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Holographic dual of free field theory

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We derive a holographic dual description of free quantum field theory in arbitrary dimensions, by reinterpreting the exact renormalization group, to obtain a higher spin gravity theory of the general type which had been proposed and studied as a dual theory. We show that the dual theory reproduces all correlation functions.

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INTRODUCTION

One of the most striking and unexpected discoveries of the 1994-98 “second superstring revolution” was the AdS/CFT correspondence [1], according to which $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions is dual to type IIb superstring theory on $AdS_5 \times S^5$. Since then, the correspondence has been much generalized and has found many applications, especially in providing simple models exhibiting nonperturbative physical phenomena such as confinement, dissipation and quantum phase transitions. However, despite a good deal of work, the microscopic workings of the duality are not well understood. In no case has there been a first principles derivation.

In this work, we derive a gravity dual to free field theory. Free scalar field theory is conjectured [2, 3] to be holographically dual to higher spin gravity as developed by M. Vasiliev and other authors [4], and nontrivial checks of this conjecture were made in [5, 6]. By standard large N arguments, the same dual formulation should describe the large N limit of the $O(N)$ model as well [3].

It is widely believed that AdS/CFT is at heart a geometric reformulation of the renormalization group (RG), in which the renormalization scale becomes an extra ‘radial’ dimension. Various explanations of this idea have been given, such as the holographic RG [7], and a mixed holographic/Wilsonian RG [8], while attempts at a precise reformulation were made in [9]. Here we begin by reviewing the exact RG.

Exact RG equations: We study the theory of N free complex scalar fields, denoted $\phi^A(x)$, in D dimensions. The bare action will be a sum of a standard (two derivative) kinetic term, and a $U(N)$-invariant interaction term with arbitrary position and momentum dependence,

$$S=\sum_A \int d^Dx\,|\partial\phi^A(x)|^2-\int d^Dx d^Dy\, B(x,y)\bar{\phi}^A(x)\phi^A(y).$$

Going to momentum variables p, q, and writing $\phi^A(p)$ for the Fourier transform, the Wilsonian effective action at energy scale Λ is

$$S=\int d^Dp\, d^Dq \Big\{ P(p,q)-B(p,q)\Big\} \bar{\phi}^A(q)\phi^A(p)\,, \quad (1)$$

with a cutoff kinetic term

$$P=p^2K^{-1}(p^2/\Lambda^2)\delta^{(D)}(p-q)\,. \quad (2)$$

The cutoff function K is chosen so that the propagator vanishes for high momenta and goes to $1/p^2$ for small momenta. We also define

$$\alpha=\frac{d_\Lambda K(p^2/\Lambda^2)}{p^2}\,\delta^{(D)}(p-q)\,. \quad (3)$$

Applying the standard derivation of Wick’s theorem from the functional integral, and taking a derivative with respect to Λ, one obtains an exact RG equation [10]. Since the theory is free, under RG flow the effective action remains quadratic in the fields. The coupling B flows as

$$\begin{aligned} d_\Lambda B(p,q) &= \quad (4)\\ &-\int d^Ds \frac{1}{s^2}\frac{\partial K(s^2/\Lambda^2)}{\partial\Lambda}\, B(p,s)\, B(s,q)\,. \end{aligned}$$

There is also a constant term F, satisfying the flow equation

$$d_\Lambda F = N\int d^Dpd^Dq\,\alpha(p,q)\,(P(q,p)+B(q,p))\,. \quad (5)$$

Integrating this flow down to $\Lambda\to 0$, one obtains the free energy.

Connected correlation functions of the bilocal operator $\bar{\phi}^A(x)\phi^A(y)$ can be obtained as functional derivatives of the free energy with respect to $B(x,y)$. Of course, since this is a free theory, there is an explicit expansion for it,

$$\begin{aligned} F &= -N\,\mathrm{Tr}\log(P-B)\\ &= -N\,\mathrm{Tr}\log P + N\sum_{n\geq 1}\frac{1}{n}\mathrm{Tr}\left(P^{-1}\cdot B\right)^n\,, \quad (6)\end{aligned}$$

where P^{-1} is the Green function, and Tr and $\cdot$ represent integration and products of kernels. Terms in this expansion correspond to one-loop diagrams with vertices taken from the interaction B. Our question is, what does this have to do with anti-de Sitter space and gravity?

RG AS EQUATIONS OF MOTION ON ADS

We now rewrite the RG flow equation (4) as an equation of motion for fields propagating in an AdS_{D+1} space with radial coordinate

$$r = \frac{1}{\Lambda}. \tag{7}$$

The other coordinates of AdS_{D+1} are 'reference coordinates' whose relation to the original space-time coordinates will be explained below. The fields on AdS_{D+1} will be a field B derived from the couplings B, and a connection W in the 'higher spin gauge algebra $hs(D-1,2)$, also to be defined below.

The first step is to reformulate (4) in terms of operators B and α with a simple multiplication law. To make contact with higher spin gravity as presented in [4], we will use an explicit representation in which operators are represented by symbols and operator products are represented by the Moyal star product. From now on we will discuss dimensionless B, i.e. $B \to B(\frac{p}{\Lambda},\frac{q}{\Lambda})\Lambda^{2-D}$. We Taylor expand the sources $B(p/\Lambda, q/\Lambda)$ in the momentum variables

$$\begin{aligned} B(p/\Lambda,q/\Lambda) &= \sum_{s,t=0}^{\infty} \Lambda^{-s-t}\, B_{a_1\ldots a_s,b_1\ldots b_t} p^{a_1}\!\ldots p^{a_s} q^{b_1}\!\ldots q^{b_t} \\ &\equiv \Lambda^{-s-t}\, B_{\underline{st}}\, p^{\underline{s}} q^{\underline{t}} \;, \end{aligned}$$

where the indices a_i and b_i take values in $\{0,\ldots,D-1\}$. We then define

$$\alpha^{\underline{st}} = \Lambda^{2-D-s-t} \int d^D p \int d^D q\ \alpha(p,q) p^{\underline{s}} q^{\underline{t}} , \tag{8}$$

so that the RG flow equation (4) becomes

$$\frac{d}{d\Lambda} B_{\underline{st}} = -B_{\underline{si}}\, \alpha^{\underline{ij}}\, B_{\underline{jt}} + \Lambda^{-1}(s+t+2d_\phi)\, B_{\underline{st}}\,, \tag{9}$$

where $d_\phi = \frac{D-2}{2}$ is the conformal dimension of ϕ^A.

Now, an RG flow equation expresses an identification between theories with the same physics, written in terms of actions defined at infinitesimally different energy scales. Mathematically, such an infinitesimal relation should be expressed by a connection on the space of actions. In fact it is simple to reinterpret (9) in this way. Define a connection one-form, whose only component is

$$\begin{aligned} (W_\Lambda)_{\underline{s}}{}^{\underline{j}} &= B_{\underline{si}}\, \alpha^{\underline{ij}} - s\Lambda^{-1} \delta_{\underline{s}}{}^{\underline{j}} \;, \\ (\widetilde{W}_\Lambda)^{\underline{k}}{}_{\underline{t}} &= -t\Lambda^{-1} \delta^{\underline{k}}{}_{\underline{t}} \;, \end{aligned} \tag{10}$$

then (9) becomes

$$0 = \frac{d}{d\Lambda}\Lambda^{-2d_\phi} B_{\underline{st}} + (W_\Lambda)_{\underline{s}}{}^{\underline{j}}\, \Lambda^{-2d_\phi} B_{\underline{jt}} + \Lambda^{-2d_\phi} B_{\underline{sj}} (\widetilde{W}_\Lambda)^{\underline{j}}{}_{\underline{t}} . \tag{11}$$

Our description of the action also depends on a choice of spatial reference point. By rewriting the position space interaction term as

$$\int d^D x' d^D x\ B(a+x, a+x') \bar{\phi}^A(a+x) \phi^A(a+x'),$$

one sees that an overall shift symmetry $x \to x+a$ acts as

$$\frac{d}{da_i}\, B_{\underline{st}} p^{\underline{s}} q^{\underline{t}} = i(p^i - q^i)\, B_{\underline{st}} p^{\underline{s}} q^{\underline{t}}. \tag{12}$$

We will also interpret this as a connection on the space of actions. Thus, we need to reinterpret the right hand sides of (9) and (12) as the action of a gauge algebra. Mathematically, this will be an algebra of pseudodifferential operators. But here, motivated by the eventual contact with higher spin gravity, we will define the gauge algebra as the Lie algebra associated to an associative algebra, defined by a Moyal star product.

We introduce oscillators (formal auxiliary variables) y^α, $\bar{y}_\alpha$, z^α and $\bar{z}_\alpha$, where $\alpha \in \{\bullet, r, 0, 1, \ldots, D-1\}$, satisfying the Moyal star product [4]

$$\begin{aligned} &(f * g)(z,y) = \\ &\frac{1}{\pi^{2(D+2)}} \int ds dt e^{-2s\cdot\bar{t} - 2\bar{s}\cdot t} f(z+s, y+s) g(z-t, y+t)\ . \end{aligned} \tag{13}$$

The metric on this auxiliary space is $\hat{\eta}_\alpha^\beta = (-1, 1, \eta)$, where η is the metric on the original flat space-time. We further define for $a \in \{0,..,D-1\}$

$$\begin{aligned} y^a &= Y^a + Z^a, \qquad \bar{y}_a = \frac{1}{2}(\bar{Y}_a - \bar{Z}_a), \\ z^a &= Z^a - Y^a, \qquad \bar{z}_a = \frac{1}{2}(\bar{Y}_a + \bar{Z}_a)\,. \end{aligned} \tag{14}$$

Then the field B and the kernel α are defined to be functions of the auxiliary variables, derived from the coupling B and cutoff propagator variation α as (we trade momentum p with iY/r and q with $-iZ/r$)

$$\begin{aligned} &B(y,z,\bar{y},\bar{z}) = i^{s-t} r^{D-2}\, B_{\underline{st}}\, Y^{\underline{s}} Z^{\underline{t}} e^{-Y\bar{Y} - Z\bar{Z}}\, (\bar{z}_r - \bar{z}_\bullet)^{s+t} , \\ &\alpha_\mu(y,z,\bar{y},\bar{z}) = \\ &\quad -\frac{(-i)^{t-s}}{s!t!} r^{-D}\, \alpha_\mu^{\underline{st}}\, \bar{Y}_{\underline{t}}\, \bar{Z}_{\underline{s}}\, e^{-Y\bar{Y} - Z\bar{Z}}\, (\bar{z}_r - \bar{z}_\bullet)^{-s-t} , \end{aligned} \tag{15}$$

where $\mu \in \{r, 0, \ldots, D-1\}$. To rewrite the equations we just derived, from a starting point with an explicit translation action on the coordinates, we can take $\alpha_r = \alpha$ and $\alpha_a = 0$. This choice can be generalized as will emerge below.

Standard connection on AdS: The group of linear transformations on each set of auxiliary variables preserving the metric η_α^β is $SO(D-1,2)$, the group of isometries of AdS_{D+1}. As is well known, we can represent the corresponding Lie algebra as star commutators with generators which are quadratic functions of the oscillators. The dilatation and the translation generators are

$$\begin{aligned} P_r &= \bar{z}_r z^\bullet - \bar{z}_\bullet z^r, \\ P_a &= \bar{z}_a\, (z^\bullet - z^r) - (\bar{z}_\bullet - \bar{z}_r)\, z^a\,. \end{aligned} \tag{16}$$

These can be used to define a connection

$$W_\mu^{(0)} = \frac{1}{r}\, P_\mu\,, \tag{17}$$

which due to the commutation relations satisfied by P_μ is flat

$$dW^{(0)} + W^{(0)} \wedge * W^{(0)} = 0\,. \tag{18}$$

Now, there is a well-known way to rewrite theories of gravity, not in terms of a metric, but in terms of a connection acting on the frame bundle (see for example [11]). The connection (17) is the one corresponding to the AdS_{D+1} metric in the Poincare patch,

$$ds^2 = \frac{dr^2 + dx^a dx^a}{r^2}\,. \tag{19}$$

The standard formulations of higher spin gravity are also in terms of a connection, now living in an infinite dimensional algebra $hs(D-1,2)$ which contains $SO(D-1,2)$. By using (16) to define (17), we have postulated just this structure. Of course this is just kinematic, a particular way to describe AdS_{D+1}.

RG as connection on AdS: Returning to (15), the field B was dressed with the auxiliary variables in a way so that $[W^{(0)}, B]_*$ reproduces the linear term in B in the RG equations (9) and (12). In particular

$$[W_r^{(0)}, (\bar{z}_r - \bar{z}_\bullet)^{s+t}]_* = \frac{s+t}{r}(\bar{z}_r - \bar{z}_\bullet)^{s+t}, \tag{20}$$
$$[W_a^{(0)}, B]_* = \frac{y^a}{r}(\bar{z}_r - \bar{z}_\bullet)B - \frac{z_r - z_\bullet}{r}\frac{\partial}{\partial z^a}B.$$

The last term in the second of these equations does not appear in (12), since in the field theory the components of B do not have $\bullet$ and r indices. The cut-off kernel α we have introduced is consistent with momentum conservation. With the notations introduced above, this fact translates into the equation

$$d\alpha + W^{(0)} \wedge * \alpha + \alpha \wedge * W^{(0)} = 0\,. \tag{21}$$

In fact, this equation can be satisfied more generally, which allows using position dependent cutoffs, or working on space-times without translation invariance.

Finally, we define a fluctuation of the connection

$$(\delta\widetilde{W}_\mu)^{\underline{p}}{}_{\underline{q}} = 0, \qquad (\delta W_\mu)_{\underline{q}}{}^{\underline{p}} = B_{\underline{qr}}\,\alpha^{\underline{rp}}{}_\mu\,. \tag{22}$$

With these definitions, the RG equations take the appealing form

$$\frac{d}{dx_\mu}B + W_\mu * B - B * \widetilde{W}_\mu = 0\,, \tag{23}$$

where $W = W^{(0)} + \delta W$ and $\widetilde{W} = W^{(0)} + \delta\widetilde{W}$ Moreover, by right star multiplication of the above equation by α_ν and antisymmetrization with respect to the space-time indices, we obtain that the total connection is flat

$$dW + W \wedge * W = 0\,, \quad d\widetilde{W} + \widetilde{W} \wedge * \widetilde{W} = 0\,. \tag{24}$$

The equations (23) and (24) admit the standard gauge transformations

$$\delta W = d\epsilon + [W, \epsilon]_*\,, \qquad \delta\widetilde{W} = d\widetilde{\epsilon} - \left[\widetilde{W}, \widetilde{\epsilon}\right]_*\,, \tag{25}$$
$$\delta B = B * \widetilde{\epsilon} - \epsilon * B\,\,.$$

and this is a gauge theory with gauge algebra $hs(D-1,2)$ as defined above. In gauge theory terms, these equations express the covariant constancy of a bulk field B on AdS_{D+1} under transport by a flat connection W. Conceptually, their field theoretic origin is clear. The AdS space parameterizes choices which must be made to define the RG; an infinitesimal relation between equivalent actions should be expressed by transport by a connection. If we vary the RG scale and the reference point along a closed loop in AdS we must recover the same action, so the connection must be flat.

In the standard formulations of higher spin gravity [4], one has the same connection W_μ, and the higher spin fields are obtained by expanding it in the auxiliary variables. The equation (24) is an equation of motion, whose linearization describes propagation of higher spin fields in AdS_{D+1}. The field B encodes, among others, the matter field coupled to the higher spin gauge fields and satisfies (23) or a similar equation of motion. Thus, we have reformulated the RG flow for D-dimensional free field theory in the terms of higher spin gravity in $D+1$ dimensions.

Finally, we obtain the solutions which correspond to RG flows by imposing (22), *i.e.* the relation $W = B * \alpha$. This relation is not gauge invariant; while formally one can postulate a transformation law for α which would make it so, this requires taking the inverse B^{-1}, which may not exist. As a relation between one-forms, some of it may correspond to a gauge fixing condition, while other parts have suggestive analogs in higher spin gravity.

Action: One may reasonably ask what action gives rise to these equations. Actually this problem has not been solved for the standard higher spin gravity theories; the equations (23) and (24) do not naturally come from an action in dimensions $D+1 \geq 4$. One can still postulate an action whose variational equations include these equations, most simply by postulating a Lagrange multiplier λ for each equation. Such an action will be zero evaluated on a solution, *i.e.* on-shell.

From our derivation, the on-shell action of our dual theory is the sum of such a zero on-shell action, and the holonomy of the $U(1)$ part of the connection,

$$S_{bulk} = -N\,\mathrm{Tr}\int dx^\mu\,\delta W_\mu + S_{on-shell}, \tag{26}$$

integrated along a contour which runs from a point on the boundary to $r = \infty$. This follows from the expression (5) for the free energy of the field theory and the identification (22). Mathematically, it follows from the identification of $\mathrm{Tr}W_\mu$ as a connection on the determinant line bundle of operators $-P + B$.

Since the connection is flat, one can deform the contour and obtain the same result. One can also vary the choice of base point on the boundary; this corresponds to a gauge transformation.

Correlators: Correlators in AdS/CFT correspondence are obtained by varying the bulk action (26) with respect to the sources. We take the contour to extend between two boundary points and take the limit of one point going to infinity.

When evaluated on-shell the action of our bulk theory is equal to a holonomy. The gauge transformations change W_μ and thus change the sources. To compute the holonomy one has to evaluate the connection W_μ in the bulk.

This task can be achieved by solving the equations of motion perturbatively in the sources (i.e. by Witten diagrams). We define a perturbative expansion of B as

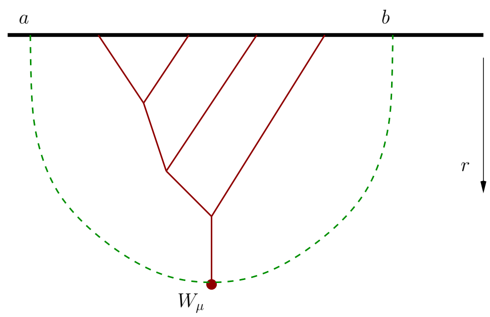

FIG. 1. A graphic representation of the calculation of a four point correlator. The dashed line is the holonomy contour. a and b are two points on the boundary; one of which is taken to infinity. The (brown) lines represent boundary-to-bulk and bulk-to-bulk propagators.

$$B = B^{(0)} + B^{(1)} + B^{(2)} + \ldots . \tag{27}$$

Here $B^{(0)}$ is the solution to the linearized equation (23),

$$\frac{d}{dx^\mu} B^{(0)} + W^{(0)}_\mu * B^{(0)} - B^{(0)} * \widetilde{W}^{(0)}_\mu = 0 . \tag{28}$$

One defines the boundary-to-bulk propagator, $K(x, x', r)$ as a solution to the above equation with δ-function boundary condition, and the bulk-to-bulk propagator, $G^\nu(x, x', r, r')$, as a solution with $\delta^{(D)}(x - x')\delta(r - r')\eta^\nu{}_\mu$ source on the right-hand side.

The solution to the linearized equations of motion for the fluctuations (22) is given by $\delta B = g^{-1} * b * dg$ with

$$g(x; z) = P \exp_* \left(- \int_x^{x_0} W^{(0)}_\mu dx'^\mu \right) , \tag{29}$$

with $W^{(0)}$ as in (17). By taking a straight contour with base point $x^\mu_0 = (r_0, x^a_0)$ we get rid of the path ordering

$$g(x; y) = \exp_* \left(P_\mu \frac{(x - x_0)^\mu}{r - r_0} \ln \frac{r}{r_0} \right) . \tag{30}$$

The boundary conditions are given by specifying the boundary sources $\hat{B}$. Then,

$$B^{(0)}(x, r) = \int d^D x' \, K(x, x', r) * \hat{B}(x'), \tag{31}$$

$$B^{(1)}(x, r) =$$
$$- \int dr' d^D x' \, G^\nu(x, x', r, r') * \left[B^{(0)} * \alpha_\nu * B^{(0)} \right] (x', r'),$$

and the higher corrections are obtained in a similar manner. Since α appears in the interaction vertex, it should be chosen to be regular at $p = 0$, *e.g.* $\alpha \sim exp(-p^2/\Lambda^2)/\Lambda^3$. The correlators are then given by varying the holonomy integral (26) with respect to $\hat{B}$ and by construction reproduce the field theory results. This procedure is illustrated in figure 1. Summing the diagrams and doing the r integrals, one reproduces the expansion (6), thus answering the question of our introduction.

DISCUSSION

Starting from free bosonic field theory, with an arbitrary position or momentum-dependent kinetic term (dispersion relation), we have derived a dual description as a higher spin gravity in anti-de Sitter space, and argued that it can reproduce all correlation functions. Although the higher spin gravity we arrived at is not of the standard form, it contains the structure tested in explicit comparisons such as [5] and seems as well motivated from this point of view as the standard theories. One point which could be improved is to rephrase the relation (22) in a more covariant way.

One of the outstanding conceptual questions about AdS/CFT is to understand the relation between the two dual space-times. There is a common though not universal belief that the relation is nonlocal away from the boundary and that any microscopic derivation must include some sort of nonlocal transformation. On the other hand, our derivation did not do this; rather, we moved all of the nonlocality into the interactions and dependence on the auxiliary variables. If the same can be done in gauge theory, in which higher derivative operators have large anomalous dimension at strong coupling, then it seems reasonable to look for a local understanding of the duality there as well.

In the end, the significance of this result depends on the extent to which these ideas and techniques apply to interacting theories. The only case for which this generalization will be direct is the interacting $O(N)$ model, as we will discuss elsewhere.

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