The Random Discrete Action for 2-Dimensional Spacetime

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Abstract

A one-parameter family of random variables, called the Discrete Action, is defined for a 2-dimensional Lorentzian spacetime of finite volume. The single parameter is a discreteness scale. The expectation value of this Discrete Action is calculated for various regions of 2D Minkowski spacetime, $\mathbb{M}^2$. When a causally convex region of $\mathbb{M}^2$ is divided into subregions using null lines the mean of the Discrete Action is equal to the alternating sum of the numbers of vertices, edges and faces of the null tiling, up to corrections that tend to zero as the discreteness scale is taken to zero. This result is used to predict that the mean of the Discrete Action of the flat Lorentzian cylinder is zero up to corrections, which is verified. The "topological" character of the Discrete Action breaks down for causally convex regions of the flat trousers spacetime that contain the singularity and for non-causally convex rectangles.

Contents

1 The 2-dimensional action of a causal set

The twin hypotheses that spacetime is fundamentally discrete and that, of all the structures of classical General Relativity, it is the causal structure of spacetime that will persist in the deep quantum regime gives rise to the idea that spacetime is a discrete order [1, 2, 3]. Indeed the basic proposal of the causal set approach to quantum gravity is that the sum-over-histories for quantum gravity is a sum over discrete orders or *causal sets*. To define such a sum-over-histories, it will be necessary to give the amplitude for each causal set (or *pair* of causal sets if the path integration is conducted in Schwinger-Kel'dysh manner) and progress has recently been made on the question of what these amplitudes might be: a 2-dimensional action and a 4-dimensional action for a causal set have been proposed [4] and actions in 3, 5 and higher dimensions can also be defined [5].

Recall that a *causal set* (*causet* for short) is a locally finite partial order, *i.e.* it is a pair $(\mathcal{C}, \preceq)$ where $\mathcal{C}$ is a set and $\preceq$ is a relation on $\mathcal{C}$ which is reflexive ($x \preceq x$), acyclic ($y \preceq x \preceq y \Rightarrow y = x$) and transitive ($z \preceq y \preceq x \Rightarrow z \preceq x$). Local finiteness is the condition that the cardinality of any *order interval* is finite, where the (inclusive) order interval between a pair of elements $y \preceq x$ is defined to be $I(x, y) := \{z \in \mathcal{C} \,|\, y \preceq z \preceq x\}$. We call x the *top element* and y the *bottom element* of $I(x, y)$. We write $y \prec x$ when $y \preceq x$ and $y \neq x$. We define $n(x, y) := |I(x, y)|$ and call a relation $y \prec x$ a *link* if $n(x, y) = 2$. A *chain* is a totally ordered subset of $\mathcal{C}$.

Sprinkling is a random process that produces a causet which is a discretisation of a d-dimensional, causal, Lorentzian manifold $(\mathcal{M}, g)$. It is a Poisson process of selecting points in $(\mathcal{M}, g)$, independently at random, with density ρ so that the expected number of points sprinkled in a region of spacetime volume V is ρV. In quantum gravity we expect that the density is Planckian so that $\rho = l^{-d}$ where l is of order the Planck length, but in this paper we treat ρ as a parameter to be varied. This process generates a causet whose elements are (identified with) the sprinkled points and whose order is that induced by the manifold's causal order restricted to the sprinkled points. If $(\mathcal{M}, g)$ is of finite volume, the causet generated is almost surely finite and so the process defines a probability distribution $\mathbb{P}_{\mathcal{M},g,\rho}$ on the set of finite causets (*aka* the set of finite partial orders). Henceforth, for ease of notation, we will drop the explicit reference to the metric g and refer, for example, to a spacetime as $\mathcal{M}$ and the probability distribution above as $\mathbb{P}_{\mathcal{M},\rho}$.

Sprinkling is not a physical process. It plays a purely kinematical role and expresses the discrete-continuum correspondence: a causet $\mathcal{C}$ is well approximated by a Lorentzian manifold $\mathcal{M}$ if it could have been generated, with relatively high probability, by sprinkling into $\mathcal{M}$. In other words, $\mathcal{C}$ is well approximated by a manifold $\mathcal{M}$ if there exists an embedding $i : \mathcal{C} \hookrightarrow \mathcal{M}$ such that (i) $x, y \in \mathcal{C}$, $y \preceq x$ iff $i(y) \in J^-(i(x))$ and (ii) the number of elements embedded in any sufficiently nice, large region of volume V is approximately ρV. Strictly, this is a *conjecture*, the "Hauptvermutung" of the causal set approach, but it is supported by much evidence including the result that a distinguishing Lorentzian geometry is fully determined by its causal structure and spacetime volume measure [6, 7, 8].

We define the 2D action, S, of a finite causal set $\mathcal{C}$ to be [4]

$$S[\mathcal{C}] = N - 2N_1 + 4N_2 - 2N_3 \tag{1.1}$$

where N is the cardinality of $\mathcal{C}$, and N_m is the number of inclusive order intervals in $\mathcal{C}$ of cardinality $m+1$. N_1 therefore is the number of links in $\mathcal{C}$, N_2 is the number of order intervals that are 3-chains (3 element chains) and N_4 is the number of order intervals that are 4-chains plus the number that are "diamonds" (with two mutually unrelated elements between the top and bottom elements). Note that N_3 is not the number of subcausets that are 3-chains but the number of *order intervals* that are 3-chains. The form of S as an alternating sum of (weighted) numbers of things is intriguingly reminiscent of certain topological indices.

The action (1.1) defines an integer valued random variable, the *Discrete Action* $\mathbf{S}_{\mathcal{M},\rho}$, for each finite volume spacetime $\mathcal{M}$ and density ρ via the sprinkling process: $\mathbf{S}_{\mathcal{M},\rho}$ takes the value $S[\mathcal{C}]$ with probability $\mathbb{P}_{\mathcal{M},\rho}(\mathcal{C})$. We also define the random variable $\mathbf{S}_{\mathcal{M},N}$ which takes the value $S[\mathcal{C}]$ with the probability that causet $\mathcal{C}$ arises in the process of selecting exactly N elements uniformly at random – according to the spacetime volume measure – from $\mathcal{M}$. We then have

$$\langle \mathbf{S}_{\mathcal{M},\rho} \rangle = \sum_{N=0}^{\infty} \frac{(\rho V)^N}{N!} e^{-\rho V} \langle \mathbf{S}_{\mathcal{M},N} \rangle \tag{1.2}$$

where $\langle \cdot \rangle$ denotes the expected value, V is the spacetime volume of $\mathcal{M}$, and $\frac{(\rho V)^N}{N!} e^{-\rho V}$ is the probability that N elements are selected in the Poisson process of sprinkling into $\mathcal{M}$ at density ρ.

The Poisson distribution gives for the mean,

$$\langle \mathbf{S}_{\mathcal{M},\rho} \rangle = \rho V - 2\rho^2 \int_{\mathcal{M}} d^d y \sqrt{-g(y)} \int_{\mathcal{M} \cap J^+(y)} d^d x \sqrt{-g(x)} \left(1 - 2\rho V_{xy} + \frac{1}{2}(\rho V_{xy})^2\right) e^{-\rho V_{xy}} \tag{1.3}$$

where V_{xy} is the volume of the spacetime causal interval, $[x, y] := J^+(y) \cap J^-(x)$, between x and y and d is the dimension of $\mathcal{M}$. This can be understood thus: $\rho \, d^d x \sqrt{-g(x)}$ is the probability that an element is sprinkled in an elemental volume at x and similarly for y; $e^{-\rho V_{xy}}$, $\rho V_{xy} e^{-\rho V_{xy}}$ or $\frac{1}{2}(\rho V_{xy})^2 e^{-\rho V_{xy}}$ is the probability that there is no element, one element or two elements, respectively, sprinkled in $[x, y]$.

Note that the double integration may be done in either order:

$$\langle \mathbf{S}_{\mathcal{M},\rho} \rangle = \rho V - 2\rho^2 \int_{\mathcal{M}} d^d x \sqrt{-g(x)} \int_{\mathcal{M} \cap J^-(x)} d^d y \sqrt{-g(y)} \left(1 - 2\rho V_{xy} + \frac{1}{2}(\rho V_{xy})^2\right) e^{-\rho V_{xy}} \, . \tag{1.4}$$

Indeed, the causet action (1.1) is invariant under reversal of the order relation on $\mathcal{C}$, and so the Discrete Action (DA) for any spacetime $(\mathcal{M}, g)$ is equal to the DA of its time-orientation-reverse.

$\mathbf{S}_{\mathcal{M},\rho}$ is defined for any finite volume (causal) spacetime of any dimension so we can ask in what sense it is 2-dimensional. Each realisation of $\mathbf{S}_{\mathcal{M},\rho}$ is the action $S[\mathcal{C}]$ of some finite causet $\mathcal{C}$ and

$$S[\mathcal{C}] = \sum_{e_i \in \mathcal{C}} L(e_i) \tag{1.5}$$

where

$$L(e_i) = 1 - 2n_1(e_i) + 4n_2(e_i) - 2n_3(e_i) \tag{1.6}$$

and $n_m(e_i)$ is the number of inclusive order intervals in $\mathcal{C}$ with cardinality $m+1$ and with top element e_i. $L(\cdot)$ itself defines a random variable, $\mathbf{L}_{\mathcal{M},\rho,y}$, for each spacetime $\mathcal{M}$, each point $y \in \mathcal{M}$ and each ρ in the following way. Fix $y \in \mathcal{M}$, sprinkle into $\mathcal{M}$ at density ρ and add an element at y to the sprinkled causet to form causet $\mathcal{C}'$ which has a marked element, call it e_y. The value of $\mathbf{L}_{\mathcal{M},\rho,y}$ is then $L(e_y)$ evaluated in $\mathcal{C}'$. If $\mathcal{M}$ is 2-dimensional, the mean of $\mathbf{L}_{\mathcal{M},\rho,y}$ tends to $\frac{1}{4\rho}R(y)$, where $R(y)$ is the Ricci scalar, as ρ tends to infinity [4]. It approaches its limit when the discreteness length scale $l := \rho^{-\frac{1}{2}}$ is much smaller than the curvature scale $R^{-\frac{1}{2}}$. If $\mathcal{M}$ is *not* 2-dimensional, there is no apparent reason for $\mathbf{L}_{\mathcal{M},\rho,y}$ to have anything to do with the continuum geometry $\mathcal{M}$.

Since L is thus related to the Ricci scalar when the causal set is a 2D sprinkling and S is a sum of $L(\cdot)$ over the causal set, this implies that when $\mathcal{M}$ is 2-dimensional and as $\rho \to \infty$, $\langle \mathbf{S}_{\mathcal{M},\rho} \rangle$ will tend to something that contains a term $\frac{1}{4}\int_{\mathcal{M}} d^2x\sqrt{-g}R$ plus terms arising from boundary effects. We will investigate this and in particular the nature of the boundary terms. In doing so we will be exploring whether the 2D Discrete Action is topological in character. The standard gravitational action for 2D Euclidean gravity, with its Einstein-Hilbert term and the (2D analogue of the) Gibbons-Hawking boundary term, is known to be a topological invariant, due to the Gauss-Bonnet theorem. The Gauss-Bonnet Theorem has been extended to Lorentzian manifolds [9, 10], so for ordinary (Lorentzian) 2D gravity, the action with an appropriate boundary term is also topological and a question arises: to what extent is the 2D causal set action topological?

2 Intervals in $\mathbb{M}^2$

Consider a causal interval in 2D Minkowski spacetime, $\mathcal{I} := [p, q] \subset \mathbb{M}^2$. For definiteness consider the interval to have fixed volume (area), V.

Following a conjecture of R. Sorkin, G. Brightwell proved that the mean $\langle \mathbf{S}_{\mathcal{I},N} \rangle = 1$, for any $N \neq 0$ [11]. This implies that the mean of $\mathbf{S}_{\mathcal{I},\rho}$ is

$$\begin{aligned} \langle \mathbf{S}_{\mathcal{I},\rho} \rangle &= \textstyle\sum_{N=1}^{\infty} \frac{(\rho V)^N}{N!} e^{-\rho V} \\ &= 1 - e^{-\rho V} \end{aligned} \tag{2.1}$$

where $\frac{(\rho V)^N}{N!} e^{-\rho V}$ is the probability, in the Poisson process, that N elements are sprinkled into $\mathcal{I}$.

We use (1.3) to prove this result in a different way:

$$\langle \mathbf{S}_{\mathcal{I}} \rangle = \rho V - 2\rho^2 \int_{\mathcal{I}} d^2y \int_{\mathcal{I}\cap J^+(y)} d^2x \;\; p(\rho V_{xy}) \tag{2.2}$$

where $p(\xi) = (1 - 2\xi + \frac{1}{2}\xi^2)\exp(-\xi)$ and we have suppressed the subscript ρ on the random variable $\mathbf{S}_{\mathcal{I},\rho}$.

We use coordinates in which p and q lie on the time axis and q is at the origin. We consider null coordinates $u_x = \frac{1}{\sqrt{2}}(x^0 - x^1)$, $v_x = \frac{1}{\sqrt{2}}(x^0 + x^1)$ and similarly for u_y, v_y. Then the interval is defined by $u, v \in [0, a]$ for $a = \sqrt{V}$.

$$\langle \mathbf{S}_{\mathcal{I}} \rangle = \rho V - 2 \int_0^a du_x \int_0^a dv_x \int_0^{u_x} du_y \int_0^{v_x} dv_y \, \rho^2 \, p(\rho \, \Delta u \, \Delta v) \tag{2.3}$$

where $\Delta u = u_x - u_y$, $\Delta v = v_x - v_y$.

$$\begin{aligned}
\langle \mathbf{S}_{\mathcal{I}} \rangle &= \rho V - 2 \int_0^a du_x \int_0^a dv_x \int_0^{u_x} d\Delta u \int_0^{v_x} d\Delta v \, \rho^2 \, p(\rho \, \Delta u \, \Delta v) \\
&= \rho V - 2 \int_0^a du_x \int_0^a dv_x \left[[\text{integrand } 1]_{\Delta v=0}^{\Delta v=v_x} \right]_{\Delta u=0}^{\Delta u=u_x}
\end{aligned}$$

where

$$\begin{aligned}
&\text{integrand } 1 = -\frac{\rho}{2}\,(1 - \rho \, \Delta u \, \Delta v) \, \exp(-\rho \, \Delta u \, \Delta v) \\
&[g(\xi)]_{\xi=\beta}^{\xi=\alpha} = g(\alpha) - g(\beta).
\end{aligned}$$

Hence

$$\langle S_{\mathcal{I}} \rangle = 1 - \exp(-\rho \, a^2) = 1 - \exp(-\rho V)\,. \tag{2.4}$$

As $\rho \to \infty$, $\langle S_{\mathcal{I}} \rangle \to 1$ and we write $\langle S_{\mathcal{I}} \rangle \approx 1$ to denote this.

Consider now splitting up the interval $\mathcal{I}$ into four smaller intervals $\mathcal{I}_i$, $i = 1, \ldots 4$, as shown in Fig. 1a. When computing the expected value of $\mathbf{S}_{\mathcal{I}}$ one can split the integral up into the means of the actions of the four subintervals plus the "bilocal" contributions when x and y lie in two different subintervals. More concretely, given any subcausets, A and B of a causal set $\mathcal{C}$, we define the bilocal action,

$$S[\mathcal{C}; A, B] = N(A, B) - 2N_1(A, B) + 4N_2(A, B) - 2N_3(A, B) \tag{2.5}$$

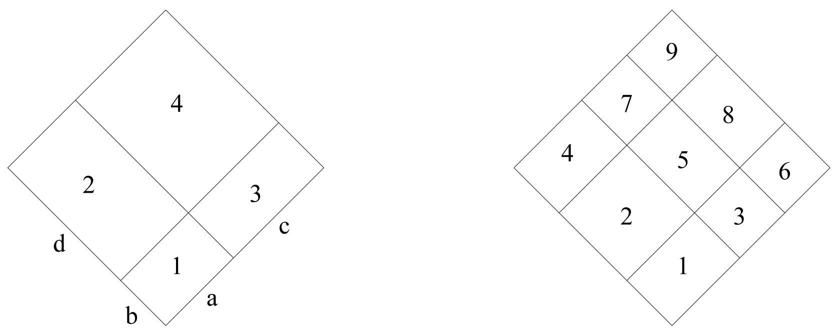
where $N(A, B)$ is the number of elements in $A \cap B$ and $N_m(A, B)$ is the number of inclusive order intervals in $\mathcal{C}$ of cardinality $m + 1$ with top element in A and bottom element in B. Now let X and Y be submanifolds of spacetime $\mathcal{M}$. We define the random variable, $\mathbf{S}_{\mathcal{M};X,Y}$, the *Discrete Bilocal Action*, via the sprinkling process: sprinkle into $\mathcal{M}$ at density ρ [1] to obtain causet $\mathcal{C}$ with subcauset $A(B)$ being that sprinkled into $X(Y)$. For that realisation, $\mathbf{S}_{\mathcal{M};X,Y}$ takes the value $S[\mathcal{C}; A, B]$.

Note that $\mathbf{S}_{\mathcal{M};X,X} = \mathbf{S}_X$ if X is a *causally convex* subset of $\mathcal{M}$.[2]

Now, consider $\mathcal{I}$ and its subintervals. If we adopt $\mathbf{S}_{ij}$ as simplified notation for the bilocal action $\mathbf{S}_{\mathcal{I};\mathcal{I}_i,\mathcal{I}_j}$, then we have

$$\langle \mathbf{S}_{\mathcal{I}} \rangle = \sum_{i=1}^{4} \langle \mathbf{S}_{\mathcal{I}_i} \rangle + \sum_{\substack{i,j=1 \\ j<i}}^{4} \langle \mathbf{S}_{ij} \rangle \,. \tag{2.6}$$

The bilocal summands can be computed using the integral in Eq. (2.3) and adjusting



(a) a and c (b and d) are the v-coordinate (u-coordinate) lengths of the sides of the subintervals

(b)

Figure 1: Splitting up a causal interval in 2D Minkowski to compute the action

the boundaries. This yields

$$\begin{aligned}
\langle \mathbf{S}_{21} \rangle &= -2 \int_0^a dv_x \int_b^{b+d} du_x \int_0^{v_x} dv_y \int_0^b du_y \, \rho^2 \, p(\rho \, \Delta u \, \Delta v) \\
&= -2 \int_0^a dv_x \int_b^{b+d} du_x \left[[\text{integrand } 1]_{\Delta u = u_x - b}^{\Delta u = u_x} \right]_{\Delta v = 0}^{\Delta v = v_x} \\
&= -1 + \exp(-a\, b\, \rho) + \exp(-a\, d\, \rho) - \exp(-a\,(b+d)\,\rho) \\
&\approx -1
\end{aligned} \tag{2.7}$$

[1] To simplify notation, we don't make the dependence on the density explicit.

[2] A causally convex region, X, of $\mathcal{M}$ is one such that $x, y \in X$ implies that the causal interval in $\mathcal{M}$ between x and y is a subset of X.

and

$$
\begin{aligned}
\langle \mathbf{S}_{41} \rangle &= -2 \int_a^{a+c} dv_x \int_b^{b+d} du_x \int_0^a dv_y \int_0^b du_y \, \rho^2 \, p(\rho \, \Delta u \, \Delta v) \\
&= -2 \int_a^{a+c} dv_x \int_b^{b+d} du_x \left[[\text{integrand } 1]_{\Delta u = u_x - b}^{\Delta u = u_x} \right]_{\Delta v = v_x - a}^{\Delta v = v_x} \\
&= 1 - \exp(-(a+c)\,(b+d)\,\rho) \\
&\quad + \exp(-a\,(b+d)\,\rho) + \exp(-c\,(b+d)\,\rho)) \\
&\quad + \exp(-(a+c)\,b\,\rho) + \exp(-(a+c)\,d\,\rho) \\
&\quad - \exp(-a\,b\,\rho) - \exp(-a\,d\,\rho) - \exp(-c\,b\,\rho) - \exp(-c\,d\,\rho) \qquad (2.8) \\
&\approx 1\,.
\end{aligned}
$$

The three other bilocal contributions $\langle \mathbf{S}_{ij} \rangle$ can be obtained from $\langle \mathbf{S}_{21} \rangle$ by changing the parameters appropriately. Putting together all parts of Eq. (2.6) one exactly recovers Eq. (2.1).

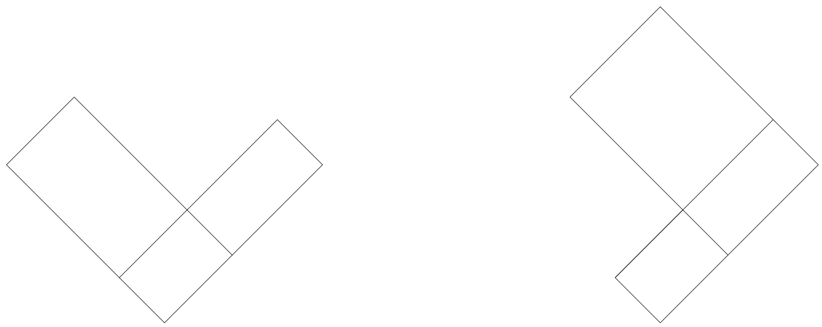
Now, one can continue this game and split up the interval even further as in Fig. 1b. To compute the mean of the action one must again calculate

$$
\langle \mathbf{S}_{\mathcal{I}} \rangle = \sum_{i=1}^{9} \langle \mathbf{S}_{\mathcal{I}_i} \rangle + \sum_{\substack{i,j=1 \\ j<i}}^{9} \langle \mathbf{S}_{ij} \rangle \,. \tag{2.9}
$$

We already know the contributions $\langle \mathbf{S}_{I_i} \rangle \approx 1$ and the bilocal contributions from two intervals that either share an edge or lie above and below a shared vertex (*e.g.* $\langle \mathbf{S}_{21} \rangle$ and $\langle \mathbf{S}_{51} \rangle$ in Fig. 1b). It remains to compute the bilocal contributions from pairs of intervals such as (4,1),(7,1) and (9,1) in Fig. 1b. It turns out they consist only of exponential terms that are small when intervening intervals are large on the discreteness scale. In the limit of large density, we are left with a contribution of 1 for every subinterval, -1 for every edge and 1 for every vertex. One could write

$$
\langle \mathbf{S} \rangle \approx F - E + V \tag{2.10}
$$

where F denotes the number of faces *i.e.* intervals, E the number of edges and V the number of vertices. $F-E+V$ is the formula for the Euler character of a polyhedron and motivates the question: Is the expected action (to some extent) a topological invariant? It is obvious that the formula can be applied to arbitrary causally convex regions of $\mathbb{M}^2$ that can be tiled by causal intervals as long as each interval is large enough for the corrections to be negligible. It is not hard to verify that any such region will have a mean Discrete Action $\langle \mathbf{S} \rangle \approx 1$. So for example the region shown in Fig. 2a will give $\langle \mathbf{S} \rangle \approx 1$ but the region in Fig. 2b will not.

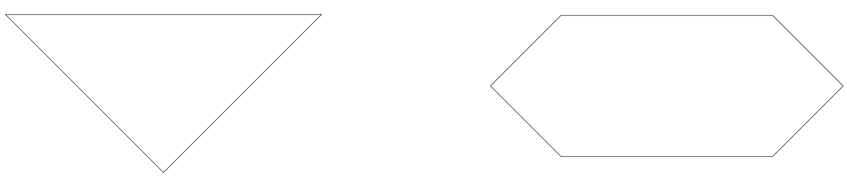


(a) causally convex (b) not causally convex

Figure 2: Different regions constructed from causal intervals in $\mathbb{M}^2$

3 Causally convex regions in $\mathbb{M}^2$

The boundary of a causally convex region of $\mathbb{M}^2$ can be spacelike in parts, but never timelike. If the region's boundary comprises straight line segments, such as the hexagon shown in Fig. 3b, then it can be divided up by null lines into a collection of intervals and causally convex triangles such as Fig. 3a. Then the formula (2.10) will apply if the mean of the Discrete Action for a causally convex triangle tends to 1 in the infinite density limit.



(a) causally convex triangle (b) causally convex hexagon

Figure 3: Causally convex regions with boundaries formed from null and spacelike line segments

First note that by Poincaré invariance we can choose coordinates so that the spacelike edge of the triangle is at t =constant, and the apex lies at the origin.

Using null coordinates, as before, we have

$$\langle \mathbf{S}_\triangle \rangle = \rho V - 2\rho^2 \int_0^L dv_x \int_0^{v_x} du_x \int_0^{v_x} dv_y \int_0^{u_x} du_y \, p(\rho \Delta u \Delta v) \tag{3.1}$$

where $L = \sqrt{2V}$ and V is the area of the triangle. This gives

$$\langle \mathbf{S}_\triangle \rangle = 1 + \frac{1}{\rho V} + O\left((\rho V)^{-2}\right) \approx 1\,. \tag{3.2}$$

We see that the mean DA of the triangle does indeed tend to 1 as $\rho \to \infty$, though the corrections are not exponentially small.

Now, consider a general causally convex region with a boundary whose spacelike portion is curved. So long as the discreteness scale is small enough – small compared to the radius of curvature of the boundary – we can tile the region with intervals and with causally convex approximate triangles along the spacelike boundary, all of which are large enough compared to the discreteness scale for the Formula (2.10) to hold approximately. We conclude that the mean of the DA for any causally convex region of $\mathbb{M}^2$ will tend to 1 in the limit of infinite density.

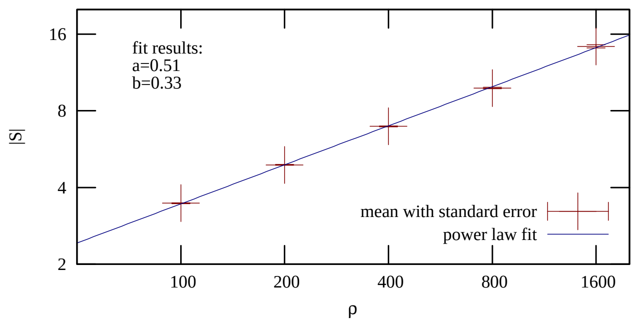
Is causal convexity necessary for the mean of the DA to be approximately 1? When a region, $R \subset \mathbb{M}^2$, is not causally convex, there will exist pairs of points $x, y \in R$ such that the causal interval in R between x and y is smaller than the causal interval between them in $\mathbb{M}^2$ (the "diamond"). Since it is the volume of the causal interval in R which appears in the expression for the mean of the DA, one might expect this to disrupt the result and indeed it does.

Consider the Discrete Action, $\mathbf{S}_\square$ of a rectangle with edges parallel to the t and x axes. Analytic computation of the expectation value $\langle \mathbf{S}_\square \rangle$ is hard exactly because of the lack of causal convexity: the integral (1.3) breaks up into several subintegrals depending on the positions of x and y relative to the boundary. Therefore we use simulations to estimate the value. A sprinkling into a rectangle has three independent parameters that fully characterise the problem. One choice is the spatial width w, the height along the time-axis h and the sprinkling density ρ.[3] The expectation value $\langle \mathbf{S}_{\square,w,h,\rho} \rangle$ must be invariant under rescaling

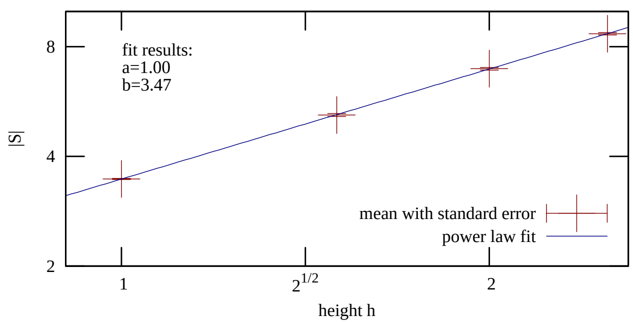
$$\begin{aligned} w &\to \lambda \cdot w \\ h &\to \lambda \cdot h \\ \rho &\to \lambda^{-2} \cdot \rho. \end{aligned} \tag{3.3}$$

Fig. 4 shows simulation data for two different setups with power-law fits. Fig. 4a

[3] Width w, height h and expected number of sprinkled elements N would be another choice.



(a) Simulation data for the action of a rectangle in $\mathbb{M}^2$ for $w = h = 1$, varying density ρ with a power-law fit. Data averaged over 10^6 to 10^7 runs. Fit function: $\rho^a \cdot b$.



(b) Simulation data for the action of a rectangle in $\mathbb{M}^2$ for $w = 1, \rho = 100$, varying height h with a power-law fit. Data averaged over 10^6 to 10^7 runs. Fit function: $h^a \cdot b$.

Figure 4: Numerical results for the action of a rectangle in $\mathbb{M}^2$.

shows $\langle \mathbf{S} \rangle$ for constant w and h and varying ρ, Fig. 4b for constant w and ρ and for varying h. Given the small relative error bars the power-law fits look quite convincing and we will assume that $\langle \mathbf{S}_\square \rangle$ can, at least in the regime covered by the simulations, be written in the form

$$\langle \mathbf{S}_\square \rangle = \text{const} \cdot h^\alpha w^\beta \rho^\gamma . \tag{3.4}$$

The scale invariance (3.3) demands $\alpha + \beta - 2\gamma = 0$. From simulation 1 (Fig. 4a) one is tempted to deduce $\gamma = 1/2$ and from simulation 2 (Fig. 4b) that $\alpha = 1$. It follows $\beta = 0$.

The fact that for constant ρ the width does not affect the value of the action whereas $\langle \mathbf{S}_\square \rangle \propto h$ suggests that in general $\langle \mathbf{S}_\square \rangle$ contains boundary terms from timelike boundaries only. We return to this question in the discussion section.

4 The flat cylinder

In order to apply formula Eq. (2.10) to a causal interval, $\mathcal{I}_c$, of height T on a cylinder with circumference L with $L \leq T \leq 2L$ one might come up with a tiling into subintervals, $\mathcal{I}_i$, $i = 1, \ldots 8$, as shown in Fig. 5. Taking into account the topological

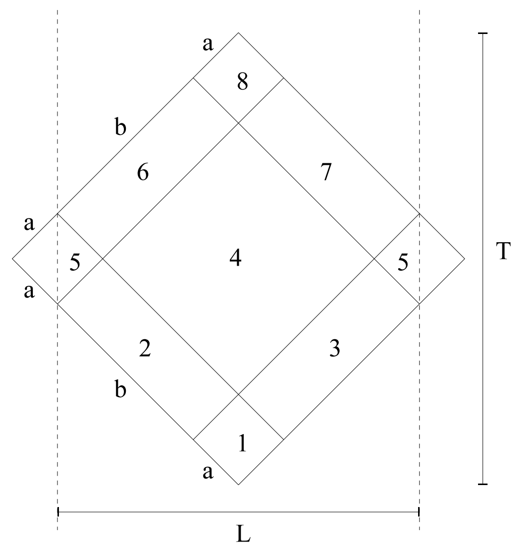


Figure 5: Tiling of the interval $\mathcal{I}_c$ with $L \leq T \leq 2L$. a and b are the u and v coordinate lengths of the sides of the subintervals shown.

identification, we have $F = 8, E = 12, V = 4$ thus yielding a predicted high-density expectation value of $\langle \mathbf{S}_{\mathcal{I}_c} \rangle \approx 0$. However we have not shown yet that formula Eq. (2.10) is applicable to the cylinder. The division of the causal interval in Fig. 5 has been chosen such that formula Eq. (2.1) for the faces and formulae Eq. (2.7) and (2.8) for the bilocal contributions of two intervals that share an edge or lie above and below a vertex can still be applied as the cylinder topology does not affect these cases. But the computation of contributions like (5,1),(6,1) and (8,1) differs from the Minkowski setup due to the nontrivial topology.

Recall

$$\langle \mathbf{S}_{\mathcal{I}_c} \rangle = \rho V - 2K \tag{4.1}$$

where

$$K = \rho^2 \int_{\mathcal{I}_c} d^2y \int_{\mathcal{I}_c \cap J^+(y)} d^2x \; p(\rho V_{xy}) \,. \tag{4.2}$$

In general, K can be split into a sum of terms, $K = \sum_{\alpha=1}^{\infty} K_\alpha$ depending on how many homotopy classes of causal curves there are from y to x:

$$K_\alpha := \rho^2 \int_{\mathcal{I}_c} d^2y \int_{\mathcal{I}_c \cap J^+_\alpha(y)} d^2x \; p(\rho V_{xy}) \tag{4.3}$$

where

$$J^+_\alpha(y) := \{x \in J^+(y) \,|\, \exists \text{ exactly } \alpha \text{ homotopy classes of causal curves from } y \text{ to } x\}\,. \tag{4.4}$$

This split is motivated by the fact that V_{xy} strongly depends on the number of homotopy classes of causal paths between x and y. For our interval, $K_\alpha = 0$ for $\alpha > 3$.

From Fig. 5 we see the relation between a, b, T and L is:

$$\begin{aligned} a &= (T-L)/\sqrt{2} \\ b &= (2L-T)/\sqrt{2} \end{aligned} \tag{4.5}$$

The causal volume V_{xy} for $x \in J^+_\alpha(y)$ for $\alpha \geq 3$ is at least $(a+b)^2$ so K_3 is suppressed by at least $\exp(-\rho(a+b)^2)$ and can thus be neglected as $L = \sqrt{2}(a+b)$ is assumed to be large in discreteness units of $\rho^{-\frac{1}{2}}$.

The values for K_1 and K_2 are [12]

$$\begin{aligned} K_1 &= \frac{\rho V}{2} + \frac{1}{2}\exp(-\rho a^2) - (1+\rho ab)\exp(-\rho a(a+b)) + \text{corr.} \\ K_2 &= -\frac{2}{(a+b)^2\rho} + \exp(-\rho a(a+b))\,[1+\rho ab \\ &\quad + \frac{1}{(a+b)^4\rho^2}\big((6 + 2\rho(a+b)(2a+b) - \rho^2(a+b)^2b^2 + \rho^3(a+b)^3ab^2) \\ &\quad -2\exp(-\rho a(a+b))(3 + 4\rho a(a+b) + 2\rho^2a^2(a+b)^2)\big)\big] + \text{corr.} \end{aligned} \tag{4.6}$$

where "+ corr." stands for neglected terms suppressed by $\exp(-\rho(a+b)^2)$. However we will keep terms with factors $\exp(-\rho\, a^2)$ and $\exp(-\rho\, a\,(a+b))$ since for T only slightly larger than L the value of a will be very small and these terms are then significant.

The overall action is

$$\begin{aligned} \langle \mathbf{S}_{\mathcal{I}_c} \rangle &= -\exp(-\rho a^2) + 2(1+\rho ab)\exp(-\rho a(a+b)) \\ &\quad + \frac{4}{(a+b)^2\rho} + \exp(-\rho a(a+b))\,[1+\rho ab \\ &\quad + \frac{1}{(a+b)^4\rho^2}\big((6 + 2\rho(a+b)(2a+b) - \rho^2(a+b)^2b^2 + \rho^3(a+b)^3ab^2) \\ &\quad -2\exp(-\rho a(a+b))(3 + 4\rho a(a+b) + 2\rho^2a^2(a+b)^2)\big)\big] + \text{corr.}\,. \end{aligned} \tag{4.7}$$

For $T > 2L$ consider a division of the interval into regions 1 and 2 as shown in Fig. 6. The expected DA is the sum of the expected actions for regions 1 and 2 and the bilocal contribution $\langle \mathbf{S}_{21} \rangle$.

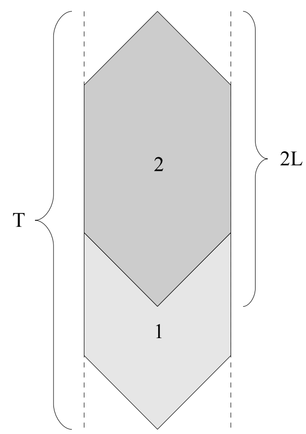


Figure 6: Division of interval when $T > 2L$.

It can be shown [12] that the expected action for region 1 and the bilocal contribution cancel (up to exponentially small terms) and the result is just given by the expected action of region 2 which can be obtained from Eq.(4.7) by setting $a = L/\sqrt{2}, b = 0$ (and now neglecting all exponentials as a is no longer close to 0):

$$\langle \mathbf{S}_{\mathcal{I}_c} \rangle = \frac{8}{L^2 \rho} + \text{corr.}\,. \tag{4.8}$$

Fig. 7 shows a plot of the analytic expectation value for the cylinder action compared to simulation results. For $T \to L$ the action approaches the Minkowskian limit 1. For T only slightly greater than L the exponential terms dominate and cause a downwards spike. As $T \to 2L$ the non-exponential correction, $\frac{8}{\rho L^2}$ (which comes from K_2) dominates. However this also tends to zero in the limit $\rho \to \infty$ so $\langle \mathbf{S}_{\mathcal{I}_c} \rangle \approx 0$ as initially predicted. Indeed it can be shown explicitly that the bilocal contributions from pairs of intervals that do not share and edge or vertex tend to zero as $\rho \to \infty$ and so the formula $F - E + V$ can be applied to intervals of the cylinder. More generally, the previous argument regarding null tilings of causally convex regions of $\mathbb{M}^2$ can be given here, and we conclude that $\langle \mathbf{S} \rangle \approx 0$ for general topologically non-trivial causally convex regions of the cylinder.

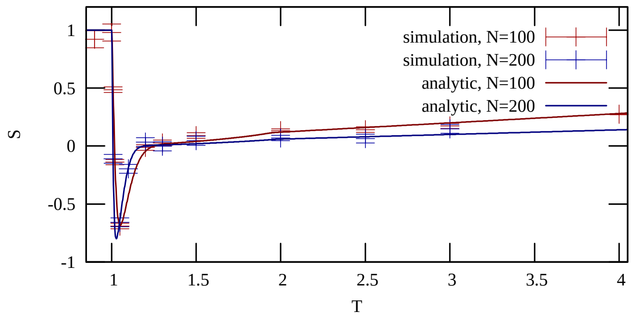


Figure 7: The expected action of a cylinder-interval for $L = 1, \langle N \rangle = 100$ and $\langle N \rangle = 200$ compared with simulation results.

5 The flat trousers

We investigate now a causally convex neighbourhood of the flat 1+1 trousers spacetime in which two S^1's join to form a single S^1. The trousers spacetime is a piece of $\mathbb{M}^2$ with cuts and identifications as shown in Fig. 8. Although the singularity, P, at which the topology changes is by some definitions not strictly in the spacetime since the metric degenerates there, nevertheless the causal order is well defined at the singularity: it is clear what the causal past and causal future of P are. Therefore we will consider P as a point of the manifold. Note that in any sprinkling into the trousers almost surely no element will be sprinkled at P.

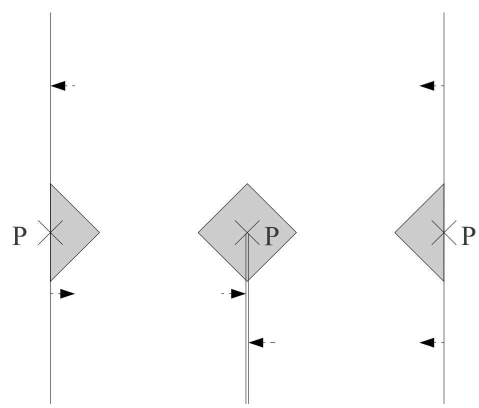


Figure 8: The trousers spacetime. P is the singularity – all three instances of P are identified – and the shaded region is a neighbourhood of P. There is a vertical cut down from the central copy of P with the two legs identified as shown.

Let $\mathcal{N}$ denote the neighbourhood of P shown as the shaded region in Fig. 8. It consists of two flat intervals each with P as their midpoint, identified across "branch cuts" from P to their past tips. $\mathcal{N}$ is topologically a disc if P is in included the manifold and if the formula (2.10) holds then the expected DA of $\mathcal{N}$ would be equal to 1 in the limit of large density.

Let the volume (area) of each of the two intervals be $4a^2$ and consider the null tiling into 8 intervals, $\mathcal{I}_i$, $i = 1, \ldots, 8$, shown in Fig. 9. The interval $\mathcal{I}_1$ comprises the two triangles labelled $1'$ and $1''$ and the interval $\mathcal{I}_2$ comprises the triangles labelled $2'$ and $2''$. Adopting the same notation for the bilocal discrete action of two intervals used in (2.6) we have

$$\langle \mathbf{S}_{\mathcal{N}} \rangle = \sum_{i=1}^{8} \langle \mathbf{S}_{\mathcal{I}_i} \rangle + \sum_{\substack{i,j=1 \\ j<i}}^{8} \langle \mathbf{S}_{ij} \rangle \,. \tag{5.1}$$

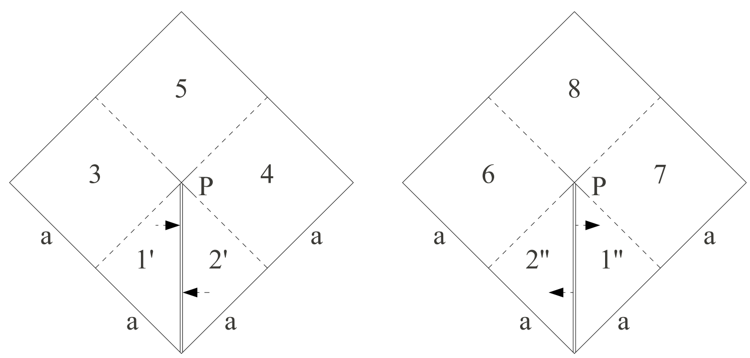


Figure 9: Null tiling of $\mathcal{N}$ into 8 intervals.

For each i, $\langle \mathbf{S}_{\mathcal{I}_i} \rangle \approx 1$. The bilocal terms are nonzero when the intervals $\mathcal{I}_i$ and $\mathcal{I}_j$ share an edge and in that case $\langle \mathbf{S}_{ij} \rangle \approx -1$. There are 8 edges so these contributions cancel the contributions of the 8 individual intervals. The only other nonzero bilocal terms are $\langle \mathbf{S}_{ij} \rangle$ where $i = 5, 8$ and $j = 1, 2$ and their sum is the contribution of the vertex at the singularity. These 4 terms are equal by symmetry so we have $\langle \mathbf{S}_{\mathcal{N}} \rangle = 4 \langle \mathbf{S}_{51} \rangle$.

The causal interval between $x \in \mathcal{I}_5$ and $y \in \mathcal{I}_1$ is shown in Fig. 10 and we deduce that

$$\langle \mathbf{S}_{51} \rangle = -2 \int_a^{2a} du_x \int_a^{2a} dv_x \int_0^a du_y \int_0^a dv_y \, \rho^2 \, p(\rho V_{xy}) \,. \tag{5.2}$$

where

$$V_{xy} = \Delta u \Delta v - (v_x - a)(a - u_y) \,. \tag{5.3}$$

This gives

$$\langle \mathbf{S}_{\mathcal{N}} \rangle = 4 \ln(\rho a^2) + 4(\gamma - 1) + \mathrm{O}\left(\frac{1}{\rho a^2} \right) \tag{5.4}$$

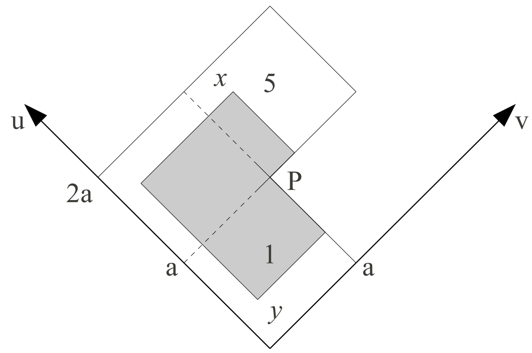


Figure 10: The causal interval between $x \in \mathcal{I}_5$ and $y \in \mathcal{I}_1$ is depicted in grey.

where γ is Euler's constant. We see that the expected DA of the neighbourhood of the singularity does not tend to 1 or any constant but grows logarithmically with the density.

6 Discussion

We have shown that in the limit of infinite density, the mean of the Discrete Action will be 1 for any causally convex region of $\mathbb{M}^2$ including regions whose past and/or future boundaries contain spacelike segments. Since these spacelike segments may have nonzero geodesic curvature, the constancy of the mean of the DA suggests that it contains no contribution from the past or future boundaries.

Indeed a handwaving argument can be given as to why this should be so, even when $\mathcal{M}$ is curved. The boundary of a causally convex region $\mathcal{U} \subset \mathcal{M}$ consists of a future boundary and a past boundary intersecting in a co-dimension 2 spacelike surface. There is no timelike portion of the boundary. The mean of the DA is a double integral over $\mathcal{U}$ which can be done in either order. The integrand is a retarded 2-point function,

$$\rho\mathcal{L}(x,y) = \frac{\rho}{\sqrt{-g}}\delta^{(2)}(x,y) - 2\rho^2\, p(\rho V_{xy})C(x,y) \tag{6.1}$$

where $C(x,y) = 1$ if $y \in J^-(x)$ and 0 otherwise. Let us assume the density is high enough that a sprinkled causal set can capture the curvature of $\mathcal{M}$, *i.e.* at each point $y \in \mathcal{M}$ there is a local inertial frame in which the curvature components are small compared to the density. If we do the x integration first, at fixed y, then the resulting function $\rho L(y)$ is approximately $\frac{1}{4}R(y)$ unless y is too close to the future boundary. If it is within length $\rho^{-\frac{1}{2}}$ of the boundary then the range of the x integration will not be large enough for the approximation to hold [4]. Then we do the integration over y to get approximately the usual Einstein-Hilbert bulk term together possibly with some contribution from the integral over the points y close to the future boundary, *i.e.* possibly some kind of future boundary term. But there is no contribution from the past boundary at all. Now reverse the order of integration: do y first and then x. Now there appears to be no contribution from the future boundary. This can only happen if neither boundary contributes. So the only points where some boundary contribution can come in, is from the points which are close to both past and future boundaries *i.e.* from the spacelike co-dimension 2 "corners" where the past and future boundaries intersect. The argument holds when the past and future boundaries are partly spacelike as well as when they are wholly null. There is no reason, from this argument, that timelike boundaries could not contribute however and we saw evidence that they do from the results for the rectangle.

This heuristic reasoning would have to be backed up with further evidence from simulations of the Discrete Action but it suggests that there is no Gauss-Bonnet formula for the 2D Discrete Action. The 2D Gauss-Bonnet Theorem can hold because, as the geometry of the bulk surface is varied, the extrinsic curvature of the boundary changes and the right combination of bulk and boundary terms can remain constant. In 2D the co-dimension 2 "corner" is an S^0, *i.e.* 2 points, and if the only boundary contributions are from these 2 points, these couldn't compensate for the changing bulk term. Another reason not to expect the DA to satisfy a Gauss-Bonnet formula is that it appears that the appropriate Lorentzian analogue of the Euclidean formula is of the form "bulk term + boundary term + corner terms" $= 2\pi i\chi$ rather than $2\pi\chi$ [13, 14] (see also [15, 16]). Both the bulk and boundary terms are real but the formula can hold because the corner contributions are Lorentzian angles which can be complex. However, the Discrete Action is real.

This putative lack of boundary terms could explain why the expected DA for any causally convex region of $\mathbb{M}^2$ is the same. The continuum bulk term is zero. If the mean of the DA is indeed close to the continuum bulk term plus only a contribution from the S^0 corners then that should be the same for all causally convex regions. Presumably, the difference for the neighbourhood of the singularity of the trousers comes from a boundary effect of the non-standard causal structure around the singularity which has a double lobed past and future. These issues all remain to be investigated.

There are a large number of open questions. What does happen in 2D curved spacetimes? Will the results bear out the conjecture that the expected DA is approximately the Einstein-Hilbert term plus a constant from the S^0 corner? What happens in higher dimensions? There are analogues of the 2D Discrete Action in 4D [4] and 3,5,6,7D and higher [5]. One would expect, for example, that the mean of the DA of an interval in $\mathbb{M}^d$ would be proportional to the volume of the S^{d-2} corner.

Although one need not take any position on quantum gravity to find interest in the Discrete Action as a random variable defined for a continuum spacetime – one need not consider the discreteness of the causal sets that arise in the definition of the Discrete Action to have any physical basis – its main application is likely to be in the causal set approach to quantum gravity. So, what is the significance in quantum gravity of the results for the interval, trousers and rectangle? For example, the result for the rectangle suggests that the expected DA contains boundary contributions proportional to the length of any timelike boundary. Can we use the DA to give an argument against the appearance of "holes" and "edges" in spacetime? Or for or against topology changing processes such as the trousers?

A major open question is how the fluctuations in the DA behave as the density gets large: we should stress that the results reported here are all concerning the mean of the DA. For a typical sprinkled causet, how far is the DA from the mean? Preliminary results show the fluctuations grow as the density gets large [12], contrary to the hope expressed in [4] and this needs to be studied further. To tame the fluctations it may be necessary to introduce a mesoscale between the discreteness scale and the observation scale [17, 4]. Further work is needed to illuminate these issues.

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