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Bel-Robinson for TMG

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Abstract

We construct, and establish the (covariant) conservation of, a 4-index "super-stress tensor" for TMG. Separately, we discuss its invalidity in quadratic curvature models and suggest a generalization.

The 4-index Bel-Robinson tensor $B_{\gamma\mu\nu\rho}$, quadratic in the Riemann tensor and (covariantly) conserved on Einstein shell, has received much scrutiny in its original $D = 4$ habitat (see references in [1]). There, B is the nearest thing to a covariant gravitational stress-tensor, for example playing essentially that role in permitting construction of higher ($L > 2$) loop local counter-terms in SUGRA [2,3]. It also generalizes to $D > 4$, at the minor price of losing tracelessness, like its spin 1 model, the Maxwell stress-tensor.

In this note, we turn to lower D, asking whether B survives in $D = 3$ and if so, to what question is it the answer–in what theory, if any, is it conserved? Since the hallmark of $D = 3$ is the identity of Riemann and Einstein tensors (they are double-duals), it is obvious that B vanishes identically on pure Einstein (i.e., flat space) shell[1], and becomes the trivial (and removable) constant tensor $\sim (\Lambda^2 g_{\gamma\mu}\, g_{\nu\rho} + \text{symm})$ in cosmological GR [4]. This leaves the dynamical hallmark of $D = 3$, TMG [5], and the new quadratic curvature models [6,7], as the other possible beneficiaries. Our main result is that B both survives dimensional reduction and is conserved on TMG shell, in accord with the similar mechanism ensuring the Maxwell tensor's conservation on TME shell. Separately, a simple argument shows why it does not work for generic quadratic curvature actions.

One obtains B in $D = 3$ by inserting the Riemann-Ricci identites (we use de-densitized $\epsilon^{\mu\nu\alpha}$ throughout)

$$R^{\mu\alpha\nu\beta} \equiv (g^{\mu\nu} R^{\alpha\beta} + \text{symm}) \equiv \epsilon^{\mu\alpha\sigma}\, G_{\sigma\rho}\, \epsilon^{\nu\beta\rho}$$

[1]Actually, B can already be made trivial on $D = 4$ GR shell, by adding suitable terms [8].

into a $D = 4$ B. The resulting combination is:

$$B_{\gamma\mu\nu\rho} = \bar{R}_{\mu\nu}\,\bar{R}_{\gamma\rho} + \bar{R}_{\mu\rho}\,\bar{R}_{\gamma\nu} - g_{\mu\gamma}\,\bar{R}_{\nu\beta}\,\bar{R}^{\beta}_{\ \rho}\,, \qquad \bar{R}_{\mu\nu} \equiv R_{\mu\nu} - 1/4\, g_{\mu\nu}\, R \tag{1}$$

where the Schouten tensor $\bar{R}$ also defines the Cotton tensor below; B is manifestly symmetric under $(\gamma\mu, \nu\rho)$ pair interchanges (but not totally symmetric here because that depended on special $D = 4$ identities). Clearly, B vanishes identically for $\bar{R}_{\mu\nu} = 0$, and reduces to a constant tensor for the cosmological $\bar{R}_{\mu\nu} = \Lambda\, g_{\mu\nu}$ extension, a term which may even be removed by suitably adding to the definition of B there. Turning to TMG, its field equation is [5]

$$G^{\mu\nu} = \mu^{-1}\, C^{\mu\nu} \equiv \mu^{-1}\, \epsilon^{\mu\rho\gamma}\, D_\rho\, \bar{R}_\gamma^{\ \nu} \tag{2}$$

The Cotton tensor $C^{\mu\nu}$ is identically (covariantly) conserved, symmetric and traceless, so tracing (2) implies $R = 0$, which simplifies on-shell calculations; μ is a constant with dimension of mass. [Our results will also apply to cosmologically extended TMG [9], much as they do for cosmological GR.] Our question then is whether B of (1) is conserved by virtue of (2). The reason we expect this is the close analogy between TMG and its vector version, TME. The latter model's abelian version (its non-abelian extension is similar), has (flat space) field equations resembling (2),

$$\partial_\beta\, F^{\alpha\beta} = \frac{1}{2}\,\mu\, \epsilon^{\alpha\gamma\beta} F_{\gamma\beta} \equiv \mu\ {}^*F^\alpha, \tag{3}$$

while the analog of B is the Maxwell stress tensor

$$T_{\mathrm{M}\,\mu\nu} = F_\mu^{\ \beta}\, F_{\nu\beta} - 1/4\, g_{\mu\nu}\, F_{\alpha\beta}\, F^{\alpha\beta}. \tag{4}$$

It is indeed conserved on TME shell, as follows:

$$\partial_\nu\, T^{\mu\nu} = F^{\mu\beta}\,\partial_\nu\, F^\nu_{\ \beta} = \mu\, F^{\mu\beta}\ {}^*F_\beta \equiv \mu\,\epsilon^{\mu\alpha\beta}\ {}^*F_\alpha\ {}^*F_\beta \equiv 0. \tag{5}$$

This success motivates seeking a TMG chain similar to (5), schematically,

$$D\,B \equiv R\ (DR - DR) \equiv R\,\epsilon\, C = \mu^{-1}\epsilon\, C\, C \stackrel{?}{\equiv} 0; \tag{6}$$

that is, we are hoping to set up a curl so as to use the algebraic identity $D_\alpha \bar{R}_{\beta\gamma} - D_\gamma\, \bar{R}_{\beta\alpha} \equiv \epsilon_{\mu\alpha\gamma}\, C^\mu_{\ \beta}$ as indicated. [There is a major distinction between the two models, however. The Maxwell tensor is also the stress tensor of TME since its Chern-Simons term, being metric-independent, does not contribute. Hence conservation is guaranteed a priori here [5], unlike the very existence, let alone conservation, of a B for TMG.] Taking the divergence of (1) and using (2) indeed yields

$$D_\gamma\, B^{\gamma\mu\nu\rho} = \left[D^\gamma\, \bar{R}^{\mu\nu} - D^\mu\, \bar{R}^{\nu\gamma}\right] \bar{R}_\gamma^{\ \rho} + \left[D^\gamma\, \bar{R}^{\mu\rho} - D^\mu\, \bar{R}^{\rho\gamma}\right]\ \bar{R}_\gamma^{\ \nu} = \mu\,\epsilon^{\sigma\gamma\mu}\ \left(C_\sigma^{\ \nu}\, C_\gamma^{\ \rho} + C_\sigma^{\ \rho}\, C_\gamma^{\ \nu}\right) \equiv 0 \tag{7}$$

where the identity follows by the symmetry under $(\sigma\gamma)$. This establishes the nontrivial role of B as a "covariant" conserved gravitational tensor for TMG. It may thus find uses here similar to those of the original B in classifying GR solutions. Whether it is relevant to the quantum extensions of these theories is unclear, since $D = 3$ GR is finite [10] and TMG may be [11].

The other gravitational model of special interest in $D = 3$ is the "new quadratic curvature" theory. Its $L = a\,R + b\,\bar{R}^2$, or even its pure $\bar{R}^2$ variant, does not conserve B. The reason is obvious and applies as well to all quadratic curvature actions in $D=4$. The divergence of (any) B behaves as $R\,DR$, while the R^2 field equations read $DD\,R+R\,R=0$, hence they do not tell us anything about DR. So unless $R\,DR$ vanishes for algebraic reasons, and it does not, there is no hope already at linearized, $DD\,R$, level, quite apart from the RR terms. A clear example is the $\bar{R}^2$ field equation itself,

$$\Box\,\bar{R}_{\mu\nu}+(\eta_{\mu\nu}\,\Box-3\,D_\mu\,D_r)\;R+\left(\bar{R}_{\mu\alpha}\,\bar{R}^\alpha{}_\nu+\bar{R}_{\nu\alpha}\,\bar{R}^\alpha{}_\mu-g_{\mu\nu}\,R\right)=0. \tag{8}$$

B-nonconservation also makes physical sense: one would expect the correct candidate (if any) to have the form $B'=DR\,DR$ to reflect the extra derivatives in R^2 actions.

In summary, we have obtained a conserved Bel-Robinson tensor for $D=3$ TMG, despite TMG's third derivative order. It is, gratifyingly, the reduction of one originally defined for $D=4$ GR, and fits nicely with the Maxwell stress tensor's conservation in TME. We also noted the unsuitability of B as a conserved tensor in quadratic curvature models, suggesting instead that a modified $B'\sim DR\,DR$ might succeed.

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References

[1] S.Deser, gr-qc/9901007.

[2] S. Deser, K.S. Stelle and J.H. Kay, *Phys. Rev. Lett.*, **38** 527 (1977).

[3] N. Beisert et al, *Phys. Lett. B*, **694** 265 (2010), hep-th/1009.1643.

[4] S. Deser and R. Jackiw, *Ann. Phys.* **153** 405 (1984).

[5] S.Deser, R.Jackiw and S.Templeton, *Ann. Phys.* **140** 372 (1982); *Phys. Rev. Lett.*, **48** 975 (1982).

[6] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Phys. Rev. Lett.* **102** 201301(2009), hep-th/0901.1766.

[7] S. Deser, *Phys. Rev. Lett.* **103** 101302 (2009), hep-th/0904.4473.

[8] R.K. Sachs, *Z. Phys.* **157**, 462 (1960).

[9] S. Deser, in "Quantum Theory of Gravity" (ed S.M.Christensen), Hilger London (1984).

[10] E. Witten, *Nucl. Phys. B.*, **311** 46 (1988); S.Deser, J.McCarthy and Z.Yang, *Phys. Lett. B*, **222** 61 (1989).

[11] S. Deser and Z. Yang, *Class. Quant. Grav.*, **7** 1603 (1990).