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Quantum of volume in de Sitter space

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Abstract: We apply the nonstandard loop quantum cosmology method to quantize a flat FRW cosmological model with a free scalar field and the cosmological constant $\Lambda > 0$. Modification of the Hamiltonian in terms of loop geometry parametrized by a length λ introduces a scale dependance of the model. The spectrum of the volume operator is discrete and depends on Λ. Relating quantum of the volume with an elementary lattice cell leads to an explicite dependance of Λ on λ. Based on this assumption, we investigate the possibility of interpreting Λ as a running constant.

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Contents

1. Introduction

The cosmological observations, interpreted in terms of the Friedmann-Robertson-Walker (FRW) model, suggest that the Universe underwent an accelerated expansion at least twice in its history. The first one is called the cosmic inflation and it is specific to the very early Universe [1]. The second one is the presently observed dark energy domination area [2]. In both cases the evolution can be explained, to some extent, by a constant or nearly-constant contribution to the energy density modeled by the so-called cosmological constant Λ. In gravitational physics one usually treats Λ as a constant, like Newton's constant G, that cannot be derived from first principles [3]. In cosmology one usually calls Λ, in the case of FRW model, the simplest model of dark energy [2] and one tries to find its nature.

In this paper, we study a possible link between Λ and *discrete* structure of space associated with quantum theories of gravity. Loop quantum cosmology (LQC) is a suitable framework to address this issue. In particular, we quantize the space *volume* function to find the relation between Λ and the length λ specifying loop geometry underlying LQC. In what follows, we consider a flat FRW model with the cosmological constant and a free scalar field.

In standard LQC one applies the Dirac quantization method [4, 5], which postpones solution of the Hamiltonian constraints to the quantum level. In the nonstandard LQC one solves constraints already at the classical level and quantization is carried out on the *physical* phase space. It is called the reduced phase space quantization method. This approach has been successfully applied to the FRW [6, 7, 8] and the Bianchi I [9, 10] models with a free scalar field. Recently, the constraints for the FRW model with a free scalar field and the cosmological constant have been solved too [11]. The physical observables like the volume and the energy density of matter field have been analysed, for both positive and negative Λ, at the classical level. In what follows we present quantization of the volume observable for the case of $\Lambda > 0$.

2. Hamiltonian

The Hamiltonian of the model *modified* by the holonomy around a loop takes the form [11, 12]

$$\begin{aligned} H^{(\lambda)} = & -\frac{vN}{32\pi^2 G^2 \gamma^3 \lambda^3} \sum_{ijk} \epsilon^{ijk} \mathrm{tr}\left[h_{\square_{ij}} h_k \left\{(h_k)^{-1}, v\right\}\right] \\ & + N\frac{p_\phi^2}{2v} + N\frac{v\Lambda}{8\pi G} \approx 0, \end{aligned} \tag{2.1}$$

where '$\approx$' reminds that Hamiltonian is a *constraint* of the system, γ is the Barbero-Immirzi parameter, G is Newton's constant, p_ϕ is the conjugate momentum of the scalar field ϕ, N is the lapse function, $h_{\square_{ij}} = h_i h_j (h_i)^{-1} (h_j)^{-1}$ is the holonomy around the square loop $\square_{ij}$, $h_i = \cos(\lambda\beta/2)\,\mathbb{I} + 2\sin(\lambda\beta/2)\,\tau_i$ is holonomy in the i-th direction, and where $\tau_i = -\frac{i}{2}\sigma_i$ (σ_i are the Pauli matrices). The variable v is the volume of a piece of space $\mathcal{V} \subset \mathbb{R}^3$ (we assume that the spacelike part of spacetime has $\mathbb{R}^3$ topology) defined as follows: $v = \int_{\mathcal{V}} dx_1 dx_2 dx_3 \sqrt{det\, q_{ab}}$, where $q_{ab} dx^a dx^b := a^2(dx_1^2 + dx_2^2 + dx_3^2)$ defines the FRW metric (a is the scale factor). The variable β is related with the Hubble parameter. The variable λ, having the dimension of a length, is a *free* parameter of the theory [13]. The Hamiltonian (2.1) may be seen as the *lattice* discretized version of the classical expression, where λ plays the role of a lattice constant and λ^3 is the volume of an elementary cubic cell. While $\lambda \to 0$, the non-modified general relativity Hamiltonian is recovered.

In LQC the gravitational degrees of freedom are represented by holonomies and fluxes. A holonomy used in LQC is a connection integrated along spatial elementary loop. Making use of holonomies enables, roughly speaking, the resolution of the cosmological singularity problem [8]. Thus, they are of primary importance. In the flat FRW model it is enough to use one holonomy parametrized by λ. Equation (2.1) *does not* define, in the nonstandard LQC [6, 8], an effective semi-classical Hamiltonian, but a classical *modified* Hamiltonian. The modification is specified by the value of λ. The bigger λ, the bigger the smearing of the holonomy variable in the Hamiltonian (2.1). Due to the form of (2.1), we propose to interpret Λ to be a *coupling* constant depending on the *smearing* λ. We push forward this interpretation in section 6 to relate Λ with λ.

The *kinematical* phase space is parametrized by four canonical variables (v, β, p_ϕ, ϕ). The imposition of constraint (2.1) leads to the *physical* phase space parametrized by two elementary observables $\mathcal{O}_1$ and $\mathcal{O}_2$, satisfying the algebra $\{\mathcal{O}_2, \mathcal{O}_1\} = 1$. The dynamics of the model is traced by the scalar field ϕ which plays the role of an intrinsic time (see, [11] for more details).

3. Observables

It is shown in [11] that the elementary observables of our model are

$$\mathcal{O}_1 = p_\phi, \tag{3.1}$$

$$\mathcal{O}_2 = \phi + \frac{\mathrm{sgn}(p_\phi)}{\sqrt{12\pi G\delta}} \frac{1}{i} \left[F\left(\beta\lambda \,\middle|\, \frac{1}{\delta}\right) - F\left(\frac{\pi}{2} \,\middle|\, \frac{1}{\delta}\right) \right]. \tag{3.2}$$

Here $F(\cdot|\cdot)$ is the Jacobi elliptic function of the first kind, and $\delta := \frac{1}{3}\Lambda\gamma^2\lambda^2$. For this model the allowed values of the parameter δ are in the set $[0, 1]$. The lower limit correspond to $\Lambda = 0$ case while the upper limit is the dynamically allowed value corresponding to $\Lambda = \frac{3}{\gamma^2\lambda^2} =: \Lambda_{\mathrm{c}}$ (see, [14] for more details).

An expression for the volume function reads [11]

$$v = \frac{|\mathcal{O}_1|}{\sqrt{2\rho_c(1-\delta)}} \frac{1}{\left|\mathrm{sn}\left(u \,\middle|\, 1 - \frac{1}{\delta}\right)\right|} =: |w|, \tag{3.3}$$

where

$$u := \sqrt{12\pi G\delta}(\mathcal{O}_2 - \phi) + K, \qquad \rho_{\mathrm{c}} := \frac{3}{8\pi G\gamma^2\lambda^2}, \tag{3.4}$$

and where $K := F(\pi/2\,|1 - 1/\delta)$. Function $\mathrm{sn}(\cdot|\cdot)$ is the elliptic sinus function defined as $\mathrm{sn}(u|m) := \sin \mathrm{am}(u|m)$, where $\mathrm{am}(u|m) := F^{-1}(u|m)$ is the amplitude of the F elliptic function. We have also used here a definition of the critical energy density. In Fig. 1 we plot an evolution of the volume given by equation (3.3).

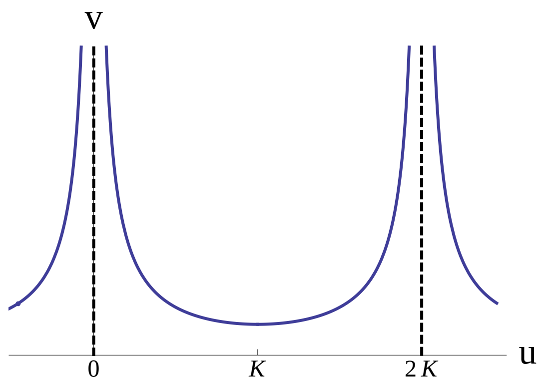

Figure 1: Evolution of volume v as a function of the parameter u. The function is periodic in variable u with the period equal to $2K$. The solutions in different periods are separated by vertical asymptotes.

4. Quantization

Quantization of the system follows the method presented in [8, 7]. We choose the Schrödinger reprezentation for the classical algebra $\{\mathcal{O}_2, \mathcal{O}_1\} = 1$ in the form

$$\hat{\mathcal{O}}_1 \psi(x) := -i\hbar \frac{d}{dx}\psi(x), \quad \hat{\mathcal{O}}_2 \psi(x) := x\psi(x), \tag{4.1}$$

where $\psi \in L^2(\mathbb{R})$, so we get $\left[\hat{\mathcal{O}}_2, \hat{\mathcal{O}}_1\right] = i\hbar\,\mathbb{I}$. Quantization of w may be done in a standard way as follows

$$\hat{w} := \frac{1}{\sqrt{2\rho_c(1-\delta)}}\frac{1}{2}\Big(\hat{\mathcal{O}}_1\; g(\hat{\mathcal{O}}_2 - \phi) + g(\hat{\mathcal{O}}_2 - \phi)\; \hat{\mathcal{O}}_1\Big), \tag{4.2}$$

where

$$g(\hat{\mathcal{O}}_2 - \phi) := \frac{1}{\mathrm{sn}\left(\sqrt{12\pi G\delta}(\hat{\mathcal{O}}_2 - \phi) + K \left|1 - \frac{1}{\delta}\right.\right)}. \tag{4.3}$$

We are looking for the solution to the eigenvalue problem of the operator $\hat{w}$

$$\hat{w}\; \psi_b(x) = b\; \psi_b(x), \quad b \in \mathbb{R}. \tag{4.4}$$

This leads to the equation for eigenfunctions in the form

$$\begin{aligned} \frac{d}{dx}\left(g(x-\phi)\psi_b(x)\right) \,+\, g(x-\phi)\frac{d\psi_b(x)}{dx} \\ = b\frac{i}{\hbar}2\sqrt{2\rho_c(1-\delta)}\psi_b(x). \end{aligned}$$

A general solution to this equation reads

$$\psi_b(u) = \psi_0 \sqrt{\mathrm{sn}\left(u \,\middle|\, 1 - \frac{1}{\delta}\right)} \exp\left\{\frac{i}{\hbar}\frac{b\sqrt{\rho_c}}{\sqrt{6\pi G}}\Theta(u)\right\}, \tag{4.5}$$

where

$$\Theta(u) := \arctan\left\{-\sqrt{\frac{1-\delta}{\delta}}\frac{\mathrm{cn}\left(u\,|1-\frac{1}{\delta}\right)}{\mathrm{dn}\left(u\,|1-\frac{1}{\delta}\right)}\right\}, \tag{4.6}$$

and where the elliptic functions are defined as follows: $\mathrm{cn}(u|m) = \cos\mathrm{am}(u|m)$, and $\mathrm{dn}(u|m) = \sqrt{1 - m\ \mathrm{sn}^2(u|m)}$. The normalization condition for the eigenfunctions reads

$$1 = \langle\psi_b|\psi_b\rangle = \frac{|\psi_0|^2}{\sqrt{12\pi G\delta}}\int_0^{2K} du\ \mathrm{sn}\left(u \,\middle|\, 1 - \frac{1}{\delta}\right). \tag{4.7}$$

We integrate over one period of evolution as other periods correspond to the same model of the universe [11]. Thus, the normalization factor is found to be

$$\psi_0 = \sqrt{\frac{\sqrt{3\pi G(1-\delta)}}{\arctan\left(\sqrt{\frac{1}{\delta}-1}\right)}}. \tag{4.8}$$

To find an orthonormal set of eigenfunctions, we calculate

$$\begin{aligned}\langle\psi_b|\psi_a\rangle &= \frac{1}{\sqrt{12\pi G\delta}}\int_0^{2K} du\ \psi_b^*(u)\psi_a(u) \\ &= \frac{|\psi_0|^2}{\sqrt{3\pi G\delta}}\frac{\sin\left[C\arctan\sqrt{\frac{1}{\delta}-1}\right]}{C\sqrt{\frac{1}{\delta}-1}},\end{aligned} \tag{4.9}$$

where

$$C := \frac{1}{\hbar}\frac{(b-a)\sqrt{\rho_c}}{\sqrt{6\pi G}}. \tag{4.10}$$

5. Quanta of volume

The orthogonality condition $\langle\psi_b|\psi_a\rangle = 0$ leads to

$$b = a + m\ \Delta_\delta, \quad a \in \mathbb{R}, \quad m \in \mathbb{Z}, \tag{5.1}$$

where

$$\Delta_\delta := 8\pi G\gamma\lambda\hbar\frac{\pi/2}{\arctan\sqrt{\frac{1}{\delta}-1}}. \tag{5.2}$$

Therefore, the eigenvalues of $\hat{v}$, due to (3.3), are: $c = |a + m\ \Delta_\delta|$. Thus, the spectrum is *discrete*, which is of a basic importance for the rest our paper.

The space $\mathcal{F}_a := \{\ \psi_b \mid b = a + m\ \Delta_\delta;\ m \in \mathbb{Z};\ b \in \mathbb{R}\ \}$ is *orthonormal.* Each subspace $\mathcal{F}_a \subset L^2(\mathbb{R})$ spans a pre-Hilbert space. The completion of $D_a(\hat{w}) :=$ *span* $\mathcal{F}_a$, $\forall a \in \mathbb{R}$, leads to $L^2(\mathbb{R})$. One may prove (in analogy to the corresponding proof in [8]) that the operator $\hat{w}$ is essentially *self-adjoint* on $D_a(\hat{w})$, $\forall a \in \mathbb{R}$.

The expression (5.2) specifies the minimum gap in the spectrum of the volume operator (for $a = 0$) in terms of the cosmological constant Λ so it defines a *quantum* of the volume. It is interesting to examine the limit when $\delta \to 0$ ($\Lambda \to 0$). Due to the relation $\lim_{\delta\to 0^+} \arctan\sqrt{\frac{1}{\delta} - 1} = \frac{\pi}{2}$, we get $\lim_{\delta\to 0^+} \Delta_\delta = 8\pi G\gamma\lambda\hbar =: \Delta$. This is precisely an expression that has been found earlier [8] in the case $\Lambda = 0$, which proves the consistency of our results. While approaching $\delta \to 1$ ($\Lambda \to \Lambda_c$) we find that $\Delta_\delta \to \infty$.

In Fig. 2 we plot the ratio Δ_δ/Δ as a function of δ. One may easily verify

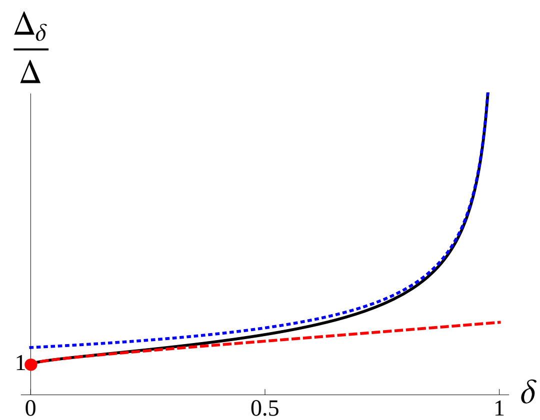

Figure 2: Ratio Δ_δ/Δ as a function of δ (black line). We see that $\lim_{\delta\to 0} \Delta_\delta/\Delta = 1$ as well as $\lim_{\delta\to 1} \Delta_\delta/\Delta = \infty$. The (red) dot represents the case with $\Lambda = 0$.

that for $\delta \to 0$ we have

$$\frac{\Delta_\delta}{\Delta} = 1 + \frac{2\sqrt{\delta}}{\pi} + \frac{4\delta}{\pi^2} + \mathcal{O}(\delta^{3/2}), \tag{5.3}$$

whereas for $\delta \to 1$ we get

$$\frac{\Delta_\delta}{\Delta} = \frac{\pi/2}{\sqrt{1-\delta}} + \mathcal{O}(\sqrt{1-\delta}). \tag{5.4}$$

These approximations are also plotted in Fig. 2. The dashed (red) line is an approximation (5.3) while dotted (blue) line is an approximation (5.4).

It is instructive to check what the value of Δ_δ is in the observed Universe. The cosmological constant Λ can be related with the observed dark energy, which dominates the energy density of the Universe. In this case one can rewrite the definition of δ parameter in the form

$$\delta = \Omega_\Lambda H_0^2 \gamma^2 \lambda^2, \tag{5.5}$$

where Ω_Λ is the fractional density of the cosmological constant, H_0 is the present value of the Hubble parameter. The five years observations of the WMAP satellite yield $\Omega_\Lambda = 0.742 \pm 0.030$ and $H_0 = 71.9^{+2.6}_{-2.7}$ km s^{-1} Mpc^{-1} [15]. Assuming that $\lambda = l_{\mathrm{Pl}}$, where l_{Pl} is the Planck length and $\gamma = 0.2375$ [16], we find

$$\delta = 6.6 \cdot 10^{-124}. \tag{5.6}$$

Because this value is extremely small, the value of Δ_δ with the high precision overlaps with Δ, obtained in the $\Lambda = 0$ limit. Therefore $\Delta_\delta = 8\pi\gamma l_{\mathrm{Pl}}^3$, where we have assumed $\lambda = l_{\mathrm{Pl}}$, as previously. With $\gamma = 0.2375$, the quanta of volume takes a value $\Delta_\delta \approx 6\ v_{\mathrm{Pl}}$, where $v_{\mathrm{Pl}} := l_{\mathrm{Pl}}^3$ is a Planck volume.

6. Running of Λ ?

We propose to relate the quantum of the volume Δ_δ, defined by (5.2), with the volume λ^3 of an *elementary* lattice cell as follows

$$\Delta_\delta = \lambda^3, \tag{6.1}$$

which leads to the equation

$$\lambda^2 = 8\pi\gamma l_{\mathrm{Pl}}^2 \frac{\pi/2}{\arctan\sqrt{\frac{1}{\delta} - 1}}. \tag{6.2}$$

Because $0 \leq \arctan\sqrt{\frac{1}{\delta} - 1} \leq \pi/2$, it is clear that $\lambda \in [\lambda_0, \infty[$, where $\lambda_0 := \sqrt{8\pi\gamma}\ l_{\mathrm{Pl}}$. Therefore (6.2) leads to the constraint on the value of λ from below. The minimum value λ_0 corresponds to the case $\Lambda = 0$.

Equation (6.1) is analogous to the equation postulated in standard LQC for the determination of the *minimum* length of a loop along which the holonomy is defined. One requires that the area of the minimum loop in LQC equals the smallest nonzero eigenvalue of the area operator of loop quantum gravity (LQG) (see, e.g. [4, 17]). However, in our case we make the postulate at the *physical* sector of LQC, contrary to the case of [17] where one compares the corresponding quantities from LQC and LQG in the *kinematical* sector of both theories.

Equation (6.2) can be inverted into the form

$$\Lambda = \Lambda_c \cos^2\left(\frac{\pi}{2}\frac{\lambda_0^2}{\lambda^2}\right), \tag{6.3}$$

where $\lambda \in [\lambda_0, \infty[$ as shown previously.

This way we have turned the cosmological constant into a *variable* cosmological constant, a function depending explicitly on λ. We propose to examine the relation (6.3) by making use of the renormalization group technics to get some cosmological implications. Thus, we introduce the flow equation of the renormalization group [18] via the $\beta-$function (it should be not confused with the canonical variable β) as follows

$$\beta(\Lambda) := \lambda \frac{d\Lambda}{d\lambda}. \tag{6.4}$$

We plot the resulting $\beta-$function in Fig. 3.

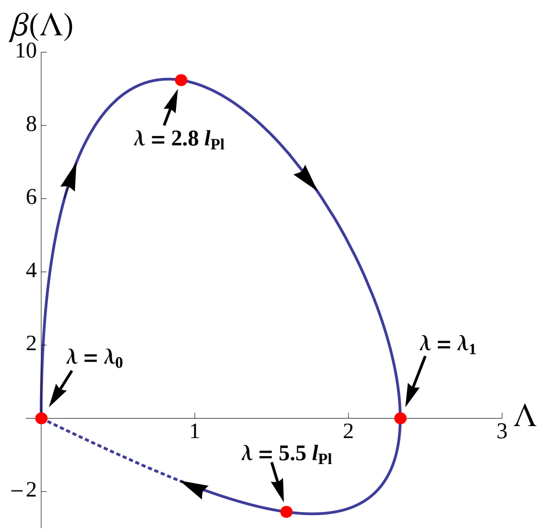

Figure 3: Renormalization group flow for Λ. The arrows indicate flow from UV ($\lambda = \lambda_0$) to IR limit ($\lambda \to \infty$). The dotted part of the curve denotes the region of large λ ($\lambda \gg \lambda_1$) where considerations become speculative. In the plot $\lambda_0 = 2.4\, l_{\rm Pl}$ and $\lambda_1 = 3.8\, l_{\rm Pl}$.

One can see that on the very small scale, Λ grows from 0 at the first fixed point λ_0, and approaches the maximum $\Lambda \approx 2.3\, m_{\rm Pl}^2$ at the second fixed point

λ_1. In the high energy domain (small λ) the value of Λ grows from zero to the Planck scale level. While approaching the largest scales with growing λ, the value of the cosmological constant tends to zero again, $\lim_{\lambda\to\infty}\Lambda(\lambda)=0$. However, we should keep in mind that as λ grows, our lattice interpretation becomes less and less justified, and it does make sense as $\lambda\to\infty$.

7. Conclusions

In this paper we present quantization of the FRW model with the cosmological constant and the free scalar field in the framework of the nonstandard loop quantum cosmology. We investigate spectral properties of the volume operator. We find the eigenfunctions as well as corresponding spectrum which is shown to be discrete. Based on this, we find an expression on the elementary quanta of volume as a function of cosmological constant Λ. In the limit $\Lambda\to 0$, the formula we have found simplifies to the expression that was found earlier in [7] for the $\Lambda=0$ case.

Postulating the relation of the quantum of a space volume with the volume of a lattice cell, we express the cosmological constant Λ in terms of λ. Based on this we *suggest* that Λ may play the role of a running constant similar to the coupling constant in QCD which decreases with increasing energy scale. In the IR limit (where our model is little justified), Λ behaves similarly to the fine structure constant in QED. We are conscious that our dependence of Λ on λ is not a renormalization group flow in any *standard* sense. The renormalization group in quantum field theory comes from integrating out high energy degrees of freedom and requiring that this be compensated for by changes in the coupling constants. We simply put forward a *hypothesis* of interpreting Λ as a running constant to be verified by future studies of the nature of the cosmological constant.

The free parameter λ, of the classical level, can be restricted after quantization by making use of (6.2). As it was shown earlier, the *lowest* allowed value of λ equals λ_0. For this vale, the corresponding critical energy density $\rho_c(\lambda_0)=\frac{3m_{\rm Pl}^4}{64\pi^2\gamma^3}\approx 0.35\ m_{\rm Pl}^4$, for $\gamma=0.2375$ (see, [16]). This is a bit lower than the value obtained within the standard LQC, $\rho_c\approx 0.82\ m_{\rm Pl}^4$ [17]. The value of Λ_c at $\lambda=\lambda_0$ is given by $\Lambda_c(\lambda_0)=\frac{3m_{\rm Pl}^2}{8\pi\gamma^3}\approx 8.9\ m_{\rm Pl}^2$, that is comparable with the value obtained in the standard LQC, $\Lambda_c\approx 10.3\ m_{\rm Pl}^2$ [19].

It is clearly seen that we could carry out the analyses due to an application of LQC method in the reduced phase space quantization version. In this approach an elementary length λ is a free parameter. Such an interpretation cannot be done within LQC which is based on Dirac's quantization with the parameter λ having a fixed value [17].

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