Unconstrained Variables and Equivalence Relations for Lattice Gauge Theories

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Abstract

We write the partition function for a lattice gauge theory exactly in terms of unconstrained variables and show that in the calculation for the effective potential, which involves uniform field configurations, only the singlet representation is relevant. We further show that this implies that the phase diagrams of the (an)isotropic $SU(2)$ theory and the (an)isotropic $U(1)$ theory in any dimension are identical, within this approximation, up to a re-evaluation of the numerical values of the coupling constants at the transitions. Corrections to the mean field approximation that lie within the singlet representation will not see the difference between the two either. Only nonuniform field configurations, that can be sensitive to higher dimensional representations for Yang–Mills fields, will be able to probe the difference between them.

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The effective potential is a standard tool for obtaining the phase diagram of a field theory in the continuum [1]. To set up the corresponding computation for lattice gauge theories we insert in the partition function

$$Z[J] = \int [\mathscr{D}U] e^{-S[U] + a^D \sum_n (J^\dagger \cdot U + J \cdot U^\dagger)} \equiv e^{-W[J]} \tag{1}$$

the following expression [2, 3]

$$1 = \int \prod_{\text{links}} dV_l^R dV_l^I \delta(\mathrm{Re}(U_\mu(n)) - V_l^R) \delta(\mathrm{Im}(U_\mu(n)) - V_l^I) \tag{2}$$

and use the integral representation of the delta functions

$$\delta(\mathrm{Re}(U_\mu(n)) - V_l^R) = \int_{-\infty}^{\infty} \frac{d\alpha_l^R}{2\pi} e^{\mathrm{i}\alpha_l^R (\mathrm{Re}(U_\mu(n)) - V_l^R)} \tag{3}$$

(and similarly for the imaginary part). The partition function takes the form

$$Z[J] = \int \left[\prod_{\text{links}} dV_l^R dV_l^I \frac{d\alpha_l^R}{2\pi} \frac{d\alpha_l^I}{2\pi} \right] e^{-S[V_l^R, V_l^I] - \mathrm{i} \sum_l (\alpha_l^R V_l^R + \alpha_l^I V_l^I) + \sum_l (J_l^R V_l^R + J_l^I V_l^I)} \int \mathscr{D}U e^{\mathrm{i}(\alpha_l^R U_l^R + \alpha_l^I U_l^I)} \tag{4}$$

The action, $S[V_l^R, V_l^I]$ becomes a function of independent links. We recognize the 1–link integral over the gauge group [4, 5]

$$e^{w(\alpha_l^R, \alpha_l^I)} \equiv \int \mathscr{D}U e^{\mathrm{i}(\alpha_l^R U_l^R + \alpha_l^I U_l^I)} \tag{5}$$

and we can identify the effective action,S_{eff}, over the *unconstrained* variables, $(\alpha_l^R, \alpha_l^I, V_l^R, V_l^I)$,

$$S_{\text{eff}}[\alpha_l^R, \alpha_l^I, V_l^R, V_l^I] \equiv S[V_l^R, V_l^I] + \mathrm{i} \sum_l \left(\alpha_l^R V_l^R + \alpha_l^I V_l^I \right) - \sum_l w(\alpha_l^R, \alpha_l^I) \tag{6}$$

This expression is an *exact* rewriting of the partition function (for Yang–Mills fields the trace over the appropriate representation is implicit). It is invariant under *local* gauge transformations (we notice that it highlights the invariant combination $\alpha_l^R V_l^R + \alpha_l^I V_l^I$ but it is, of course, possible to show that the other two terms will only depend on the locally invariant combinations) and its advantage is that it uses unconstrained variables, instead of the group valued gauge links. Therefore it is suited for analytical approximations (e.g. saddle point)–but could also be useful for numerical simulations. In this note we focus on the former issue and defer a study of the latter to future work.

The effective action appears to be complex. This is, however, not an unavoidable conclusion. The theory of Fourier transforms teaches us that the Fourier transform of a function that is reflection–positive is a real function. The Wilson action, we consider here, is, indeed, reflection-positive [6], thus the partition function is real and we may find a contour that renders this explicit [3], for instance, a Wick rotation in the α_l integrals, $\alpha_l^R = -\mathrm{i}\widehat{\alpha}_l^R, \alpha_l^I = -\mathrm{i}\widehat{\alpha}_l^I$ (for the case of $U(1)$). The effective action then takes the form

$$S_{\text{eff}}[\widehat{\alpha}_l^R, \widehat{\alpha}_l^I, V_l^R, V_l^I] \equiv S[V_l^R, V_l^I] + \sum_l \left(\widehat{\alpha}_l^R V_l^R + \widehat{\alpha}_l^I V_l^I \right) - \sum_l w(\widehat{\alpha}_l^R, \widehat{\alpha}_l^I) - \sum_l \left(J_l^R V_l^R + J_l^I V_l^I \right) \tag{7}$$

and the partition function is given by

$$Z[J] = \int_{-\infty}^{\infty} \left[\prod_{\text{links}} dV_l^R dV_l^I \right] \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \left[\frac{d\widehat{\alpha}_l^R}{2\pi\mathrm{i}} \frac{d\widehat{\alpha}_l^I}{2\pi\mathrm{i}} \right] e^{-S_{\text{eff}}[V_l^R, V_l^I, \widehat{\alpha}_l^R, \widehat{\alpha}_l^I, J_l^R, J_l^I]} \tag{8}$$

If the lattice action is not reflection positive, or is even complex itself, then it is an interesting question, whether this representation is more, equally or less useful than the original one in terms of the constrained variables(i.e. the group valued gauge links). In our case the advantages are clearly apparent.

We wish to evaluate the partition function in eq. (8) for field configurations that are uniform over the lattice, i.e. by setting $V_l^R = V^R$, $V_l^I = V^I, \widehat{\alpha}_l^R = \widehat{\alpha}^R$ and $\widehat{\alpha}_l^I = \widehat{\alpha}^I$. This restriction is called the mean field approximation and in the continuum has quite drastic consequences. On the lattice, where the fields take values in the group, which is compact and not the algebra, which is not, the approximation has subtler effects: the local constraints imply that the effective action retains a non-trivial dependence on the coupling(s) even at this level.

In this case the effective action is invariant under *global* gauge transformations only. But this also means that one can find a transformation that will "rotate" *all* fields to the identity in group space, *independently* of the representation the gauge fields were taken in in the original Wilson action! This holds for any gauge group. Therefore it isn't an *assumption* within but, rather, a *consequence* of the mean field approximation, on the lattice, to take the fields in the singlet representation [3]. We also note that the expression obtained in this case should be properly called the effective potential, in analogy to the continuum, where the corresponding quantity is also obtained from the effective action by considering uniform, in spacetime, field configurations. We shall keep the designation S_{eff} for it in this note in order to avoid confusion with the unconstrained variable for the link (and because corrections that are *not* uniform along the lattice would lead us back to the effective action itself).

Let us now write the partition function for $SU(2)$, using this result. We will set $V_l^R = V^R I_{r\times r}$, $V_l^I = V^I I_{r\times r}$, $\widehat{\alpha}_l^R = \widehat{\alpha}^R I_{r\times r}$ and $\widehat{\alpha}_l^I = \widehat{\alpha}^I I_{r\times r}$, where r is the size of the representation. The effective potential will now involve traces over the representation, which the identity will render trivial–they will simply give a global factor equal to the size, r, which we can absorb by a redefinition. So the effective action will take the form

$$S_{\text{eff}}[V^R, V^I, \widehat{\alpha}^R, \widehat{\alpha}^I] = S[V^R, V^I] + D(\widehat{\alpha}^R V^R + \widehat{\alpha}^I V^I) - D w_{SU(2)}(\widehat{\alpha}^R, \widehat{\alpha}^I) \tag{9}$$

and the contribution of the plaquettes, the term $S[V^R, V^I]$, will be identical to that of the $U(1)$ action. So the *only* term, that is different is the term $w_{SU(2)}(\widehat{\alpha}^R, \widehat{\alpha}^I)$, which, since $SU(2)$ has rank 1 can only be a function of the unique invariant combination $[\widehat{\alpha}^R]^2 + [\widehat{\alpha}^I]^2$. Once more we notice that the effective action depends only on the moduli of the "vectors" (V^R, V^I) and $(\widehat{\alpha}^R, \widehat{\alpha}^I)$ and their relative orientation on the 3-sphere, over which we integrate. So we can *choose* to set $V^I = 0, \widehat{\alpha}^I = 0$ (by an appropriate $SU(2)$ transformation) and obtain an effective action that looks exactly like that for $U(1)$–except that, instead of $w_{U(1)}(\widehat{\alpha})$, it contains $w_{SU(2)}(\widehat{\alpha})$ in its place. One is now tempted to change variables: to set

$$d\widehat{\alpha} e^{-D w_{SU(2)}(\widehat{\alpha})} = d\widehat{\eta} e^{-D w_{U(1)}(\widehat{\eta})}$$

thus

$$\widehat{\eta} = \left[w_{U(1)} \right]^{-1} \left(w_{SU(2)}(\widehat{\alpha}) \right) \tag{10}$$

which is well-defined, since $w_{U(1)}(\cdot)$ is a monotonic function of its argument. In fact, so is the relation

$$\widehat{\alpha} = \left[w_{SU(2)}\right]^{-1} \left(w_{U(1)}(\eta)\right) \tag{11}$$

since $w_{SU(2)}(\widehat{\alpha}) = \log(I_0(\widehat{\alpha}) - I_2(\widehat{\alpha}))$ [7] is also monotonic. The partition function for the $SU(2)$ theory thus becomes

$$Z = \int \prod_{\text{links}} \left[dV^R \frac{d\widehat{\eta}}{2\pi \mathrm{i}} \right] e^{-S[V^R] + D(\widehat{\alpha}(\widehat{\eta}) V^R - w_{U(1)}(\widehat{\eta}))} \tag{12}$$

where the function $\widehat{\alpha}(\widehat{\eta})$ is defined by eq. (11). Since $\widehat{\alpha}(\widehat{\eta})$ is a monotonic function of its argument, the *qualitative* aspects of the phase diagram of the theory are identical to those of the phase diagram of the $U(1)$ theory. The *quantitative* aspects, i.e. the value of the coupling, β_c, where the transition from the confining to the Coulomb phase occurs, will, of course, depend on the numerical details of the function $\widehat{\alpha}(\widehat{\eta})$–but its existence, as well as the order of the transition depend only on the "topological" aspects, namely on the shape of the functional relation.

The same arguments hold when anisotropic couplings are introduced [8, 9]. It suffices to replace, in the effective potential for the anisotropic $U(1)$ theory, $\widehat{\alpha}$ and $\widehat{\alpha}'$ by $\widehat{\alpha}(\widehat{\eta})$, respectively $\widehat{\alpha}'(\widehat{\eta}')$ and $d\widehat{\alpha}d\widehat{\alpha}'$ by $d\widehat{\eta}d\widehat{\eta}'$ to obtain the partition function for the anisotropic $SU(2)$ theory!

This observation allows us to understand the results of ref. [10, 11] where the anisotropy, introduced for the compact $U(1)$ case, was studied, under different approximation schemes (among them the mean field approximation) for the case of $SU(2)$ and $SU(3)$ gauge groups and a layered phase was also found in five dimensions. This sounds indeed very surprising, since the layered phase requires the existence of a Coulomb phase within the layer for its intrinsic definition–and four–dimensional Yang–Mills theories don't have a Coulomb phase [12]: they go from confinement at strong coupling to asymptotic freedom at weak coupling. We now realize that, in the mean field approximation, these results are an inevitable consequence of the equivalence of the $U(1)$ theory, that *does* have a Coulomb phase, with the $SU(2)$ theory, that, in this approximation, is, indeed, "color-blind". For the $SU(3)$ case one is tempted to conjecture that the equivalence with the $U(1) \times U(1)$ theory is responsible and that the equivalence provides, indeed, a realization of the "Abelian projection" proposed by 't Hooft [13] many years ago and since studied for understanding confinement [14]. It is interesting to remark here that we haven't fixed the gauge in any way–and we don't need to, in principle, for compact gauge groups. Of course we need to in practice, when we try to sample the configuration space by Monte Carlo methods (and we will need to when computing the corrections [7]). However the importance of the gauge condition is much less "visible" than it is in the continuum.

Let us now discuss whether corrections to the mean field approximation can affect this equivalence. The corrections are of two kinds: (a) those that stem from the fact that the effective potential is *not* a quadratic function of its variables. Thus there will be "loop" corrections, that will still be uniform along the lattice. These will remain within the singlet representation and thus cannot affect this equivalence: Eq. (11) provides the dictionary. This implies that this class of corrections, computed in ref. [11], in fact tests the robustness of the layered phase in the $U(1)$ theory itself to such corrections; (b) those that are sensitive to the spacetime variations of the fields. Among them are (b-1) fields in the singlet representation and (b-2) fields that transform in higher dimensional representations. The $U(1)$ theory will only be sensitive to fields

in the (a) and (b-1) categories–since it possesses only one-dimensional representations, while Yang–Mills theories will receive contributions from all categories, but only the corrections from category (b-2) will affect the correspondance with the abelian theory. While they have been computed in the past, this issue was never raised and it would be interesting to look at them in this light and compute the corrections for the anisotropic case (these were studied in ref. [8] under certain assumptions that would be interesting to study further).

In conclusion we have used an exact transcription of a lattice gauge theory, which is interesting in its own right, to obtain an equivalence between the dynamics of all pure gauge theories that are invariant under groups of the same rank in the mean field approximation, including a certain class of corrections to it. For the case of rank 1 the equivalence is explicitly realized and the effect of the corrections can be identified. It might be interesting to study the modulated phase in five–dimensional gauge theories [15] in this context in the presence of anisotropy [9]. For the case of higher rank groups (such as $SU(3)$ which is rank 2) the 1–link integral will be a function of as many variables as its rank. This time the change of variables involves a determinant, whose size is that of the rank of the gauge group. If the determinant is positive–definite, then one would establish the exact equivalence between $SU(3)$, $U(2)$ and $U(1) \times U(1)$ for uniform field configurations, a much stronger realization of the "Abelian projection". It is worth noting that such a determinant has a natural interpretation as the area of a surface in "group space" and it would be, perhaps, not devoid of meaning to ask whether the fluctuations of such a surface could be related to the corrections that the correspondance receives, when nonuniform configurations are taken into account.

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