On the finite spectral triple of an almost-commutative geometry*

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Abstract

In this paper we examine the relevance of the signature of the metric of space-time on constructing the product of the pseudo-Riemannian spectral triple with a finite triple describing the internal geometry.

1 Non-commutative geometry and the Standard Model of Particle Physics

In A.Connes' non-commutative geometry **NCG** approach to the standard model **SM** of elementary particle physics, at least at the Lagrangian level, is described as the tensor product of two real spectral triples :

1. The spectral triple associated to the commutative geometry of a Riemannian spin mannifold "Euclidean space-time" with its algebra of functions $\mathcal{A}_1$, its Clifford structure with self-adjoint Dirac operator $\mathbf{D}_1$, hermitian chirality Ω_1 with $\Omega^2 = \mathbf{Id}$ and anti-unitary charge conjugation or real structure $\mathbf{J}_1$, all represented on the bona-fide Hilbert space $\mathcal{H}_1$ of square-integrable spinors;
2. A finite spectral triple describing the **NCG** of the "Eigenschaften" algebra related to internal quantum numbers. This algebra $\mathcal{A}_2$ is a direct sum of matrix algebra over the real associative division algebras. It acts on a finite dimensional module $\mathcal{H}_2$, with a scalar product, an hermitan Dirac operator $\mathbf{D}_2$, a chirality Ω_2 and an anti-unitary real structure $\mathbf{J}_2$.

However, this "almost commutative geometry" **ACG** approach, was mainly plagued by two problems :

1. There is a fermion quadruple overcounting [4].
2. No neutrino mixing is allowed and the neutrino remains strictly massless.

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The classification of the real finite spectral triples was studied in [5] and, more recently in [12]. The different real structures are classified by the so-called KO-dimension or signature, which is an integer defined modulo eight which corresponds to the modulo eight periodicity of the real Clifford algebras as established in [2]. When the space-time geometry is Lorentzian, having p positive and q negative eigenvalues of the metric tensor, not all of the axiomatics of Connes' **NCG** are met. One of the apparent differences is that the space of square integrable spinors does not form a Hilbert space, but rather a Kreĭn space with indefinite scalar product. The Clifford algebra is denoted by $\mathcal{C}\ell(p,q)$ and the signature is $\sigma = p - q$ modulo eight. Besides this signature dimension, the **NCG** has a metric dimension, which in case of spin manifolds, coincide with the usual geometric dimension $n = p + q$. In the case of a finite spectral triple with metric dimension 0, the signature is not necessary zero. Using this fact, A. Connes [9] and independently J.Barrett [10] could remedy these shortcomings allowing the signature of the finite spectral triple to be six instead of zero.

2 Where the Sign of the Metric makes a Difference

Having in mind the relevance of this signature concept[1] , we observe that, in general, the Clifford algebra $\mathcal{C}\ell(p,q)$ is different from $\mathcal{C}\ell(q,p)$. Their even subalgebras are isomorphic $\mathcal{C}\ell^+(p,q) \cong \mathcal{C}\ell^+(q,p)$ and so are the **SPIN** groups, but the **PIN** groups are not. This was already observed by Yang and Tiomno [1] and a different signature σ corresponds to a different (s)pinor. A more recent study was made by De Witt-Morette et al.[8]. If the KO-dimension is physically significant, so will be the sign of the metric. The physically relevant cases are :

$$\begin{aligned} \mathcal{C}\ell(1,3) &\cong \mathbf{M}_2(\mathbf{H}) \; ; \; \sigma = -2 = +6 \; mod \; 8 \\ \mathcal{C}\ell(3,1) &\cong \mathbf{M}_4(\mathbf{R}) \; ; \; \sigma = +2 \end{aligned}$$

with the even subalgebra $\mathcal{C}\ell^+(1,3) \cong \mathcal{C}\ell^+(3,1) \cong \mathbf{M}_2(\mathbf{C})$.
and for an Euclidean space-time :

$$\mathcal{C}\ell(4,0) \cong \mathcal{C}\ell(0,4) \cong \mathbf{M}_2(\mathbf{H}) \; ; \; \sigma = +4 = -4 \; mod \; 8$$

and the even subalgebra $\mathcal{C}\ell^+(4,0) \cong \mathcal{C}\ell^+(0,4) \cong \mathbf{H} \oplus \mathbf{H}$.
In earlier work [6] and [7], where the distinction of the metric dimension and the KO-dimension was not so clear, we already pointed out some difficulties in the definition of the product of real spectral triples.
The signature of a real, even, spectral triple is determined by the set of three ± 1-valued numbers $\{\epsilon, \, \epsilon', \, \epsilon''\}$, given in table **1.** and defined by :

$$\mathbf{J}^2 = \epsilon \, \mathbf{Id} \; , \; \mathbf{J} \, \mathbf{D} = \epsilon' \, \mathbf{D} \, \mathbf{J} \; , \; \mathbf{J} \, \Omega = \epsilon'' \, \Omega \, \mathbf{J}$$

[1] The above title was borrrowed from [3].

Table 1: The epsilon's

$\sigma =$	0	1	2	3	4	5	6	7
ϵ	+1	+1	-1	-1	-1	-1	+1	+1
ϵ'	+1	-1	+1	+1	+1	-1	+1	+1
ϵ''	+1	$\star$	-1	$\star$	+1	$\star$	-1	$\star$

In [6] the product of two spectral triples was discussed but no special attention was paid to the different parts the metric- and KO-dimension had to play. However most of the calculations of [6] are still applicable to solve the following problem :
Given an even spectral triple $\mathcal{T}_1$, what are the possible spectral triples $\mathcal{T}_2$ such that their product, $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$, is an even spectral triple with certain required properties.

3 Defining the Product $\mathcal{T}_1 \otimes \mathcal{T}_2$

Let $\mathcal{T}_1 = \{\mathcal{A}_1, \mathcal{H}_1, \mathbf{D}_1, \Omega_1, \mathbf{J}_2\}$ and $\mathcal{T}_2 = \{\mathcal{A}_2, \mathcal{H}_2, \mathbf{D}_2, \Omega_2, \mathbf{J}_2\}$ be two even spectral triples, $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2 = \{\mathcal{A}, \mathcal{H}, \mathbf{D}, \mathbf{\Omega}, \mathbf{J}\}$ with :

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \ , \ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \ ,$$
$$\mathbf{D} = \mathbf{D}_1 \otimes \mathbf{I}_2 + \Omega_1 \otimes \mathbf{D}_2 \ , \ \Omega = \Omega_1 \otimes \Omega_2$$

The Dirac operator $\mathbf{D}$ and the chirality Ω are constructed such that the resulting Dirac operator $\mathbf{D}$ is odd : $\mathbf{D}\,\Omega + \Omega\,\mathbf{D} = 0$ and the metric dimensions add : $n = n_1 + n_2$, since $\mathbf{D}^2 = {\mathbf{D}_1}^2 + {\mathbf{D}_2}^2$.

3.1 The natural choice : $\mathbf{J} = \mathbf{J}_1 \otimes \mathbf{J}_2$

We compute

$$\begin{aligned} \mathbf{J}^2 &= \epsilon_1\,\epsilon_2\,\mathbf{I} = \epsilon\,\mathbf{I} \\ \mathbf{J}\,\mathbf{D} &= \left(\epsilon_1'\,\mathbf{D}_1 \otimes \mathbf{I}_2 + \epsilon_1''\,\epsilon_2'\,\Omega_1 \otimes \mathbf{D}_2\right)\,(\mathbf{J}_1 \otimes \mathbf{J}_2) = \epsilon'\,\mathbf{D}\,\mathbf{J} \\ \mathbf{J}\,\Omega &= \epsilon_1''\,\epsilon_2''\,\Omega\,\mathbf{J} = \epsilon''\,\Omega\,\mathbf{J} \end{aligned}$$

which are consistent if we require that

$$\epsilon = \epsilon_1\,\epsilon_2 \ , \ \epsilon' = \epsilon_1' = \epsilon_1''\,\epsilon_2' \ , \ \epsilon'' = \epsilon_1''\,\epsilon_2''$$

The signatures of the spectral triples will be denoted respectively by σ_1 , σ_2 and σ. Consider the different possibilities for σ_1 :

1. $\sigma_1 = 0$

$$\epsilon = +\,\epsilon_2 \ , \ \epsilon' = +1 = +\,\epsilon_2' \ , \ \epsilon'' = +\,\epsilon_2''$$

and we have the possibility of having $\sigma_2 \in \{0, 2, 4, 6\}$. Some odd values are also allowed : $\sigma_2 \in \{3, 7\}$ but there will be no chirality. In each case $\sigma = \sigma_1 + \sigma_2$.

2. $\sigma_1 = 2$
$$\epsilon = -\,\epsilon_2\ ,\ \epsilon' = +1 = -\,\epsilon'_2\ ,\ \epsilon'' = -\,\epsilon''_2$$
The only possibilities are $\sigma_2 \in \{1,5\}$ and again $\sigma = \sigma_1 + \sigma_2$, without a chirality.

3. $\sigma_1 = 4$
$$\epsilon = -\,\epsilon_2\ ,\ \epsilon' = +1 = +\,\epsilon'_2\ ,\ \epsilon'' = +\,\epsilon''_2$$
All even cases are allowed $\sigma_2 \in \{0,2,4,6\}$ and also the odd values $\sigma_2 \in \{3,7\}$. We still have : $\sigma = \sigma_2 + \sigma_2$.

4. $\sigma_1 = 6$
$$\epsilon = +\,\epsilon_2\ ,\ \epsilon' = +1 = -\,\epsilon'_2\ ,\ \epsilon'' = -\,\epsilon''_2$$
Only the odd values $\sigma_2 \in \{1,5\}$ are allowed. There is no chirality but the rule $\sigma = \sigma_2 + \sigma_2$ still holds.

Connes [9] requires a product with $\sigma = 2$: $\mathbf{J}^2 = -\mathbf{I}$, $\mathbf{J}\,\mathbf{D} = \mathbf{D}\,\mathbf{J}$ and $\mathbf{J}\,\Omega = -\,\Omega\,\mathbf{J}$. Since its first factor is Euclidean, $\sigma_1 = 4$, according to our analysis, the second factor must be a $\sigma_2 = 6$.

3.2 The Minkowski real spectral triple

As seen in the above analysis **3.1**, if we have a Minkowski signature $\sigma_1 \in \{2,4\}$, the product can only be defined with a second odd factor and the chirality paradigm is lost. This problem was already noticed in [6]. If we restrict to the product of two even factors, a solution is provided with a modified tensor product of the real structures :

$$\mathbf{J} = \mathbf{J}_1 \otimes \mathbf{J}_2\,\Omega_2$$

Again, we compute

$$\begin{aligned}
\mathbf{J}^2 &= \epsilon_1\ \epsilon_2\,\epsilon''_2\,\mathbf{I} = \epsilon\,\mathbf{I} \\
\mathbf{J}\,\mathbf{D} &= \left(\epsilon'_1\,\mathbf{D}_1 \otimes \mathbf{I}_2 - \epsilon''_1\,\epsilon'_2\,\Omega_1 \otimes \mathbf{D}_2\right)\,\left(\mathbf{J}_1 \otimes \mathbf{J}_2\,\Omega_2\right) = \epsilon'\,\mathbf{D}\,\mathbf{J} \\
\mathbf{J}\,\Omega &= \epsilon''_1\,\epsilon''_2\,\Omega\,\mathbf{J} = \epsilon''\,\Omega\,\mathbf{J}
\end{aligned}$$

and, for consistency, we require that

$$\epsilon = \epsilon_1\,\epsilon_2\,\epsilon''_2\ ,\ \epsilon' = \epsilon'_1 = -\,\epsilon''_1\,\epsilon'_2\ ,\ \epsilon'' = \epsilon''_1\,\epsilon''_2$$

Again we consider the different possibilities σ_1 :

1. $\sigma_1 = 0$
$$\epsilon = \epsilon_2\,\epsilon''_2\ ,\ \epsilon' = +1 = -\,\epsilon'_2\ ,\ \epsilon'' = \epsilon''_2$$
But, $\epsilon'_2 = -1$ implies only odd values of σ_2, but in this case no chirality Ω_2 is available to define $\mathbf{J}$.

2. $\sigma_1 = 2$
$$\epsilon = -\,\epsilon_2\,\epsilon''_2\ ,\ \epsilon' = +1 = +\,\epsilon'_2\ ,\ \epsilon'' = -\,\epsilon''_2$$
Now all even values of σ_2 are allowed and again $\sigma = \sigma_1 + \sigma_2$. In particular, if $\sigma_2 = 6$ we obtain $\sigma = 0$.

3. $\sigma_1 = 4$
$$\epsilon = -\,\epsilon_2\,\epsilon_2'' \ , \ \epsilon' = +1 = -\,\epsilon_2' \ , \ \epsilon'' = \epsilon_2''$$
Again the second triple must be odd and $\mathbf{J}$ is not defined.

4. $\sigma_1 = 6$
$$\epsilon = \epsilon_2\,\epsilon_2'' \ , \ \epsilon' = +1 = +\,\epsilon_2' \ , \ \epsilon'' = -\,\epsilon_2''$$
All even values are allowed and the rule $\sigma = \sigma_1 + \sigma_2$ still holds. In particular if $\sigma_2 = 2$, the product has signature $\sigma = 0$

Barrett [10] requires a product with $\sigma = 0$, which is obtained : $\mathbf{J}^2 = -\mathbf{I}$, $\mathbf{J}\,\mathbf{D} = \mathbf{D}\,\mathbf{J}$ and $\mathbf{J}\,\Omega = -\,\Omega\,\mathbf{J}$. Since its first factor is Euclidean, $\sigma_1 = 4$, according to our analysis, the second factor must be a $\sigma_2 = 6$. When the signature of the resulting product is zero or eight we have :

$$\mathbf{J}^2 = \mathbf{I} \ ; \ \mathbf{J}\,\Omega = \Omega\,\mathbf{J}$$

We may then restrict the representation space to the common eigenstates of $\mathbf{J}$ and Ω with eigenvalues ± 1.

4 Summary

In this work we examined the signature of the finite spectral triple which leads to a consistent product. In the Minkowski case, the structure of the finite spectral triple depends on the signature i.e. on the sign of the metric and this appears to imply the distinction of two kind of pinors with different Eigenschaften-algebra.

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