Conformal symmetry breaking effects in gauge theories: new perturbative expressions

A. L. Kataev[a*] and S. V. Mikhailov[b†]

[a] *Institute for Nuclear Research of the Academy of Sciences of Russia, 117312, Moscow, Russia*

[b] *Bogoliubov Laboratory of Theoretical Physics, JINR 141980 Dubna, Russia*

Abstract

We propose a conjecture about the detailed structure of the conformal symmetry breaking term in the QCD generalization of Crewther relation and demonstrate its validity at the α_s^4-level. We conclude that this new structure leads to formulation of the constrains on the QCD expansion coefficients of the Adler D-function and the polarized Bjorken sum rule $S_{\rm Bjp}$. Using this formulation a new relation between 5-loop coefficients of these two quantities is derived. It gives a possibility to present an additional check of the advanced results of order α_s^4.

1. The concept of conformal symmetry forms the basis for important theoretical studies in various massless quantum field models [1], [2], including QED [3] and QCD [4]. The attraction of the notion on this symmetry in the process of consideration of the axial-vector-vector (AVV) triangle amplitude revealed the existence of special relation between basic characteristics of two main inclusive processes. These characteristics are the normalised expression D of flavour non-singlet part of the e^+e^--annihilation Adler function, D_A^{NS}, and the non-singlet coefficient function C^{Bjp} of the Bjorken sum rule of the polarised lepton-hadron deep-inelastic scattering (DIS), which also enters into the non-singlet part of the Ellis-Jaffe sum rule of the polarised lepton-hadron DIS. This relation, discovered by Crewther [5] and independently in Ref. [6] within quark-parton model, can be extended to the case of the conformal invariant (CI) limit of QCD as well. In both cases this relation reads

$$D \cdot C^{Bjp}|_{CI} = 1 \ , \tag{1}$$

the entries in the l.h.s of this basic equation are defined as

$$D_A^{NS}(a_s) = \left(N_c \sum_f Q_f^2\right) \cdot D(a_s) \tag{2}$$

$$S_{\rm Bjp}(a_s) = \left(\frac{1}{6}\frac{g_A}{g_V}\right) \cdot C^{Bjp}(a_s) \tag{3}$$

It is known that conformal symmetry is broken by the renormalization of the coupling constant in the renormalized massless quantum field models (for details see e.g. [7]). The latter leads to the non-zero renormalization-group β–function. Moreover, the factor $\beta(a_s)/a_s$, here $a_s = \alpha_s/\pi$, appears as the result of renormalization of the trace of energy-momentum tensor [8–12] and generates the conformal anomaly.

*e-mail: kataev@ms2.inr.ac.ru
†e-mail: mikhs@theor.jinr.ru

The status of a generalisation of the original Crewther relation (CR) [5] to the case of gauge theories with fermions, like QED and QCD, was unclear before the appearance of paper [13]. In this work the colour group structure of massless perturbative predictions for the l.h.s. of Eq.(1) was analysed using the perturbative QCD expressions for the $D-$ and $C^{Bjp}-$ functions evaluated in the $\overline{\rm MS}$-scheme up to $\mathcal{O}(a_s^3)$ corrections. These studies discovered that the r.h.s. of Eq.(1) can be presented at $\mathcal{O}(a_s^3)$ in the following form:

$$D(a_s) \cdot C^{Bjp}(a_s) = 1 + \Delta_{\rm csb}(a_s), \tag{4}$$

where the "Crewther unity" of CR is modified by the conformal–symmetry breaking (CSB) term $\Delta_{\rm csb}$. This term is expressed as the product of the factor $\beta(a_s)/a_s$ and the polynomial $P(a_s)$, namely

$$\Delta_{\rm csb}(a_s) = \left(\frac{\beta(a_s)}{a_s}\right)P(a_s) = \left(\frac{\beta(a_s)}{a_s}\right)\sum_{m\geq 1} K_m a_s^m\,. \tag{5}$$

The polynomial in the r.h.s of Eq.(5) has the form of power expansion in terms of the coupling constant a_s where the first coefficients K_i at $i \leq 3$ depend on quadratic Casimir operators $\rm C_F, C_A$ of the $SU(N_c)$ colour gauge group and on the number of fermion flavours n_f. The first two coefficients were fixed in [13] using the expressions for the NNLO approximations of D and C^{Bjp} coefficient functions and the scheme-independent 2-loop expression of the QCD β–function.

The discovery of the NNLO $\overline{\rm MS}$-scheme QCD generalization of the CR of [13] was the first independent theoretical check of the validity of the $\mathcal{O}(a_s^3)$ results for the C^{Bjp} [14] and D functions, obtained in [15] and later on in [16] using the same calculations machinery. The result of these calculations was then checked in [17] with the help of different theoretical techniques, described in Ref. [17]. Note that the NLO perturbative QCD corrections to D-function were evaluated analytically in [18] and numerically in [19]. The results were confirmed analytically in [20]. In the case of C^{Bjp} similar perturbative corrections were obtained in [21] and confirmed later on in [22] using another calculation method.

To understand better the origin of this form of the QCD generalization of CR, discovered in [13], in [23] the method of the operator-product expansion was applied to the AVV triangle diagram in the momentum space (for some extra details see [24]). The first indications that the factor $\beta(a_s)/a_s$ may be factorized in front of the CSB contribution to Eq. (5) were obtained there in all orders of perturbation theory. Moreover, in [23] the understanding was gained that this generalization of CR will take the form of Eq. (5) with coefficients K_m of the polynomial $P(a_s)$ unfixed from the theory. This property was proved in the coordinate space in [25], [26]. The proof of [26] was published only recently [4]. The subsequent more phenomenological QCD studies of the variant of the $\overline{\rm MS}$-scheme generalization of the CR, discovered in [13], were performed in [27] and [28] within the framework of the certain multiloop extension [29] of the BLM approach [30], and the "restored" CR was formulated.

The calculation of the $O(a_s^4)$–correction to the $C^{Bjp}(a_s)$–function was performed recently in [31]. They confirmed (derived by the back-of-envelope calculations) the ζ_3-containing QED contribution to conformal-invariant approximation for the Bjorken polarized sum rule, performed in [32]. This allowed one to fix the whole expression of the coefficient K_3 in the polynomial $P(a_s)$ in Eq.(5).

The main purpose of this work is to give clear theoretical arguments in favour of our new more detailed form of the generalized CR, proposed by us previously in [33]. In this new form of CR the conformal symmetry breaking term $\Delta_{\rm csb}$ on the r.h.s. of Eq.(4) has the form of double expansion in terms of powers of $(\beta(a_s)/a_s)$ and powers of the coupling constant a_s, namely,

$$\Delta_{\rm csb}(a_s) = \sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n \mathcal{P}_n(a_s) \;=\; \sum_{n\geq 1}\sum_{r\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n^{(r)} a_s^r \tag{6}$$

$$= \sum_{n\geq 1}\sum_{r\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n^{(r)}[\mathrm{k},\mathrm{m}] C_F^{\rm k} C_A^{\rm m} a_s^r, \tag{7}$$

where $\mathrm{k}+\mathrm{m}=r$ and the coefficients $P_n^{(r)}[\mathrm{k},\mathrm{m}]$ contain rational numbers and the terms proportional to odd ζ-functions. On the contrary to the coefficients of $P(a_s)$ in the previous form of Eq.(5), the coefficients of $\mathcal{P}_n(a_s)$ in Eq.(6) do not contain the dependence on n_f.

2. We start our consideration with perturbative expansion of the normalized flavour non-singlet part of the Adler function D from Eq.(2) and normalized C^{Bjp} function from Eq.(3),

$$D = 1 + \sum_{n=1} d_n\, a_s^n\ ; \quad C^{Bjp} = 1 + \sum_{l=1} c_l\, a_s^l\ . \tag{8}$$

The general expressions of the results d_1, d_2, d_3, and c_1, c_2, c_3 with the colour group factors are already known for a rather long period of time. The analytical QCD expression of the 5–loop coefficient d_4 was obtained recently in [34]. This result was extended to the case of general $SU(N_c)$ colour group in [31]. The result for the 5-loop coefficient c_4 for C^{Bjp} can be extracted from the related order a_s^4 expression, obtained in [31], and reads:

$$\begin{aligned}
c_4 \;=\;& \left[-\frac{3}{16}+\frac{1}{4}\zeta_3+\frac{5}{4}\zeta_5\right]\frac{\mathrm{d_F^{abcd}d_A^{abcd}}}{\mathrm{d_R}} + \left[\frac{13}{16}+\zeta_3-\frac{5}{2}\zeta_5\right]\frac{\mathrm{d_F^{abcd}d_F^{abcd}}}{\mathrm{d_R}}\mathrm{n_f} - \left[\frac{4823}{2048}+\frac{3}{8}\zeta_3\right]\mathrm{C_F^4} \\
+& \left[\frac{839}{2304}+\frac{451}{96}\zeta_3-\frac{145}{24}\zeta_5\right]\mathrm{C_F^3T_Fn_f} + \left[-\frac{265}{576}+\frac{29}{24}\zeta_3\right]\mathrm{C_F^2T_F^2n_f^2} + \left[\frac{605}{972}\right]\mathrm{C_FT_F^3n_f^3} \\
+& \left[-\frac{3707}{4608}-\frac{971}{96}\zeta_3+\frac{1045}{48}\zeta_5\right]\mathrm{C_F^3C_A} + \left[-\frac{87403}{13824}-\frac{1289}{144}\zeta_3+\frac{275}{144}\zeta_5+\frac{35}{4}\zeta_7\right]\mathrm{C_F^2C_AT_Fn_f} \\
+& \left[-\frac{165283}{20736}-\frac{43}{144}\zeta_3+\frac{5}{12}\zeta_5-\frac{1}{6}\zeta_3^2\right]\mathrm{C_FC_AT_F^2n_f^2} \\
+& \left[\frac{1071641}{55296}+\frac{1591}{144}\zeta_3-\frac{1375}{144}\zeta_5-\frac{385}{16}\zeta_7\right]\mathrm{C_F^2C_A^2} \\
+& \left[\frac{1238827}{41472}+\frac{59}{64}\zeta_3-\frac{1855}{288}\zeta_5+\frac{11}{12}\zeta_3^2-\frac{35}{16}\zeta_7\right]\mathrm{C_FC_A^2T_Fn_f} \\
+& \left[-\frac{8004277}{248832}+\frac{1069}{576}\zeta_3+\frac{12545}{1152}\zeta_5-\frac{121}{96}\zeta_3^2+\frac{385}{64}\zeta_7\right]\mathrm{C_FC_A^3}\ .
\end{aligned} \tag{9}$$

In the representation of $SU(N_c)$-group one has $\mathrm{C_F} = (N_c^2-1)/(2N_c)$, $\mathrm{C_A} = N_c$, $\mathrm{T_F} = 1/2$, $\mathrm{d_F^{abcd}d_A^{abcd}/d_R = N_c(N_c^2+6)/18}$, $\mathrm{d_F^{abcd}d_F^{abcd}/d_R = (N_c^4-6N_c^2+18)/(36N_c^2)}$, while in the case of QCD $\mathrm{C_F}=4/3$, $\mathrm{C_A}=3$, $\mathrm{d_R}=3$ and $\mathrm{d_F^{abcd}d_A^{abcd}}=15/2$, $\mathrm{d_F^{abcd}d_F^{abcd}}=5/12$ **.**

The $\mathrm{C_F^4}$–term in Eq.(9) coincides with the expression, obtained in Ref. [32] using CR of Eq.(1) and the perturbatively quenched QED 5-loop part of the D-function (which was first presented in Ref. [35]). This agreement provides the first strong check of the validity of the calculations in [34] and gives the positive answer to the question "Is it possible to check urgently the 5-loop analytical results for the e^+e^--annihilation Adler function ?"[1] raised in [32]. The second, even stronger, confirmation of the self-consistency of the results in [34], [31] follows from the explicit demonstration of the validity of the 4-loop $\overline{\rm MS}$-scheme QCD generalization of CR of Ref. [13] (see Eq. (5)) at the 5-loop level [31], in agreement with the general proof in Ref. [25].

[1]For a possible explanation of the appearance of the "puzzling" ζ_3-term in the 5-loop perturbatively quenched QED results see Ref. [36]

The NNLO expression for the polynomial $P(a_s)$ in Eq.(5), can be expressed as [13]:

$$\begin{aligned} K_1 &= K_1[1,0,0]\mathrm{C_F}, & (10)\\ K_2 &= K_2[2,0,0]\mathrm{C_F^2} + K_2[1,1,0]\mathrm{C_F C_A} + K_2[1,0,1]\mathrm{C_F T_F n_f}, & (11) \end{aligned}$$

The 5-loop calculations of [31] express the third term in the polynomial $P(a_s)$ as the sum of six gauge structures of $SU(N_c)$-group, namely

$$\begin{aligned} K_3 &= K_3[3,0,0]\mathrm{C_F^3} + K_3[2,1,0]\mathrm{C_F^2 C_A} + K_3[1,2,0]\mathrm{C_F C_A^2} + K_3[2,0,1]\mathrm{C_F^2 T_F n_f} \\ &+K_3[1,1,1]\mathrm{C_F C_A T_F n_f} + K_3[1,0,2]\mathrm{C_F (T_F n_f)^2}, \end{aligned} \quad (12)$$

The expression for the last coefficient $K_3[1,0,2]$ in (12) coincides with the 5-loop term obtained in [13] in the process of "large $\mathrm{n_f}$" calculations.

The 5-loop approximation of the generalized CR of Refs. [13], [25] of Eqs.(4–5) (see Ref. [23] as well) contains the three-loop approximation of the $\overline{\mathrm{MS}}$-scheme β-function defined as

$$\mu^2 \frac{d}{d\mu^2} a_s = \beta(a_s) = -a_s^2 \left(\beta_0 + \beta_1 a_s + \beta_2 a_s^2\right) \quad (13)$$

where the β_i-terms can be expressed in the following form:

$$\begin{aligned} \beta_0 &= \beta_0[0,1,0]\mathrm{C_A} + \beta_0[0,0,1]\mathrm{T_F n_f}, & (14)\\ \beta_1 &= \beta_1[0,2,0]\mathrm{C_A^2} + \beta_1[0,1,1]\mathrm{C_A T_F n_f} + \beta_1[1,0,1]\mathrm{C_F T_F n_f} & (15)\\ \beta_2 &= \beta_2[0,3,0]\mathrm{C_A^3} + \beta_2[0,2,1]\mathrm{C_A^2 T_F n_f} + \beta_2[1,1,1]\mathrm{C_F C_A T_F n_f} \\ &+\beta_2[0,1,2]\mathrm{C_A T_F^2 n_f^2} + \beta_2[2,0,1]\mathrm{C_F^2 T_F n_f} + \beta_2[1,0,2]\mathrm{C_F T_F^2 n_f^2}\ , & (16) \end{aligned}$$

with the following coefficients

$$\begin{aligned} \beta_0[0,1,0] &= \frac{11}{12}\ ,\ \beta_0[0,0,1] = -\frac{1}{3} & (17)\\ \beta_1[0,2,0] &= \frac{17}{24}\ ,\ \beta_1[0,1,1] = -\frac{5}{12}\ ,\ \beta_1[1,0,1] = -\frac{1}{4} & (18)\\ \beta_2[0,3,0] &= \frac{2857}{3456}\ ,\ \beta_2[0,2,1] = -\frac{1415}{1728}\ ,\ \beta_2[1,1,1] = -\frac{205}{576} & (19)\\ \beta_2[0,1,2] &= \frac{79}{864}\ ,\ \beta_2[2,0,1] = \frac{1}{32}\ ,\ \beta_2[1,0,2] = \frac{11}{144}\ . & (20) \end{aligned}$$

They are known from the three-loop analytical calculations performed in Ref. [37] and confirmed later on [38].

3. The question now arises whether it is possible to obtain uniquely a more detailed generalization of CR from Eq.(6), which reveals factorization of multiple powers of the β-function. In this section, we support our initial guess, made in [33], and give more substantiated arguments in favour of the existence of this "multiple-power" generalization of the CR using the results of the 5-loop approximation [31] for the initial single-power β-function factorizable extension of CR, given in Eqs.(4–5).

The derivation of our more detailed generalization of CR is based on the requirement that the coefficients of polynomials $\mathcal{P}_n$ in Eq.(6) should not depend on the β–function coefficients, and on the number of fermion flavours $\mathrm{n_f}$, in particular. This property may be realized by extending the expression of the conformal symmetry breaking term in Eq.(4) represented in the "single-power" β-function factorizable form in Eq.(5) to the "multiple-power" β–function one in Eq.(6). This was proposed in [33] before the appearance of the results of 5-loop analytical calculations in [31].

To get this new expression, one should equate the r.h.s. of both representations for $\Delta_{\rm csb}$ to each other. At the α_s^3-level the coefficients in the r.h.s. of Eq.(5) are related to the ones in the r.h.s. of Eq.(7) by the following system of linear equations:

$$
\begin{aligned}
K_1[1,0,0] &= P_1^{(1)}[1,0] \\
K_2[2,0,0] &= P_2^{(2)}[2,0] \\
K_2[1,1,0] &= P_1^{(2)}[1,1] - \beta_0[0,1,0]P_2^{(1)}[1,0] \\
K_2[1,0,1] &= P_1^{(1)}[1,0] + \beta_0[0,0,1]P_2^{(1)}[1,0] \\
K_3[3,0,0] &= P_1^{(3)}[3,0] \\
K_3[2,1,0] &= P_1^{(3)}[2,1] - \beta_0[0,1,0]P_2^{(2)}[2,0] \\
K_3[1,2,0] &= P_1^{(3)}[1,2] - \beta_0[0,1,0]P_2^{(2)}[1,1] - \beta_1[0,2,0]P_2^{(1)}[1,0] + (\beta_0[0,1,0])^2P_3^{(1)}[1,0] \\
K_3[2,0,1] &= -\beta_1[1,0,1]P_2^{(1)}[1,0] - \beta_0[0,0,1]P_2^{(2)}[2,0] \\
K_3[1,1,1] &= -\beta_1[0,1,1]P_2^{(1)}[1,0] - \beta_0[0,0,1]P_2^{(2)}[1,1] + 2\beta_0[0,1,0]\beta_0[0,0,1]P_3^{(1)}[1,0] \\
K_3[1,0,2] &= (\beta_0[0,0,1])^2P_3^{(1)}[1,0]
\end{aligned}
\tag{21}
$$

The unique solution of this system determines the explicit expressions of three polynomials $\mathcal{P}_n(a_s)$ in Eqs.(22-25) with flavour independent coefficients $P_n^{(r)}[\mathrm{k},\mathrm{m}]$, namely,

$$
\mathcal{P}_1(a_s) = \Big(-\frac{21}{8}+3\zeta_3\Big)\mathrm{C_F}a_s + \bigg[\Big(\frac{397}{96}+\frac{17}{2}\zeta_3-15\zeta_5\Big)\mathrm{C_F^2} + \Big(-\frac{47}{48}+\zeta_3\Big)\mathrm{C_F C_A}\bigg]a_s^2 \tag{22}
$$

$$
\begin{aligned}
&+\bigg[\Big(\frac{2471}{768}+\frac{61}{8}\zeta_3-\frac{715}{8}\zeta_5+\frac{315}{4}\zeta_7\Big)\mathrm{C_F^3} \\
&+\Big(\frac{16649}{1536}-\frac{11183}{192}\zeta_3+\frac{1015}{24}\zeta_5-\frac{105}{8}\zeta_7+\frac{99}{4}\zeta_3^2\Big)\mathrm{C_F^2C_A} \\
&+\Big(\frac{2107}{192}+\frac{2503}{72}\zeta_3-\frac{355}{18}\zeta_5-33\zeta_3^2\Big)\mathrm{C_F C_A^2}\bigg]a_s^3 + O(a_s^4);
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\mathcal{P}_2(a_s) &= \Big(\frac{163}{8}-19\zeta_3\Big)\mathrm{C_F}a_s + \bigg[\Big(-\frac{13597}{384}-\frac{2523}{16}\zeta_3+\frac{375}{2}\zeta_5+27\zeta_3^2\Big)\mathrm{C_F^2} \\
&+\Big(\frac{1433}{32}-\frac{1}{4}\zeta_3-\frac{170}{4}\zeta_5-6\zeta_3^2\Big)\mathrm{C_F C_A}\bigg]a_s^2 + O(a_s^3);
\end{aligned}
\tag{24}
$$

$$
\mathcal{P}_3(a_s) = \Big(-\frac{307}{2}+\frac{203}{2}\zeta_3+45\zeta_5\Big)\mathrm{C_F}a_s + O(a_s^2). \tag{25}
$$

Note that the 4-loop term β_3 of the $SU(N_c)$ group $\beta(a_s)$-function, analytically evaluated in [39] and confirmed in [40], contains three new group structures $\mathrm{d_A^{abcd}d_A^{abcd}}$, $\mathrm{d_F^{abcd}d_A^{abcd}n_f}$ and $\mathrm{d_F^{abcd}d_F^{abcd}n_f^2}$, which did not appear in lower order expressions of Eqs.(14–16). Due to the validity of "single-power" β–function factorization, Eq.(5), in all orders of perturbation theory (see Ref. [25], [4]), we conclude that the appearance of these extra group terms will not spoil the β–function factorization property in both the "single-power" and the "multiple-power" Eq.(6) expansions. More detailed arguments in favour of this statement will be presented elsewhere.

One more conclusion comes from "large-$\mathrm{n_f}$" calculations (or calculations of the terms proportional to largest powers of β_0), performed in [13]. In fact, the results there contain all leading coefficients in polynomials of the "multiple-power" factorizable expression for the $\Delta_{\rm csb}$ term of Eq.(6)

$$
\mathcal{P}_n(a_s) = \frac{S_n}{4^n}3^{(n-1)}\mathrm{C_F}a_s + O(a_s^2). \tag{26}
$$

The first nine coefficients S_n ($1 \leq n \leq 9$) are fixed explicitly by the results of [13].

4. The expansion (6) can be obtained in a different way with the help of the β-expansion formalism developed in [41]. Within this approach, instead of commonly used expansions in powers of $\mathrm{T_F n_f}$ and the colour group factors, one should consider the expansions of the coefficients d_n and c_n in powers of the β_0, $\beta_1 \ldots$ of the β-function with the coefficients $d_n[n_0, n_1, \ldots]$, $c_n[n_0, n_1, \ldots]$. Their first arguments correspond to n_0 powers of β_0, the second one – n_1 powers of β_1 and so on. The elements $d_n[0, 0, \ldots, 0]$, $c_n[0, 0, \ldots, 0]$ represent "genuine" corrections with powers $n_i = 0$ of all coefficients β_i. The latter elements coincide with expressions for the standard coefficients d_n, c_n in the imaginary limit of nullification of the QCD β–function in all orders of perturbation theory. This limit corresponds to restoration of the conformal symmetry of some quantum field model and will be considered here as the technical trick. If all arguments n_i after index m of the elements $d_n[\ldots, m, 0, \ldots, 0]$, $c_n[\ldots, m, 0, \ldots, 0]$ are equal to zero, then, for the sake of a simplified notation, we will omit these arguments and write instead $d_n[\ldots, m]$. As a result, we obtain the following representation for several coefficients of Eq.(8), namely

$$d_2 = \beta_0 \, d_2[1] + d_2[0] \,, \tag{27}$$

$$d_3 = \beta_0^2 \, d_3[2] + \beta_1 \, d_3[0,1] + \beta_0 \, d_3[1] + d_3[0] \,, \tag{28}$$

$$d_4 = \beta_0^3 \, d_4[3] + \beta_1 \, \beta_0 \, d_4[1,1] + \beta_2 \, d_4[0,0,1] + \beta_0^2 \, d_4[2] + \beta_1 \, d_4[0,1] + \beta_0 \, d_4[1] + d_4[0] \,. \tag{29}$$

The same ordering in the β-function elements may be applied for all higher coefficients d_n and c_l as well. The presentations like Eq.(27–29) are unique. The coefficients $d_n[n-1]$ are identical to the terms generated by the renormalon chain insertions and can be obtained, e.g., from Eq.(26). For others elements it is a separate and not straightforward task. The diagrammatic meaning of the different contributions to the expansion was discussed in [41]. We shall consider later on the way to obtain the results at the level of order a_s^3-corrections.

The expansion (6) together with Eq.(27–29) provides the relation between the unknown yet elements of the 5-loop terms d_4, c_4, and already known elements of the 4-loop results. Indeed, Eq.(1) is satisfied at $\beta = 0$, when all the coefficients have genuine content only, d_n $(c_n) \equiv d_n[0]$ $(c_n[0])$. This provides evident relation between the genuine elements in any loops, namely,

$$c_n[0] + d_n[0] + \sum_{l=1}^{n-1} d_l[0] c_{n-l}[0] = 0. \tag{30}$$

They express the sum of the n–loop elements through the ones resulting from $(n-1)$–loop calculations. In particular, the following expression of the sum of 5-loop coefficients $c_4[0]$, $d_4[0]$ can be obtained:

$$c_4[0] + d_4[0] = 2d_1 d_3[0] - 3d_1^2 d_2[0] + (d_2[0])^2 + d_1^4. \tag{31}$$

This equation contains contributions proportional to $\mathrm{C_F}$ and $\mathrm{C_A}$. Note that the projection of the relation (31) onto the maximum power of $\mathrm{C_F}$, $\mathrm{C_F^4}$, was suggested in [32] to check the QED results for d_4, available from [35].

The β–expansion of d_3 was obtained in [41] on the basis of the result for the Adler function $D(a_s, n_f, n_{\tilde{g}})$ with the $n_{\tilde{g}}$ MSSM gluino multiplets in [17]. The 3-loop contribution of light gluinos coincide with the numerical result of Ref. [42], while at the 4-loop the analytical result for gluino contribution, evaluated in Ref. [17], was confirmed in Ref. [43]. The element $d_3[2]$, which is proportional to the maximum power β_0^2 in (28), can be obtained in a straightforward way. Then, one should separate in d_3 the contributions from the terms $\beta_1 \, d_3[0,1]$ and $\beta_0 \, d_3[1]$. They both are linear in the number of quark flavours n_f. Their separation is possible if one uses additional degrees of freedom – the gluino contributions mentioned above and labelled here by their $n_{\tilde{g}}$ multiplet number.[2]. In this way one can obtain expressions for the functions

[2] Note that the possibility that gluino with $m_{\tilde{g}} \geq 195$ GeV is lighter than the MSSM scalar quark is not excluded by the existing Tevatron limits [44]

$n_f = n_f(\beta_0, \beta_1)$ and $n_{\tilde{g}} = n_{\tilde{g}}(\beta_0, \beta_1)$. These expressions can be obtained also after taking into account gluino contributions to the first two coefficients of QCD β-functions known from the two-loop calculations performed in [45]. Finally, one arrives at the evident expressions for the coefficients in Eqs.(27–28),

$$d_1 = \frac{3}{4}\mathrm{C_F};\ d_2[1] = \left(\frac{33}{8} - 3\zeta_3\right)\mathrm{C_F};\ d_2[0] = -\frac{3}{32}\mathrm{C_F^2} + \frac{1}{16}\mathrm{C_F C_A}; \quad (32)$$

$$d_3[2] = \left(\frac{151}{6} - 19\zeta_3\right)\mathrm{C_F}; d_3[1] = \left(-\frac{27}{8} - \frac{39}{4}\zeta_3 + 15\zeta_5\right)\mathrm{C_F^2} - \left(\frac{9}{64} - 5\zeta_3 + \frac{5}{2}\zeta_5\right)\mathrm{C_F C_A}; \quad (33)$$

$$d_3[0,1] = \left(\frac{101}{16} - 6\zeta_3\right)\mathrm{C_F};\ d_3[0] = -\frac{69}{128}\mathrm{C_F^3} + \frac{71}{64}\mathrm{C_F^2 C_A} + \left(\frac{523}{768} - \frac{27}{8}\zeta_3\right)\mathrm{C_F C_A^2}\ . \quad (34)$$

The $c_3[\ldots]$ elements there can be obtained in the same way as the $d_3[\ldots]$ ones, taking into account the relation like Eq.(31) for the $c_3[0]$ and the known $d_3[0]$,

$$\begin{aligned} c_1 &= -\frac{3}{4}\mathrm{C_F};\ c_2[1] = -\frac{3}{2}\mathrm{C_F};\ c_2[0] = \frac{21}{32}\mathrm{C_F^2} - \frac{1}{16}\mathrm{C_F C_A}; && (35)\\ c_3[2] &= -\frac{115}{24}\mathrm{C_F};\ c_3[1] = \left(\frac{83}{24} - \zeta_3\right)\mathrm{C_F^2} + \left(\frac{215}{192} - 6\zeta_3 + \frac{5}{2}\zeta_5\right)\mathrm{C_F C_A}; && (36)\\ c_3[0,1] &= \left(-\frac{59}{16} + 3\zeta_3\right)\mathrm{C_F};\ c_3[0] = -\frac{3}{128}\mathrm{C_F^3} - \frac{65}{64}\mathrm{C_F^2 C_A} - \left(\frac{523}{768} - \frac{27}{8}\zeta_3\right)\mathrm{C_F C_A^2}. && (37)\end{aligned}$$

Note that the expansions similar to those of Eqs.(27–28), where only the terms proportional to powers of β_0 were taken into account both in the case of the Adler function and Bjorken polarized sum rule, were proposed and analyzed in Ref. [46].

Substituting now the expansions in Eqs.(27-29) and similar ones for c_i into the general relations of Eq.(6) one arrives at the following expressions:

$$\begin{aligned} \mathcal{P}_1(a_s) &= -a_s\mathrm{C_F}\Big\{P_1^{(1)} + a_s P_1^{(2)} + a_s^2 P_1^{(3)}\Big\}\\ &= -a_s\Big\{c_2[1] + d_2[1] + a_s\Big(c_3[1] + d_3[1] + d_1\big(c_2[1] - d_2[1]\big)\Big)\\ &\quad + a_s^2\Big(c_4[1] + d_4[1] + d_1\big(c_3[1] - d_3[1]\big) + d_2[0]c_2[1] + d_2[1]c_2[0]\Big)\Big\} && (38)\\ \mathcal{P}_2(a_s) &= a_s\mathrm{C_F}\{P_2^{(1)} + a_s P_2^{(2)}\}\\ &= a_s\Big\{c_3[2] + d_3[2] + a_s\Big(c_4[2] + d_4[2] - d_1(c_3[2] - d_3[2])\Big)\Big\} && (39)\\ \mathcal{P}_3(a_s) &= a_s\mathrm{C_F}P_3^{(1)} = -a_s\{c_4[3] + d_4[3]\} = a_s\mathrm{C_F}\left(\frac{307}{2} - \frac{203}{2}\zeta_3 - 45\zeta_5\right) && (40)\\ & a_s\mathrm{C_F}P_n^{(1)} = (-1)^n a_s\{c_n[n-1] + d_n[n-1]\} && (41)\end{aligned}$$

Let us stress that though the numerical content of Eqs.(38–40) coincides with Eqs.(22–25), it is expressed in the above notation. The elements $d_n[n-1]$ ($c_n[n-1]$) are partially formed by the leading renormalon chain insertions and they can be obtained from [13], while the elements $d_n[l], (l < n-1)$ stem from the subleading renormalon chains. Using the expression of Eq.(6), checked by us at the 5-loop level, different relations between the elements d_4 (d_n) and c_4 (c_n) can be obtained. Indeed, the term $\mathcal{P}_1(a_s)$ in (6) generates the following chain of equations:

$$\begin{aligned} P_1^{(1)} &= -c_2[1] - d_2[1] = -c_3[0,1] - d_3[0,1] = -c_4[0,0,1] - d_4[0,0,1] = \ldots\\ &= -c_n[\underbrace{0,0,\ldots,1}_{n-1}] - d_n[\underbrace{0,0,\ldots,1}_{n-1}] = \mathrm{C_F}\left(-\frac{21}{8} + \zeta_3\right) && (42)\end{aligned}$$

that fixes the universal first term $P_1^{(1)}$ in the polynomial $\mathcal{P}_1$. The second term of $\mathcal{P}_1$ in Eq.(38) defines a similar chain of equations

$$P_1^{(2)} = c_3[1] + d_3[1] + d_1\left(c_2[1] - d_2[1]\right) = c_4[0,1] + d_4[0,1] + d_1\left(c_3[0,1] - d_3[0,1]\right) = \ldots \quad (43)$$

where the explicit expression for $P_1^{(2)}$ is already known, see, e.g., the r.h.s. of Eq.(22).

However, in order to verify similar expressions for $P_1^{(3)}$ and $P_2^{(2)}$ terms independently, it is necessary to obtain the coefficients of β-expansion for the 5-loop contributions to the D and C^{Bjp} functions. This may be done after analytical evaluation of the gluino contributions to 5-loop perturbative coefficients for these important quantities and taking into account the 3-loop gluino effects in the QCD β-function, already calculated in Ref. [47].

The relations obtained previously allow us to derive a new constraint for 5-loop results for d_4+c_4. To get it, we fix the number of fermions, $\mathrm{n_f}$, by the Banks–Zaks ansatz : $\beta_0(\mathrm{n_f} = n_0) = 0$ [48] which is equivalent to the following condition $\mathrm{T_F} n_0 = 11/4\mathrm{C_A}$. As the result, we get

$$c_4(n_0)+d_4(n_0) = c_4[0]+d_4[0]-\beta_2(n_0)\left(c_4[0,0,1] + d_4[0,0,1]\right)-\beta_1(n_0)\left(c_4[0,1] + d_4[0,1]\right). \quad (44)$$

The terms in the r.h.s. of Eq.(44) are already known from the r.h.s. of Eq.(31), Eq.(42) $(-c_4[0,0,1] - d_4[0,0,1])$ and Eq.(43) $(-c_4[0,1] - d_4[0,1])$ correspondingly. Thus we get

$$\begin{aligned} \mathrm{d}_4(n_0) + \mathrm{c}_4(n_0) \quad = \quad & -\frac{333}{1024}\mathrm{C_F^4} + \mathrm{C_A}\mathrm{C_F^3}\left(-\frac{1661}{3072} + \frac{1309}{128}\zeta_3 - \frac{165}{16}\zeta_5\right) \\ & +\mathrm{C_A^2}\mathrm{C_F^2}\left(-\frac{3337}{1536} + \frac{7}{2}\zeta_3 - \frac{105}{16}\zeta_5\right) + \mathrm{C_A^3}\mathrm{C_F}\left(-\frac{28931}{12288} + \frac{1351}{512}\zeta_3\right). \quad (45) \end{aligned}$$

Then, applying the condition $\mathrm{T_F} n_0 = 11/4\mathrm{C_A}$ to the concrete analytical expression for $c_4(n_0)+d_4(n_0)$, which follows from the result of Ref. [31], we reproduce the r.h.s. of Eq.(45).

Thus, the application of the Bankz-Zaks ansatz together with the β-expansion approach of [41] give the extra argument in favour of the correctness of the results of distinguished analytical calculations of the INR-Karlsruhe-SINP group [31]. Moreover, having a look at the r.h.s. of Eq.(45) we observe the absence of the ζ_7 and ζ_3^2-terms, which exist in analytical expressions of both d_4 and c_4 (see Ref. [31] and Eq.(9). This nullification confirms the observation, made in Ref. [31], on the proportionality of these transcendences to the first coefficient β_0 of the QCD β-function.

Acknowledgements. This work comes from the talk of one of us (ALK), presented at the Quarks–2010 International Seminar (6–12 June 2010, Kolomna), and at the HSQCD–2010 Workshop (5–9 July 2010, Gatchina). We are grateful to K. G. Chetyrkin for productive discussions and A. G. Grozin for the fruitful discussion of the consequences of our results in higher orders. We also wish to thank D. I. Kazakov and O. V. Teryaev for their interest in the results of our work. The work of both of us was supported by the Russian Foundation for Fundamental Research (grant No. 08-01-00686). The work of ALK is also supported in part by the Ministry of Science and Education, State Contract 02.740.11.0244.

References

[1] V. A. Matveev, R. M. Muradyan and A. N. Tavkhelidze, Fiz. Elem. Chast. Atom. Yadra **2** (1971) 5.

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323** (2000) 183.

[3] M. Baker and K. Johnson, Physica A **98** (1979) 120.

[4] V. M. Braun, G. P. Korchemsky and D. Mueller, Prog. Part. Nucl. Phys. **51** (2003) 311, Sect. 5.

[5] R. J. Crewther, Phys. Rev. Lett. **28** (1972) 1421.

[6] S. L. Adler, C. G. Callan, D. J. Gross and R. Jackiw, Phys. Rev. D **6** (1972) 2982.

[7] N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, IV edition (1984), v 10 in N. N. Bogoliubov, Collection of Scientific Works in 12 volumes, Moscow, Nauka, 2008, ed.A.D. Sukhanov.

[8] M. S. Chanowitz and J. R. Ellis, Phys. Lett. B **40** (1972) 397.

[9] N. K. Nielsen, Nucl. Phys. B **120** (1977) 212.

[10] S. L. Adler, J. C. Collins and A. Duncan, Phys. Rev. D **15** (1977) 1712.

[11] J. C. Collins, A. Duncan and S. D. Joglekar, Phys. Rev. D **16** (1977) 438.

[12] P. Minkowski, Berne Print-76-0813 (1976)

[13] D. J. Broadhurst and A. L. Kataev, Phys. Lett. B **315** (1993) 179.

[14] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **259** (1991) 345.

[15] S. G. Gorishny, A. L. Kataev and S. A. Larin, Phys. Lett. B **259** (1991) 144.

[16] L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66** (1991) 560 [Erratum-ibid. **66** (1991) 2416].

[17] K. G. Chetyrkin, Phys. Lett. B **391** (1997) 402.

[18] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, Phys. Lett. B **85** (1979) 277.

[19] M. Dine and J. R. Sapirstein, Phys. Rev. Lett. **43** (1979) 668.

[20] W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. **44** (1980) 560.

[21] S. G. Gorishny and S. A. Larin, Phys. Lett. B **172** (1986) 109.

[22] E. B. Zijlstra and W. L. van Neerven, Phys. Lett. B **297** (1992) 377.

[23] G. T. Gabadadze and A. L. Kataev, JETP Lett. **61** (1995) 448 [Pisma Zh. Eksp. Teor. Fiz. **61** (1995) 439].

[24] A. L. Kataev, "The generalized Crewther relation: The peculiar aspects of the analytical perturbative QCD calculations," Preprint INR-0926-96, arXiv:hep-ph/9607426, in Proceedings of the 2nd Workshop on Continuous Advances in QCD, Minneapolis, MN, 28-31 March, 1996, World Scientific, pp.107-132, Ed.M.I.Polikarpov

[25] R. J. Crewther, Phys. Lett. B **397** (1997) 137.

[26] D. Muller (1996) — private communication to A.L.K. (unpublished)

[27] S. J. Brodsky, G. T. Gabadadze, A. L. Kataev and H. J. Lu, Phys. Lett. B **372** (1996) 133.

[28] J. Rathsman, Phys. Rev. D **54** (1996) 3420.

[29] G. Grunberg and A. L. Kataev, Phys. Lett. B **279** (1992) 352.

[30] S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D **28** (1983) 228.

[31] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, Phys. Rev. Lett. **104** (2010) 132004.

[32] A. L. Kataev, Phys. Lett. B **668** (2008) 350.

[33] A. L. Kataev and S. V. Mikhailov, PoS **RADCOR2009** (2010) 036 [arXiv:1001.0728].

[34] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, Phys. Rev. Lett. **101** (2008) 012002.

[35] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, PoS **RADCOR2007** (2007) 023 [arXiv:0810.4048 [hep-ph]].

[36] A. L. Kataev, Phys. Lett. B **691** (2010) 82.

[37] O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B **93** (1980) 429.

[38] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **303** (1993) 334.

[39] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B **400** (1997) 379.

[40] M. Czakon, Nucl. Phys. B **710** (2005) 485.

[41] S. V. Mikhailov, JHEP **0706** (2007) 009 [arXiv:hep-ph/0411397]; "Any order generalization of BLM procedure in QCD", in *Proceedings of the 13th International Seminar Quarks'2004, Vol. 2, Pushkinogorie, Russia, May 24–30, 2004*, edited by D. G. Levkov, V. A. Matveev, and V. A. Rubakov (INR RAS, Moscow, 2005), pp. 536–550; [arXiv:hep-ph/0410134]

[42] A. L. Kataev and A. A. Pivovarov, JETP Lett. **38** (1983) 369 [Pisma Zh. Eksp. Teor. Fiz. **38** (1983) 309].

[43] L. J. Clavelli and L. R. Surguladze, Phys. Rev. Lett. **78** (1997) 1632.

[44] D. I. Kazakov, Nucl. Phys. Proc. Suppl. **203-204** (2010) 118.

[45] M. E. Machacek and M. T. Vaughn, Nucl. Phys. B **236** (1984) 221.

[46] C. N. Lovett-Turner and C. J. Maxwell, Nucl. Phys. B **452** (1995) 188.

[47] L. Clavelli, P. W. Coulter and L. R. Surguladze, Phys. Rev. D **55** (1997) 4268.

[48] T. Banks and A. Zaks, Nucl. Phys. B **196** (1982) 189.