

# Twisting Adjoint Module Algebras

Petr Kulish

St.-Petersburg Department of Steklov Mathematical Institute,  
Fontanka 27, 191023 St.-Petersburg, Russia

Andrey Mudrov

Department of Mathematics, University of Leicester,  
University Road, LE1 7RH Leicester, UK

## Abstract

Transformation of operator algebras under Hopf algebra twist is studied. It is shown that that adjoint module algebras are stable under the twist. Applications to vector fields on non-commutative space-time are considered.

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## 1 Introduction

The interest in the quantum field theory on non-commutative space-time arose long before the invention of quantum groups. It was reinvigorated after the Heisenberg relations for the coordinates appeared as a special limit in the open string theory of Seiberg-Witten, [1]. Later those relations were shown to be invariant under an action of the quantum Poincaré algebra, which resulted from the classical Poincaré algebra through a quantum group twist [2].

Twist transformation of the Poincaré algebra leads to a deformation of algebraic structures on the space-time, such as the function algebra, differential operators *etc.* In this paper we study the behavior of differential operators under this transformation. In general mathematical terms, we consider a Hopf algebra  $\mathcal{H}$ , an adjoint module algebra  $\mathcal{A}$ , and how

the latter changes under a twist of  $\mathcal{H}$ . Recall that adjoint module algebra is an arbitrary algebra admitting a homomorphism  $\mathcal{H} \rightarrow \mathcal{A}$ , which defines an (adjoint) action of  $\mathcal{H}$  on  $\mathcal{A}$  compatible with the multiplication in  $\mathcal{A}$ . Typical examples come from representations of  $\mathcal{H}$ : one can take for  $\mathcal{A}$  the image of  $\mathcal{H}$ , the entire algebra of endomorphisms, or any algebra in between. We prove that twist transformation of an adjoint module algebra  $\mathcal{A}$  is isomorphic to  $\mathcal{A}$ .

Our interest in the adjoint action is motivated by geometrical applications of quantum groups. We regard  $\mathcal{H}$  as symmetry of the quantum space  $\mathcal{A}$ , whose geometry also involves operators acting on  $\mathcal{A}$  (like differential operators in the differential setting). In this picture, the algebra  $\mathcal{H}$  is a sort of thing that is external to the geometry of the quantum space. Indeed, under the twist of  $\mathcal{H}$  the entire geometry transforms accordingly, see e.g. [3, 4], while the multiplication in  $\mathcal{H}$  remains untouched.

On the other hand, the Hopf algebra  $\mathcal{H}$  "participates" in geometry through the representation  $\rho: \mathcal{H} \rightarrow \text{End}(\mathcal{A})$ . For example, the d'Alembertian  $\square = \partial_\mu \partial^\mu$  on functions on the Minkowski space belongs to the (image of) Poincaré algebra. Thus, the multiplication between the partial derivatives in  $\partial_\mu \partial^\mu$  must be deformed in  $\text{End}(\mathcal{A})$ , but at the same time it must remain the same in  $\mathcal{H}$ . This apparent controversy is a cause of certain confusion in the literature on the non-commutative gauge field theory (c.f. [5, 6]). It can be resolved in the following way. Instead of "external" object  $\mathcal{H}$  one should deal with the image  $\rho(\mathcal{H})$ . Under the twist, the algebra  $\rho(\mathcal{H})$  is undergone a deformation, however isomorphic to the original algebra  $\mathcal{H}$ . In the "new coordinates" both comultiplication and multiplication are deformed in a compatible way, and the algebra structure in  $\mathcal{H}$  is incorporated uniformly into the whole picture.

This point of view helps to avoid the confusion between the deformed and non-deformed products, as it allows to get rid of the "external" object  $\mathcal{H}$  and work with the purely geometrical object  $\rho(\mathcal{H})$ . In particular, the noncommutative star product of annihilation operators is often interpreted in the physics literature as a change of statistics. In the quantum theory  $\mathcal{A}$  can be identified with the algebra of differential operators on the space-time (or the algebra of observables) while  $\mathcal{H}$  is the symmetry (Poincaré) algebra. Then the above mentioned isomorphism means that one cannot speak about the "change of statistics" under a twist of comultiplication of  $\mathcal{H}$  and the corresponding multiplication in  $\mathcal{A}$  (see Example 2.2).

Here is the setup of the paper. Some facts about Hopf algebras and twists that are used in what follows are collected in the next part of Introduction. The second section is devoted to adjoint module algebras and their twist transformation. There, we also consider

a special case of smash product of a Hopf algebra and its general module algebra. Further we apply the general considerations of this section to universal enveloping Lie algebras and their generalizations associated with unitary quantum permutations. This is illustrated on the example of quantum vector fields in the last section.

## 1.1 Triangular Hopf algebras

For the reader's convenience we collect in this section the basic information on Hopf algebras, which will be used further on. Throughout the paper,  $\mathcal{H}$  is a *triangular* Hopf algebra over  $\mathbb{C}$ , [8]. That is,  $\mathcal{H}$  is a complex associative algebra equipped with comultiplication  $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ , counit  $\varepsilon: \mathcal{H} \rightarrow \mathbb{C}$ , antipode  $\gamma: \mathcal{H} \rightarrow \mathcal{H}$ , and a universal R-matrix  $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$  which obeys the identity  $\mathcal{R}\mathcal{R}_{21} = 1 \otimes 1$  (the subscripts indicate that the second copy of the R-matrix has the tensor legs flipped).

The comultiplication is an algebra homomorphism satisfying the coassociativity condition  $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ . We shall use the standard symbolic Sweedler notation  $\Delta(h) = h^{(1)} \otimes h^{(2)}$  for the coproduct of  $h \in \mathcal{H}$ . This notation assumes the suppressed summation over decomposable tensors.

The counit  $\varepsilon$  is an algebra homomorphism, and  $(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta$ , under the identification  $\mathbb{C} \otimes \mathcal{H} \simeq \mathcal{H} \simeq \mathcal{H} \otimes \mathbb{C}$ .

The antipode is an anti-algebra and anti-coalgebra map. The latter means that  $(\gamma \otimes \gamma) \circ \Delta = \Delta^{op} \circ \gamma$ , where  $\Delta^{op}(h) := h^{(2)} \otimes h^{(1)}$  is the opposite comultiplication. The antipode satisfies the equalities  $\gamma(x^{(1)})x^{(2)} = \varepsilon(x) = x^{(1)}\gamma(x^{(2)})$ .

The universal R-matrix  $\mathcal{R}$  is an invertible element from  $\mathcal{H} \otimes \mathcal{H}$  (may be a completed tensor product) satisfying the identities [7]

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

in  $\mathcal{H}^{\otimes 3}$ . Here the subscripts indicate the embeddings of  $\mathcal{R}$  in  $\mathcal{H}^{\otimes 3}$  in the standard way. Also,

$$\mathcal{R}\Delta(h) = \Delta^{op}(h)\mathcal{R},$$

for all  $h \in \mathcal{H}$ . As  $\mathcal{H}$  is assumed to be triangular,  $\mathcal{R}_{21} = \mathcal{R}^{-1}$ .

Note that  $(\mathcal{H}, \Delta^{op}, \varepsilon, \gamma^{-1}, \mathcal{R}^{-1})$  is again a triangular Hopf algebra. This makes sense, as the antipode  $\gamma$  is invertible in triangular Hopf algebras. We denote it by  $\mathcal{H}^{op}$ . One can also define the triangular Hopf algebra  $\mathcal{H}_{op}$  with the opposite multiplication, the same comultiplication and counit, and the inverse antipode and R-matrix.

An example of triangular Hopf algebra is a universal enveloping algebra  $U(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g}$  with

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi, \quad \varepsilon(\xi) = 0, \quad \gamma(\xi) = -\xi$$

on the elements  $\xi \in \mathfrak{g}$ . The universal R-matrix is just  $1 \otimes 1$ .

An associative algebra  $\mathcal{A}$  is called an  $\mathcal{H}$ -module algebra if it is an  $\mathcal{H}$ -module and the multiplication of  $\mathcal{A}$  is compatible with the comultiplication of  $\mathcal{H}$ :

$$h \triangleright (ab) = (h^{(1)} \triangleright a)(h^{(2)} \triangleright b), \quad h \in \mathcal{H}, \quad a, b \in \mathcal{A}.$$

In geometrical applications,  $\mathcal{H}$  acts on a (non-commutative) space,  $M$ , and  $\mathcal{A}$  is the algebra of "smooth" functions on  $M$ . Observe that for any  $\mathcal{H}$ -module  $(V, \rho)$  the vector space  $\text{End}(V)$  of all linear operators on  $V$  is an  $\mathcal{H}$ -module algebra with the action  $h \triangleright A := \rho(h^{(1)})A\rho(\gamma(h^{(2)}))$ .

Now we recall the basics of the Hopf algebra twist. Suppose there is an invertible element  $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$  satisfying the identities [8]

$$(\Delta \otimes \text{id})(\mathcal{F})\mathcal{F}_{12} = (\text{id} \otimes \Delta)(\mathcal{F})\mathcal{F}_{23},$$

$$(\varepsilon \otimes \text{id})(\mathcal{F}) = 1 \otimes 1 = (\text{id} \otimes \varepsilon)(\mathcal{F}).$$

Then  $(\mathcal{H}, \tilde{\Delta}, \varepsilon, \tilde{\gamma}, \tilde{\mathcal{R}})$  is again a triangular Hopf algebra with

$$\tilde{\Delta}(h) := \mathcal{F}^{-1}\Delta(h)\mathcal{F} \quad \tilde{\gamma}(g) := \vartheta^{-1}\gamma(h)\vartheta, \quad \tilde{\mathcal{R}} := \mathcal{F}_{21}^{-1}\mathcal{R}\mathcal{F}$$

for all  $h \in \mathcal{H}$ . The invertible element  $\vartheta$  is defined by the formula (1) below. To distinguish this new Hopf algebra from  $\mathcal{H}$ , we denote it by  $\tilde{\mathcal{H}}$  and call it twist of  $\mathcal{H}$  via the cocycle  $\mathcal{F}$ . Note that  $\tilde{\mathcal{H}}$  has the same multiplication and counit as  $\mathcal{H}$ .

A twist of  $\mathcal{H}$  transforms any  $\mathcal{H}$ -module algebra  $\mathcal{A}$  to an  $\tilde{\mathcal{H}}$ -module algebra  $\mathcal{A}_\mathcal{F}$ . As an  $\mathcal{H}$ -module,  $\mathcal{A}_\mathcal{F}$  coincides with  $\mathcal{A}$  but has a different multiplication,  $a * b := (\mathcal{F}_1 \triangleright a)(\mathcal{F}_2 \triangleright b)$ . This multiplication is compatible with the twisted comultiplication in  $\tilde{\mathcal{H}}$ . Thus a twist transforms Hopf algebras and their module algebras. To distinguish transformation of module algebras from Hopf algebras, we call  $\mathcal{A}_\mathcal{F}$  *cotwist* of  $\mathcal{A}$ .

## 2 Twisting adjoint module algebras

In the present subsection we consider module algebras of a special type. Suppose  $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$  is a twisting cocycle. Define the elements  $\vartheta, \zeta \in \mathcal{H}$  by setting

$$\vartheta = \gamma(\mathcal{F}_1)\mathcal{F}_2, \quad \zeta = \mathcal{F}_2^{-1}\gamma^{-1}(\mathcal{F}_1^{-1}). \quad (1)$$

It is known, [8], that

$$\vartheta^{-1} = \gamma(\zeta), \quad \zeta^{-1} = \gamma^{-1}(\vartheta). \quad (2)$$

As usual, the subscripts are used to label tensor factors in the symbolic notation  $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ . The elements  $\vartheta, \zeta$  participate in the antipode  $\tilde{\gamma}$  of  $\tilde{\mathcal{H}}$ , which is related to the old antipode by the formulas

$$\tilde{\gamma}(h) = \vartheta^{-1} \gamma(h) \vartheta = \gamma(\zeta^{-1} h \zeta). \quad (3)$$

Also, we have the identities

$$\mathcal{F}_1^{(1)} \otimes \gamma(\mathcal{F}_1^{(2)}) \mathcal{F}_2 = \mathcal{F}_1^{-1} \otimes \gamma(\mathcal{F}_2^{-1}) \vartheta, \quad \gamma(\mathcal{F}_1) \mathcal{F}_2^{(1)} \otimes \mathcal{F}_2^{(2)} = \vartheta \mathcal{F}_1^{-1} \otimes \mathcal{F}_2^{-1}. \quad (4)$$

These formulas can be easily derived from the definition  $\mathcal{F}$ .

Now suppose an associative algebra  $\mathcal{E}$  admits a homomorphism  $\rho: \mathcal{H} \rightarrow \mathcal{E}$ . The adjoint action  $\text{ad}_\rho: \mathcal{H} \rightarrow \text{End}(\mathcal{E})$  is defined by

$$\text{ad}_\rho(h)a = \rho(h^{(1)})a\rho(\gamma(h^{(2)})), \quad h \in \mathcal{H}, \quad a \in \mathcal{E}.$$

When  $\rho$  is clear from the context, we use for the adjoint action the dot notation  $\text{ad}_\rho(h)a = h.a$ . The adjoint action makes  $\mathcal{E}$  an  $\mathcal{H}$ -module algebra, which is further called adjoint module algebra. We also extend this shorthand notation to the action on maps between any  $\mathcal{H}$ -modules, say  $(S, \rho_S)$  and  $(T, \rho_T)$ :

$$h.f = \rho_T(h^{(1)})f\rho_S(\gamma(h^{(2)})),$$

for  $f: S \rightarrow T$ . This will help us to distinguish the action on maps from the actions on  $S$  and  $T$ .

We are going to show that  $\mathcal{E}$  is stable under twist. To simplify the formulas, we shall drop the symbol of the homomorphism  $\rho$ , as though  $\mathcal{H}$  were a subalgebra in  $\mathcal{E}$ .

**Theorem 2.1.** *Let  $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$  be a twisted cocycle and suppose  $\mathcal{E}$  is an adjoint  $\mathcal{H}$ -module algebra. Then the cotwist  $\mathcal{E}_\gamma$  is isomorphic to  $\mathcal{E}$  as an associative algebra, with the isomorphism  $\mathcal{E}_\gamma \rightarrow \mathcal{E}$  given by the assignment  $\varphi: a \mapsto \mathcal{F}_1^{-1} a \gamma(\mathcal{F}_2^{-1}) \vartheta = (\text{ad}(\mathcal{F}_1) a) \mathcal{F}_2$ .*

*Proof.* Denote the inverse mapping  $\mathcal{E} \rightarrow \mathcal{E}_\gamma$  by  $\varphi^{-1}: a \mapsto \mathcal{F}_1 a \gamma(\mathcal{F}_2 \zeta)$  and check that  $\varphi^{-1}$  is an algebra homomorphism. This immediately follows from the identity

$$\mathcal{F}_1^{(1)} \mathcal{F}_{1'} \otimes \gamma(\mathcal{F}_1^{(2)} \mathcal{F}_{2'} \zeta) \mathcal{F}_2^{(1)} \mathcal{F}_{1''} \otimes \gamma(\mathcal{F}_2^{(2)} \mathcal{F}_{2''} \zeta) = \mathcal{F}_1 \otimes 1 \otimes \gamma(\mathcal{F}_2 \zeta), \quad (5)$$

so let (5) be checked first. Applying the cocycle equation to  $(\Delta \otimes \Delta)(\mathcal{F})\mathcal{F}_{12}$  we transform the left-hand side to

$$\mathcal{F}_1 \otimes \gamma(\mathcal{F}_2^{(1)} \mathcal{F}_{1'} \zeta) \mathcal{F}_2^{(2)} \mathcal{F}_{2'}^{(1)} \mathcal{F}_{1''} \otimes \gamma(\mathcal{F}_2^{(3)} \mathcal{F}_{2'}^{(2)} \mathcal{F}_{2''} \zeta),$$

from which we easily get the expression

$$\mathcal{F}_1 \otimes \gamma(\zeta) \gamma(\mathcal{F}_{1'}) \mathcal{F}_{2'}^{(1)} \mathcal{F}_{1''} \otimes \gamma(\mathcal{F}_2 \mathcal{F}_{2'}^{(2)} \mathcal{F}_{2''} \zeta).$$

Applying the right formula from (4) and the left formula from (2), we obtain the right-hand side of (5).  $\square$

**Example 2.2.** The universal enveloping algebra  $U(\mathcal{P})$  of the Poincaré Lie algebra  $\mathcal{P}$  is realized in the algebra  $\mathcal{A}$  of differential operators on the Minkowski space, which is generated by the mutually commuting coordinates  $x^\mu$  and constant vector fields  $i\partial/\partial x_\mu$  representing the momenta  $P_\nu \in \mathcal{P}$ . Simple Abelian twist

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := \mathcal{F} = \exp\left(-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right), \quad \theta^{\mu\nu} = -\theta^{\nu\mu},$$

changes the coproduct of  $U(\mathcal{P})$  to  $\tilde{\Delta}(h) = \mathcal{F}^{-1}\Delta(h)\mathcal{F}$ , for  $h \in U(\mathcal{P})$ . The algebra  $\mathcal{A}$  is an adjoint  $U(\mathcal{P})$ -module algebra, and after the twist it has the  $*$ -product of  $\mathcal{A}_i$ :

$$x^\alpha * x^\beta - x^\beta * x^\alpha = [x^\alpha, x^\beta]_* = i\theta^{\alpha\beta}.$$

The homomorphism  $\varphi: \mathcal{A}_i \rightarrow \mathcal{A}$  from Theorem 2.1 acts by

$$x \mapsto y^\alpha \mapsto (\mathcal{F}_1 x^\alpha) \mathcal{F}_2 = x^\alpha + \frac{1}{2}\theta^{\alpha\beta} P_\beta \in \mathcal{A}.$$

One can check that the elements  $y^\alpha$  obey the relations  $[y^\alpha, y^\beta] = i\theta^{\alpha\beta}$ .

Similar construction can be done for realization of  $U(\mathcal{P})$  in the algebra, call it  $\mathcal{A}$  again, generated by the creation and annihilation operators  $a(p)^+, a(k)$  of the relativistic scalar field. The action of  $P_\mu$  on  $a(p)$  is  $P_\mu \triangleright a(p) = [P_\mu, a(p)] = -p_\mu a(p)$  giving the  $*$ -product  $a(p) * a(k) = a(k) * a(p) \exp(-i\theta^{\mu\nu} p_\mu k_\nu)$ . The isomorphism  $\varphi: \mathcal{A}_i \rightarrow \mathcal{A}$  results in

$$a(p) \mapsto b(p) := (\mathcal{F}_1 a(p)) \mathcal{F}_2 = a(p) \exp\left(\frac{i}{2}\theta^{\mu\nu} p_\mu \otimes P_\nu\right) \in \mathcal{A},$$

$b(p)b(k) = b(k)b(p) \exp(-i\theta^{\mu\nu} p_\mu k_\nu)$ . Therefore,  $\mathcal{A}_i$  is realized in the smash product of the original algebra and  $U(\mathcal{P})$  by a change of variables.

The isomorphism  $\varphi$  can be defined in the obvious way for any adjoint module, not only for  $\mathcal{H}$ . We retain for it the same notation, as the domain of  $\varphi$  is always clear from the context. The map  $\varphi$  features the following intertwining properties.

**Proposition 2.3.** *The isomorphism  $\varphi$  intertwines the adjoint actions of  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  on  $\mathcal{E}$ .*

*Proof.* First of all, remark that  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  coincide as associative algebras and enjoy the same homomorphism  $\rho$  to  $\mathcal{E}$ . We need to show

$$\rho(h^{(1)}) (\varphi(a)) \rho(\tilde{\gamma}(h^{(2)})) = \varphi\left(\rho(h^{(1)}) a \rho(\gamma(h^{(2)}))\right) \quad (6)$$

for all  $h \in \mathcal{H}$  and  $a \in \mathcal{E}$ . The twiddled indices on the left-hand side designate the twisted coproduct. To simplify the formulas we assume  $\mathcal{E} = \mathcal{H}$  and  $\rho = \text{id}$ . For  $h \in \mathcal{H}$  and  $a \in \mathcal{E}$  we find

$$\begin{aligned} h^{(1)} (\varphi(a)) \tilde{\gamma}(h^{(2)}) &= \mathcal{F}_1^{-1} h^{(1)} \mathcal{F}_{1'} (\mathcal{F}_{1''}^{-1} a \gamma(\mathcal{F}_{2''}^{-1}) \vartheta) \vartheta^{-1} \gamma(\mathcal{F}_2^{-1} h^{(2)} \mathcal{F}_{2'}) \vartheta \\ &= \mathcal{F}_1^{-1} h^{(1)} a \gamma(h^{(2)}) \gamma(\mathcal{F}_2^{-1}) \vartheta = \varphi\left(\rho(h^{(1)}) a \rho(\gamma(h^{(2)}))\right), \end{aligned}$$

as required. □

It follows from Theorem 2.1 that

$$\tilde{\rho}(h) := \rho(\mathcal{F}_1 h \gamma(\mathcal{F}_2 \zeta)) = \rho((\mathcal{F}_1^{-1} \cdot h) \mathcal{F}_2^{-1}) = (\varphi^{-1} \circ \rho)(h)$$

defines an algebra homomorphism  $\mathcal{H} \rightarrow \mathcal{E}_\zeta$ . Put  $\widetilde{\text{ad}} = \text{ad}_{\tilde{\rho}}$  to be the adjoint action of  $\tilde{\mathcal{H}}$  on  $\mathcal{E}_\zeta$  defined through the homomorphism  $\tilde{\rho}$ :

$$\widetilde{\text{ad}}(h)a = \tilde{\rho}(h^{(1)}) a \tilde{\rho}(\tilde{\gamma}(h^{(2)})), \quad a \in \mathcal{E}_\zeta, \quad h \in \tilde{\mathcal{H}}, \quad h^{(1)} \otimes h^{(2)} := \tilde{\Delta}(h).$$

The product on the right-hand side is taken with respect to the multiplication in  $\mathcal{E}_\zeta$ . Thus one has two representations,  $\widetilde{\text{ad}}$  and  $\text{ad}$ , of the same associative algebra  $\mathcal{H}$  on the vector space  $\mathcal{E}$ .

**Corollary 2.4.** *The representations  $\widetilde{\text{ad}}$  and  $\text{ad}$  coincide.*

*Proof.* Apply the homomorphism  $\varphi^{-1}: \mathcal{E} \rightarrow \mathcal{E}_\zeta$  to both sides of equality (6). □

Theorem 2.1 admits the following generalization. Consider the smash product  $\mathcal{A} \rtimes \mathcal{H}$  of the Hopf algebra  $\mathcal{H}$  and its module algebra  $\mathcal{A}$ . The multiplication in  $\mathcal{A} \rtimes \mathcal{H}$  is described as follows. Both  $\mathcal{A}$  and  $\mathcal{H}$  are subalgebras in  $\mathcal{A} \rtimes \mathcal{H}$ , which is the free right  $\mathcal{H}$ -module generated by  $\mathcal{A}$ . Then the multiplication is determined by the relations  $ha = (h^{(1)} \triangleright a)h^{(2)}$ ,  $h \in \mathcal{H}$ ,  $a \in \mathcal{A}$ . The subalgebra  $\mathcal{A}$  is invariant under the adjoint action of  $\mathcal{H}$  on  $\mathcal{A} \rtimes \mathcal{H}$  due to  $\text{ad}(h)(a) = h \triangleright a$ , for all  $h \in \mathcal{H}$  and  $a \in \mathcal{A}$ . Now consider a twist  $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$  and let  $\tilde{\mathcal{H}}$  be the twisted Hopf algebra. Let  $\mathcal{A}_\zeta$  be the corresponding cotwist of the  $\mathcal{H}$ -module algebra  $\mathcal{A}$ ; it is a module algebra over  $\tilde{\mathcal{H}}$  with the same action  $\triangleright$ .

**Proposition 2.5.** *The mapping  $\mathcal{A}_\wr \times \tilde{\mathcal{H}} \rightarrow \mathcal{A} \rtimes \mathcal{H}$ ,  $ah \mapsto (\mathcal{F}_1 \triangleright a)\mathcal{F}_2 h$  is an isomorphism of algebras.*

*Proof.* In particular, we need to check that the assignment  $a \mapsto (\mathcal{F}_1 \triangleright a)\mathcal{F}_2$  defines an algebra homomorphism  $\mathcal{A}_\wr \rightarrow \mathcal{A} \rtimes \mathcal{H}$ . The proof is essentially the same as of Theorem 2.1, because  $(\mathcal{F}_1 \triangleright a)\mathcal{F}_2 = \text{ad}(\mathcal{F}_1)(a)\mathcal{F}_2$ . We have yet to check that the cross-relations in the smash product are preserved. The equality  $ha = (h^{(1)} \triangleright a)h^{(2)}$  in  $\mathcal{A}_\wr \times \tilde{\mathcal{H}}$  goes over to the equality  $h(\mathcal{F}_1 \triangleright a)\mathcal{F}_2 = (\mathcal{F}_1 h^{(1)} \triangleright a)\mathcal{F}_2 h^{(2)}$  in  $\mathcal{A} \rtimes \mathcal{H}$ . The latter equality holds true, as the right-hand side is nothing but  $(h^{(1)}\mathcal{F}_1 \triangleright a)h^{(2)}\mathcal{F}_2$  (follows directly from the definition of twisted coproduct).  $\square$

Remark that Proposition 2.5 may be also considered as a specialization of Theorem 2.1. Indeed, the embedding  $\mathcal{H} \rightarrow \mathcal{A} \rtimes \mathcal{H}$  allows to consider  $\mathcal{A} \rtimes \mathcal{H}$  as an adjoint  $\mathcal{H}$ -module algebra. The factors  $\mathcal{A}$  and  $\mathcal{H}$  are invariant, and the adjoint action restricted to  $\mathcal{A}$  is simply  $\triangleright$ . This implies that the cotwist  $(\mathcal{A} \rtimes \mathcal{H})_\wr$  factorizes to the product  $\mathcal{A}_\wr \mathcal{H}_\wr$ , where  $\mathcal{H}_\wr$  is isomorphic to  $\mathcal{H}$  as an associative algebra. This factorization turns into the smash product  $\simeq \mathcal{A}_\wr \times \tilde{\mathcal{H}}$  upon transition from  $\mathcal{H}_\wr$  to  $\mathcal{H}$  (it suffices to check the cross-relations only). On the other hand,  $(\mathcal{A} \rtimes \mathcal{H})_\wr$  is isomorphic to  $\mathcal{A} \rtimes \mathcal{H}$ , by Theorem 2.1.

Adjoint module algebras naturally appear in the following situation. Suppose  $\mathcal{A}$  is an  $\mathcal{H}$ -module algebra and consider the algebra  $\mathcal{E} = \text{End}(\mathcal{A})$  of linear endomorphisms of  $\mathcal{A}$ . The action of  $\mathcal{H}$  on  $\mathcal{A}$  implies a homomorphism  $\mathcal{H} \rightarrow \mathcal{E}$ . If we define the adjoint action of  $\mathcal{H}$  on  $\mathcal{E}$  by setting  $x.A := x^{(1)}A\gamma(x^{(2)})$ , then the action of  $\mathcal{E}$  on  $\mathcal{A}$  is  $\mathcal{H}$ -equivariant (that is, the map  $\mathcal{E} \otimes \mathcal{A} \rightarrow \mathcal{A}$  is a homomorphism of  $\mathcal{H}$ -modules). Twist of  $\mathcal{H}$  induces a transformation of the algebras  $\mathcal{A}$  and  $\mathcal{E}$ , as well as of the  $\mathcal{E}$ -action on  $\mathcal{A}$ :

$$X * a := (\mathcal{F}_1.X)(\mathcal{F}_2 a), \quad X \in \mathcal{E}, \quad a \in \mathcal{A}.$$

The algebra isomorphism  $\varphi^{-1}: \mathcal{E} \rightarrow \mathcal{E}_\wr$  and the transformation of the action compensate each other in the following sense:

$$\varphi^{-1}(X) * a = Xa$$

for all  $X \in \mathcal{E}$  and  $a \in \mathcal{A}$ .

### 3 $\mathcal{H}$ -Lie algebras

In the present section, we recall braided or  $\mathcal{H}$ -Lie algebras in our terminology, assuming  $\mathcal{H}$  to be triangular.



Suppose that  $\mathcal{H}$  is triangular and let  $\mathfrak{g}$  be an  $\mathcal{H}$ -module. Let us call  $\mathfrak{g}$  an  $\mathcal{H}$ -Lie algebra if it is equipped with an  $\mathcal{H}$ -equivariant map  $[\cdot, \cdot]_{\mathcal{R}}: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying the braided Jacobi identity, [9],

$$[\xi, [\eta, \zeta]_{\mathcal{R}}]_{\mathcal{R}} = [[\xi, \eta]_{\mathcal{R}}, \zeta]_{\mathcal{R}} + [\mathcal{R}_2 \triangleright \eta, [\mathcal{R}_1 \triangleright \xi, \zeta]_{\mathcal{R}}]_{\mathcal{R}}. \quad (7)$$

plus the skew symmetry condition

$$[\mathcal{R}_2 \triangleright \xi, \mathcal{R}_1 \triangleright \eta]_{\mathcal{R}} = -[\eta, \xi]_{\mathcal{R}}.$$

For example, any associative  $\mathcal{H}$ -module algebra  $\mathcal{B}$  is an  $\mathcal{H}$ -Lie algebra via the commutator

$$[a, b]_{\mathcal{R}} := ab - (\mathcal{R}_2 \triangleright b)(\mathcal{R}_1 \triangleright a)$$

for all  $a, b \in \mathcal{B}$ . The algebra  $\mathcal{B}$  is quasi-commutative if and only if this commutator vanishes. In general, we shall call an  $\mathcal{H}$ -Lie algebra with zero commutator quasi-Abelian.

One can naturally define homomorphisms between  $\mathcal{H}$ -Lie algebras, which are equivariant maps preserving the commutators.

The quotient of the tensor algebra  $T(\mathfrak{g})$  by the relations  $\xi \otimes \eta - (\mathcal{R}_2 \triangleright \eta) \otimes (\mathcal{R}_1 \triangleright \xi) - [\xi, \eta]_{\mathcal{R}}$  is called universal enveloping algebra of  $\mathfrak{g}$  and denoted by  $U(\mathfrak{g})$ . Since the ideal is invariant, the  $\mathcal{H}$ -action on  $T(\mathfrak{g})$  produces an action on  $U(\mathfrak{g})$ .

**Proposition 3.1.** *Suppose that  $\mathcal{H}$  is triangular and let  $\mathfrak{g}$  be an  $\mathcal{H}$ -Lie algebra. Then the smash product  $U(\mathfrak{g}) \rtimes \mathcal{H}$  is a triangular Hopf algebra with the universal  $R$ -matrix  $\mathcal{R}$  and the comultiplication extended from  $\mathcal{H}$  by*

$$\Delta(\xi) := \xi \otimes 1 + \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \xi, \quad \xi \in \mathfrak{g}. \quad (8)$$

*The counit and antipode when restricted to  $\mathfrak{g}$  are given by  $\varepsilon(\xi) = 0$  and  $\gamma(\xi) = -(\mathcal{R}_2 \triangleright \xi)\mathcal{R}_1$ .*

*Proof.* The formulas for counit and antipode readily follow from the comultiplication, which is obviously coassociative. Let us prove that it is a homomorphism. First of all, the comultiplication respects the cross-relations  $h\xi = (h^{(1)} \triangleright \xi)h^{(2)}$ :

$$\begin{aligned} \Delta(h)\Delta(\xi) &= h^{(1)}\xi \otimes h^{(2)} + h^{(1)}\mathcal{R}_2 \otimes h^{(2)}\mathcal{R}_1 \triangleright \xi \\ &= (h^{(1)} \triangleright \xi)h^{(2)} \otimes h^{(3)} + h^{(1)}\mathcal{R}_2 \otimes (h^{(2)}\mathcal{R}_1) \triangleright \xi h^{(3)} \\ &= (h^{(1)} \triangleright \xi)h^{(2)} \otimes h^{(3)} + \mathcal{R}_2 h^{(2)} \otimes \mathcal{R}_1 \triangleright (h^{(1)} \triangleright \xi)h^{(3)} = \Delta(h^{(1)} \triangleright \xi)\Delta(h^{(2)}), \end{aligned} \quad (9)$$

for all  $\xi \in \mathfrak{g}$ ,  $h \in \mathcal{H}$ . Further, for any tensor  $\xi \otimes \eta \in \mathfrak{g} \otimes \mathfrak{g}$  we write

$$\begin{aligned}\Delta(\xi\eta) &= \xi\eta \otimes 1 + \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright (\xi\eta) + \xi\mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \eta + \mathcal{R}_2\eta \otimes \mathcal{R}_1 \triangleright \xi, \\ \Delta(\tilde{\eta}\tilde{\xi}) &= \tilde{\eta}\tilde{\xi} \otimes 1 + \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright (\tilde{\eta}\tilde{\xi}) + \tilde{\eta}\mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \tilde{\xi} + \mathcal{R}_2\tilde{\xi} \otimes \mathcal{R}_1 \triangleright \tilde{\eta}.\end{aligned}$$

where we put  $\tilde{\eta} \otimes \tilde{\xi} = \mathcal{R}_2 \triangleright \eta \otimes \mathcal{R}_1 \triangleright \xi = \Phi_1 \eta \Phi_2 \otimes \Phi_3 \xi \Phi_4$  for certain  $\Phi = \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4 \in \mathcal{H}^{\otimes 4}$ . Subtracting  $\Delta(\tilde{\eta}\tilde{\xi})$  from  $\Delta(\xi\eta)$  we prove the statement, provided that

$$\begin{aligned}\mathcal{R}_2\eta \otimes (\mathcal{R}_1 \triangleright \xi) &= (\Phi_1\eta\Phi_2)\mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright (\Phi_3\xi\Phi_4), \\ \xi\mathcal{R}_2 \otimes (\mathcal{R}_1 \triangleright \eta) &= \mathcal{R}_2(\Phi_3\xi\Phi_4) \otimes \mathcal{R}_1 \triangleright (\Phi_1\eta\Phi_2).\end{aligned}$$

They are satisfied indeed, because, by the definition of  $\Phi$ ,

$$(\Delta \otimes \Delta^{op})(\mathcal{R}) = \gamma_2^{-1} \gamma_3 \Phi.$$

Then the first equation holds identically. The second equation is true for triangular  $\mathcal{R}$ .  $\square$

There is a natural  $\mathcal{H}$ -invariant filtration on  $U(\mathfrak{g})$ , and the corresponding graded algebra is the quasi-commutative  $\mathcal{R}$ -symmetric algebra of the module  $\mathfrak{g}$ . Using this filtration, one can check that  $\mathfrak{g}$  is precisely the set of quasi-primitive elements in  $U(\mathfrak{g})$ , i.e. the elements satisfying the identity (8).

There is a natural action of the Hopf algebra  $U(\mathfrak{g}) \rtimes \mathcal{H}$  on  $U(\mathfrak{g})$  making the latter an  $U(\mathfrak{g}) \rtimes \mathcal{H}$ -module algebra. Namely,  $U(\mathfrak{g})$  is invariant under the adjoint action. So, for any  $\xi \in \mathfrak{g}$  and  $a \in U(\mathfrak{g})$  one has  $\text{ad}(\xi)a = [\xi, a]_{\mathcal{R}}$ .

**Proposition 3.2.** *Let  $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$  be a twisting cocycle. Suppose that  $\mathfrak{g}$  is an  $\mathcal{H}$ -Lie algebra. Then the commutator  $[\cdot, \cdot]_{\tilde{\mathcal{R}}} := [\cdot, \cdot]_{\mathcal{R}} \circ \mathcal{F}$  defines on  $\mathfrak{g}$  a structure of an  $\tilde{\mathcal{H}}$ -Lie algebra.*

*Proof.* We need to show that  $[\cdot, \cdot]_{\tilde{\mathcal{R}}}$  satisfies the Jacobi identity. Denote by  $\tau$  the ordinary flip  $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  and write for the mapping  $[\cdot, [\cdot, \cdot]_{\tilde{\mathcal{R}}}]_{\tilde{\mathcal{R}}}: \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ :

$$[\cdot, [\cdot, \cdot]_{\mathcal{R}}]_{\mathcal{R}} \circ \Delta_2(\mathcal{F})\mathcal{F}_{23} = [[\cdot, \cdot]_{\mathcal{R}}, \cdot]_{\mathcal{R}} \circ \Delta_1(\mathcal{F})\mathcal{F}_{12} + [\cdot, [\cdot, \cdot]_{\mathcal{R}}]_{\mathcal{R}} \circ \tau_{12} \circ \mathcal{R}_{12} \Delta_1(\mathcal{F})\mathcal{F}_{12}.$$

Here we applied the twist equation and the Jacobi identity to the commutator  $[\cdot, \cdot]_{\mathcal{R}}$ . The first term on the right-hand side is just  $[[\cdot, \cdot]_{\tilde{\mathcal{R}}}, \cdot]_{\tilde{\mathcal{R}}}$ . Using the standard Hopf algebra technique, we get

$$\tau_{12} \circ \mathcal{R}_{12} \Delta_1(\mathcal{F})\mathcal{F}_{12} = \tau_{12} \circ \Delta_1^{op}(\mathcal{F})\mathcal{F}_{21} \tilde{\mathcal{R}}_{12} = \Delta_1(\mathcal{F})\mathcal{F}_{12} \circ \tau_{12} \circ \tilde{\mathcal{R}}_{12},$$

which gives for the second term the expression  $[\cdot, [\cdot, \cdot]_{\tilde{\mathcal{R}}}]_{\tilde{\mathcal{R}}} \circ \tau_{12} \circ \tilde{\mathcal{R}}_{12}$ , i.e. the second summand in the Jacobi identity.  $\square$

We denote by  $\mathfrak{g}_l$  the vector space  $\mathfrak{g}$  together with this  $\tilde{\mathcal{H}}$ -Lie algebra structure.

**Proposition 3.3.** *The cotwist  $U(\mathfrak{g})_l$  is isomorphic to  $U(\mathfrak{g}_l)$ , and the isomorphism is extended from the identity map  $\mathfrak{g} \ni \xi \mapsto \xi \in \mathfrak{g}_l$ .*

*Proof.* The universal enveloping algebra  $U(\mathfrak{g})$  is a quotient of the tensor algebra  $T(\mathfrak{g})$  over the ideal  $J(\mathfrak{g})$  generated by the commutator relations. It is easy to see that the cotwist  $U(\mathfrak{g})_l$  is isomorphic to the quotient of  $T(\mathfrak{g})$  by the ideal  $J(\mathfrak{g}_l)$ , where  $\mathfrak{g}_l$  is obtained from  $\mathfrak{g}$  by rewriting the defining relations in terms of the new multiplication,  $*$ . That is,  $(\mathcal{F}_1^{-1} \triangleright \xi) * (\mathcal{F}_2^{-1} \triangleright \eta) - (\mathcal{F}_1^{-1} \mathcal{R}_2 \triangleright \eta) * (\mathcal{F}_2^{-1} \mathcal{R}_1 \triangleright \xi) = [\xi, \eta]_{\mathcal{R}}$  for all  $\xi, \eta \in \mathfrak{g}$ . As  $\mathcal{F}$  is invertible, this is equivalent to the relations  $\xi * \eta - (\tilde{\mathcal{R}}_2 \triangleright \eta) * (\tilde{\mathcal{R}}_1 \triangleright \xi) = [\mathcal{F}_1 \triangleright \xi, \mathcal{F}_2 \triangleright \eta]_{\mathcal{R}}$ , as required.

The isomorphism  $U(\mathfrak{g}_l) \rightarrow U(\mathfrak{g})_l$  is induced by a linear endomorphism of the tensor algebra  $T(\mathfrak{g})$ . On the homogeneous component of degree  $k$  this isomorphism is given by  $\xi_1 \otimes \dots \otimes \xi_k \mapsto \mathcal{F}_1^{(k)} \triangleright \xi_1 \otimes \dots \otimes \mathcal{F}_k^{(k)} \triangleright \xi_k$ , where  $\mathcal{F}^{(k)} \in \mathcal{H}^{\otimes k}$  is defined inductively by  $\mathcal{F}^{(1)} := 1$ ,  $\mathcal{F}^{(2)} := \mathcal{F}$ ,  $\mathcal{F}^{(m+n)} := (\Delta^{(m)} \otimes \Delta^{(n)})(\mathcal{F})(\mathcal{F}^{(m)} \otimes \mathcal{F}^{(n)})$ . This definition is consistent, i.e., independent of the partition  $k = m + n$ . Thus the isomorphism in question is identical on  $\mathfrak{g}$  by construction.  $\square$

**Proposition 3.4.** *Consider a twisting cocycle  $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$  as that of the Hopf algebra  $U(\mathfrak{g}) \rtimes \mathcal{H}$ . Then the assignment  $\mathfrak{g} \ni \xi \mapsto (\mathcal{F}_1 \triangleright \xi) \mathcal{F}_2 \in U(\mathfrak{g}) \rtimes \mathcal{H}$  defines an isomorphism of Hopf algebras*

$$U(\mathfrak{g}_l) \rtimes \tilde{\mathcal{H}} \rightarrow \widetilde{U(\mathfrak{g}) \rtimes \mathcal{H}} \quad (10)$$

which is identical on  $\mathcal{H}$ .

*Proof.* It follows from Proposition 2.5 that the assignment  $\varphi: \xi \mapsto (\mathcal{F}_1 \triangleright \xi) \mathcal{F}_2$  for all  $\xi \in \mathfrak{g}$  is a restriction of the associative algebra isomorphism  $U(\mathfrak{g})_l \rtimes \tilde{\mathcal{H}} \rightarrow U(\mathfrak{g}) \rtimes \mathcal{H}$ . On the other hand, the identity mapping  $\mathfrak{g}_l \rightarrow \mathfrak{g}$  extends to an  $\tilde{\mathcal{H}}$ -equivariant isomorphism  $U(\mathfrak{g}_l) \rightarrow U(\mathfrak{g})_l$ . Therefore the assignment in question extends to an isomorphism of algebras.

We need to check that the coalgebra structure is preserved by (10). That is obvious for its restriction to  $\tilde{\mathcal{H}}$ . Let us prove that  $\varphi$  is a coalgebra map when restricted to  $U(\mathfrak{g}_l)$ . To this end, we calculate  $\tilde{\Delta}(\varphi(\xi)) = \mathcal{F}^{-1} \Delta((\mathcal{F}_1 \triangleright \xi) \mathcal{F}_2) \mathcal{F}$  for  $\xi \in \mathfrak{g}_l$ . Note that all the products and coproducts here are written in terms of  $U(\mathfrak{g}) \rtimes \mathcal{H}$ . Using the twist identity we get for  $\tilde{\Delta}(\varphi(\xi))$ :

$$\mathcal{F}^{-1} \Delta(\mathcal{F}_1 \triangleright \xi) (\mathcal{F}_2^{(1)} \mathcal{F}_{1'} \otimes \mathcal{F}_2^{(2)} \mathcal{F}_{2'}) = \mathcal{F}^{-1} \Delta((\mathcal{F}_1^{(1)} \mathcal{F}_{1'}) \triangleright \xi) (\mathcal{F}_1^{(2)} \mathcal{F}_{2'} \otimes \mathcal{F}_2).$$

Now evaluate  $\Delta$  of the element from  $\mathfrak{g}$  and get the sum of two terms, of which the first is the product of  $\mathcal{F}^{-1}$  and  $((\mathcal{F}_1^{(1)}\mathcal{F}_{1'}) \triangleright \xi)\mathcal{F}_1^{(2)}\mathcal{F}_{2'} \otimes \mathcal{F}_2 = \mathcal{F}_1(\mathcal{F}_{1'}) \triangleright \xi\mathcal{F}_{2'} \otimes \mathcal{F}_2$ . Thus, the first summand is equal to  $\varphi(\xi) \otimes 1$ . The other summand is the product of  $\mathcal{F}^{-1}$  and

$$\begin{aligned} & \mathcal{R}_2\mathcal{F}_1^{(2)}\mathcal{F}_{2'} \otimes ((\mathcal{R}_1\mathcal{F}_1^{(1)}\mathcal{F}_{1'}) \triangleright \xi)\mathcal{F}_2 = \mathcal{F}_1^{(1)}\mathcal{R}_2\mathcal{F}_{2'} \otimes ((\mathcal{F}_1^{(2)}\mathcal{R}_1\mathcal{F}_{1'}) \triangleright \xi)\mathcal{F}_2 \\ & = \mathcal{F}_1^{(1)}\mathcal{F}_{1'}\tilde{\mathcal{R}}_2 \otimes ((\mathcal{F}_1^{(2)}\mathcal{F}_{2'}\tilde{\mathcal{R}}_1) \triangleright \xi)\mathcal{F}_2 = \mathcal{F}_1\tilde{\mathcal{R}}_2 \otimes ((\mathcal{F}_2^{(1)}\mathcal{F}_{1'}\tilde{\mathcal{R}}_1) \triangleright \xi\mathcal{F}_2^{(2)})\mathcal{F}_{2'} \\ & = \mathcal{F}_1\tilde{\mathcal{R}}_2 \otimes \mathcal{F}_2((\mathcal{F}_{1'}\tilde{\mathcal{R}}_1) \triangleright \xi)\mathcal{F}_{2'}. \end{aligned}$$

Multiplying this by  $\mathcal{F}^{-1}$  we get  $\tilde{\mathcal{R}}_2 \otimes \varphi(\tilde{\mathcal{R}}_1 \triangleright \xi)$  for the second summand. Note that the dot here is the adjoint action of  $\mathcal{H}$  on  $\mathfrak{g}$ , not of  $\tilde{\mathcal{H}}$ . Apply to this formula (6) where  $\rho$  is the embedding  $\mathcal{H} \hookrightarrow \mathrm{U}(\mathfrak{g}) \rtimes \mathcal{H}$  and get  $\tilde{\mathcal{R}}_2 \otimes \tilde{\mathcal{R}}_1 \triangleright \varphi(\xi)$ . This proves that  $\varphi$  respects comultiplication and when restricted to  $\mathfrak{g}_l$  is therefore a coalgebra map.  $\square$

**Remark 3.5.** Let us emphasize that the  $\tilde{\mathcal{H}}$ -Lie algebra  $\mathfrak{g}_l$  is rotated from  $\mathfrak{g}$  inside of the associative algebra  $\mathrm{U}(\mathfrak{g}) \rtimes \mathcal{H}$  by means of  $\varphi$ , that is  $\mathfrak{g}_l = \varphi(\mathfrak{g})$ . This readily follows from the proof of the above proposition.

For any triangular Hopf algebra  $\mathcal{H}$  denote by  $\mathrm{Lie}(\mathcal{H})$  the subset of elements satisfying the condition

$$\Delta(\xi) := \xi \otimes 1 + \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \xi = \xi \otimes 1 + \mathcal{R}_2\mathcal{R}_{1'} \otimes \mathcal{R}_1\xi\mathcal{R}_{2'}.$$

It is easy to check that  $\mathrm{Lie}(\mathcal{H})$  is invariant under the adjoint action of  $\mathcal{H}$ . Moreover,  $\mathrm{Lie}(\mathcal{H})$  is an  $\mathcal{H}$ -Lie algebra under the  $\mathcal{R}$ -commutator, hence there exists an algebra homomorphism  $\mathrm{U}(\mathrm{Lie}(\mathcal{H})) \rightarrow \mathcal{H}$ . This homomorphism also extends to a Hopf algebra homomorphism  $\mathrm{U}(\mathrm{Lie}(\mathcal{H})) \rtimes \mathcal{H} \rightarrow \mathcal{H}$ .

**Proposition 3.6.** Put  $\mathfrak{g} = \mathrm{Lie}(\mathcal{H})$  and denote by  $\mathfrak{g}_l$  the cotwist of  $\mathfrak{g}$  by a cocycle  $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ . Then  $\mathrm{Lie}(\tilde{\mathcal{H}}) \simeq \mathfrak{g}_l$ .

*Proof.* Denote by  $\tilde{\mathfrak{g}}$  the Lie algebra  $\mathrm{Lie}(\tilde{\mathcal{H}})$ . Apply the chain of Hopf algebra homomorphisms

$$\mathrm{U}(\mathfrak{g}_l) \rtimes \tilde{\mathcal{H}} \rightarrow \widetilde{\mathrm{U}(\mathfrak{g}) \rtimes \mathcal{H}} \rightarrow \tilde{\mathcal{H}}$$

to  $\mathfrak{g}_l$  and find that  $\varphi(\mathfrak{g}_l) \subset \tilde{\mathfrak{g}}$ , see Remark 3.5. On the other hand, twist is invertible and the role of  $\varphi$  for the inverse twist belongs to  $\varphi^{-1}$ . Hence we can do the same with  $\tilde{\mathcal{H}}$  replaced by  $\mathcal{H}$  and get the inverse inclusion  $\varphi^{-1}(\tilde{\mathfrak{g}}) \subset \mathfrak{g}$ . This implies the statement, because  $\mathfrak{g}_l$  is isomorphic to  $\mathfrak{g}$  as a vector space.  $\square$

**Proposition 3.7.** *Suppose that  $\mathcal{H}$  is generated by  $\text{Lie}(\mathcal{H})$  as an associative algebra. Then  $\tilde{\mathcal{H}}$  is generated by  $\text{Lie}(\tilde{\mathcal{H}})$ .*

*Proof.* Denote  $\mathfrak{g} = \text{Lie}(\mathcal{H})$ . It is sufficient to consider the case when  $\mathcal{H} = U(\mathfrak{g})$ . Put  $\mathcal{A} := \mathcal{H}$  and consider it as the adjoint module algebra over  $\mathcal{H}$ . Since  $\mathcal{A} = U(\mathfrak{g})$ , one has  $\tilde{\mathcal{A}} = U(\mathfrak{g})_\imath \simeq U(\mathfrak{g}_\imath)$ , by Proposition 3.3 and  $\text{Lie}(\tilde{\mathcal{H}}) \simeq \mathfrak{g}_\imath$  by Proposition 3.6. But  $\tilde{\mathcal{A}}$  is isomorphic to  $\mathcal{A}$ , by Proposition 2.1.  $\square$

Note that the Lie algebras  $\mathfrak{g}_\imath$  and  $\mathfrak{g}$  are related as subsets of  $\mathcal{H}$  by the isomorphism  $\varphi$ :  $\mathfrak{g}_\imath = \varphi(\mathfrak{g})$ , cf. Remark 3.5.

Suppose that  $\mathcal{H}$  is a classical universal enveloping algebra  $U(\mathfrak{g})$ . Given a twist of  $\mathcal{H}$ , the associative algebra  $\tilde{\mathcal{H}}$  is isomorphic to  $U(\mathfrak{g}_\imath)$ . The elements of  $\mathfrak{g}_\imath$  form a Lie algebra over  $\tilde{\mathcal{H}}$ , and  $\mathfrak{g}_\imath$  is linked to  $\mathfrak{g}$  by the transformation  $\varphi$ . In other words, there is a set of generators for  $U(\mathfrak{g})$ , such that the twisted comultiplication is given by  $\tilde{\Delta}(\xi) = \xi \otimes 1 + \mathcal{R}_1^{-1}\mathcal{R}_{1'} \otimes \mathcal{R}_2^{-1}\xi\mathcal{R}_{2'}$ , where  $\mathcal{R} = \mathcal{F}_{21}^{-1}\mathcal{F}$ .

## 4 Twisted vector fields

In this section we consider an important example of quantum Lie algebras formed by "quantum vector fields", [10]. As before, we assume that  $\mathcal{H}$  is a triangular Hopf algebra and  $\mathcal{A}$  is a left  $\mathcal{H}$ -module algebra.

**Definition 4.1.** An element  $X \in \text{End}(\mathcal{A})$  is called a first order differential operator if

$$X(ab) = (Xa)b + (\mathcal{R}_2\mathcal{R}_{2'}^{-1}a)(\mathcal{R}_1X\mathcal{R}_1^{-1}b) = (Xa)b + (\mathcal{R}_2a)(\mathcal{R}_1.Xb).$$

The subset of first order differential operators is denoted by  $\text{Der}_{\mathcal{H}}(\mathcal{A})$ . Following the geometric analogy, we shall also call elements of  $\text{Der}_{\mathcal{H}}(\mathcal{A})$  vector fields, thinking of  $\mathcal{A}$  as the function algebra on a quantum space. If  $\mathcal{A}$  is quasi-commutative, then  $\text{Der}_{\mathcal{H}}(\mathcal{A})$  is a natural  $\mathcal{A}$ -submodule in the left module  $\text{End}(\mathcal{A})$ . Clearly the  $\mathcal{A}$ -action on  $\text{Der}_{\mathcal{H}}(\mathcal{A})$  is  $\mathcal{H}$ -equivariant.

We can introduce a "comultiplication" on vector fields as a map

$$\Delta: \text{Der}_{\mathcal{H}}(\mathcal{A}) \rightarrow \text{Der}_{\mathcal{H}}(\mathcal{A}) \rtimes \mathcal{H} \otimes \text{Der}_{\mathcal{H}}(\mathcal{A}), \quad \Delta: X \mapsto X \otimes 1 + \mathcal{R}_2 \otimes \mathcal{R}_1.X.$$

This allows to use same technique to prove facts about  $\text{Der}_{\mathcal{H}}(\mathcal{A})$  as we did in the previous section for  $\mathcal{H}$ -Lie algebras.

**Proposition 4.2.** *The set  $\text{Der}_{\mathcal{H}}(\mathcal{A})$  is  $\mathcal{H}$ -invariant.*

*Proof.* Same as the proof of (9). □

If the algebra  $\mathcal{A}$  is quasi-commutative,  $\text{Der}_{\mathcal{H}}(\mathcal{A})$  is a left  $\mathcal{A}$ -module regarded as a subalgebra of  $\text{End}(\mathcal{A})$  by left multiplication.

Now consider the cotwist of  $\mathcal{A}$ , and also of  $\mathcal{E} := \text{End}(\mathcal{A})$ . The new action of  $\mathcal{E}_l$  on  $\mathcal{A}_l$  is expressed through the old action and the homomorphism  $\varphi$  by  $X * a = \mathcal{F}_1.X\mathcal{F}_2a = \varphi(X)a$ .

**Proposition 4.3.**  $\text{Der}_{\tilde{\mathcal{H}}}(\mathcal{A}_l) = \text{Der}_{\mathcal{H}}(\mathcal{A})$  as subsets in  $\mathcal{E}$ .

*Proof.* Suppose  $X \in \text{Der}_{\mathcal{H}}(\mathcal{A})$ . We need to show that

$$X * (a * b) = (X * a) * b + (\tilde{\mathcal{R}}_2a) * ((\tilde{\mathcal{R}}_1.X) * b)$$

for all  $a, b \in \mathcal{A}$ . Applying to the left-hand side the same steps as in the proof of Proposition 3.4 we arrive at the expression  $(X * a) * b + (\tilde{\mathcal{R}}_2a) * ((\mathcal{F}_1.\tilde{\mathcal{R}}_1).X\mathcal{F}_2b)$ . The dot in the second term involves the coproduct  $\mathcal{H}$ . The intertwining formula (6) allows to write it as  $(\tilde{\mathcal{R}}_2\tilde{\mathcal{R}}_1'a) * (\tilde{\mathcal{R}}_1.X * \tilde{\mathcal{R}}_2'b)$ , i.e. to interpret the dot through the new coproduct. □

The following statement asserts that the subset  $\text{Der}_{\mathcal{H}}(\mathcal{A}) \subset \text{End}(\mathcal{A})$  is a subalgebra of the commutator  $\mathcal{H}$ -Lie algebra  $\text{End}(\mathcal{A})$ .

**Proposition 4.4.** For any  $X, Y \in \text{Der}_{\mathcal{H}}(\mathcal{A})$ , the operator  $[X, Y]_{\mathcal{R}} = XY - (\mathcal{R}_2.Y)(\mathcal{R}_1.X)$  belongs to  $\text{Der}_{\mathcal{H}}(\mathcal{A})$ .

*Proof.* Same as the proof of Proposition 3.1. □

Thus, the  $\mathcal{H}$ -module  $\mathfrak{g} = \text{Der}_{\mathcal{H}}(\mathcal{A})$  is an  $\mathcal{H}$ -Lie algebra. The action of  $\mathfrak{g}$  on  $\mathcal{A}$  extends to an action of the universal enveloping algebra  $U(\mathfrak{g})$ , which together with the action of  $\mathcal{H}$  makes  $\mathcal{A}$  a module algebra over  $U(\mathfrak{g}) \rtimes \mathcal{H}$ . The representation  $\mathcal{H} \rightarrow \text{End}(\mathcal{A})$  induces an  $\mathcal{H}$ -Lie algebra homomorphism  $\text{Lie}(\mathcal{H}) \rightarrow \text{Der}_{\mathcal{H}}(\mathcal{A})$ .

Further we give examples of  $\mathcal{H}$ -Lie algebras naturally arising in geometrical applications and, in particular, in gauge field theory on non-commutative space-time.

**Example 4.5.** Let  $\mathcal{H}$  be a triangular Hopf algebra and  $\mathcal{A}$  be a quasi-commutative  $\mathcal{H}$ -module algebra. Suppose that  $\mathfrak{g}$  is an  $\mathcal{H}$ -Lie algebra. Denote by  $\mathcal{A} \bowtie \mathfrak{g}$  the  $\mathcal{H}$ -module  $\mathcal{A} \otimes \mathfrak{g}$  equipped with the bilinear operation

$$[a \otimes \xi, b \otimes \eta]_{\mathcal{R}} := a(\mathcal{R}_2 \triangleright b) \otimes [\mathcal{R}_1 \triangleright \xi, \eta]_{\mathcal{R}}, \quad (11)$$

where  $a, b \in \mathcal{A}$  and  $\xi, \eta \in \mathfrak{g}$ .

**Proposition 4.6.** *The  $\mathcal{H}$ -module  $\mathcal{A} \bowtie \mathfrak{g}$  is an  $\mathcal{H}$ -Lie algebra.*

*Proof.* Denote by the  $\mathcal{A} \bowtie U(\mathfrak{g})$  the associative algebra with the underlining vector space  $\mathcal{A} \otimes \mathfrak{g}$  and the multiplication  $(a \otimes \xi)(b \otimes \eta) := a(\mathcal{R}_2 \triangleright b) \otimes (\mathcal{R}_1 \triangleright \xi)\eta$ . This is the so called braided tensor product of  $\mathcal{A}$  and  $U(\mathfrak{g})$  and is a module algebra over  $\mathcal{H}$ . The vector space  $\mathcal{A} \bowtie \mathfrak{g}$  is a  $\mathcal{H}$ -submodule in  $\mathcal{A} \bowtie U(\mathfrak{g})$ . Let us prove that the commutator  $\mathcal{H}$ -Lie algebra restricts to  $\mathcal{A} \bowtie \mathfrak{g}$  and coincides with the pre-defined  $\mathcal{H}$ -Lie algebra structure on  $\mathcal{A} \bowtie \mathfrak{g}$ . That is obvious for the  $\mathcal{H}$ -Lie subalgebra  $\mathfrak{g} \subset \mathcal{A} \bowtie \mathfrak{g}$  and follows from the very construction. Further, every element  $\xi \in \mathfrak{g}$   $\mathcal{R}$ -commutes with every element  $b \in \mathcal{A} \subset \mathcal{A} \bowtie U(\mathfrak{g})$ , hence the commutator of  $\xi$  and  $b \otimes \eta$  has the desired form. By assumption, the algebra  $\mathcal{A}$  is quasi-commutative. From this we conclude that derivations of  $\mathcal{A} \bowtie U(\mathfrak{g})$  form a left  $\mathcal{A}$ -module (in this particular case even two-sided module). Hence the commutator of  $a \otimes \xi$  and  $b \otimes \eta$  is given by (11).  $\square$

The algebra  $\mathcal{A} \bowtie \mathfrak{g}$  is an  $\mathcal{A}$ -bimodule and the commutator is a bimodule map. The left action comes from the regular  $\mathcal{A}$ -action on itself, while the right action is defined by  $(a \otimes \xi)b = a\mathcal{R}_2 \triangleright b \otimes \mathcal{R}_1 \triangleright \xi$ . Geometrically, such Lie algebras are modeled by vector bundles whose fiber is an  $\mathcal{H}$ -Lie algebra, say, of the gauge group.

**Example 4.7.** The previous example is a special case of the following construction. Suppose that an  $\mathcal{H}$ -Lie algebra  $\mathfrak{g}$  acts on a quasi-commutative  $\mathcal{H}$ -module algebra  $\mathcal{A}$ . That means that there is a ( $\mathcal{H}$ -equivariant) homomorphism  $\mathfrak{g} \rightarrow \text{Der}_{\mathcal{H}}(\mathcal{A})$ . Regard the tensor product  $\mathcal{A} \otimes \mathfrak{g}$  as the natural left  $\mathcal{A}$ -module and identify  $\mathfrak{g}$  with  $1 \otimes \mathfrak{g} \subset \mathcal{A} \otimes \mathfrak{g}$ . There is a unique extension of the  $\mathcal{H}$ -Lie algebra structure from  $\mathfrak{g}$  to  $\mathcal{A} \otimes \mathfrak{g}$  satisfying

$$[\xi, f\eta]_{\mathcal{R}} = (\xi f)\eta + \mathcal{R}_2 f[\mathcal{R}_1 \xi, \eta]_{\mathcal{R}}, \quad \forall \xi, \eta \in \mathcal{A} \otimes \mathfrak{g}, \forall f \in \mathcal{A}.$$

This Lie algebra may be regarded as that of local transformations, as opposed to the global transformations by  $\mathfrak{g}$ . The loop algebra from the previous example is obtained from this by taking the zero action of  $\mathfrak{g}$  on  $\mathcal{A}$ .

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