Geometric approach to asymptotic expansion of Feynman integrals

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We present an algorithm that reveals relevant contributions in non-threshold-type asymptotic expansion of Feynman integrals about a small parameter. It is shown that the problem reduces to finding a convex hull of a set of points in a multidimensional vector space.

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I. INTRODUCTION

Evaluation of Feynman integrals depending on multiple parameters is a notoriously difficult task. When direct computation fails, one resorts to studying asymptotics in various limits. In practice, a few first terms in the expansion may already suffice to reach the desired precision. However, expansion of a multi-loop integral may become non-trivial due to an interplay of parameters with the integration variables (components of loop momenta). Classification of relevant sectors in the integration space is itself a challenging problem [1, 2].

One important case is the asymptotic expansion in momenta and masses in the limits typical for the Euclidean space. This problem has been completely solved in terms of sums over subgraphs [3–7]. At least one automated tool [8, 9] implements this approach in practice. For a more general situation, including the limits appearing in the Minkowski space, there exists the universal strategy of expansion by regions in the momentum space [1, 2, 10]. In all known cases, it produces correct results, but a rigorous proof is still lacking. Typically, one manually analyzes a multi-scale problem, starting from simpler examples that can be checked against known analytical results or numerical estimates, computed e.g. with `FIESTA` [11] (later versions [12] of FIESTA may also evaluate a few first terms in a given asymptotic expansion.)

An important type of non-Euclidean expansions, the so-called threshold expansion [1], requires the most careful treatment. Cancellation of dominant terms becomes obvious only in a specially chosen frame or with a certain routing of loop momenta. In what follows we try to elaborate some approach to non-threshold asymptotic expansion, based on alpha-representation of integrals, and describe a simple practical algorithm.

II. EXPANSION BY REGIONS AND ALPHA-REPRESENTATION

A thorough introduction to the expansion by regions and alpha-representation can be found elsewhere [10, 13]. Here we briefly introduce the basic notation with a trivial example.

Consider a family of one-loop propagator-type integrals in the Euclidean space:

$$I_1(a_1,a_2;p^2,m^2) \;=\; \int \frac{d^Dk}{(2\pi)^D}\frac{1}{D_1^{a_1}D_2^{a_2}}, \quad D_1 = k^2+m^2, \quad D_2=(k+p)^2+m^2. \tag{1}$$

A specific integral is determined by the exponents a_1 and a_2 and depends on the two parameters, m^2 and p^2. The structure of the expansion does not depend on a_1 and a_2 and we will not mention those exponents in the following discussion.

We consider the asymptotics of $I_1(a_1,a_2;p^2,m^2)$ in the limit when $|p^2| \gg m^2$, or $\rho = |m^2/p^2| \ll 1$. The naive Taylor expansion does not capture the complete asymptotic behaviour since the integration variables (components of k) span all values from $-\infty$ to $+\infty$, and in particular can be as small as m or as large as $\sqrt{|p^2|}$.

The prescription in this case is to find regions, or scalings of momentum components that after the expansion provide non-zero contributions. In each region, we first Taylor expand the integrand and drop the scaling restrictions. In our example, there are three non-zero regions, summarized in Tab. I. For example, in the region (c) one expands D_2 as follows:

$$\begin{aligned} I_1(a_1,a_2;p^2,m^2) \;&=\; \int_{k_\mu\sim m}\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+m^2)^{a_1}}\left[\frac{1}{(p^2)^{a_2}}-\frac{a_2(k^2+2kp+m^2)}{(p^2)^{a_2+1}}+\ldots\right]+\text{other regions} \\ &=\;\left[\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+m^2)^{a_1}(p^2)^{a_2}}+\int\frac{d^Dk}{(2\pi)^D}\frac{(...)}{(k^2+m^2)^{a_1}(p^2)^{a_2+1}}+\ldots\right]+\text{other regions.}\end{aligned} \tag{2}$$

In the last line, we dropped the restriction on k and the problem reduced to evaluation of (multiple) Feynman integrals with simpler denominator factors, various denominator exponents and possibly more complex numerators.

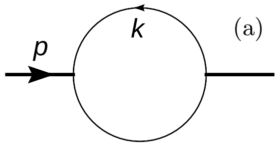
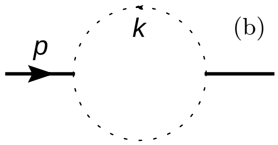
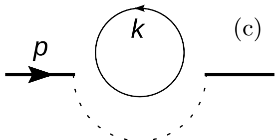
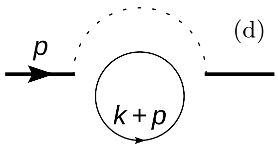
 (a)	$D_1 = k^2+m^2$, $D_2 = (k+p)^2+m^2$, $\rho = \lvert m^2/p^2 \rvert \ll 1$	$\mathcal{U} = x_1+x_2$, $\mathcal{F} = x_1x_2(p^2+2m^2)+x_1^2m^2+x_2^2m^2$
 (b)	$\lvert k^2 \rvert \sim \lvert p^2 \rvert \gg m^2$, $D_1^{(b)} = k^2$, $D_2^{(b)} = (k+p)^2$	$x_1, x_2 \sim \rho^A$, $\mathcal{U}^{(b)} = x_1+x_2$, $\mathcal{F}^{(b)} = x_1x_2$
 (c)	$\lvert k^2 \rvert \sim m^2$, $D_1^{(c)} = k^2+m^2$, $D_2^{(c)} = p^2$	$x_2 \sim \rho^A$, $x_1 \sim \rho^{A-1}$, $\mathcal{U}^{(c)} = x_1$, $\mathcal{F}^{(c)} = x_1x_2p^2+x_1^2m^2$
 (d)	$\lvert (k+p)^2 \rvert \sim m^2$, $D_1^{(d)} = p^2$, $D_2^{(d)} = (k+p)^2+m^2$	$x_2 \sim \rho^{A-1}$, $x_1 \sim \rho^A$, $\mathcal{U}^{(d)} = x_2$, $\mathcal{F}^{(d)} = x_1x_2p^2+x_2^2m^2$

TABLE I: Regions (b-d) of expansion of a double-scale integral (a) in the momentum space and in the alpha-representation.

The non-trivial statement is that the double-counting which could have been introduced disappears in the sum of all regions. For the purpose of the following discussion we assume that the tensor reduction of numerators and the proliferation of terms can be managed; we will focus on the transformations of denominator factors in every region (e.g. $(k+p)^2+m^2 \to p^2$ in the example above).

Some types of asymptotic expansion may require more elaborate choice of regions. For example, k_0 may scale differently from k_i or some combination of components may have a separate scale (this happens, e.g., in Sudakov limits). It is thus desirable to have an explicitly covariant formalism to identify regions, independent of the frame choice and the routing of momenta. For that purpose, we may switch to the alpha-representation of integrals. We re-write an integral with n lines (denominator factors) over D-dimensional loop momenta as an integral over n positive parameters $x_1, .., x_n$. Information about the graph is then encoded in the two homogeneous polynomials, $\mathcal{U}$ and $\mathcal{F}$. For example, our integral above is

$$I_1(a_1,a_2;p^2,m^2) = \frac{\Gamma(a_1+a_2-D/2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx_1dx_2\delta(1-x_1-x_2)x_1^{a_1-1}x_2^{a_2-1}\mathcal{U}^{a_1+a_2-D}\mathcal{F}^{D/2-a_1-a_2}, \qquad (3)$$
$$\mathcal{U} = x_1+x_2, \quad \mathcal{F} = x_1x_2(p^2+2m^2)+x_1^2m^2+x_2^2m^2.$$

The expansion by regions may also be formulated in the alpha-representation [2, 10]. Instead of finding the scaling behaviour of loop momentum components, here we deal with the scaling of each parameter x_i that directly corresponds to the scale of the i-th line (denominator factor) of the original integral. During the expansion, only the leading terms remain in the polynomials $\mathcal{U}$ and $\mathcal{F}$, and the resulting alpha-representation represents the integrals obtained by the expansion in the momentum space. The last column of Tab. I demonstrates the scaling of alpha-parameters and the polynomials corresponding to each region.

Note that in the language of alpha-parameters, the difference between the threshold-type and non-threshold expansion becomes clear. Let us consider integral of Eq. 1 in the threshold limit $y = m^2 + \frac{p^2}{4} \ll m^2$ (that, of course, implies that $p^2 < 0$, i.e. this limit is essentially non-Euclidean). Choosing the frame where $p = (p_0, \vec{0})$ and re-routing the loop momentum, we obtain the denominator factors $D_1 = k_0^2 + \vec{k}^2 + k_0p_0 + y$ and $D_2 = k_0^2 + \vec{k}^2 - k_0p_0 + y$.

This integral has two non-vanishing regions. The first "hard" region is characterized by $k \sim m$. The second "potential" region corresponds to $k_0 \sim y/m$, $|\vec{k}| \sim \sqrt{y}$. In the language of alpha-parameters, in the "hard" region only the second term survives in the polynomial $\mathcal{F} = y(x_1+x_2)^2 - \frac{p^2}{4}(x_1-x_2)^2$. The most troublesome "potential" region stems from a thin layer in the integration space near the surface $x_1 = x_2$, when the second term has the same scaling as the first.

In a similar way, more complex threshold expansions receive contributions which depend on cancellations between the terms in the expanded $\mathcal{F}$ which happens along some non-trivial surface and not at zero or infinity. Presently we do not know a general rule to identify such surfaces and find substitutions revealing such regions. Instead, we focus on the "usual" regions that can be determined by examining independently the monomials in $\mathcal{U}$ and $\mathcal{F}$. However limited, this problem is still important for many applications.

III. GENERAL FORMALISM

We consider an l-loop Feynman integral

$$I(a_1,...,a_n) \;=\; \int \frac{d^D k_1...d^D k_l}{(2\pi)^{lD} D_1^{a_1}...D_n^{a_n}}, \tag{4}$$

which depends on n exponents $a_1,...,a_n$, scalar products of e external momenta $p_1,...,p_e$ and parameters (such as masses) in denominator factors D_i. The latter must be quadratic in momenta but other than that may have any form, e.g. correspond to a massive, such as $-(k_i+p_j)^2+m_k^2-i0$, or a static propagator, such as $(-2k_ip_j \pm i0)$. The alpha-representation for this integral has a general structure

$$I(a_1,...,a_n) \;=\; c\int_0^1 dx_1...dx_n\ \delta(1-x_1-...-x_n)x_1^{a_1-1}...x_n^{a_n-1}\mathcal{U}^a\mathcal{F}^b, \tag{5}$$

where coefficient c and exponents a and b depend only on l, D, and a_i. $\mathcal{U}$ and $\mathcal{F}$ are homogeneous polynomials (of order l and $l+1$, respectively) of integration variables x_i, and $\mathcal{F}$ also depends on the kinematic invariants. If the denominators D_i correspond to some graph and have a standard form $-k^2+m^2-i0$ (in Minkowski space), then the functions $\mathcal{U}$ and $\mathcal{F}$ can be read off the graph in terms of trees and 2-trees [13]. In a more general case, one may obtain $\mathcal{U}$ and $\mathcal{F}$ with a tool found at `http://www-ttp.particle.uni-karlsruhe.de/~asmirnov/Tools-UF.htm`. In what follows, we will only discuss the properties of $\mathcal{U}$ and $\mathcal{F}$ that are independent of specific indices a_i.

In dimensional regularization, "scaleless" integrals (having no inherent scale) turn to zero. More specifically, an integral is scaleless if it is possible to re-scale some loop momenta or their components so that the result remains proportional to the original integral, or $D_i(\{k_j\},\{ak_i\}) = a^{u_i}D_i(\{k\})$, with some subset $\{k_i\}$ of integration momenta. In particular, massless vacuum bubbles vanish:

$$I = \int \frac{d^D k}{(k^2)^n} = \int \frac{d^D(\alpha k)}{((\alpha k)^2)^n} = \alpha^{D-2n} I = 0. \tag{6}$$

In terms of the alpha-representation Eq. 5, a similar statement applies to homogeneity of $\mathcal{U}$ and $\mathcal{F}$ with respect to a subset $\{B\}$ of integration variables x_i ($\{B\}$ should not coincide with the full set of $\{x_i\}$). Integrals vanish if $\mathcal{U}(\{x_j\},\{ax_i\}) = a^u\mathcal{U}(\{x\})$ and $\mathcal{F}(\{x_j\},\{ax_i\}) = a^f\mathcal{F}(\{x\})$, $i \in \{B\}$, with some scaling dimensions u and f.

In order to avoid separate treatment of $\mathcal{U}$ and $\mathcal{F}$, one may consider the product $\mathcal{UF}$ that incorporates the scaling and asymptotic properties of both factors (but may contain many terms).

IV. GEOMETRIC INTERPRETATION OF ASYMPTOTIC EXPANSION

Let us start with some integral in the alpha-representation Eq. 5 with integration variables $x_1,...,x_n$ and a small expansion parameter ρ. Each of M terms in $\mathcal{F}$ corresponds to a vector of $n+1$ exponents (we here neglect common factors and numeric coefficients, irrelevant to the non-threshold expansion):

$$\rho^{r_0}x_1^{r_1}...x_n^{r_n} \to (r_0,r_1,...,r_n), \tag{7}$$

and $\mathcal{F}$ corresponds to a set $\{F\}$ of M points in $(n+1)$-dimensional vector space. Due to homogeneity of $\mathcal{F}$, all these points belong to an n-dimensional hyperplane $r_1+...+r_n = l+1$, parallel to the 0-th axis (the axis of r_0).

Terms of $\mathcal{U}$ have no explicit powers of ρ in the coefficients. The corresponding set $\{U\}$ is thus confined to an $(n-2)$-dimensional hyperplane $r_0 = 0$, $r_1+...+r_n = l$. In Fig. 1 we present such points corresponding to the example in Eq. 1, where the three terms of $\{F\}$ are denoted with crossed points and the two terms of $\{U\}$ with diamonds.

If we fix the scales of alpha-parameters as $x_i \sim \rho^{v_i}$, then the scale of a monomial is $\rho^{r_0}x_1^{r_1}...x_n^{r_n} \sim \rho^{r_0+v_1r_1+...+r_nv_n} \sim \rho^{\vec{r}\vec{v}}$ with $\vec{r} = (r_0,...,r_n)$ from $\{F\}$ and $\vec{v} = (1,v_1,...,v_n)$. Graphically, $\vec{r}\vec{v}$ represents the length of a projection of the vector $\vec{r}$ on the direction $\vec{v}$.

Some special choices of directions $\vec{v}$ determine the regions of expansion that we seek. The terms in $\mathcal{F}$ that remain after the expansion are all characterized by the same scale in powers of ρ. All points of the corresponding subset $\{F'\}$ then feature the same value of the projection on $\vec{v}$, i.e. these points belong to the hyperplane orthogonal to $\vec{v}$.

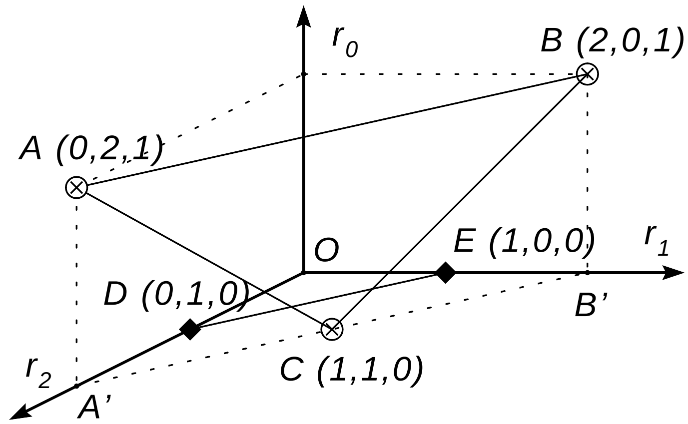


FIG. 1: Graphical representation of sets $\{F\}$ (crossed points) and $\{U\}$ (diamonds) corresponding to the integral of Eq. 1.

The points corresponding to the neglected terms will be located "above" this hyperplane (since $\vec{v}$ always points "up" with respect to the 0-th axis). In other words, $\{F'\}$ belong to a facet of the envelope, or the "convex hull" of the set $\{F\}$, while the corresponding $\vec{v}$ is the normal vector to that facet. In a similar manner we may define a subset $\{U'\}$ of the remaining terms in $\mathcal{U}$.

Relating the three expansion regions of Tab. I to graphics in Fig. 1, we find the corresponding points and vectors (points as denoted in the figure):

- (b): $\vec{v} = (1, 0, 0)$, $\{F'\} = (C)$, $\{U'\} = (D, E)$,
- (c): $\vec{v} = (1, 1, -1)$, $\{F'\} = (A, C)$, $\{U'\} = (D)$,
- (d): $\vec{v} = (1, -1, 1)$, $\{F'\} = (B, C)$, $\{U'\} = (E)$.

Here we exploit the freedom to re-scale all x_i by the same power of ρ, i.e. shift $\vec{v}$ by any vector $\vec{a} = (0, A, ..., A)$. If $\vec{v}$ corresponds to a region, then $\vec{v'} = \vec{v} + \vec{a}$ determines the same region. For example, $\vec{v'} = (1, 2, 0)$ also corresponds to the region (c) above. It is convenient to choose $\vec{v}$ parallel to the plane where points $\{F\}$ are confined, i.e. orthogonal to the vector $(0, 1, ..., 1)$.

Normally, only a few scaling choices produce non-zero regions. In our example, the choice $x_1 \sim \rho^2$, $x_2 \sim \rho^0$, leading to $\mathcal{U} = x_2$, $\mathcal{F} = x_2^2 m^2$, or $\{F'\} = (A)$, $\{U'\} = (D)$, corresponds to a scaleless integral. As discussed above, this implies an existence of a scaling leaving both $\mathcal{U}$ and $\mathcal{F}$ invariant up to a pre-factor (in this case, $x_1 \to a x_1$).

The requirement that a region does not vanish can be easily formulated in the geometrical language. Consider the polynomial $\mathcal{UF}$ and the corresponding set of points $\{UF\}$. After the expansion with the chosen scalings, we are left with its subset $\{UF'\}$. Numeric coefficients and kinematic invariants are irrelevant to the scalefulness of the region, and we get rid of them by projecting $\{UF'\}$ on the plane $r_0 = 0$. The thus obtained set of points $\{UF'_0\}$ belongs to the $(n-1)$-dimensional hyperplane.

Scalelessness implies that all terms of the polynomial $\mathcal{UF}'$ are homogeneous with respect to a certain re-scaling. We thus deduce that the points of $\{UF'_0\}$ must then belong to an orthogonal space of the corresponding re-scaling vector $\vec{v}_h$. In other words, $\{UF'_0\}$ is confined to at most $(n-2)$-dimensional subspace, and its $(n-1)$-dimensional volume is zero. The latter property can be easily checked (and used to check whether a given integral vanishes). However, it is easy to see that the "bottom" facets of the convex hull for $\{UF\}$ automatically correspond to scaleful regions: their dimension is $(n-1)$ by construction (otherwise they become "ridges" or "vertices"), and they (by selection) are not orthogonal to the plane $r_0 = 0$ (thus the projection has non-zero volume).

Finally, we may formulate the general procedure to determine the expansion regions. We start by building the set of points $\{UF\}$. Next, we find the n-dimensional convex hull $\mathcal{C}$ of the set $\{UF\}$ in the n-dimensional plane $r_1 + ... + r_n = l + 1$, using any preferred algorithm. The implementation that we chose, QHull [14], does not allow building hulls of dimensionality lower than the dimension of vector space. Thus, one has to introduce local coordinate system and deal with non-integer coordinate values. However, it is also possible to project $\{UF\}$ along any of axes r_i, $i \neq 0$, e.g. consider $(n-1)$-dimensional points $\vec{r}_{\parallel} = (r_0, r_1, ..., r_{n-1})$. Convex hull $\mathcal{C}'$ built for this projection will be the projection of the "true" convex hull $\mathcal{C}$. Its dimensions will be stretched but the correspondence of the points to the facets and the vertices will persist.

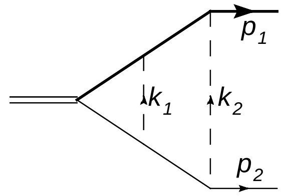


FIG. 2: Double-scale two-loop vertex integral.

From the $(n-1)$-dimensional facets of $\mathcal{C}$ we then select the "bottom", i.e. facets with normal vectors $\vec{v}$ pointing "up", with $v_0 > 0$. For each of those "bottom" facets, we choose the normal vector $\vec{v}$ such that $v_0 = 1$. Its components 1 to n represent the relative scales of alpha-parameters x_i and thus uniquely determine an expansion region.

V. LESS TRIVIAL EXAMPLE

Let us consider the integral in Fig. 2, this time defined in the Minkowski space (this example was first considered in [2] and [10], Chapter 10):

$$\begin{aligned}
&I_2(a_1, ..., a_6; s, m^2, M^2) = \int \frac{d^D k_1 d^D k_2}{(2\pi)^{2D} D_1^{a_1} ... D_6^{a_6}}, \qquad (8)\\
&D_1 = (p_1 - k_1 - k_2)^2 - M^2, \quad D_2 = (p_1 - k_2)^2 - M^2, \quad D_3 = (p_2 + k_1 + k_2)^2 - m^2,\\
&D_4 = (p_2 + k_2)^2 - m^2, \quad D_5 = k_1^2, \quad D_6 = k_2^2,\\
&p_1^2 = M^2, \quad p_2^2 = m^2, \quad (p_1 + p_2)^2 = s, \quad s, M^2 \gg m^2.
\end{aligned}$$

With $S = m^2 + M^2 - s = -2p_1p_2$, its alpha-representation reads:

$$\begin{aligned}
\mathcal{U} &= x_1x_2 + x_3x_2 + x_5x_2 + x_1x_4 + x_3x_4 + x_1x_5 + x_3x_5 + x_4x_5 + x_1x_6 + x_3x_6 + x_5x_6, \qquad (9)\\
\mathcal{F} &= M^2x_1^2x_2 + M^2x_1^2x_4 + M^2x_1^2x_5 + M^2x_1x_2^2 + M^2x_2^2x_3 + M^2x_2^2x_5 + M^2x_1^2x_6 + 2M^2x_1x_2x_5\\
&+ m^2x_2x_3^2 + m^2x_3^2x_4 + m^2x_1x_4^2 + m^2x_3x_4^2 + m^2x_3^2x_5 + m^2x_4^2x_5 + m^2x_3^2x_6 + 2m^2x_3x_4x_5\\
&+ Sx_1x_2x_3 + Sx_1x_2x_4 + Sx_1x_3x_4 + Sx_1x_3x_5 + Sx_1x_4x_5 + Sx_1x_3x_6 + Sx_2x_3x_4 + Sx_2x_3x_5 + Sx_2x_4x_5.
\end{aligned}$$

For simplicity, let us analyze $\mathcal{F}$ instead of the product $\mathcal{UF}$. Choosing m as the small parameter and preserving the order of terms, in the 7-dimensional space we have 25 points: $\{F\}$ = (0,2,1,0,0,0,0), (0,2,0,0,1,0,0), (0,2,0,0,0,1,0), ...

The projection of $\{F\}$ along the 6-th axis (i.e. $\{F\}_p$ = (0,2,1,0,0,0), (0,2,0,0,1,0), ...) has a six-dimensional convex hull with 18 facets. Of them, four belong to the "bottom". Restoring the 7-dimensional normal vectors with unit 0-th component, we find: $\vec{v}_1 = (1, 0, -2, -2, -2, -2, -4)$, $\vec{v}_2 = (1, 0, 0, -2, -2, -2, -2)$, $\vec{v}_3 = (1, 0, 0, -2, 0, 0, 0)$, $\vec{v}_4 = (1, 0, 0, 0, -2, 0, -2)$. Since $\mathcal{U}$ is scaleful, we also have to add the "hard" region: $\vec{v}_0 = (1, 0, 0, 0, 0, 0, 0)$. (The latter would appear automatically, had we used the product $\mathcal{UF}$.)

For illustration, let us consider the most non-trivial "ultrasoft-collinear" region corresponding to $\vec{v}_1$, or the scaling of alpha-parameters $x_1 \sim m^0$, $x_2 \sim 1/m^2$, $x_3 \sim 1/m^2$, $x_4 \sim 1/m^2$, $x_5 \sim 1/m^2$, and $x_6 \sim 1/m^4$. First, we may check that $\vec{v}_1$ is indeed orthogonal to the plane containing the points 5, 6, 15, 22, 23, 24, and 25 from $\{F\}$ (in the order as in Eq. 9). Those points correspond to the terms remaining in $\mathcal{F}'$ after the expansion: $M^2x_2^2x_3 + M^2x_2^2x_5 + \ldots$.

In the momentum space, the interpretation becomes clear only in the special reference frame, where $p_1 = (M, \vec{0}, 0)$, $p_2 = Mn_+ + \left(\frac{m^2}{M}\right) n_-$, and $n_\pm = (1/2, \vec{0}, \mp 1/2)$. We also decompose the first loop momentum in plus- and minus-parts, $k_1 = (k_+ + k_-, \vec{k}, k_+ - k_-)$. To reproduce the "ultrasoft-collinear" region, we should prescribe the following scales to the components of loop momenta: $k_+ \sim m^2/M$, $k_- \sim M$, $\vec{k} \sim m$, $k_2 \sim m^2/M$.

After the expansion, the denominator factors scale as: $D_1 \sim M^2$, $D_2, D_3, D_4, D_5 \sim m^2$, $D^6 \sim m^4/M^2$. One can easily see how the powers of m here correspond to the components of $-\vec{v}_1$.

VI. IMPLEMENTATION

We wrote a Mathematica program that determines the expansion regions of a given Feynman integral based on the procedure described above. The general problem of building a convex hull of M points in n dimensions is well-known in computational geometry; we employ the algorithm `quickhull` [14] that has complexity $\mathcal{O}\left(M^{\lfloor d/2 \rfloor}\right)$. It is sufficient when the number of lines is not too large; for example, finding 11 expansion regions of a 4-loop integral with 10 lines takes about 10 seconds on a laptop PC.

The program has been checked against some non-trivial examples discussed in [10] and [15]. The code can be downloaded from `http://www-ttp.particle.uni-karlsruhe.de/~asmirnov/Tools-Regions.htm`

In order to run the program, one has to install the open-source package QHull [14]. If the executable is not in the current directory, `Options[QHull]` must be updated in the file asy.m. The program is loaded with command `<<asy.m`. The main function is `AlphaRepExpand[ks,ds,cs,hi]`, where `ks` is the list of loop momenta (e.g., `{v1,v2}`), `ds` are the denominators (e.g., `{(p1-v1-v2)^2-M^2,(p1-v2)^2-M^2,(p2+v1+v2)^2-m^2,(p2+v2)^2-m^2,v1^2,v2^2}`), `cs` contains the kinematic constraints (e.g., `{p1^2->M^2,p2^2->m^2,p1*p2->-S/2}`), and `hi` represents the scalings of kinematic invariants with respect to the small parameter `x` (e.g., `{M->x^0,S->x^0,m->x^1}`).

The output is a list of vectors specifying the scales of the alpha-parameters factors, or the non-zero components of vectors $\vec{v}_i$. For the last example above, the output is `{{0,-2,-2,-2,-2,-4},{0,0,-2,-2,-2,-2},{0,0,-2,0,0,0},{0,0,0,-2,0,2},{0,0,0,0,0,0}}`, corresponding to the regions (us-1c), (1c-h), (h-1c), (1c-1c), and (h-h). These regions can be understood by analogy to the example of Section V.

VII. CONCLUSION

We present an algorithm to find the relevant regions of expansion for a Feynman integral in a given limit of momenta and masses. The algorithm is implemented in `Wolfram Mathematica` language and uses open-source package `QHull`. The program and examples can be downloaded from our web-page. In the future, we plan to extend the code in order to apply it to more general parametric integrals.

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