THE LOGIC BEHIND FEYNMAN'S PATHS[*]

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The classical notions of continuity and mechanical causality are left in order to reformulate the Quantum Theory starting from two principles: I) the intrinsic randomness of quantum process at microphysical level, II) the projective representations of symmetries of the system. The second principle determines the geometry and then a new logic for describing the history of events (Feynman's paths) that modifies the rules of classical probabilistic calculus. The notion of classical trajectory is replaced by a history of spontaneous, random an discontinuous events. So the theory is reduced to determining the probability distribution for such histories according with the symmetries of the system. The representation of the logic in terms of amplitudes leads to Feynman rules and, alternatively, its representation in terms of projectors results in the Schwinger trace formula.

Keywords: Projective geometry; logic; quantum probabilities.

1. Introduction

...when one does not try to tell which way the electron goes, when there is nothing in the experiment to disturb the electrons, then one may not say that an electron goes either through hole 1 or hole 2. If ones does say that, and starts to make any deductions from the statement, he will make errors in the analysis. This is the logical tightrope on which we must walk if we wish to describe nature successfully. (R. P. Feynman, Caltech Lectures, 1965).

The purpose of this work is the quest for the first principles of the Quantum Theory. If we carefully take a look at the axioms of the mathematical formalisms,[1–6] we will realize that they are organized by two fundamental ideas: the projective linear representation of the symmetries of the system and the intrinsic aleatory behavior of the events which happen in it.[a] The ideas mentioned can be summarized

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[a]In the development of Quantum Theory, the concept of chance and spontaneity of transition, much to our surprise, merges in Einstein's[7] derivation of Planck's formula which finally crystallizes in

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in the following table:

CHANCE (Distribution of probabilities)	• Born rule • Projection postulate • Sum rule for intermediate events .
SYMMETRIES (Projective representation of generators algebra)	• Schroedinger equation (Galilei Group) • $[Q, P] = i\hbar$ (Heisenberg-Weyl Group) • $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ (Rotation Group) ...

In this way, on the one hand, we know that the basic rules, as the Schroedinger equation and the algebra of generators of the symmetries of the system, are a consequence of assuming the second descriptive principle (projective representations of the symmetries of the system).

On the other hand, we shall see that the rules such as the projection postulate of von Neumann[10] and Born rule,[8] or alternatively Feynman[11] rules for combining amplitudes, are a consequence of assuming randomness as the first principle, in the framework of the second descriptive one. As we explain in this work, such rules for quantum probabilities are the democratic way of assigning a distribution of probabilities for the random transitions between the events of ray space. It means that all the rays have an equal statistical weight, avoiding any kind of privileged direction. Therefore, a priori, we will to assume an isotropic probability distribution in ray space.

Feynman rules, summarized in the next table, represent an equivalent way of providing the rules for quantum distribution probabilities. They encode, in terms of amplitudes, the underlying logic of ray space:[11]

Born's[8] interpretation of wave function. However the first antecedent of introducing chance as physical principle was in kinetic gas theory through Boltzmann's[9] molecular chaos hypothesis.

FEYNMAN RULES

- **I -Generalization of Born rule:** *the probability that a particle arrives at a* given *position x, departing from the source s, can be represented by the square of the absolute value of a complex number called probability amplitude.*
- **II -Sum rule for intermediate alternative events:** *when a particle can reach a state taking two possible alternative paths, the total amplitude of the process is the sum of the amplitudes of each path considered independently.*
- **III- Product rule for consecutive amplitudes** (which implicitly contains the actualization rule of conditional probabilities): *when a particle follows a path, the amplitude of such path can be written as the amplitude of advancing the first part of the way times the amplitude of advancing the second one.*

Generally people think that such rules follow an enigmatic logic, because they conceive them from the point of view of the classical rules of logic and probabilities. But, we will see that Feynman rules are not contradictory, they only encode a natural logic for ray space.

As was recognized by its founding fathers,[1] the essential characteristics of the quantum processes are:

- spontaneity
- randomness
- discontinuity
- bifurcation

This picture sharply contrasts with the one corresponding to classical processes, that are inertially stable, deterministic, continuous, and univocal. In other words, at a quantum level, after any event occurs, it can spontaneously follow any of the potential events that the interaction permits. That is, there is no unique history determined by the initial conditions; precisely because we have a range of possibilities measured by the probabilities of the aleatory transitions. In this way, Quantum Theory gives us a physical image of aleatory processes, as if they were interconnected in a network that takes into account all the possibilities.

Quantum theory often surprises and confuses us since it implies a radical change of the classical logic. It happens because, in a subtle way, all our ordinary languages hide epistemological, ontological and logical assumptions taken from the mechanical conception of the Universe.

In the following section, the prejudices of classical logics will be discussed and removed. Von Neumann was the pioneer in this enterprise, realizing that, in order to understand Quantum Theory, we also must move away from classical logics.

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2. Logic

In order to understand the logic of quantum processes, first we have to realize that the theory does not speak about classical probabilities of sets! Only in the case of Classical Statistical Mechanics, the probabilities are defined on sets of points of phase space. Wigner-Moyal representation of the theory perhaps can bring us this illusion, but it is certainly not the case. Wigner's distributions are not probabilities.

In order to understand it better, let us briefly review the descriptive framework of Quantum Theory. Such theory describes the transformations of symmetries (T) of the systems. As these symmetries have a group structure, they admit a linear representation $U(T)$ in terms of matrices or linear operators in a vector space.[b] And this is the reason why Quantum Mechanics works in linear vector spaces (Principle of linear superposition[3]). In particular, Quantum Theory works with projective representations in a vector space with a scalar complex field,[12] that are representations up to a phase factor

$$U(T_1 \circ T_2) = e^{i\alpha(T_1,T_2)}U(T_1)U(T_2) \tag{1}$$

due to the fact that all the vectors belonging to the same ray are physically equivalent.[c] Although phases will be irrelevant in the geometrical examples discussed in this work, they have crucial importance[d] in the projective representations of Galilei Group[14,15] and in the group of translations in phase space (Heisenberg-Weyl group.[12]) The phases make that the projective representations of the momentum and the spatial coordinate do not commute. This is to say that they are the reason for the uncertainty principle.

So we will focus on the idea that Quantum Theory works on linear vector spaces. With the sole aim of illustrating it, here we will consider the simplest non trivial case: The vector space corresponding to the Euclidean plane. In this case, rotations $R(\theta)$ are represented by square matrices of the form

$$U\left[R(\theta)\right] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

This matrix rotates vectors in a two dimensional space. However, as the physical events of the theory $a, b, ...$ do not correspond to vectors $|a\rangle, |b\rangle, ...$ but to rays or directions in such space, all the vectors are equivalent up to a scale factor. Then, in some sense, it is not the Euclidean Geometry who plays the game but Projective Geometry.

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[b]Notice that the product of the non-singular square matrix is closed, associative, has the identity matrix as the neutral element, and inverse matrix for each element. Therefore, for any group we can find an homomorphism with square non-singular matrices. Such homomorphism is a linear representation of the group.
[c]All the vectors are equivalent up to a complex scale factor, but working with normalized vectors only the phase is relevant.
[d]A relative phase of π between intrinsic parities of particles and antiparticles, determines Dirac equation.[13]

Then, with this geometrical picture in mind, notice the following. If we affirm that the event a "is" the set of points of ray a and similarly in the case of b, as soon as we try to find the event corresponding to "simultaneously being" in a and b, we obtain the null ray, which has no sense. The problem appears no sooner than we try to use the set logic theory which only works well for Classical Mechanics. The words between inverted commas are used to point out that the problem rests on the fact that the logic of quantum processes is not combinational but sequential. Quantum logic is not a logic of states, but one of processes or transitions. As it was originally pointed out by Bohr and Heisenberg,[1,2] the theory does not describe "states" but processes or transitions between them.[17,18] Bear in mind that Quantum Theory was developed for describing quantum jumps between energy levels. With these ideas, we have to reformulate the problem. So if we imagine that the two rays represent a process like an aleatory jump of the event a to b, the picture recovers its sense.

Returning to the geometry of Euclidean plane, notice that the successive projections of a ray a onto ray b, forming a relative angle θ_{ab}, and again onto a result[e]

$$P_a P_b P_a = \cos^2 \theta_{ab} P_a \tag{3}$$

because the operation $P_a P_b P_a$ has to be proportional to P_a, $P_a P_b P_a = \lambda P_a$ (this is a projection composed with a dilatation) with $\lambda = \cos^2 \theta_{ab}$. Notice that the contraction factor λ defines a geometrical probability which coincides with the Born's one. This is a general property of vector spaces which can be easily verified using Dirac's bra-ket notation ($P_a = |a\rangle \langle a|$)

$$P_a P_b P_a = \langle a \mid b \rangle \langle b \mid a \rangle P_a \tag{4}$$

Von Neumann[10] was the first in associating certainties with projection operators in his bi-valued logic. According to him, the projector's eigenvalues 1 and 0 of a projector P_a indicate if the system "is" in a given ray a or in the orthogonal one $\overline{a}$. So the projector P_a itself represents a logical proposition of sharp true values 0 and 1. Likewise, the projector onto the orthogonal complement $\overline{P_a} = 1 - P_a$ represents the negation of the proposition P_a. It is a nice evocative idea but with an ontology still contaminated by Classical Mechanics. We will try to reformulate it in the following sense: the eigenvalues 1 and 0 indicate that we have the certainty of making a transition to the same ray, but rule out the possibility of doing it to the orthogonal one. In general, the projection of a ray b onto a gives as a result the operator $P_a P_b P_a$ of eigenvalues λ and 0, being the first one a number (between 0 and 1) which measures the degree of certainty of making the transition. In this way, sharp true values 0 and 1 of von Neumann's idea are extended to the real number interval $[0, 1]$.

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[e]For example in the base that $P_a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal, the projector $P_b = U(\theta)P_a U(-\theta) = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$, so $P_a P_b P_a = \cos^2 \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\theta = \theta_{ab}$.

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Summarizing, as we have explained above, the operation $P_a P_b P_a$, that represents the projection of ray b onto a, keeps close analogy with the intersection of sets ($\cap$) which we will denote as

$$P_b \sqcap P_a = P_a P_b P_a \qquad (5)$$

This will be our definition of quantum "intersection" or AND ($\sqcap$). However, it is important to remember that projections in general do not commute. So our AND between rays is, in general, non commutative. And, this is precisely the essential distinctive character of the logics for the quantum processes proposed here, no taken into account by earlier attempts.

If we have a history γ in which three consecutive events a, b, and c appear in this order, then it represents a logical proposition that can be rewritten as

$$\gamma = (a \sqcap b) \sqcap c \qquad (6)$$

In terms of Feynman rules for combining amplitudes, the total amplitude of the history can be obtained by multiplying the partial amplitudes of the chain (in Dirac's notation)

$$A(\gamma) = \langle a \mid b \rangle \langle b \mid c \rangle \qquad (7)$$

where we have followed the opposite of Feynman's convention, who prefers to draw the amplitudes writing the initial event on the right.

If we have a path which bifurcates into two alternative (mutually excluded) intermediate events, according to Feynman we must add the amplitudes. This defines the logical operation XOR ($\sqcup$), which in terms of projectors reads

$$P_a \sqcup P_b = P_a + P_b \qquad (8)$$

In the following table we summarize the basic operations of the logic of quantum processes, alternatively represented in terms of operators and amplitudes:[16]

History	Operator	Amplitude
γ	$\Gamma(\gamma)$	$A(\gamma)$
$a \sqcap b$	$P_b P_a P_b$	$\langle a \mid b \rangle$
$a \sqcup b$	$P_a + P_b$	——
$(a \sqcap b) \sqcap c$	$P_c P_b P_a P_b P_c$	$\langle a \mid b \rangle \langle b \mid c \rangle$
$(a_1 \sqcup a_2) \sqcap b$	$P_b (P_{a_1} + P_{a_2}) P_b$	$\langle a_1 \mid b \rangle + \langle a_2 \mid b \rangle$
$[a \sqcap (b_1 \sqcup b_2)] \sqcap c$	$P_c (P_{b_1} + P_{b_2}) P_a (P_{b_1} + P_{b_2}) P_c$	$\langle a \mid b_1 \rangle \langle b_1 \mid c \rangle$ $+ \langle a \mid b_2 \rangle \langle b_2 \mid c \rangle$

Any history γ has associated a degree of certainty $\lambda(\gamma)$ (the non trivial eigenvalue of $\Gamma(\gamma)$), that, in terms of projectors, can be written as[f] $tr(\Gamma)$ and, in terms of

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[f]For example, consider the history $\Gamma = (P_a \sqcap P_b) ... \sqcap P_z$ that starts at a and finishes at z, the non trivial eigenvalue of Γ satisfies $\Gamma P_z = \lambda P_z$; then taking the trace in both members we have $\lambda(\gamma) = tr(\Gamma P_z) = tr(\Gamma)$, because $tr(P_z) = 1$.

amplitudes, as $A(\gamma)A(\gamma^{-1})$, where γ^{-1} is the inverse path of γ. The equivalent of the representation of the second and third columns is guaranteed by[g]

$$tr(\Gamma) = A(\gamma)A(\gamma^{-1}) \tag{9}$$

Summarizing, quantum histories admit two representations that are mathematically equivalent: Feynman's[11] representation in terms of amplitudes which has its roots in the path integral formalism[19–21] and the representations in terms of projectors whose first ideas can be found in the works by Schwinger on the algebra of measurements.[22,14,15] Von Neumann[10] wrote a similar expression in his *Mathematical Foundations of Quantum Mechanics*, for the case of commuting projectors. However, neither Feynman nor Schwinger realized of the logic behind quantum processes. Even though von Neumann became conscious that the problem lays on the logic, he finally failed in finding the right one. He did not succeed, because the quantum logic is not a black and white's one, is a logic of greys.

The third row of the table represents the history of a polarization experiment, and the fifth one the history corresponding to the double slit experiment. These are the two basic experiments chosen by Dirac[3] for introducing the theory in his *Principles of Quantum Mechanics*. Feynman believed that these experiments encapsulate all the mystery of the Quantum.[11,23]

The paradox that brings to light the polarization experiment is the following: suppose that, in the history of the third row, a and c represent the events corresponding to the passage of a photon through two filters that polarize light in two orthogonal directions ($c = \overline{a}$); and that b is the event associated with the passage through an intermediate filter which polarizes light in a direction on the same plane, forming an angle θ_{ab} with the one determined by the first polarizer, and an angle $\theta_{bc} = \pi/2 - \theta_{ab}$ with the second one. The corresponding quantum history

$$(a \sqcap b) \sqcap \overline{a} \tag{10}$$

has its analog in the Boolean expression (capital letters indicate sets)

$$(A \cap B) \cap \overline{A} = \emptyset \tag{11}$$

But, from the point of view of the classical reasoning, the light would not pass through the third polarizer. But, in spite of the fact that the projection of ray a onto the orthogonal ray $\overline{a}$ is null, the magic of quantum logic rests on the fact that the projection of a onto b and then onto $\overline{a}$ is different from zero. So, the history has a degree of certainty

$$\lambda = \cos^2(\theta_{ab})\sin^2(\theta_{ab}) \tag{12}$$

The explanation is analogous to the tunnel effect. A classically forbidden history, as the passage through a potential barrier, is only possible at quantum level because intermediate events that have non null projections.

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[g]For example, in the case of the third row, it is easy to verify that $tr\,(P_c P_b P_a P_b P_c) = \langle c \mid b \rangle \langle b \mid a \rangle \langle a \mid b \rangle \langle b \mid c \rangle$ so, we have $tr\,[(P_a \sqcap P_b) \sqcap P_c] = A(a \sqcap b \sqcap c)A(c \sqcap b \sqcap a)$.

In the case of the double slits experiment, the mystery is the generation of an interference pattern. But, if we follow the logic of Quantum processes, the interference is an unavoidable consequence. In fact, the operator corresponding to the history in which the electron can pass "alternatively"[h] by the two slits (event $b_1 \sqcup b_2$) is the the sum of the operators representing the histories in which the electron effectively passes by each slit plus an interference term (I) :

$$\Gamma_{[a \sqcap (b_1 \sqcup b_2)] \sqcap c} = (P_a \sqcap P_{b_1}) \sqcap P_c + (P_a \sqcap P_{b_2}) \sqcap P_c + I \tag{13}$$

with

$$I = P_c P_{b_1} P_a P_{b_2} P_c + P_c P_{b_2} P_a P_{b_1} P_c \tag{14}$$

Again the paradox only appears when we insist in reasoning classically. In this case, the analogous Boolean expression would be

$$A \cap (B_1 \cup B_2) \cap C = (A \cap B_1 \cap C) \cup (A \cap B_2 \cap C) \tag{15}$$

which leads to think in terms of Kolmogorov's additive probabilities for disjoint events.[24]

The crossed terms between projectors $P_{b_1} P_a P_{b_2} + P_{b_2} P_a P_{b_1}$, associated to the orthogonal rays b_1 and b_2, are responsible for the characteristic interference of the quantum phenomena. For instance, crossed terms between positive (electron) and negative energy levels (positron) are responsible for the trembling motion of the positron-electron charge (*Zitterbewegung*) which, in each subspace of definite sign, acquires a magnetic moment.[i] It is immediate to see that, if P_a commutes with P_{b_1} and P_{b_2}, then the interference vanishes. But, again, it is a property of vectorial spaces that the successive action of three projectors such as $P_{b_1} P_a P_{b_2}$ can be different from zero. In other words, the mystery of the quantum behavior rests on the logic of projections, since the event space is the ray space. In other words, the slits do not play as filters in ordinary space but in ray space.

For those who feel the vertigo of *making equilibrium over the logic tightrope of Quantum Theory*,[11] the amplitude representation offers a momentary relaxation that maintains some parts of the reasoning in terms of Boolean logic. In this case, it is enough to follow Feynman rules II and III for computing the total amplitude of the path $\gamma = [a \sqcap (b_1 \sqcup b_2)] \sqcap c$. For counting all possible paths and calculating partial and total amplitudes, only ordinary logic is needed. But the calm is just temporary, because at the end of the day, for getting the probabilities for amplitudes, we have to use rule I, and multiply this amplitude by the amplitude of the reversed path $\gamma = [c \sqcap (b_1 \sqcup b_2)] \sqcap a$. In general, as can be easily verified, this procedure is

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[h]It is important not to fall in the trap of ordinary language, that interprets "alternatively" in Boolean sense. Here "alternatively" must be read in terms of the connective $\sqcup$. In other words, from the proposition $[a \sqcap (b_1 \sqcup b_2)] \sqcap c$ (the electron passed by $(b_1 \sqcup b_2)$) it does not follows that it passed by the slit b_1 or $(\sqcup)$ by the slit b_2 because the interference term I is different from zero.

[i]See Ref. 25 and references cited therein.

equivalent to taking into account all the closed paths that start from the initial event and come back. In fact, in our example, it is easy to check that[j]

$$tr(\Gamma) = A(a \sqcap b_1 \sqcap c \sqcap b_1 \sqcap a) + A(a \sqcap b_2 \sqcap c \sqcap b_2 \sqcap a) + tr(I) \tag{16}$$

where

$$tr(I) = A(a \sqcap b_1 \sqcap c \sqcap b_2 \sqcap a) + A(a \sqcap b_2 \sqcap c \sqcap b_1 \sqcap a). \tag{17}$$

Paying attention to the last expressions, it immediately follows that we not only have to compute the closed path which comes back by taking the same path in the opposite sense, but we have also to take into account the closed loops that go through one slit and come back through the other one. These two paths $\gamma_\circlearrowleft = a \sqcap b_1 \sqcap c \sqcap b_2 \sqcap a$ and $\gamma_\circlearrowright = a \sqcap b_2 \sqcap c \sqcap b_1 \sqcap a$, that circulate in reverse sense, are responsible for the interference

$$tr(I) = A(\gamma_\circlearrowright) + A(\gamma_\circlearrowleft) \tag{18}$$

In other words, as the probability of a path is the amplitude of going forward and coming backward, this simple rule for the "logic of paths" open the possibility of having closed loops which enclose "area" different from zero.[k] This is a general property and, perhaps, the most striking example is the Aharonov-Bohm[26] effect, in which the amplitude of the closed loop which encircles the solenoid acquires a phase proportional to the magnetic flux. But, probably, the most relevant one is the path integral in phase space itself. In this case, closed paths enclose an area

$$S(\gamma_\circlearrowleft) = \oint p\,dx - H dt \tag{19}$$

in an extended phase space of canonical variables (p, H) and (x, t), which represent the action of these paths.[l] This area in units of $\hbar$ $\left(\frac{S}{\hbar}\right)$ is the phase of the closed amplitude which contributes to the interference terms. In the classical regime we have big phases which highly oscillate; thus, they destructively interfere. The main contribution to the total probability comes from the close surroundings of the path, whose phase is stationary ($\delta S = 0$), that is to say, the path corresponding to the classical trajectory derived from the principle of least action.[19,20,21,28] [m] In this way, the choice among all the possibilities in the network of the whole potential events (the path integral picture) is taken by the "laws" of chance,[n] recovering the determinism at a classical scale.

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[j]When in order to abbreviate, we omit the parenthesis, it is assumed the operation $\sqcap$ is taken in sequential order. At this point is instructive to calculate probability in both representations. That is taking the trace of Γ and squaring the absolute value of the amplitude $A(\gamma)$.

[k]In general there is no area in the ordinary sense (the space can be discrete), unless the ray space has a continuum spectrum as is the case of momentum and coordinate representations.

[l]In general H is not the classical Hamiltonian, but the Wigner's equivalent of the corresponding operator.[27]

[m]See also Ref. 23 for a nice derivation of geometrical optics starting from Feynman rules.

[n]The law of big numbers for quantum probabilities was proved by Finkelstein,[29] departing form Born Rule.

3. Probability

To conclude, we desire to emphasize that Born rule in Quantum Theory plays the same role as the Pythagoras' theorem in Euclidean geometry. In a framework such as Analytic Geometry, one can decide to take it as starting point to introduce the notion of distance. Otherwise, one looks for another framework based on basic assumptions, in order to derive it as a theorem, as is the case of the synthetic Euclidean Geometry. This last point of view was adopted by the author in Ref. 16 where the logic of quantum processes was taken as the basic assumption. For this purpose, it is necessary to generalize Kolmogorov's axioms,[24] originally developed for the classical theory of sets. Here, using Laplace's notion of probability, we show in a more heuristic way, how its rule can be derived. First, we will argue that, in general, the absolute probability $p(\gamma)$ of a history is given by:

Probability	Operator representation	Amplitude representation
$p(\gamma/I)$	$\frac{1}{N}tr(\Gamma)$	$\frac{1}{N}A(\gamma)A(\gamma^{-1})$

where $N = trI$ is the dimension of the space ray, and I, the identity matrix, represents the universe of sample space.

For turning the ideas more concrete, we will consider the idealized model of a *quantum* die, essentially a six-level system. A quantum die differs from a classical one in the sense that all its faces represent orthogonal rays in ray space, because the corresponding events are mutually exclusive.

We are going to assume that when playing with Einstein, *the Lord is subtle but not malicious*, so the die is "perfectly balanced"; therefore all faces have the same a priori probability (isotropy of ray space). In this case, Laplace would say that the probability of obtaining any face is $1/6$ (number of cases divided the number of total possible ones). The reasoning can be formalized as follows: the event A, e.g. obtaining the face 5, is given by the set $A = \{5\}$, a subset of the sample space $E = \{1, 2, 3, 4, 5, 6\}$. So, under equal *a priori* probabilities for any elements of the sample space, the probability of obtaining A is directly calculated as the ratio of the cardinal of set A $(card(A))$ to the cardinal of the sample space

$$p(A/E) = \frac{card(A)}{card(E)} = \frac{1}{6} \tag{20}$$

Similarly, in the quantum case, events are represented by projectors, for instance, the face f_5 is represented by the operator P_{f5}, and the sample space by the development of the identity $I = P_{f_1} + P_{f_2} + P_{f_3} + P_{f_4} + P_{f_5} + P_{f_6}$. But, in order to obtain a probability equal to $\frac{1}{6}$, we cannot take cardinals because projectors are not sets! Therefore, we have to generalize this notion in ray space. It is easy to convince oneself that the analogous operation is taking the trace

$$p(P_{f_5}/I) = \frac{tr(P_{f_5})}{tr(I)} = \frac{1}{6} \tag{21}$$

In fact, tracing is the natural generalization of counting. For example: each elementary projector has a unit trace and the trace of the identity is the dimension of the space. Moreover, as the trace is invariant under rotations, so it is the only invariant counting we can define in ray space.

However, in contrast with the classical dice, quantum dice admit more than one representation of the sample space ("Bohr's complementary principle"). We can rotate the die and obtain a new base of orthogonal faces represented by the projectors $P_{R(f_1)} = U(R)P_{f_2}U^{-1}(R), P_{R(f_2)}, ...P_{R(f_6)}$. Then we can ask the following question: if we know, for example, that the face $a = f_2$ has gone out, which is the conditional probability of obtaining the face $b = R(f_2)$? In this case, this is the history $\gamma = a \sqcap b$, which is represented by the operator $\Gamma(\gamma) = P_b P_a P_b$. But, as we have seen

$$P_b P_a P_b = \cos^2 \theta_{ab} P_b. \tag{22}$$

So tracing both sides of the equality

$$\frac{tr\,(P_b P_a P_b)}{tr P_b} = \cos^2 \theta_{ab} \tag{23}$$

or equivalently

$$\frac{\frac{1}{6}tr\,(P_b P_a P_b)}{\frac{1}{6}tr P_a} = \cos^2 \theta_{ab} \tag{24}$$

since the trace of any elementary projector is one.

The first member of the equality is the conditional probability of obtaining the face b, having obtained the face a before,

$$p(a \sqcap b/a) = \frac{p(a \sqcap b)}{p(a)} = \cos^2 \theta_{ab}, \tag{25}$$

which coincides with the expression originally proposed by Born.□

In general, the probability of a history γ which begins at an initial event a, results in

$$p(\gamma/a) = \frac{p(\gamma \sqcap a)}{p(a)} = tr(\Gamma) \tag{26}$$

which coincides with the degree of certainty $\lambda(\gamma)$ of the history. The right side of last expression was originally proposed by Schwinger[22] in the context of his algebra of measurements.[14,15] Later, the trace formula was rediscovered, but consistent conditions were imposed *ad hoc* in order to keep Boolean logic. However, the only logic that reproduces the standard formalism of Quantum Theory is the one exposed in this paper.

Let us consider again the history of three events $\lambda = a \sqcap b \sqcap c$. Then, if we compute the conditional probability of the last event c given the two first $a \sqcap b$ we

have

$$p(c/a \sqcap b) = \frac{p\left[(a \sqcap b) \sqcap c\right]}{p(a \sqcap b)}$$

(27)

$$= \frac{tr(P_c P_b P_a P_b P_c)}{tr(P_b P_a P_b)}$$

(28)

Now, if we look this expression carefully, we observe it can be rewritten as

$$p(c/a \sqcap b) = tr(P_c P_\Psi P_c)$$

(29)

where the operator

$$P_\Psi = \frac{P_b P_a P_b}{tr(P_b P_a P_b)}$$

(30)

is equivalent to the projection operator onto ray b. In this way, the expression is reduced to the one corresponding to the conditional probability for the history of two events $\Psi \sqcap c$, that is

$$p(c/a \sqcap b) = p(c/\Psi)$$

(31)

and, as $\Psi = b$, the last expression indicates us that: after that event b has happened, the probability of the original history does not remember the event a.□ All works as if the history, after projection onto b, begins in this event. This actualization property of conditional probabilities is equivalent to the von Neumann rule.

4. Final remarks and conclusions

Physics has inherited the notions of Classical Mechanics. The physical explanation of the problem of motion is essentially deterministic. So, we have seen that is necessary to dig deeply in order to find the roots of the problem. And, above all, it is imperative to cut these roots in order to understand Quantum Theory. This is not an easy enterprise due to the fact that mechanic philosophy is ubiquitous in our language and has installed in our minds. The most difficult point to interpret is understanding the reason of movement. Notice that, contrary to what happens in the mechanical Universe, at quantum scale, the interactions do not cause changes; they only open the door so as the transitions can take place. The quantum processes occur spontaneously by chance.

The interpretations of Quantum Theory that try to conserve the ontology of Classical Mechanics exacerbate the role of Schroedinger's equation in the theory. It is claimed that the state of the system evolves continuously, as if after an event the system followed by "inertia" the continuous evolution of probabilities determined by this equation.º But this path takes us to a dead end: explaining the discontinuity

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ºSometimes people confuse the symmetry of the temporal evolution with the actual history of the system. However, the so called "temporal evolution of the system" is not the history of the events that really happened but a probabilistic description of the potential events that could have happened.

through continuity, and chance by determinism.

So, the conceptual revolution of Quantum Theory will only be completed after the new conception of the problem of motion will be generally accepted. On the whole, quantum phenomena show us that aleatory processes are spontaneous and chance plays the role of the "inertia principle" in the theory (the spontaneous persistence of motion). Any other attempt would be another way of returning to the theory of "hidden variables".

At present, the big question is the opposite one. How to explain the apparent determinism we observe at classical level from the aleatory behavior quantum phenomena. The first answer to this question was outlined by Dirac;[30,3] then Feynman[19,20] completed this elegant idea developing his path integral formalism. In this framework, all paths, continuous or discontinuous, are allowed and the classical trajectory is just the most probable sequence in the network of potential events.

In conclusion, the paradigm of a legal Universe is absent in Quantum Theory. Physical "laws" are just symmetries or reduced to the "law of big numbers". Chance is the true physical principle of the theory. The second principle, symmetry, only plays the role of a descriptive one. In fact, it proposes a framework for the theory from which the logic and probabilistic distributions for the physical processes can be deduced.[p] In fact, as the processes are random, we are just limited (as in the case of the die) to find the symmetries in the system which allow us to determine the probability for these processes.

As symmetries have the group mathematical structure, we can establish an homomorphism with the non singular square matrix. That is a linear representation of the group of symmetries of the system. In particular, the mathematical framework of Quantum Theory is the projective representation in a complex ray space.[q] This is to say that the representation of rays and matrices are defined up to a phase factor, because, at the end of the day, probabilities only depend on the square of the amplitudes. This phase factor is a non trivial element. This one is responsible for the non commutativity of translations in phase space and, in general, for the interference effects characteristic of quantum processes. Moreover, it also explains the classical limit of the theory.

To sum up, we have seen that the geometry of the ray space determines the logic of the quantum processes and, as a consequence, the quantum probabilistic rules such as Born's and von Neumann's ones, historically postulated *ad hoc*. This logic and probabilistic rules a priori sound enigmatic, since we have to leave aside the familiar Boolean logic and Kolmogorov's notion of probabilities. However, if we

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[p] The suggestion that there may not be any fundamental dynamical laws was also made by Anandan.[31] He also claimed that the non existence of laws imply that there can be neither deterministic laws nor fundamental probabilistic ones. So, transition probabilities can be just determined by symmetries. Therefore Born rule has not any fundamental status.

[q] The non trivial point is if we are limited to use complex numbers as the scalar fields. In fact instead of having a group $U(1)$ for the phases we can generalize it to another internal group of symmetry.

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finally accept that the ray space is the true scenery where the events occur, the new logic is inherently derived from geometry. The propositions of this logic are chains of sequential paths of events (histories) that have two equivalent representations. One uses of projector operators, and the other one works in terms of complex amplitudes which follow Feynman rules. In this rules, which describe the basic experiments of polarization and interference, is encapsulated the mysterious logic of the quantum processes.

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