24}}{z_{23} z_{14}} \right|^2 - \frac{\pi}{z_{34} z_{12} z_{23} z_{24}}. \quad (\text{D.2})$$

We obtain

$$\begin{aligned}
A_4^{(1)} &= \frac{(1-\lambda)}{1-\lambda^2} \left(\frac{-\lambda}{k\pi} \right) (-1)^{a(b+c+d)} \int \frac{d^2w}{z_1-w} i f_{re}^a (-1)^{b(c+d)} \frac{k\kappa^{be}}{(z_2-w)^2} \langle J^c(z_3) J^d(z_4) \bar{J}^r(\bar{w}) \rangle_\mu \\
&= \frac{(1-\lambda)}{1-\lambda^2} \left(\frac{\lambda}{k\pi} \right) (-1)^{a(b+c+d)} k^2 \kappa^{be} (-i f^{cdr}) \frac{\lambda(1-\lambda)}{(1-\lambda^2)^3} (i f_r^a)^b (-1)^b I_1 \\
&= -k \frac{\lambda^2(1-\lambda)^2}{(1-\lambda^2)^4} f_{re}^{ab} f^{rcd} \left[\frac{1}{z_{34}^2 z_{12}^2} \ln \left| \frac{z_{13} z_{24}}{z_{23} z_{14}} \right|^2 - \frac{1}{z_{34} z_{12} z_{23} z_{24}} \right], \tag{D.3}
\end{aligned}$$

Furthermore a straightforward computation yields

$$\begin{aligned}
A_4^{(2)} &= \frac{\lambda^2(1-\lambda)}{1-\lambda^2} (-1)^{a(b+c+d)} i f_{re}^a \int \frac{d^2w}{z_1-w} (-1)^{b(c+d+e+r)} \kappa^{rb} \delta(z_2-w) \\
&\quad \times \langle J^c(z_3) J^d(z_4) J^e(w) \rangle_\lambda = k \frac{\lambda^2(1-\lambda^3)(1-\lambda)}{(1-\lambda^2)^4} \frac{f_{re}^{ab} f^{ecd}}{z_{12} z_{23} z_{34} z_{42}}. \tag{D.4}
\end{aligned}$$

Up to now all formulas were exact. Now we will start using the abelian approximation for the remaining correlators entering $A_4^{(3)}$, $A_4^{(4)}$, $A_4^{(5)}$, $A^{(6)}$, $A^{(7)}$. In our approximation we have

$$\langle J^c(z_3) J^d(z_4) : J^s(w) \bar{J}^r(\bar{w}) : \rangle_\lambda = -\frac{\pi k^2 \lambda}{(1-\lambda^2)^2} \left[\kappa^{cr} \kappa^{ds} \frac{\delta(z_3-w)}{(z_4-w)^2} + (-1)^{cd} \kappa^{cs} \kappa^{dr} \frac{\delta(z_4-w)}{(z_3-w)^2} \right] \tag{D.5}$$

Using this formula we obtain

$$A_4^{(3)} = -k \frac{(1-\lambda)\lambda^2}{(1-\lambda^2)^3} \left[\frac{(-1)^{ab} f_{ac}^e f^{bed}}{z_{13} z_{23} z_{43}^2} - \frac{(-1)^{a(b+c)} f_{ad}^e f^{bce}}{z_{14} z_{24} z_{34}^2} \right], \tag{D.6}$$

$$A_4^{(4)} = k \frac{(1-\lambda)\lambda^3}{(1-\lambda^2)^3} \left[\frac{(-1)^{ab} f_{ac}^e f^{bed}}{z_{14} z_{24} z_{43}^2} - \frac{(-1)^{a(b+c)} f_{ad}^e f^{bce}}{z_{13} z_{23} z_{34}^2} \right]. \tag{D.7}$$

To compute the remaining term $A_4^{(5)}$ we need to compute the correlator

$$\langle J^c(z_3) J^d(z_4) : J^e \bar{J}^r : (w, \bar{w}) : J^s \bar{J}^q : (w_2, \bar{w}_2) \rangle_\lambda \tag{D.8}$$

in the abelian approximation. There are eight non-vanishing contraction schemes contributing to $A_4^{(5)}$:¹²

$$A = \langle (J^c [J^d : J^e] \{ \bar{J}^r : : J^s \} \bar{J}^q) : \rangle \quad B = \langle (J^c [J^d : J^e] \{ \bar{J}^r : : J^s \} \bar{J}^q) : \rangle \tag{D.9}$$

$$C = \langle (J^c [J^d : J^e] \{ \bar{J}^r : : J^s \} \bar{J}^q) : \rangle \quad D = \langle (J^c [J^d : J^e] \{ \bar{J}^r : : J^s \} \bar{J}^q) : \rangle \tag{D.10}$$

$$E = \langle (J^c [J^d : \{ J^e \bar{J}^r \} : J^s] \bar{J}^q) : \rangle \quad F = \langle (J^c [J^d : \{ J^e \bar{J}^r \} : J^s] \bar{J}^q) : \rangle \tag{D.11}$$

$$G = \langle (J^c [J^d : \{ J^e \bar{J}^r \} : J^s] \bar{J}^q) : \rangle \quad H = \langle (J^c [J^d : \{ J^e \bar{J}^r \} : J^s] \bar{J}^q) : \rangle, \tag{D.12}$$

where each of the 3 contractions we mark by a pair of brackets: $\{ J J \}$, $(J J)$, $[J, J]$. The corresponding contributions to $A_4^{(5)}$ are

$$A_4^{(5)} = \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D} + \tilde{E} + \tilde{F} + \tilde{G} + \tilde{H}. \tag{D.13}$$

Evaluating the integrals in $\tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}, \tilde{G}, \tilde{H}$ is straightforward because each integrand contains two delta functions. Evaluating \tilde{A} involves

$$\begin{aligned}
I_2 &= \iint \frac{d^2w d^2w_2}{(z_1-w)(z_2-w_2)(z_4-w)^2(z_3-w_2)^2(\bar{w}-\bar{w}_2)^2} \\
&= \frac{\pi^2}{z_{14} z_{24} z_{34}^2} - \frac{\pi^2}{z_{14}^2 z_{23}^2} \ln \left| \frac{z_{24} z_{13}}{z_{12} z_{34}} \right|^2 + \frac{\pi^2}{z_{14} z_{32} z_{31} z_{34}}. \tag{D.14}
\end{aligned}$$

¹²Some contraction schemes drop out upon contraction with the f_{bc}^a tensors present in $A_4^{(5)}$.

The integral emerging in the \tilde{B} contribution is obtained by interchanging z_4 and z_3 . We obtain

$$\begin{aligned} \tilde{A} + \tilde{B} = k \frac{\lambda^2(1-\lambda)^2}{(1-\lambda^2)^4} & \left[(-1)^{a(b+c)} f^{ad}_r f^{bcr} \left(\frac{1}{z_{14}z_{24}z_{34}^2} - \frac{1}{z_{14}^2z_{23}^2} \ln \left| \frac{z_{24}z_{13}}{z_{12}z_{34}} \right|^2 + \frac{1}{z_{14}z_{32}z_{31}z_{34}} \right) \right. \\ & \left. + (-1)^{ab} f^{ac}_r f^{brd} \left(\frac{1}{z_{13}z_{23}z_{34}^2} - \frac{1}{z_{13}^2z_{24}^2} \ln \left| \frac{z_{23}z_{14}}{z_{12}z_{34}} \right|^2 + \frac{1}{z_{31}z_{42}z_{41}z_{34}} \right) \right] \end{aligned} \quad (\text{D.15})$$

We further obtain

$$\tilde{E} + \tilde{G} = k \frac{\lambda^4(1-\lambda)^2}{(1-\lambda^2)^4} \frac{(-1)^{a(b+c)} f^{ad}_s f^{bcs}}{z_{14}z_{23}z_{34}z_{42}}, \quad (\text{D.16})$$

$$\tilde{H} + \tilde{F} = k \frac{\lambda^4(1-\lambda)^2}{(1-\lambda^2)^4} \frac{(-1)^{ab} f^{ac}_s f^{bsd}}{z_{13}z_{23}z_{34}z_{42}}, \quad (\text{D.17})$$

$$\tilde{C} + \tilde{D} = k \frac{\lambda^4(1-\lambda)^2}{(1-\lambda^2)^4} \left[\frac{(-1)^{a(b+c)} f^{ad}_s f^{bcs}}{z_{13}z_{23}z_{43}^2} - \frac{(-1)^{ab} f^{ac}_s f^{bsd}}{z_{14}z_{24}z_{43}^2} \right]. \quad (\text{D.18})$$

Collecting all terms proportional to $1/(z_{14}z_{24}z_{34}^2)$ and $1/z_{13}z_{23}z_{34}^2$ from $A_4^{(3)}$, $A_4^{(4)}$, $A_4^{(5)}$ we obtain

$$\begin{aligned} & k \frac{\lambda^3(1-\lambda)^2}{(1-\lambda^2)^4} \left[(-1)^{ab} f^{ac}_s f^{bsd} + (-1)^{a(b+c)} f^{ad}_s f^{bcs} \right] \left(\frac{1}{z_{14}z_{24}z_{43}^2} - \frac{1}{z_{13}z_{23}z_{43}^2} \right) \\ & = -k \frac{\lambda^3(1-\lambda)^2}{(1-\lambda^2)^4} \left[(-1)^{ab} f^{ac}_s f^{bsd} + (-1)^{a(b+c)} f^{ad}_s f^{bcs} \right] \left(\frac{1}{z_{13}z_{24}z_{23}z_{34}} + \frac{1}{z_{24}z_{34}z_{14}z_{13}} \right). \end{aligned} \quad (\text{D.19})$$

Here we used the identities

$$\frac{\lambda^2(1-\lambda)^2}{(1-\lambda^2)^4} - \frac{(1-\lambda)\lambda^2}{(1-\lambda^2)^3} = -\frac{\lambda^3(1-\lambda)^2}{(1-\lambda^2)^4}, \quad (\text{D.20})$$

$$\frac{\lambda^4(1-\lambda)^2}{(1-\lambda^2)^4} - \frac{(1-\lambda)\lambda^3}{(1-\lambda^2)^3} = -\frac{\lambda^3(1-\lambda)^2}{(1-\lambda^2)^4}. \quad (\text{D.21})$$

We also compute

$$A_4^{(6)} = k \frac{\lambda^2(1-\lambda)}{(1-\lambda^2)^3} (-1)^{a(b+c)} \frac{f^{ad}_s f^{bcs}}{z_{23}z_{34}z_{13}z_{14}}, \quad (\text{D.22})$$

$$A_4^{(7)} = k \frac{\lambda^2(1-\lambda)}{(1-\lambda^2)^3} (-1)^{ab} \frac{f^{ac}_s f^{bsd}}{z_{24}z_{34}z_{13}z_{14}}. \quad (\text{D.23})$$

Collecting all terms we obtain

$$(1-\lambda^2)A_4 = A_4^{(0)} + L_4 + R_4 \quad (\text{D.24})$$

where the logarithmic part L_4 is given by

$$\begin{aligned} L_4 = k \frac{\lambda^2(1-\lambda)^2}{(1-\lambda^2)^4} & \left[-f^{ab}_r f^{rcd} \frac{1}{z_{34}^2z_{12}^2} \ln \left| \frac{z_{13}z_{24}}{z_{23}z_{14}} \right|^2 \right. \\ & \left. + (-1)^{a(b+c)} f^{ad}_r f^{bcr} \frac{1}{z_{14}^2z_{23}^2} \ln \left| \frac{z_{24}z_{13}}{z_{12}z_{34}} \right|^2 - (-1)^{ab} f^{ac}_r f^{brd} \frac{1}{z_{13}^2z_{24}^2} \ln \left| \frac{z_{23}z_{14}}{z_{12}z_{34}} \right|^2 \right] \end{aligned} \quad (\text{D.25})$$

and the additional rational part R_4 is

$$R_4 = k \frac{\lambda^3(1-\lambda)}{(1-\lambda^2)^3} \left[\frac{f^{ab}_r f^{rcd}}{z_{12}z_{23}z_{34}z_{42}} + \frac{(-1)^{ab} f^{ac}_s f^{bsd}}{z_{13}z_{23}z_{34}z_{42}} + \frac{(-1)^{a(b+c)} f^{ad}_s f^{bcs}}{z_{14}z_{23}z_{34}z_{42}} \right] \quad (\text{D.26})$$

where the first term in the square brackets contains contributions from $A_4^{(1)}$ and $A_4^{(2)}$. These terms combine together with $A_4^{(0)}$ so that we finally get (5.16).

E The OPE closure in the $G \times G$ -preserving deformation

In this appendix we give further considerations regarding the issue of the current algebra closure for the $G \times G$ -preserving deformation. This deformation is generated by the operator $:J^a \phi_{ab} \bar{J}^b:$ that involves the adjoint representation primary field ϕ_{ab} . Such a field satisfies the following OPEs with the WZW currents

$$J^a(z) \phi_{bc}(w, \bar{w}) \sim \frac{i f^a{}_{bd} \phi_c^d}{z-w}, \quad \bar{J}^a(\bar{z}) \phi_{bc}(w, \bar{w}) \sim (-1)^{ab} \frac{i f^a{}_{cd} \phi_b^d}{\bar{z}-\bar{w}}. \quad (\text{E.1})$$

In the deformed theory the global symmetry group is $G \times G$. The components of the Noether current associated with the left action of G on itself can be classically written as $K_z = -k \partial g g^{-1}$ and $K_{\bar{z}} = k \bar{\partial} g g^{-1}$. At the quantum level these components reduce to the operators

$$K_z^a(z) = J^a(z), \quad K_{\bar{z}}^a = \kappa^{ab} : \phi_c^a \bar{J}^c : (z, \bar{z}) \quad (\text{E.2})$$

at the WZW point. The Knizhnik-Zamolodchikov equations for the field ϕ_{ab} read

$$\partial \phi_{ab}(z, \bar{z}) = \frac{i}{k} f_{aep} : J^p \phi_b^e : (z, \bar{z}), \quad (\text{E.3})$$

$$\bar{\partial} \phi_{ab}(z, \bar{z}) = \frac{i}{k} f_{bep} (-1)^{ap} : \bar{J}^p \phi_a^e : (z, \bar{z}). \quad (\text{E.4})$$

Equation (E.3) implies the Maurer-Cartan equation

$$\partial K_{\bar{z}}^a = \frac{i}{k} f^a{}_{bc} : K_z^c K_{\bar{z}}^b : . \quad (\text{E.5})$$

The assumption of the OPE closure in the current algebra leads to the OPE (formula (2.21) of [23])

$$K_{\bar{z}}^a(z, \bar{z}) K_{\bar{z}}^b(0) \sim \frac{\kappa^{ab} k}{\bar{z}^2} - 2i f^{ab}{}_c \frac{K_{\bar{z}}^c(0)}{\bar{z}} + i \frac{(z-w)}{(\bar{z}-\bar{w})^2} f^{ab}{}_c J^c(w) + \text{less singular terms}. \quad (\text{E.6})$$

We can check this OPE using formula (E.2). The OPE of the field ϕ_{ab} with itself is not known in detail, however we do know the group-theoretic content and can estimate the singularities present in the OPE in the large k limit. One finds that possible power singularities go as $|z-w|^{-1/k}$ and thus are very mild for large k 's. The leading and subleading singularities in the OPE of $K_{\bar{z}}^a$ with $K_{\bar{z}}^b$ then come from the singularities in the OPEs of \bar{J}^c with themselves and with the fields ϕ_{ab} . We obtain

$$\begin{aligned} : \phi_e^a \bar{J}^e : (z, \bar{z}) : \phi_p^b \bar{J}^p : (w, \bar{w}) &\sim \frac{k}{(\bar{z}-\bar{w})^2} \kappa^{pe} (-1)^{eb} : \phi_e^a \phi_p^b : (w, \bar{w}) \\ &\quad - 2i \frac{1}{\bar{z}-\bar{w}} f_{er}^p (-1)^{p(p+b)} : (\phi^{ar} \bar{J}^e) \phi_p^b : (w, \bar{w}) \\ &\quad + i \frac{(z-w)}{(\bar{z}-\bar{w})^2} f_{rs}^a (-1)^{pe} \kappa^{pe} : (J^s \phi_e^r) \phi_p^b : (w, \bar{w}) + \dots, \end{aligned} \quad (\text{E.7})$$

where we used (A.9) to get rid of the terms containing two factors of the structure constants. Matching the singularities in (E.7) with those in (E.6) we obtain the equations¹³

$$\kappa^{pe} (-1)^{eb} : \phi_e^a \phi_p^b : (w, \bar{w}) = \kappa^{ab} \mathbf{1} \quad (\text{E.8a})$$

$$f_{er}^p (-1)^{p(p+e)} : (\phi^{ar} \bar{J}^e) \phi_p^b : (w, \bar{w}) = f_{ec}^{ab} : \phi_e^c \bar{J}^e : (w, \bar{w}), \quad (\text{E.8b})$$

$$f_{rs}^a (-1)^{pe} \kappa^{pe} : (J^s \phi_e^r) \phi_p^b : (w, \bar{w}) = f_{ec}^{ab} J^c(w). \quad (\text{E.8c})$$

¹³An equation rather similar to our equation (E.8a) also appeared in [25]. See formulae (3.8), (3.9) in that paper.

In equation (E.8b) we can rearrange the normal ordering on the left hand side as

$$f_{er}^p (-1)^{p(p+b)} :(\phi^{ar} \bar{J}^e) \phi_p^b : (w, \bar{w}) = f_{er}^p (-1)^{r(p+b)} :(\phi^{ar} \phi_p^b) \bar{J}^e : (w, \bar{w}). \quad (\text{E.9})$$

This can be done because by virtue of (A.9) the operator \bar{J}^e has no singularities with the other two operators in that expression. This suggests the following relation

$$- f^{sr} (-1)^{rb} : \phi_r^a \phi_s^b : (w, \bar{w}) = : f^{ab} \phi_p^c : (w, \bar{w}). \quad (\text{E.10})$$

Relations (E.8a) and (E.10) have classical analogues. The classical analogue of operator ϕ_{ab} is the matrix Ad_g . This matrix satisfies the equations $\text{Ad}_g \text{Ad}_{g^{-1}} = 1$ and $\text{Ad}_g[X, Y] = [\text{Ad}_g(X), \text{Ad}_g(Y)]$ which are analogous to quantum equations (E.8a) and (E.10). On the quantum level, however, their validity is far from obvious. It is easy to observe that such equations can only hold when the field ϕ_{ab} has dimension zero, and thus do not hold for the WZW theories built on ordinary semisimple Lie groups or supergroups with non-vanishing Killing form. The meaning of relation (E.8c) is less clear to us.¹⁴

We conclude that identities (E.8a), (E.8b), (E.8c) are non-trivial necessary conditions for the closure of the OPEs of the currents $K_z, K_{\bar{z}}$. It would be interesting to verify these identities directly as this would test the bootstrap approach suggested in [23, 25].

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¹⁴ If one could rearrange the terms as in (E.9) equation (E.8c) would then follow from (E.8a). However direct investigation of singularities shows that such rearrangement is not possible in this case.

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