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Differential K-theory. A survey.

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Abstract

Generalized differential cohomology theories, in particular differential K-theory (often called "smooth K-theory"), are becoming an important tool in differential geometry and in mathematical physics.

In this survey, we describe the developments of the recent decades in this area. In particular, we discuss axiomatic characterizations of differential K-theory (and that these uniquely characterize differential K-theory). We describe several explicit constructions, based on vector bundles, on families of differential operators, or using homotopy theory and classifying spaces. We explain the most important properties, in particular about the multiplicative structure and push-forward maps and will state versions of the Riemann-Roch theorem and of Atiyah-Singer family index theorem for differential K-theory.

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1 Introduction

The most classical differential cohomology theory is ordinary differential cohomology with integer coefficients. It has various realizations, e.g. as smooth Deligne cohomology (compare [18]) or as Cheeger-Simons differential characters [32]. In the last decade, differential extensions of generalized cohomology theories, in particular of K-theory, have been studied intensively. In part, this is motivated by its application in mathematical physics, for the description of fields with quantization anomalies in abelian gauge theories, suggested by Freed in [35], compare also [43].

The basic idea is that a differential cohomology theory should combine cohomological information with differential form information. More precisely, given a generalized cohomology theory E with together with a natural transformation $\mathrm{ch}\colon E(X) \to H(X; N)$, to cohomology with coefficients in a graded real vector space N, and using an appropriate setup one can define the differential refinement $\hat{E}$ of E as a *homotopy* pullback

$$\begin{array}{ccc} \hat{E}(X) & \xrightarrow{I} & E(X) \\ \big\downarrow{\scriptstyle R} & & \big\downarrow{\scriptstyle \mathrm{ch}} \\ \Omega_{d=0}(X;N) & \xrightarrow{\mathrm{Rham}} & H(X;N). \end{array}$$

The natural transformations I (the underlying cohomology class) and R (the characteristic closed differential form) are essential parts of the picture. With slight abuse of notation, we call R the *curvature* homomorphism. This is a bit of a misnomer, as in a geometric situation R will be determined by the honest curvature, but not vice versa. $\hat{E}$ is *not* a generalized cohomology theory and not meant to be one: it contains differential form information and as a consequence is not homotopy invariant.

If E is ordinary integral cohomology, ch is just induced by the inclusion of coefficients $\mathbb{Z} \to \mathbb{R}$. For K-theory, the situation we are mainly discussing in this article, ch is the ordinary Chern character.

The *flat part* $\hat{E}_{\mathrm{flat}}(X)$ of $\hat{E}(X)$ is defined as the kernel of the curvature morphism:

$$\hat{E}_{\mathrm{flat}}(X) := \ker\Big(R\colon \hat{E}(X) \to \Omega(X;N)\Big) \ .$$

It turns out that $\hat{E}_{\mathrm{flat}}(X)$ is a cohomology theory, usually just $E\mathbb{R}/\mathbb{Z}[-1]$, the generalized cohomology with $\mathbb{R}/\mathbb{Z}$-coefficients with a degree shift: $\hat{E}^k_{\mathrm{flat}}(X) = E\mathbb{R}/\mathbb{Z}^{k-1}(X)$. An original interest in differential K-theory (before it even was introduced as such) was its role as a geometric model for $K\mathbb{R}/\mathbb{Z}$. Karoubi in [45, Section 7.5] defined $K^{-1}\mathbb{C}/\mathbb{Z}$ using essentially the flat part of a cycle model for $\hat{K}^0$, compare also Lott [51, Defintion 5, Definition 7] where also $K\mathbb{R}/\mathbb{Z}^{-1}$ is introduced. Homotopy theory provides a universal construction of $E\mathbb{R}/\mathbb{Z}$ for a generalized cohomology theory E. However, this is in general hard to combine with geometry.

K-theory is the home for index theory. The differential K-theory (in particular its different cycle models) and also its flat part naturally are the home for index problems, taking more of the geometry into account. Indeed, in suitable models it is built into the definitions that geometric families of Dirac operators, parametrized by X, give rise to classes in $\hat{K}^*(X)$, where $*$ is the parity of the dimension of the fiber. A submersion $p\colon X \to Z$ with closed fibers with fiberwise geometric spin$^{\mathrm{c}}$-structure (the precise meaning of geometry will be discussed below) is oriented for differential K-theory and one has an associated push-forward

$$\hat{p}_!\colon \hat{K}^*(X) \to \hat{K}^{*-d}(Y)$$

(with $d = (\dim(X) - \dim(Z))$). The same data also gives rise to a push-forward in Deligne cohomology

$$\hat{p}_! \colon \hat{H}^*(X) \to \hat{H}^{*-d}(Y) \ .$$

There is a unique lift of the Chern character to a natural transformation

$$\hat{\mathrm{ch}} \colon \hat{K}^*(X) \to \hat{H}^{*+2\mathbb{Z}}_{\mathbb{Q}}(X) \ .$$

Here, the right hand side is the differential extension of $H^*(X;\mathbb{Q})$ and one of the main results of [26] is a refinement of the classical Riemann-Roch theorem to a differential Riemann-Roch theorem which identifies the correction for the compatibility of $\hat{\mathrm{ch}}$ with the push-forwards. In [38], Bismut superconnection techniques are used to define the analytic index of a geometric family: it can be understood as a particular representative of the differential K-theory class of a geometric family as above, determined by the analytic solution of the index problem. Moreover, they develop a geometric refinement of the topological index construction of Atiyah-Singer [3] based on geometrically controlled embeddings into Euclidean space (which does not require deep spectral analysis) and prove that topological and analytic index in differential K-theory coincide.

Finally, we observe that, in suitable special situations, we can easily construct classes in differential K-theory which turn out not to depend on the special geometry, but only on the underlying differential-topological data. Typically, these live in the flat part of differential K-theory and are certain (generalizations of) secondary index invariants. Examples are rho-invariants of the Dirac operator twisted with two flat vector bundles (and family versions hereof), or the $\mathbb{Z}/k\mathbb{Z}$-index of Lott [51] for a manifold W whose boundary is identified with the disjoint union of k copies of a given manifold M.

Similar to smooth Deligne cohomology, there is a counterpart of differential K-theory in the holomorphic setting [40] and there is an arithmetic Riemann-Roch for these groups. This, however, will not be discussed in this survey.

1.1 Differential cohomology and physics

A motivation for the introduction of differential K-theory comes from quantum physics. The fields of abelian gauge theories are described by objects which carry the local field strength information of a closed differential form (assuming that there are no sources). Dirac quantization, however, requires that their de Rham classes lie in an integral lattice in de Rham cohomology. For Maxwell theory the field strength simply is a 2-form which is the curvature of a complex line bundle and therefore lies in the image of ordinary integral cohomology. For Ramond-Ramond fields in type II string theories it is a differential form of higher degree which lies in the image under the Chern character of K-theory, as suggested by [54, 36]. Indeed, Witten suggests that D-brane charges in the low energy limit of type IIA/B superstring theory are classified by K-theory. In this case, even if the field strength differential form is zero, the fields or D-brane charges can contain some *global* information, corresponding to torsion in K-theory.

It is suggested by Freed [35] that these Ramond-Ramond fields are described by classes in *differential* K-theory (or other generalized differential cohomology theories, depending on the particular physical model). Given a space-time background X and a field represented by a class $F \in \hat{E}^*(X)$, this field contains the differential form information $R(F)$ (as expected for an abelian gauge field). The field equations (generalizing Maxwell's equations) require that $dR(F) = 0$ if there are no sources (which we assume here). However, there is a quantization condition: the de Rham class represented by $R(F)$ is not arbitrary, but lies in an integral lattice, namely in $\mathrm{im}(ch)$. Indeed, F also contains the integral (and possibly torsion) information of the class $I(F) \in E^*(X)$. Finally, even $I(F)$ and $R(F)$ together don't determine F entirely, there is extra information, corresponding to a physically significant potential or holonomy. More precisely, $\hat{E}^*(X)$ is the configuration space with a gauge group action. Details of such a gauge field theory are studied e.g. in [47], where it is shown that the free part of $E^*(X)$ is an obstruction to a global gauge fixing. Nonetheless, [47] proposes a partition function and among others computes the vacuum expectation value.

All discussed so far describes the situation without any background field or flux. However, such background fields are an important ingredient of the theory. Depending on the chosen model and the precise situation, a background field can be defined in many different ways. In the classical situation where fields are just given by differential forms, a background field is a closed 3-form Ω. It creates an extra term in the field equations. Correspondingly, the relevant charges are even or odd forms (depending on the type of the theory) which are closed for the differential d^Ω with $d^\Omega \omega := d\omega + \omega \wedge \Omega$. And they are classified up to equivalence by the Ω-*twisted de Rham cohomology*

$$H^{*+\Omega}_{dR}(X) := \ker(d^\Omega)/\operatorname{im}(d^\Omega).$$

When looking at charges in the presence of a background B-field (producing an H-flux) which are classified topologically by K-theory, we need to work with *twisted K-theory*, compare in particular [64, 17]. We will give a short introduction to twisted K-theory in Section 7.1. The role of twisted K-theory is discussed a lot in the case of T-duality. T-duality predicts an isomorphism of string theories on different background manifolds which are T-dual to each other, and in particular an isomorphism of the K-theory groups which classify the D-brane charges.

It turns out, however, that the topology of one of the partners in duality dictates a background B-field on the other, and the required isomorphism can only hold in twisted K-theory, compare e.g. [16, 23]. In those papers, mainly the *topological* classification of D-brane charges is considered. A new picture now arises when one wants to move to T-duality for Ramond-Ramond fields described by *differential K-theory* as explained above. One has to construct and study *twisted* differential K-theory. A first step toward this is carried out in [31]. Now, physicists try to understand T-duality at the level of Ramond-Ramond fields, compare e.g. [9] where the ideas are discussed explicitly without mathematical rigor. With mathematical rigor, the T-duality isomorphism in (twisted) differential K-theory has been worked out by Kahle and Valentino in [44]. We will describe these results in more detail in Section 7.4.

2 Axioms for differential cohomology

A fruitful approach to generalized cohomology theories is based on the Eilenberg-Steenrod axioms. It turns out that many of the basic properties of smooth

Deligne cohomology and differential K-theory also are captured by a rather small set of axioms, proposed in [26, Section 1.2.2] (and motivated by [35]). Therefore, we want to base our treatment of differential K-theory on those axioms, as well.

The starting point is a generalized cohomology theory E, together with a natural transformation $\mathrm{ch}\colon E(X) \to H(X;N)$, where N is a graded coefficient $\mathbb{R}$-vector space. The two basic examples are

- $E(X) = H(X;\mathbb{Z})$, ordinary cohomology with integer coefficients, where $N = \mathbb{R}$ and ch is induced by the inclusion of coefficients $\mathbb{Z} \to \mathbb{R}$. More generally, $\mathbb{Z}$ can be replaced by any subring of $\mathbb{R}$, e.g. by $\mathbb{Q}$.
- $E(X) = K(X)$, K-theory, where ch is the usual Chern character, and $N = \mathbb{R}[u, u^{-1}]$ with u of degree 2. Multiplication with u corresponds to Bott periodicity.

2.1 Definition. A differential extension of the pair (E, ch) is a functor $X \to \hat{E}(X)$ from the category of compact smooth manifolds (possibly with boundary) to $\mathbb{Z}$-graded groups together with natural transformations

(1) $R\colon \hat{E}(X) \to \Omega_{d=0}(E;N)$ (curvature)

(2) $I\colon \hat{E}(X) \to E(X)$ (underlying cohomology class)

(3) $a\colon \Omega(X;N)/\operatorname{im}(d) \to \hat{E}(X)$ (action of forms) .

Here $\Omega(E;N) := \Omega(E) \otimes_{\mathbb{R}} N$[1] denote the smooth differential forms with values in N, $d\colon \Omega(E;N) \to \Omega(E;N)[1]$ the usual de Rham differential and $\Omega_{d=0}(E;N)$ the space of closed differential forms.

The transformations I, a, R are required to satisfy the following axioms:

(1) The following diagram commutes

$$\begin{array}{ccc} \hat{E}(X) & \xrightarrow{I} & E(X) \\ \downarrow R & & \downarrow \mathrm{ch} \\ \Omega_{d=0}(X,N) & \xrightarrow{\mathrm{Rham}} & H(X;N) . \end{array} \tag{2.2}$$

(2)
$$R \circ a = d . \tag{2.3}$$

(3) a is of degree 1.

(4) The following sequence is exact:

$$E^{*-1}(X) \xrightarrow{ch} \Omega^{*-1}(X,N)/\operatorname{im}(d) \xrightarrow{a} \hat{E}^{*}(X) \xrightarrow{I} E^{*}(X) \to 0 . \tag{2.4}$$

Alternatively, when dealing with K-theory one can and often (e.g. in [26]) does consider the whole theory as $\mathbb{Z}/2\mathbb{Z}$-graded with the obvious adjustments. Note that with $N = K^*(pt) \otimes \mathbb{R} = \mathbb{R}[u, u^{-1}]$ we have natural and canonical isomorphism $\Omega^*(X;N) = \oplus_{k\in\mathbb{Z}} \Omega^{*+2k}(X)$ and $H^*(X;N) = \oplus_{k\in\mathbb{Z}} H^{*+2k}(X;\mathbb{R})$. The associated $\mathbb{Z}/2\mathbb{Z}$-graded ordinary cohomology is therefore given by the direct sum of even or odd degree forms.

[1] This definition has to be modified in a generalization to non-compact manifolds!

2.5 Corollary. *If $\hat{E}$ is a differential extension of (E, ch), then we have a second exact sequence*

$$E^{*-1}(X) \xrightarrow{\mathrm{ch}} H^{*-1}(X;N) \xrightarrow{a} \hat{E}^*(X) \xrightarrow{R} \Omega^*_{d=0}(X;N) \times_{\mathrm{ch}} E^*(X) \to 0, \quad (2.6)$$

where $\Omega_{d=0}(X;N) \times_{\mathrm{ch}} E(X) = \{(\omega, x) \mid \mathrm{Rham}(\omega) = \mathrm{ch}(x)\}$ is the pullback of abelian groups.

Proof. This is a direct consequence of (2.4), (2.2) and (2.3). □

2.7 Definition. Given a differential extension $\hat{E}$ of a cohomology theory (E, ch), we define the associated *flat functor*

$$\hat{E}_{flat}(X) := \ker(R\colon \hat{E}(X) \to \Omega_{d=0}(X;N)).$$

2.8 Remark. The naturality of R indeed implies that $X \mapsto \hat{E}_{flat}(X)$ is a contravariant functor on the category of smooth manifolds. Actually, this functor by Corollary 2.11 is always homotopy invariant and extends to a cohomology theory in many examples, as we will discuss in Section 2.3. Typically, there is a natural isomorphism $\hat{E}^*_{flat}(X) \cong E\mathbb{R}/\mathbb{Z}^{*-1}(X)$, but we still don't know whether this is neccessarily always the case (compare the discussion in [27, Section 5 and Section 7]).

The most interesting cases are not just group valued cohomology functors, but multiplicative cohomology theories, for example K-theory and ordinary cohomology. We therefore want typically a differential extension which carries a compatible product structure.

2.9 Definition. Assume that E is a multiplicative cohomology theory, that N is a $\mathbb{Z}$-graded algebra over $\mathbb{R}$, and that ch is compatible with the ring structures. A differential extension $\hat{E}$ of (E, ch) is called multiplicative if $\hat{E}$ together with the transformations R, I, a is a differential extension of (E, ch), and in addition

(1) $\hat{E}$ is a functor to $\mathbb{Z}$-graded rings,

(2) R and I are multiplicative,

(3) $a(\omega) \cup x = a(\omega \wedge R(x))$ for all $x \in \hat{E}(B)$ and $\omega \in \Omega(B;N)/\mathrm{im}(d)$.

Deligne cohomology is multiplicative [18, Chapter 1], [34, Section 6], [21, Section 4]. In this paper, we will consider multiplicative extensions $\hat{K}$ of K-theory.

2.1 Variations of the axiomatic approach

Our list of axioms for differential cohomology theories seems particularly natural: it allows for efficient constructions and to derive the conclusions we are interested in. However, for the differential refinements of integral cohomology, a slightly different system of axioms has been proposed in [59]. The main point there is that the requirement of a given natural isomorphism $\hat{E}^*_{flat}(X) \to E\mathbb{R}/\mathbb{Z}^{*-1}(X)$ between the flat part of Definition 2.7 and E with coefficients in $\mathbb{R}/\mathbb{Z}$. It turns out that, for differential extensions of ordinary cohomology, both sets of axioms imply that there is a unique natural isomorphism to Deligne cohomology (compare [59] and [27, Section 7]). In particular, for ordinary cohomology they are equivalent. The corresponding result holds in general under extra assumptions, which are satisfied for K-theory, compare Section 2.3.

2.2 Homotopy formula

A simple, but important consequence of the axioms is the homotopy formula. If one differential cohomology class can be deformed to another, this formula allows to compute the difference of the two classes entirely in terms of differential form information. In a typical application, one will deform an unknown class to one which is better understood and that way get one's hands on the complicated class one started with.

2.10 Theorem (Homotopy formula)**.** *Let $\hat{E}$ be a differential extension of (E, ch). If $x \in \hat{E}([0,1] \times B)$ and $i_k\colon B \to [0,1] \times B; b \mapsto (k, b)$ are the inclusions then*

$$i_1^* x - i_0^* x = a\left(\int_{[0,1]\times B/B} R(x)\right),$$

where $\int_{[0,1]\times B/B}$ denotes integration of differential forms over the fiber of the projection $p\colon [0,1] \times B/B$ with the canonical orientation of the fiber $[0,1]$.

Proof. Note that, if $x = p^* y$ for some $y \in \hat{E}(B)$ then by naturality the left hand side of the equation is zero. Moreover, in this case $R(x) = p^* R(x)$ so that by the properties of integration over the fiber the right hand side vanishes, as well.

In general, observe that p is a homotopy equivalence, so that we always find $\bar{y} \in E(B)$ with $I(x) = p^* \bar{y}$. Using surjectivity of I, we find $y \in \hat{E}(B)$ with $I(y) = \bar{y}$, and then $I(p^* y - x) = p^* \bar{y} - \bar{y} = 0$. Since the sequence (2.4) is exact, there is $\omega \in \Omega(B; N)$ with $a(\omega) = x - p^* y$. Stokes' theorem applied to ω yields

$$i_1^* \omega - i_0^* \omega = \int_{[0,1]\times B/B} d\omega.$$

On the other hand, because of (2.3), $d\omega = R(a(\omega))$. Substituting, we get

$$\int_{[0,1]\times B/B} d\omega = \int_{[0,1]\times B/B} R(a(\omega)) = \int_{[0,1]\times B/B} R(x - p^* y) = \int_{[0,1]\times B/B} R(x)$$

and (using again vanishing of our expressions for $p^* y$)

$$i_1^* x - i_0^* = i_1^* a(\omega) = i_0^* a(\omega) = a(i_1^* \omega - i_0^* \omega) = a\left(\int_{[0,1]\times B/B} R(x)\right).$$

□

2.11 Corollary. *Given a differential extension $\hat{E}$ of (E, ch), the associated flat functor $\hat{E}_{flat}$ of Definition 2.7 is homotopy invariant.*

Proof. Let $H\colon [0,1] \times X \to Y$ be a homotopy between $f = H_0$ and $g = H_1$. We have to show that $f^* = g^*\colon \hat{E}_{flat}(Y) \to \hat{E}_{flat}(X)$. By functoriality, it suffices to show that $i_0^* = i_1^*\colon \hat{E}_{flat}([0,1] \times X) \to \hat{E}_{flat}(X)$, as $f^* = i_0^* \circ H^*$ and $g^* = i_1^* \circ H^*$. This, however, follows immediately from Theorem 2.10 once $R(x) = 0$. □

We have seen above that E^*_{flat} is a homotopy invariant functor. Ideally, it should extend to a generalized cohomology theory (compare the discussion of Section 2.1). For this, we need a bit of extra structure which corresponds to the suspension isomorphism, and which is typically easily available. We formulate this in terms of integration over the fiber for $X \times S^1$, originally defined in [27, Definition 1.3]. Note that the projection $X \times S^1 \to X$ is canonically oriented for an arbitrary cohomology theory because the tangent bundle of S^1 is canonically trivialized. For a push-down in differential cohomology, in general we expect that one has to choose geometric data of the fibers, which again we can assume to be canonically given for the fiber S^1. Orientations and push-forward homomorphisms for differential cohomology theories, in particular for differential K-theory are discussed in Section 5.

2.12 Definition. We say that a differential extension $\hat{E}$ of a cohomology theory (E, ch) has S^1-*integration* if there is a natural transformation $\int_{X\times S^1/X} \colon \hat{E}^*(X \times S^1) \to \hat{E}^{*-1}(X)$ which is compatible with the transformations R, I and the "integration over the fiber" $\Omega^*(X \times S^1; N) \to \Omega^{*-1}(X; N)$ as well as $E^*(X \times S^1) \to E^{*-1}(X)$. In addition, we require that $\int_{X\times S^1/X} p^* x = 0$ for each $x \in \hat{E}^*(X)$ and $\int_{X\times S^1/X} (\mathrm{id} \times t)^* x = -x$ for each $x \in \hat{E}^*(X \times S^1)$, where $t \colon S^1 \to S^1$ is complex conjugation.

In [27, Corollary 4.3] we prove that in many situations, e.g. for ordinary cohomology or for K-theory, there is a canonical choice of integration transformation.

2.13 Theorem. *If $\hat{E}$ is a multiplicative differential extension of (E, ch) and if $E^{-1}(pt)$ is a torsion group, then $\hat{E}$ has a canonical S^1-integration as in Definition 2.12.*

2.3 $\hat{E}^*_{flat}$ as generalized cohomology theory $E\mathbb{R}/\mathbb{Z}$

Let E be a generalized cohomology theory. In the present section we consider a universal differential extension $\hat{E}$, i.e. we take $N := E(*) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\mathrm{ch} : E(X) \to H(X; N)$ be the canonical transformation.

To E there is an associated generalized cohomology theory $E\mathbb{R}/\mathbb{Z}$ (E with coefficients in $\mathbb{R}/\mathbb{Z}$). It is constructed with the help of stable homotopy theory: the cohomology theory E is given by a spectrum (in the sense of stable homotopy theory) $\mathbf{E}$, and $E\mathbb{R}/\mathbb{Z}$ is given by the spectrum $\mathbf{E}M(\mathbb{R}/\mathbb{Z})$, where $M(\mathbb{R}/\mathbb{Z})$ is the Moore spectrum of the abelian group $\mathbb{R}/\mathbb{Z}$. $E\mathbb{R}/\mathbb{Z}$ is constructed in such a way that one has natural long exact sequences

$$\to E^*(X) \to E^*(X) \otimes \mathbb{R} \to E\mathbb{R}/\mathbb{Z}^*(X) \to E^{*+1}(X) \to \dots$$

Note that for a finite CW-complex X one can alternatively write $E^*(X) \otimes \mathbb{R} = E\mathbb{R}^*(X)$ with $E\mathbb{R}$ defined by smash product with $M(\mathbb{R})$.

In the fundamental paper [43], Hopkins and Singer construct a specific differential extension for any generalized cohomology theory E. For this particular construction, one has by [43, (4.57)] a natural isomorphism

$$E\mathbb{R}/\mathbb{Z}^{*-1}(X) \to E^*_{flat}(X).$$

However, this is a consequence of the particular model used in [43]. It is therefore an interesting question to which extend the axioms alone imply that

$\hat{E}_{flat}$ is a generalized cohomology theory. Here, some extra structure about the suspension isomorphism seems to be necessary, implemented by the transformation "integration over S^1" of Definition 2.12. With a surprisingly complicated proof one gets [27, Theorem 7.11]:

2.14 Theorem. *If $(\hat{E}, R, I, a, \int)$ is a differential extension of E with integration over S^1, then $\hat{E}^*_{flat}$ has natural long exact Mayer-Vietoris sequences. It is equivalent to the restriction to compact manifolds of a generalized cohomology theory represented by a spectrum. Moreover, $a\colon E\mathbb{R}^* \to \hat{E}^*_{flat}$ and $I\colon \hat{E}^*_{flat} \to E\mathbb{R}/\mathbb{Z}^{*+1}$ are natural transformations of cohomology theories and one obtains a natural long exact sequence for each finite CW-complex X*

$$\to E\mathbb{R}^{*-1}(X) \xrightarrow{a} \hat{E}^*_{flat}(X) \xrightarrow{I} E^*(X) \xrightarrow{ch} E\mathbb{R}^*(X) \to . \tag{2.15}$$

Proof. The long exact sequence (2.15) is not stated like that in [27] and we will need it below, therefore we explain how this is achieved. Because of (2.3), the restriction of a to $E\mathbb{R}^{*-1}(X)$ (realized as de Rham cohomology, i.e. as $\ker(d\colon \Omega^{*-1}(X;N)/\operatorname{im}(d) \to \Omega^*(X;N)))$ hits exactly $\hat{E}_{flat}(X)$. The exactness at $\hat{E}_{flat}(X)$ therefore is a direct consequence of the exactness of (2.4). (2.4) also implies immediately that $\ker(a) = \operatorname{im}(ch)$ in (2.15). Because of (2.2), $ch \circ I = 0$ in (2.15). Finally, as $I\colon \hat{E}^*(X) \to E(X)$ is surjective, any $x \in \ker(ch)$ can be written as $I(y)$ with $y \in \hat{E}^*(X)$ and such that $R(y) \in \operatorname{im}(d)$, i.e. $R(y) = d(\omega)$ for some $\omega \in \Omega^{*-1}(X)/\operatorname{im}(d)$. But then $x = I(y - d\omega)$ and $y - d\omega \in E^*_{flat}(X)$, which implies that (2.15) is also exact at $E^*(X)$. □

2.16 Corollary. *Let $(\hat{E}, R, I, a, \int)$ be a differential extension of E with integration over S^1, or more generally assume that $\hat{E}_{flat}$ has natural long exact Mayer-Vietoris sequences. Assume that $X_1, X_2, X_0 := X_1 \cap X_2 \subseteq X$ are closed submanifolds of codimension 0 with boundary (and corners) such the interiors of X_1 and X_2 cover X. Then one has a long exact Mayer-Vietoris sequence*

$$\hat{E}^{n+1}_{flat}(X_0) \xrightarrow{i\circ\delta} \hat{E}^n(X) \to \hat{E}^n(X_1) \oplus \hat{E}^n(X_2) \to \hat{E}^n(X_0) \xrightarrow{\delta\circ I} E^{n-1}(X) \cdots$$

which continues to the right with the Mayer-Vietoris sequence for E and to the left with the Mayer-Vietoris sequence for $\hat{E}$.

Proof. The proof is an standard diagram chase, using the Mayer-Vietoris sequences for E and E_{flat}, the short exact sequence

$$0 \to \Omega^*(X) \to \Omega^*(X_1) \oplus \Omega^*(X_2) \to \Omega^*(X_0) \to 0,$$

the homotopy formula, and the exact sequences (2.4). □

2.17 Theorem. *If, in addition to the assumption of Theorem 2.14, E^k is finitely generated for each $k \in \mathbb{Z}$ and the torsion subgroup $E^*_{tors}(pt) = 0$ then there is an isomorphism of cohomology theories $\hat{E}^*_{flat} \xrightarrow{\cong} E\mathbb{R}/\mathbb{Z}^{*+1}$.*

Proof. We claim this statement as [27, Theorem 7.12]. However, the proof given there is not correct, and the assertion of [27, Theorem 7.12] unfortunately is slightly stronger than the one we can actually prove, namely Theorem 2.17.

As by Theorem 2.14 $\hat{E}_{flat}$ is a generalized cohomology theory, it is represented by a spectrum U, and the natural transformation a by a map of spectra.

We extend this to a fiber sequence of spectra $F \to E\mathbb{R} \xrightarrow{a} U$, inducing for each compact CW-complex X an associated long exact sequence

$$\to F^*(X) \to E\mathbb{R}^*(X) \xrightarrow{a} \hat{E}^*_{flat}(X) \to . \tag{2.18}$$

Comparison with (2.15) implies that the image of $F(X)$ in $E\mathbb{R}(X)$ coincides with the image of $E(X)$. This means by definition that the composed map of spectra $F \to E\mathbb{R} \to E\mathbb{R}/\mathbb{Z}$ is a phantom map. However, under the assumption that the E^k is finitely generated for each k, it is shown in [27, Section 8] that such a phantom map is automatically trivial. Using the triangulated structure of the homotopy category of spectra, we can choose ϕ, ϕ_F to obtain a map of fiber sequences (distinguished triangles)

$$\begin{array}{ccccc} F & \longrightarrow & E\mathbb{R} & \xrightarrow{a} & U \\ \downarrow{\scriptstyle \phi_F} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \phi} \\ E & \longrightarrow & E\mathbb{R} & \longrightarrow & E\mathbb{R}/\mathbb{Z}. \end{array}$$

with associated diagram of exact sequences which because of the knowledge about image and kernel of a specializes to

$$\begin{array}{ccccccc} E^*(X) & \longrightarrow & E\mathbb{R}^*(X) & \xrightarrow{a} & \hat{E}^{*+1}_{flat}(X) & \longrightarrow & E^{*+1}_{tors}(X) \to 0 \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \phi} & & \\ E^*(X) & \longrightarrow & E\mathbb{R}^*(X) & \longrightarrow & E\mathbb{R}/\mathbb{Z}^*(X) & \longrightarrow & E^{*+1}_{tors}(X) \to 0. \end{array} \tag{2.19}$$

Note that we do not claim (and don't know) whether the diagram can be completed to a commutative diagram by $\mathrm{id}\colon E^{*+1}_{tors}(X) \to E^{*+1}_{tors}(X)$.

If $E^*_{tors}(pt) = 0$, the 5-lemma implies that $\phi\colon U \to E\mathbb{R}/\mathbb{Z}$ induces an isomorphism on the point and therefore for all finite CW-complexes. □

3 Uniqueness of differential extensions

Given the many different models of differential extensions of K-theory, many of which we are going to described in Section 4, it is reassuring that the resulting theory is uniquely determined. The corresponding statement for differential extensions of ordinary cohomology has been established in [59] by Simons and Sullivan. For K-theory and many other generalized cohomology theories we will establish this in the current section.

As in Subsection 2.3 we consider universal differential extensions of E. Given two extensions $\hat{E}$ and $\hat{E}'$ of E with corresponding natural transformations a, I as in 2.1, we are looking for a natural isomorphism $\Phi\colon \hat{E} \to \hat{E}'$ compatible with the natural transformation. Provided such a natural transformation exists, we ask whether it is unique. We have seen that it often is natural to have additionally a transformation "integration over S^1" as in Definition 2.12, and we require that Φ is compatible with this transformation, as well.

To construct the transformation Φ in degree k there is the following basic strategy:

- find a classifying space B for E^k with universal element $u \in E^k(B)$. This means that for any space X and $x \in E^k(X)$, there is a map $f\colon X \to B$ (unique up to homotopy) such that $x = f^*u$.

- lift this universal element to $\hat{u} \in \hat{E}^k(B)$ and show that this class is universal for $\hat{E}^k$, or at least that for a class $\hat{x} \in \hat{E}^k(X)$ one can find $f\colon X \to B$ such that the difference $\hat{x} - f^*\hat{u}$ is under good control.
- Obtain similarly $\hat{u}' \in (\hat{E}')^k(B)$. Define the transformation by $\Phi(\hat{u}) = \hat{u}'$ and extend by naturality.
- Check that Φ has all desired properties.
- Uniqueness of Φ does follow if the lifts $\hat{u}, \hat{u}'$ are uniquely determined by u once their curvature is also fixed.

There are a couple of obvious difficulties implementing this strategy. The first is the fact, that the classifying space B almost never has the homotopy type of a finite dimensional manifold. Therefore, $\hat{E}(B)$ is not defined. This is solved by replacing B by a sequence of manifolds approximating B with a compatible sequence of classes in $\hat{E}^k$ which replace $\hat{u}$. Then, the construction of Φ as indicated indeed is possible. However, a priori this has a big flaw. Φ is not necessarily a transformation of abelian groups. Because of the compatibilities with a, R and I the deviation from being additive is rather restricted and in the end is a class in $E\mathbb{R}^{k-1}(B \times B)/\operatorname{im}(ch)$. The assumption that E is rationally even then implies that this group is $\{0\}$ for k even. The different possible transformation are by naturality and the compatibility conditions determined by the different lift $\hat{u}'$ with fixed curvature. This indetermancy is given by an element in $E\mathbb{R}^{k-1}(B)/\operatorname{im}(ch)$. If k is even and E is rationally even, then so is B as classifying space of E^k. It then follows that $E\mathbb{R}^{k-1}(B \times B)/\operatorname{im}(ch) = \{0\}$ and $E\mathbb{R}^{k-1}(B)/\operatorname{im}(ch) = \{0\}$, i.e. Φ automatically is additive and unique.

For k odd, the transformation can then be defined (and is uniquely determined) by the requirement that it is compatible with $\int_{X\times S^1/X}$. This construction has been carried out in detail in [27] and we arrive at the following theorem.

3.1 Theorem. *Assume that $(\hat{E}, R, I, a, \int)$ and $(\hat{E}', R', I', a', \int')$ are two differential extensions with S^1-integration of a generalized cohomology theory E which is rationally even, i.e. $E^{2k+1}(pt) \otimes \mathbb{Q} = 0$ for all $k \in \mathbb{Z}$. Assume furthermore that $E^k(pt)$ is a finitely generated abelian group for each $k \in \mathbb{Z}$. Then there is a unique natural isomorphism between these differential extensions compatible with the S^1-integrations.*

If no S^1-orientation is given, the natural isomorphism can still be constructed on the even degree part.

If $\hat{E}$ and $\hat{E}'$ are multiplicative, the transformation is automatically multiplicative. Note that the assumptions imply by Theorem 2.13 that then there is a canonical integration.

If $\hat{E}$ and $\hat{E}'$ are defined on all manifold, not only on compact manifolds (possibly with boundary), then it suffices to require that $E^k(pt)$ is countably generated for each $k \in \mathbb{Z}$, and the same assertions hold.

Proof. The proof is given in [27] and we don't plan to repeat it here. However, there we made the slightly stronger assumption that $E^{2k+1}(pt) = 0$ if $\hat{E}$ is only defined on the category of compact manifolds. Let us therefore indicate why the stronger result also holds. As described in the strategy, the first task is to approximate the classifying spaces B for E^{2k} by spaces on which we can evaluate $\hat{E}^*$. These approximations are constructed inductively by attaching handles to obtain the correct homotopy groups (introducing new homotopy, but also killing superfluous homotopy). To construct a compact manifold, we are only allowed to attach finitely many handles. Therefore, we have to know a priori that we have to kill only a finitely generated homotopy groups. In [27] we assume that $\pi_1(B) = E^{2k-1}(pt)$ is zero, to be allowed to start with a simply connected approximation. Then we use that all homotopy groups of a simply connected finite CW-space are finitely generated. However, exactly the same holds for finite CW-spaces with finite fundamental group, because the higher homotopy groups are the homotopy groups of the universal covering, which in this case is a finite simply connected CW-complex. Note that a finitely generated abelian group A with $A \otimes \mathbb{Q} = 0$ is automatically finite. □

3.2 Remark. The general strategy leading toward Theorem 3.1 has been developed by Moritz Wiethaup 2006/07. However, his work has not been published yet. This was then taken up and developed further in [27].

3.1 Uniqueness of differential K-theory

We observe that all the assumptions of Theorem 3.1 are satisfied by K-theory, and also by real K-theory. Therefore, we have the following theorem:

3.3 Theorem. *Given two differential extensions $\hat{K}$ and $\hat{K}'$ of complex K-theory, there is a unique natural isomorphism $\hat{K}^{ev} \to \hat{K}'^{ev}$ compatible with all the structure. If the extensions are multiplicative, this transformation is compatible with the products.*

If both extensions come with S^1-integration as in Definition 2.12 there is a unique natural isomorphism $\hat{K} \to \hat{K}'$ compatible with all the structure, including the integration.

In other words, all the different models for differential K-theory of Section 4 define the same groups —up to a canonical isomorphism.

3.4 Remark. In Theorem 3.3 we really have to require the existence of S^1-integration. In [27, Theorem 6.2] an infinite family of "exotic" differential extensions of K-theory are constructed. Essentially, the abelian group structure is modified in a subtle way in these examples to produce non-isomorphic functors which all satisfy the axioms of Section 2.

4 Models for differential K-theory

4.1 Vector bundles with connection

The most obvious attempt to construct differential cohomology (at least for $\hat{K}^0$) is to use vector bundles *with connection.* It is technically convenient to also add an odd differential form to the cycles. This, indeed is the classical picture already used by Karoubi in [45, Section 7] for his definition of "multiplicative K-theory", which we would call the flat part of differential K-theory.

4.1 Definition. A cycle for vector bundle K-theory $\hat{K}^0(M)$ is a triple (E, ∇, ω), where E is a smooth complex Hermitean vector bundle over M, ∇ a Hermitean connection on E and $\omega \in \Omega^{odd}(M)/\mathrm{im}(d)$ a class of a differential form of odd degree. The *curvature* of a cycle essentially is defined as the Chern-Weil representative $\mathrm{ch}(\nabla) := \mathrm{tr}(e^{-\frac{\nabla^2}{2\pi i}})$ of the Chern character of E, computed using the connection ∇:

$$R(E, \nabla, \omega) := \mathrm{ch}(\nabla) - d\omega.$$

4.2 Remark. We require the use of Hermitean connections to obtain real valued curvature forms. Alternatively, one would have to use as target of *ch* cohomology with complex instead of cohomology with real coefficients. A slightly more extensive discussion of this matter can be found in [51, Section 2].

We define in the obvious way the sum $(E, \nabla_E, \omega) + (F, \nabla_F, \eta) := (E \oplus F, \nabla_E \oplus \nabla_F, \omega + \eta)$. Two cycles (E, ∇, ω) and (E', ∇', ω') are equivalent if there is a third bundle with connection (F, ∇_F) and an isomorphism $\Phi \colon E \oplus F \to E' \oplus F$ such that

$$d\widetilde{\mathrm{ch}}(\nabla \oplus \nabla_F, \Phi^{-1}(\nabla' \oplus \nabla_F)\Phi) = \omega - \omega',$$

where $\widetilde{\mathrm{ch}}(\nabla, \nabla')$ denotes the transgression Chern form between the two connections such that $\mathrm{ch}(\nabla) - \mathrm{ch}(\nabla') = d\widetilde{\mathrm{ch}}(\nabla, \nabla')$.

4.3 Remark. In [38, Section 9], a model for differential $\hat{K}^1$ is given where the cycles are Hermitean vector bundles with connection and a unitary automorphism, and an additional form (modulo the image of d) of even degree. The relations include in particular a rule for the composition of the unitary automorphisms: if U_1 and U_2 are two unitary automorphisms of E then $(E, \nabla, U_1, \omega_1) + (E, \nabla, U_2, \omega_2) = (E, \nabla, U_2 \circ U_1, CS(\nabla, U_1, U_2) + \omega_1 + \omega_2)$, where $CS(\nabla, U_1, U_2)$ is the Chern-Simons form relating ∇, $U_2 \nabla U_2^{-1}$ and $U_2 U_1 \nabla U_1^{-1} U_2^{-1}$. We do not discuss this model in detail here.

4.2 Classifying maps

Hopkins and Singer, in their ground breaking paper [43] give a cocycle model of a differential extension $\hat{E}$ of any cohomology theory (with transformation) (E, ch), based on classifying maps. Here, for the construction of $\hat{E}^n(X)$ one has to choose two fundamental ingredients:

(1) a classifying space X_n for E^n; note that no smooth structure for X_n is required (and could be expected)

(2) a cocycle c representing ch_n, so that, whenever $f \colon X \to X_n$ represents a class $x \in E^n(X)$, then f^*c represents $\mathrm{ch}(x) \in H^n(X; N)$. We can think of this cocycle as an N-valued singular cocycle, although variations are possible.

A Hopkins-Singer cycle for $\hat{E}(X)$ then is a so called *differential function*, which by definition is a triple (f, h, ω) consisting of a continuous map $f \colon X \to X_n$, a closed differential n-form ω with values in N and an $(n-1)$-cochain h satisfying

$$\delta h = \omega - f^*c.$$

In other words: f is an explicit representative for a class $x \in E^n(X)$, and we are of course setting $I(f, h, \omega) = [f] \in E^n(X)$. ω is a de Rham representative of $\mathrm{ch}(x)$, and we are setting $R(f, h, \omega) := \omega$. This data actually gives two explicit representatives for $\mathrm{ch}(x)$, namely ω and f^*c (here, we have to map both ω and

f^*c to a common cocycle model for $H^n(X;N)$ like smooth singular cochains. ω defines such a cocycle by the de Rham homomorphism "integrate ω over the chain", f^*c by restriction).

By definition, (f_1, h_1, ω_1) and (f_0, h_0, ω_0) are equivalent if $\omega_0 = \omega_1$ and there is $(f, h, \mathrm{pr}^* \omega_1)$ on $X \times [0,1]$ which restricts to the two cycles on $X \times \{0\}$ and $X \times \{1\}$.

The advantage of this approach is its complete generality. A disadvantage is that the cycles don't have a nice geometric interpretation. Moreover, operations (like the addition and multiplication) rely on the choice of corresponding maps between the classifying spaces realizing those. These maps have then to be used to define the same operations on the differential cohomology groups. This is typically not very explicit. Moreover, properties like associativity, commutativity etc. will not hold on the nose for these classifying maps but are implemented by homotopies which have to be taken into account when establishing the same properties for the generalized differential cohomology. This can quickly get quite cumbersome and we refrain from carrying this out in any detail.

4.3 Geometric families of elliptic operators

In [26], a cycle model for differential K-theory (there called "smooth K-theory") is developed which is based on local index theory. In spirit, it is similar to the passage of the classical model of K-theory via vector bundles to the Kasparov KK-model, where all families of generalized index problems are cycles.

Similarly, the cycles of [26] are geometric families of Dirac operators. It is clear that a lot of differential structure has to be present to obtain *differential* K-theory, so the definition has to be more restrictive than in Kasparov's model. There are many advantages of an approach with very general cycles:

- first and most obvious, it is very easy to construct elements of differential K-theory if one has a broad class of cycles.
- the approach allows for a unified treatment of even and odd degrees
- the flexibilities of the cycles allows for explicit constructions in many contexts. In particular, it is easy to explicitly define the product and also the push-forward along a fiber-bundle.

It might seems as a disadvantage that one necessarily has a broad equivalence relation and that it is therefore hard to construct homomorphisms out of differential K-theory. To do this, one has to use the full force of local index theory. However, this is very well developed and one can make use of many properties as black box and then efficiently carry out the relevant constructions.

4.4 Definition. Let X be a compact manifold, possibly with boundary. A cycle for a $\hat{K}(X)$ is a pair $(\mathcal{E}, \rho)$, where $\mathcal{E}$ is a geometric family, and $\rho \in \Omega(X)/\operatorname{im}(d)$ is a class of differential forms.

A geometric family over X (introduced in [20]) consists of the following data:

(1) a proper submersion with closed fibers $\pi\colon E \to X$,

(2) a vertical Riemannian metric $g^{T^v\pi}$, i.e. a metric on the vertical bundle $T^v\pi \subset TE$, defined as $T^v\pi := \ker(d\pi\colon TE \to \pi^*TX)$.

(3) a horizontal distribution $T^h\pi$, i.e. a bundle $T^h\pi \subseteq TE$ such that $T^h\pi \oplus T^v\pi = TE$.

(4) a family of Dirac bundles $V \to E$,

(5) an orientation of $T^v\pi$.

Here, a family of Dirac bundles consists of

(1) a Hermitean vector bundle with connection (V, ∇^V, h^V) on E,

(2) a Clifford multiplication $c\colon T^v\pi \otimes V \to V$,

(3) on the components where $\dim(T^v\pi)$ has even dimension a $\mathbb{Z}/2\mathbb{Z}$-grading z.

We require that the restrictions of the family Dirac bundles to the fibers $E_b := \pi^{-1}(b)$, $b \in X$, give Dirac bundles in the usual sense (see [20, Def. 3.1]):

(1) The vertical metric induces the Riemannian structure on E_b,

(2) The Clifford multiplication turns $V_{|E_b}$ into a Clifford module (see [11, Def.3.32]) which is graded if $\dim(E_b)$ is even.

(3) The restriction of the connection ∇^V to E_b is a Clifford connection (see [11, Def.3.39]).

A geometric family is called even or odd, if $\dim(T^v\pi)$ is even-dimensional or odd-dimensional, respectively.

4.5 Example. The cycles for $\hat{K}^0(X)$ of Definition 4.1 are special cases of the cycles of Definition 4.4: $p\colon E \to X$ is just $\mathrm{id}\colon X \to X$, i.e. the fibers consist of the 0-dimensional manifold $\{pt\}$.

There are obvious notions of isomorphism (preserving all the structure) and of direct sum of cycles. We now introduce the structure maps I and R and the equivalence relation on the semigroup of isomorphism classes.

4.6 Definition. The opposite $\mathcal{E}^{op}$ of a geometric family $\mathcal{E}$ is obtained by reversing the signs of the Clifford multiplication and the grading (in the even case) of the underlying family of Clifford bundles, and of the orientation of the vertical bundle.

4.7 Definition. The usual construction of Dirac type operators of a Clifford bundle (compare [50, 11]), applied fiberwise, assigns to a geometric family $\mathcal{E}$ over X a family of Dirac type operators parametrized by X, and this is indeed the main idea behind the geometric families. Then, the classical construction of Atiyah-Singer assigns to this family its (analytic) index $\mathrm{ind}(\mathcal{E}) \in K^*(B)$, where $*$ is equal to the parity of the dimension of the fibers. In the special case of Example 4.5 —a vector bundle with connection over X— this is exactly the K-theory class of the underlying $\mathbb{Z}/2$-graded vector bundle.

We define $I(\mathcal{E}, \omega) := \mathrm{ind}(\mathcal{E}) \in K^*(X)$.

4.8 Remark. We define $\mathcal{E}^{op}$ in such a way that $\mathrm{ind}(\mathcal{E}^{op}) = -\mathrm{ind}(\mathcal{E})$. Moreover, ind is additive under sums of geometric families. The equivalence relation we are going to define will be compatible with ind.

We now proceed toward the definition of $R(\mathcal{E})$. It is based on the notion of local index form, an explicit de Rham representative of $\mathrm{ch}(\mathrm{ind}(\mathcal{E})) \in H^*_{dR}(X)$. It is one of the important points of the data collected in a geometric family that such a representative can be constructed canonically. For a detailed definition we refer to [20, Def. 4.8], but we briefly formulate its construction as follows. The vertical metric $T^v\pi$ and the horizontal distribution $T^h\pi$ together induce a connection $\nabla^{T^v\pi}$ on $T^v\pi$. Locally on E we can assume that $T^v\pi$ has a spin structure. We let $S(T^v\pi)$ be the associated spinor bundle. Then we can write the family of Dirac bundles V as $V = S \otimes W$ for a twisting bundle (W, h^W, ∇^W, z^W) with metric, metric connection, and $\mathbb{Z}/2\mathbb{Z}$-grading which is determined uniquely up to isomorphism. The form $\hat{A}(\nabla^{T^v\pi}) \wedge \mathrm{ch}(\nabla^W) \in \Omega(E)$ is globally defined, and we get the local index form by applying the integration over the fiber $\int_{E/B} \colon \Omega(E) \to \Omega(B)$:

$$\Omega(\mathcal{E}) := \int_{E/B} \hat{A}(\nabla^{T^v\pi}) \wedge \mathrm{ch}(\nabla^W) \ . \tag{4.9}$$

The characteristic class version of the index theorem for families is

4.10 Theorem ([4])**.** $\mathrm{ch}_{dR}(\mathrm{ind}(\mathcal{E})) = [\Omega(\mathcal{E})] \in H^*_{dR}(X)$.

A proof using methods of local index theory has been given in [12], compare [11].

The equivalence relation we impose is based on the following idea: a geometric family $\mathcal{E}$ with index 0 should potentially be equivalent to the cycle 0, but in general only up to some differential form (with degree of shifted parity). Moreover, in this case the local index form $R(\mathcal{E})$ will be exact, but it is important to find an explicit primitive η with $d\eta = R(\mathcal{E})$. Therefore, we identify geometric *reasons* why the index is zero and which provide such a primitive.

4.11 Definition. A pre-taming of $\mathcal{E}$ is a family $(Q_b)_{b\in B}$ of self-adjoint operators $Q_b \in B(H_b)$ given by a smooth fiberwise integral kernel $Q \in C^\infty(E \times_B E, V \boxtimes V^*)$. In the even case we assume in addition that Q_b is odd with respect to the grading. The pre-taming is called a taming if $D(\mathcal{E}_b) + Q_b$ is invertible for all $b \in B$. In this case, by definition $\mathrm{ind}(D(\mathcal{E}) + Q)$ is zero. However, as the index is unchanged by the smoothing perturbation Q, also $\mathrm{ind}(\mathcal{E}) = 0$ if $\mathcal{E}$ admits a taming.

Let $\mathcal{E}_t$ be the notation for geometric family $\mathcal{E}$ with a chosen taming. In [20, Def. 4.16], the η-form $\eta(\mathcal{E}_t) \in \Omega(X)$ is defined. By [20, Theorem 4.13]) it satisfies

$$d\eta(\mathcal{E}_t) = \Omega(\mathcal{E}) \ . \tag{4.12}$$

We skip the considerable analytic difficulties in the construction of the eta-form and use it and its properties as a black box.

4.13 Definition. We call two cycles $(\mathcal{E}, \rho)$ and $(\mathcal{E}', \rho')$ paired if there exists a taming $(\mathcal{E} \sqcup_B \mathcal{E}'^{op})_t$ such that

$$\rho - \rho' = \eta((\mathcal{E} \sqcup_B \mathcal{E}'^{op})_t) \ .$$

We let $\sim$ denote the equivalence relation generated by the relation "paired".

We define the differential K-theory $\hat{K}^*(B)$ of B to be the group completion of the abelian semigroup of equivalence classes of cycles as in Definition 4.4 with fiber dimension congruent $*$ modulo 2.

4.14 Theorem. *Definition 4.13 of $\hat{K}^*$ indeed defines (with the obvious notion of pullback) a contravariant functor on the category of smooth manifolds which is a differential extension of K-theory.*

The necessary natural transformations are induced by

(1) $I\colon \hat{K}^*(X) \to K^*(X); \quad (\mathcal{E}, \omega) \mapsto \operatorname{ind}(\mathcal{E})$ *of Definition 4.7*

(2) $a\colon \Omega^{*-1}(X)/\operatorname{im}(d) \to \hat{K}^*(X); \quad \omega \mapsto (0, -\omega)$

(3) $R\colon \hat{K}^*(X) \to \Omega^*_{d=0}(X); \quad (\mathcal{E}, \omega) \mapsto \Omega(\mathcal{E}) + d\omega$ *of* (4.9).

Proof. This is carried out in [26, Section 2.4]. □

4.15 Definition. Given two geometric families $\mathcal{E}, \mathcal{F}$ over a base X, there is an obvious geometric way to define their fiber product $\mathcal{E} \times_B \mathcal{F}$, with underlying fiber bundle the fiber product of bundles of manifolds etc. Details are spelled out in [26, Section 4.1]. We then define the product of two cycles $(\mathcal{E}, \rho)$ and $(\mathcal{F}, \theta)$ (homogeneous of degree e and f, respectively) as

$$(\mathcal{E}, \rho) \cup (\mathcal{F}, \theta) := [\mathcal{E} \times_B \mathcal{F}, (-1)^e \Omega(\mathcal{E}) \wedge \theta + \rho \wedge \Omega(\mathcal{F}) - (-1)^e d\rho \wedge \theta] \ .$$

4.16 Theorem. *The product of Definition 4.15 turns $\hat{K}^*$ into a multiplicative differential extension of K-theory.*

Proof. This is carried out in [26, Section 4]. □

4.17 Remark. Note that it is much more cumbersome to define a product structure with the other models described: for the vector bundle model the inclusion of the odd part is problematic, for the homotopy theoretic model one would have to realize the product structure using maps between (products of) classifying spaces. This is certainly a worthwhile task, but does not seem to be carried out in detail so far.

In the geometric families model, it is also rather easy to construct “integration over the fiber” and in particular S^1-integration as defined in 2.12. This will be explained in Section 5.

4.4 Differential characters

One of the first models for a differential extension of integral cohomology are the differential characters of Cheeger-Simons [32]. A differential character is a pair (ϕ, ω), consisting of a homomorphism $\phi\colon Z_*(X) \to \mathbb{R}/\mathbb{Z}$ defined on the group of (smooth) singular cycles on X such that for a boundary $b = dc$, $c \in C_{*+1}(X)$ $\phi(b) = [\int_c \omega]_{\mathbb{R}/\mathbb{Z}}$. The set of differential characters on $C_*(X)$ is a model for $\hat{H}^{*+1}(X)$. Along similar lines, in [10] the cycle model for K-homology due to Baum and Douglas [7] —recently worked out in detail in [8]— is used to define $\hat{K}^*(X)$ as the group of $\mathbb{R}/\mathbb{Z}$-valued homomorphism from the set of Baum-Douglas cycles for K-homology which, on boundaries, are given as pairing with a differential form. [10] also define this pairing of a cycle and a differential form. To do this, in contrast to [7] the cycles have to carry additional geometric structure.

4.5 Currential K-theory

Freed and Lott in [38, 2.28] introduce a variant of differential K-theory where they replace the differential forms throughout by currents. This *changes* the theory in us much as the curvature homomorphism then also takes values in currents, i.e. forms are replaced by currents throughout. Because the multiplication of currents is not well defined, in this new theory one loses the ring structure. The advantage of this variant, on the other hand, is that push-forward maps can be described very directly also for embeddings: for the differential form part, the image under push-forward will be a current supported on the submanifold. The currential theory has the advantage that push-forwards can be defined easily. Moreover, they can be defined for arbitrary maps. However, pullbacks are not always defined (and not discussed at all for this theory in [38]). Consequently, the theory should (up to a degree shift given by Poincaré duality) better be considered as differential K-homology, twisted by some kind of spin^c-orientation twist.

To describe differential extensions of bordism theories as in [28, 21], currents are also used —but there only as an intermediate tool. The axiomatic setup for those theories is the one described in 2.1. In particular, the curvature homomorphism takes values in smooth differential forms.

4.6 Differential K-theory via bordism

In [28, Section 4], a geometric model of the differential extension $\hat{MU}^*$ of MU^* is constructed, where MU^* is cohomology theory dual to complex bordism. Cycles are given by pairs $(\tilde{c}, \alpha)$. Here $\tilde{c}$ is a geometric cycle for $MU^n(X)$, i.e. a proper smooth map $W \to X$ with $n = \dim(X) - \dim(W)$ and an explicit geometric model of the stable normal bundle with $U(N)$-structure. Moreover, α is a differential $(n-1)$-form with distributional coefficients whose differential differs from a certain characteristic current of $\tilde{c}$ by a smooth differential form. We impose on these cycles the equivalence relation generated by an obvious notion of bordism together with stabilization of the normal bundle data.

The functor $\hat{MU}^*$ is a multiplicative extension of complex cobordism. In particular, it takes values in algebras over $\hat{MU}^{ev}(*)$, as $\hat{MU}^{ev}(*)$ is a subring of $\hat{MU}^*(*)$. However, as $MU^{odd}(*) = 0$, the long exact sequence (2.4) give the canonical isomorphism

$$\hat{MU}^{ev}(*) \xrightarrow{\cong} MU^{ev}(*) = MU^*(*).$$

Assume now that E^* is a generalized cohomology theory which is *complex oriented*, i.e. which comes with a natural transformation $MU^* \to E^*$. An example is complex K-theory, with the usual orientation $MU \to MSpin^c \to K$. If the complex oriented theory E^* is Landweber exact (i.e. the condition of [49, Theorem 2.6_{MU}] is satisfied), then Landweber [49, Corollary 2.7] proves that $E^*(X) \cong MU^*(X) \otimes_{MU} E^*$ is obtained from $MU^*(X)$ by just taking the tensor product with the coefficients E^* (in the graded sense). Complex K-theory is Landweber exact [49, Example 3.3] so that this principle gives a simple bordism definition of K-theory.

In [28, Theorem 2.5] it is shown that Landweber's results extends directly to differential extensions. If E^* is a Landweber exact complex oriented generalized cohomology theory, then

$$\hat{E}^*(X) := \hat{MU}^*(X) \otimes_{MU} E$$

is a multiplicative differential extension of $E^*(X)$. Moreover, as explained above, we can apply this to complex K-theory and obtain a bordism description of differential K-theory:

$$\hat{K}^*(X) = \hat{MU}^*(X) \otimes_{MU} K.$$

By [28, Section 2.3], this is indeed a multiplicative differential extension of complex K-theory. Furthermore, it has a natural S^1-integration as in Definition 2.12 and even general "integration over the fiber" as discussed in Section 5.

4.7 Geometric cycle models for $\mathbb{R}/\mathbb{Z}$-K-theory via the flat part of differential K-theory

As K-theory satisfies the conditions of Theorem 2.13, any multiplicative differential extension canonically comes with a transformation "integration over S^1". Therefore, by Theorem 2.14 its flat subfunctor is a generalized cohomology theory. Moreover, K-theory satisfies the conditions of Theorem 2.17 and we therefore get a natural isomorphism

$$\hat{K}_{flat} \to K\mathbb{R}/\mathbb{Z}.$$

In other words, any of the models for differential K-theory described above provides a model for $\mathbb{R}/\mathbb{Z}$-K-theory. In particular, we recover the original result [45, Section 7.21] that the flat part of the vector bundle model of Section 4.1 describes $K\mathbb{R}/\mathbb{Z}^{-1}$.

4.8 Real K-theory

It seems to be not difficult to modify the above models to obtain differential extensions $\hat{KO}$ of real K-theory. This is particularly clear with the vector bundle model of Section 4.1. Complex Hermitean vector bundles with Hermitean connection have to be replaced by real Euclidean vector bundles with metric connection. This describes $\hat{KO}^0$. Of course, $\hat{KO}^*$ now is 8-periodic. The full model in terms of geometric families, following Section 4.3 replaces the families of Dirac bundles by their real analogs. For the analysis of η-forms it seems to be most suitable to implement the real structures via additional conjugate linear operations on the complex Dirac bundle (as explained e.g. in [24, Sec. 2.2]). Alternatively one could possibly work with $\dim(T^v\pi)$-multigradings in the sense of e.g. [42, A.3] (in other words, a compatible Cl_k-module structure, compare also [50, Section 16]). The details of a model for $\hat{KO}$ based on geometric families have still to be worked out. It is actually worthwhile to work out a Real differential K-theory in the usual sense (i.e. for spaces with additional involution). However, we will not carry this out here.

4.9 Differential K-homology and bivariant differential K-theory

An important aspect of any cohomology theory is the dual homology theory and the pairing between the cohomology theory and the homology theory. This applies in particular to K-theory, where the dual homology theory can be described with three different flavors:

- homotopy theoretically, using the K-theory spectrum
- with a geometric cycle model introduced by Baum and Douglas [7]
- an analytic model proposed by Atiyah [2] and made precise by Kasparov [46], compare also [42].

4.18 Definition. The cycles for $K_*(X)$ are triples (M, E, f) where M is a closed manifold with a given spin^c-structure with dimension congruent to $*$ modulo 2, $f\colon M \to X$ is a continuous map and $E \to X$ is a complex vector bundle. On the set of isomorphism classes of triples we put the equivalence relation generated by bordism, *vector bundle modification* and the relation that $(M, E_1, f) \amalg (M, E_2, f) \sim (M, E_1 \oplus E_2, f)$.

Here, vector bundle modification means that, given a complex vector bundle $V \to M$, $(M, E, f) \sim (S(V \oplus \mathbb{R}), T(E), f \circ p)$ where $S(V \oplus \mathbb{R})$ is the sphere bundle of the real bundle $V \oplus \mathbb{R}$, $p\colon V \oplus \mathbb{R} \to M$ is the bundle projection and $T(E)$ is a vector bundle which (up to addition of a bundle which extends over the disc bundle) represents the push-forward of E along the "north pole inclusion" $i\colon M \to S(E \oplus \mathbb{R}); x \mapsto (x, 0, 1)$. There is an explicit clutching construction of this bundle.

It took over 20 years before in [8] the equivalence of the geometric cycle model with the other models has been worked out in full detail.

In the geometric model, the pairing between K-homology and K-theory has a very transparent description, related to index theory. Given a K-homology cycle (M, E, f) and a K-theory cycle represented by the vector bundle $V \to X$, the pairing is the index of the spin^c Dirac operator $D_{E \otimes f^*V}$ on M, twisted with $E \otimes f^*V$. In the analytic model, this Dirac operator by itself essentially already is a K-homology cycle.

The analytic theory has two very important extensions: first of all, it extends from a (co)homology theory for spaces to one for arbitrary C^*-algebras. "Extension" here in the sense that the category of compact Hausdorff spaces is equivalent to the opposite category of commutative unital C^*-algebras via the functor $X \mapsto C(X)$. Secondly, Kasparov's theory really is bivariant, with $KK(X, *) = K_*(X)$ being K-homology, and $KK(*, X) = K^*(X)$ K-theory. Most important is that KK-theory comes with an associative composition product $KK(X, Y) \otimes KK(Y, Z) \to KK(X, Z)$. This is an extremely powerful tool with many applications in index theory and beyond.

In this context, it is very desirable to have differentiable extensions also of K-homology and ideally of bivariant KK-theory (of course only restricted to manifolds). This should e.g. provide a powerful home for refined index theory.

4.19 Definition. Indeed, guided by the model of differential K-theory, we see that differential K-homology or more general $\hat{KK}$, differential bivariant KK-theory, should be a functor from smooth manifolds to graded abelian groups with the following additional structure:

(1) A transformation $I\colon \hat{KK}(X, Y) \to KK(X, Y)$

(2) a transformation R to the appropriate version of differential forms. For K-homology, the best bet for this are differential currents (essentially the dual space of differential forms) and for the bivariant theory the closed (i.e. commuting with the differentials) continuous homomorphisms from differential forms of the first space to differential forms of the second: $R\colon \hat{KK}(X,Y) \to \mathrm{Hom}(\Omega(X),\Omega(Y))$.

(3) action of "forms": a degree-1 transformation $a\colon \mathrm{Hom}(\Omega(X),\Omega(Y)) \to \hat{KK}(X,Y)$.

These have to satisfy relations and exact sequences which are direct generalizations of those of Definition 2.1:

$$\hat{KK}(X,Y) \xrightarrow{R} [-1]\,\mathrm{Hom}(\Omega^*(X),\Omega^*(Y))[-1]/\,\mathrm{im}(d) \xrightarrow{a} \hat{KK}(X,Y) \xrightarrow{I} KK(X,Y) \to 0$$

$$\alpha \circ R = d.$$

Here, we adopt the tradition that KK-theory is $\mathbb{Z}/2\mathbb{Z}$-graded and use the even/odd grading of forms throughout.

Additionally, there should be a natural associative "composition product" $\hat{KK}(X,Y) \otimes \hat{KK}(Y,Z) \to \hat{KK}(X,Z)$ such that I maps this product to the Kasparov product and R to the composition.

Note that (up to Poincaré duality degree shifts), the currential K-theory of Section 4.5 actually is a model for differential K-homology, which is the specialization of the bivariant theory to $\hat{KK}(X,*)$.

An early preprint version of [26] contains a section which develops a bivariant theory as in Definition 4.19 and its applications. The groups are defined in a cycle model where a cycle ϕ for $KK(X,Y)$ consists of

(1) a $\hat{K}$-oriented bundle $p\colon W \to X$, using $\hat{K}$-orientations as in Section 5.1,

(2) a class $x \in \hat{K}(W)$,

(3) a smooth map $f\colon W \to Y$

(4) a continuous homomorphism $\Phi \in \mathrm{Hom}(\Omega^*(X),\Omega^*(Y))[-1]$.

On the set of isomorphism classes of cycles one puts the equivalence relation generated by

(1) compatibility of the sum operations (one given by disjoint union, the other by addition in $\hat{K}(W)$ and $\mathrm{Hom}(\Omega^*(X),\Omega^*(Y))$).

(2) bordism

(3) a suitable definition of vector bundle modification,

(4) change of Φ by the image of d.

Details of such a bivariant construction, or a construction of differential K-homology seem not to exist in the published literature. Note that this approach is not tied to K-theory. It provides a construction of a bivariant differential theory from a differential extension of a cohomology theory together with a theory of differential orientation and integration satisfying a natural set of axioms.

5 Orientation and integration

5.1 Differential K-orientations

Let $p\colon W \to B$ a proper submersion with fiber-dimension n. An important aspect of cohomology is a map "integration along the fiber" or "push-forward" $p_!\colon E^*(W) \to E^{*-n}(B)$. There is a general theory for this, and it requires the extra structure of an *E-orientation* of p (one could also say an E-orientation of the fibers of p). If E is ordinary cohomology, this comes down to an ordinary orientation and in de Rham cohomology $p_!$ literally is "integration along the fiber".

For K-theory, it is a spin^c-structure of π which gives rise to a K-theory orientation, which in turn defines a topological push-forward map $p_!\colon K^*(W) \to K^{*-n}(B)$.

To extend this to *differential* K-theory, one has to add additional geometric data to the spin^c-structure.

This is a pattern which applies to a general differential cohomology theory $\hat{E}$: a differential orientation will consist of an E-orientation in the usual sense together with extra differential and geometric data. There is one notable exception, though. In ordinary integral differential cohomology (Cheeger-Simons differential characters), a topological orientation (which is an ordinary orientation) lifts uniquely to a differential orientation. Therefore, in the literature treating ordinary differential cohomology and push-forward in that context, differential orientations are not discussed [34, 41].

We now describe a model for differential K-orientations for a proper submersion $p\colon W \to B$ which particularly suits the analytic model for $\hat{K}$ of Section 4.3. We will describe the $\mathbb{Z}$-graded version of orientations in order to make clear in which precise degrees the form constituents live. This is important if one wants to set up a similar theory in the case of other, non-two-periodic theories like complex bordism.

Fix an underlying topological K-orientation of p by choosing a spin^c-reduction of the structure group of the vertical tangent bundle T^vp of p. In order to make this precise we choose along the way an orientation and a metric on T^vp.

We now consider the set $\mathcal{O}$ of tuples $(g^{T^vp}, T^hp, \tilde{\nabla}, \sigma)$ where

(1) g^{T^vp} is the Riemannian metric on the vertical bundle $\ker Tp$ (and the K-orientation is given by a spin^c-reduction of the resulting $O(n)$-principal bundle P, i.e. a spin^c-principal bundle Q lifting P).

(2) T^hp is a horizontal distribution.

(3) From the horizontal distribution, we get a connection ∇^{T^vp} which restricts to the Levi-Civita connection along the fibers as follows. First one chooses a metric g^{TB} on B. It induces a horizontal metric g^{T^hp} via the isomorphism $dp_{|T^hp}\colon T^hp \xrightarrow{\sim} p^*TB$. We get a metric $g^{T^vp} \oplus g^{T^hp}$ on $TW \cong T^vp \oplus T^hp$ which gives rise to a Levi-Civita connection. Its projection to T^vp is ∇^{T^vp}. $\tilde{\nabla}$ is a spin^c-reduction of ∇^{T^vp}, i.e. a connection on the spin^c-principal bundle Q which reduces to ∇^{T^vp}. If we think of the connections ∇^{T^vp} and $\tilde{\nabla}$ in terms of horizontal distributions $T^hSO(T^vp)$ and T^hQ, then we say that $\tilde{\nabla}$ reduces to ∇^{T^vp} if $d\pi(T^hQ) = \pi^*(T^hSO(T^vp))$.

(4) $\sigma \in \Omega^{-1}(W; \mathbb{R}[u, u^{-1}])/\operatorname{im}(d)$.

We introduce the globally defined complex line bundle

$$L^2 := Q \times_\lambda \mathbb{C} \to W \tag{5.1}$$

associated to the spin^c-bundle Q via the representation $\lambda\colon Spin^c(n) \to U(1)$. The connection $\tilde\nabla$ thus induces a connection on ∇^{L^2}.

We introduce the form

$$c_1(\tilde\nabla) := \frac{1}{2}(2\pi i u)^{-1} R^{L^2} \in \Omega^0(W;\mathbb{R}[u,u^{-1}]). \tag{5.2}$$

Let $R^{\nabla^{T^vp}} \in \Omega^2(W, \mathrm{End}(T^vp))$ denote the curvature of ∇^{T^vp}. The closed form

$$\hat{\mathbf{A}}(\nabla^{T^vp}) := \det{}^{1/2}\left(\frac{\frac{u^{-1}R^{\nabla^{T^vp}}}{4\pi}}{\sinh\left(\frac{u^{-1}R^{\nabla^{T^vp}}}{4\pi}\right)}\right) \in \Omega^0(W;\mathbb{R}[u,u^{-1}])$$

represents the $\hat{\mathbf{A}}$-class of T^vp.

5.3 Definition. The relevant differential form for local index theory in the spin^c-case is

$$\hat{\mathbf{A}}^c(\tilde\nabla) := \hat{\mathbf{A}}(\nabla^{T^vp}) \wedge e^{c_1(\tilde\nabla)} \ . \tag{5.4}$$

If we consider $p\colon W \to B$ with the geometry $(g^{T^vp}, T^hp, \tilde\nabla)$ and the Dirac bundle $S^c(T^vp)$ as a geometric family $\mathcal{W}$ over B, then

$$\int_{W/B} \hat{\mathbf{A}}^c(\tilde\nabla) = \Omega(\mathcal{W}) \in \Omega^{-n}(B;\mathbb{R}[u,u^{-1}]) \ ,$$

is the local index form of $\mathcal{W}$.

We introduce a relation $\sim$ on $\mathcal{O}$: Two tuples $(g_i^{T^vp}, T_i^hp, \tilde\nabla_i, \sigma_i)$, $i=0,1$ are related if and only if $\sigma_1 - \sigma_0 = \tilde{\hat{\mathbf{A}}}^c(\tilde\nabla_1, \tilde\nabla_0)$, the transgression form of $\hat{\mathbf{A}}^c(\tilde\nabla)$. In [26, Section 3.1], it is show that $\sim$ is an equivalence relation.

5.5 Definition. The set of differential K-orientations which refine a fixed underlying topological K-orientation of $p\colon W \to B$ is the set of equivalence classes $\mathcal{O}/\sim$. The vector space $\Omega^{-1}(W;\mathbb{R}[u,u^{-1}])/\operatorname{im}(d)$ acts on the set of differential K-orientations by translation of the differential form entry.

In [26, Corollary 3.6] we show:

5.6 Proposition. *The set of differential K-orientations refining a fixed underlying topological K-orientation is a torsor over $\Omega^{-1}(W;\mathbb{R}[u,u^{-1}])/\operatorname{im}(d)$.*

5.7 Remark. Proposition 5.6 should generalize to differential extensions of a general cohomology theory E: the differential orientations refining a fixed E-orientation should form a torsor over $\Omega^{-1}(W;E\mathbb{R})$. If we apply this to the special case of ordinary cohomology, this group is trivial as the coefficients are concentrated in degree 0. This explains why there is a unique lift of an ordinary orientation to a differential orientation in case of ordinary cohomology.

5.2 Integration

We have set up our model for differential K-orientations in such a way that we have a natural description of the integration homomorphism in differential K-theory using the analytic model of Section 4.3. From now on we reduce to the two-periodic case (by setting $u = 1$).

Given $p\colon W \to B$ with a differential K-orientation as in Section 5.1 and a class $x = [\mathcal{E}, \rho] \in \hat{K}(W)$, the basic idea for the construction of $p_!(x) \in \hat{K}(B)$ is to form the representative for $p_!(x)$ as the geometric family $p_!\mathcal{E}$ obtained by simply composing the family over W with p to obtain a family over B. The geometry on p given by the differential orientation allows to put the required geometry on the composition in a canonical way. For example, the fiberwise metric is obtained as direct sum of the fiberwise metric on $\mathcal{E}$ and on the fibers of p. We omit the details which are given in [26, Section 3.2].

The main remaining question is to define the correct differential form; this requires the study of adiabatic limits. Indeed, in the construction of the geometry on $p_!\mathcal{E}$ we can introduce an additional parameter $\lambda \in (0, \infty)$ by scaling the metric on the fibers of $\mathcal{E}$ by λ^2 (and adjusting the remaining geometry) to obtain $p_!^\lambda\mathcal{E}$. In total, this gives an adiabatic deformation family $\mathcal{F}$ over $(0, \infty) \times B$ which restricts to $p_!^\lambda\mathcal{E}$ on $\{\lambda\} \times B$.

Although the vertical metrics of $\mathcal{F}$ and $p_!^\lambda\mathcal{E}$ collapse as $\lambda \to 0$ the induced connections and the curvature tensors on the vertical bundle $T^v q$ converge and simplify in this limit. This fact is heavily used in local index theory, and we refer to [11, Sec 10.2] for details. In particular, the integral

$$\tilde{\Omega}(\lambda, \mathcal{E}) := \int_{(0,\lambda)\times B/B} \Omega(\mathcal{F}) \tag{5.8}$$

converges.

We now define

$$\hat{p}_!(\mathcal{E}, \rho) := [p_!\mathcal{E}, \int_{W/B} \hat{\mathbf{A}}^c(\tilde{\nabla}) \wedge \rho + \tilde{\Omega}(1, \mathcal{E}) + \int_{W/B} \sigma \wedge R([\mathcal{E}, \rho])] \in \hat{K}(B)\ . \tag{5.9}$$

This push-forward has all the expected properties, listed below. The proofs use the explicit model and a heavy dose of local index theory in order to verify that this cycle-level construction is compatible with the equivalence relation involving tamings and η-forms. They can be found in [26, Sections 3.2, 3.3, 4.2].

5.10 Theorem. *The push-forward/integration for differential K-theory defined in* (5.9) *has the following properties.*

(1) Given a proper submersion $p\colon W \to B$ with differential K-orientation and with fiber dimension n, the differential push-forward is $\hat{p}_!\colon \hat{K}^*(W) \to \hat{K}^{*-n}(B)$ a well defined homomorphism which only depends on the differential orientation, not its particular representative.

(2) If $q\colon Z \to W$ is another proper submersion with differential K-orientation, there is a canonical way to put a composed differential K-orientation on the composition $p \circ q$, and push-forward is functorial: $(p \hat{\circ} q)_! = \hat{p}_! \circ \hat{q}_!$.

(3) Fix a Cartesian diagram

$$\begin{array}{ccc} W' & \xrightarrow{F} & W \\ \downarrow{\scriptstyle f^*p} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

where as before $p\colon W \to B$ is a proper submersion with differential K-orientation. There is a canonical way to pull back the differential K-orientation of p to $p' := f^*p$. One then has compatibility of pull-back and push-forward

$$f^*\hat{p}_! = \hat{p}'_! F^*.$$

(4) The projection formula relating push-forward and product holds in differential K-theory

$$\hat{p}_!(p^*y \cup x) = y \cup \hat{p}_!(x); \qquad x \in \hat{K}(W),\ y \in \hat{K}(B).$$

(5) The differential push-forward is via the structure maps compatible with the push-forward in ordinary K-theory and on differential forms. However, for the differential forms we have to define the modified integration, depending on the differential K-orientation o, as

$$p_!^o\colon \Omega(W) \to \Omega(B);\ \omega \mapsto \int_{W/B} (\hat{\mathbf{A}}^c(\tilde{\nabla}) - d\sigma) \wedge \omega.$$

Indeed, this does not depend on the representative of the differential K-orientation. Moreover, it induces $p_!^o\colon \Omega(W)/\operatorname{im}(d) \to \Omega(B)/\operatorname{im}(d)$. With this map, we get commutative diagrams

$$\begin{array}{ccccccc} K(W) & \xrightarrow{\mathrm{ch}} & \Omega(W)/\operatorname{im}(d) & \xrightarrow{a} & \hat{K}(W) & \xrightarrow{I} & K(W) \\ \downarrow{\scriptstyle p_!} & & \downarrow{\scriptstyle p_!^o} & & \downarrow{\scriptstyle \hat{p}_!} & & \downarrow{\scriptstyle p_!} \\ K(B) & \xrightarrow{\mathrm{ch}} & \Omega(B)/\operatorname{im}(d) & \xrightarrow{a} & \hat{K}(B) & \xrightarrow{I} & K(B) \end{array} \tag{5.11}$$

$$\begin{array}{ccc} \hat{K}(W) & \xrightarrow{R} & \Omega_{d=0}(W) \\ \downarrow{\scriptstyle \hat{p}_!} & & \downarrow{\scriptstyle p_!^o} \\ \hat{K}(B) & \xrightarrow{R} & \Omega_{d=0}(B) \end{array} \tag{5.12}$$

5.3 S^1-integration

We consider the projection $\mathrm{pr}_1\colon B \times S^1 \to B$.

The projection pr_1 fits into the Cartesian diagram

$$\begin{array}{ccc} B \times S^1 & \xrightarrow{\mathrm{pr}_2} & S^1 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{r} & \{*\}\ . \end{array}$$

We choose the metric g^{TS^1} of unit volume and the bounding spin structure on TS^1. This spin structure induces a spin^c structure on TS^1 together with the connection $\tilde{\nabla}$. In this way we get a representative o of a differential K-orientation of p. By pull-back we get the representative r^*o of a differential K-orientation of pr_1 which is used to define $(\hat{\mathrm{pr}}_1)_!$.

5.13 Definition. We define S^1-integration for differential K-theory as in Definition 2.12 simply by setting

$$\int_{B\times S^1/B} := (p_1)_! \colon \hat{K}^*(B\times S^1) \to \hat{K}^{*-1}(B)$$

where we use the differential K-orientation of $p_1\colon B\times S^1 \to B$ just described. Note that by Theorem 5.10 it has the properties required of S^1-integration.

By [26, Corollary 4.6] we get

$$(\hat{\mathrm{pr}}_1)_! \circ \mathrm{pr}_1^* = 0.$$

6 Index theory and natural transformations

It is well established that index theory of elliptic operators is closely related to K-theory and K-homology. Indeed, this is the reason why the analytic model of Section 4.3 can work at all. For a fiber bundle $p\colon W \to B$ with K-orientation, the push-forward in K-theory can be interpreted as the family index of the fiberwise spin^c Dirac operator, twisted with the bundle representing the K-theory class. Of course, one might define the push-forward in a different way —then this is a somewhat abstract statement.

Continuing on this formal level, the Chern character provides a way to compute K-theory in terms of cohomology. Now there is also the push-forward in cohomology. It is well known that Chern character and push-forward are not compatible. The Riemann-Roch formula provides the appropriate correction. In some sense, a Riemann-Roch theorem therefore is a cohomological index theorem.

On this level, we will now find a lift of index theory to differential K-theory. In Section 5 we have discussed the push-forward in differential K-theory. We will now describe how to lift the Chern character to a natural transformation from differential K-theory to differential cohomology and will then discuss a differential Riemann-Roch theorem correcting the defect that this Chern character is not compatible with integration.

A further refinement of this theorem is obtained by a direct analytic definition of a (family) index with values in differential K-theory, given in a natural way for geometric families of elliptic index problems. The goal of an index theorem is then to find a topological formula for this index, i.e. a formula which does not involve the explicit solution of differential equations. This has indeed been achieved by Lott and Freed in [38] and we discuss the details in Section 6.3.

6.1 Differential Chern character

The classical Chern character has two fundamental properties. First, it is a certain characteristic class of vector bundles. As such, it is a certain explicit (rational) polynomial $p_{\mathrm{ch}}(c_1, c_2, \dots)$ in the Chern classes of the vector bundle. Secondly, it tuns out that this characteristic class is compatible with direct sum and stabilization and with Bott periodicity in the appropriate way to define a natural transformation of cohomology theories

$$\mathrm{ch}\colon K^*(\cdot) \to H^*(\cdot; \mathbb{Q}[u, u^{-1}]); \qquad (u \text{ of degree } 2).$$

Finally, after tensor product with $\mathbb{Q}$, $\mathrm{ch} \otimes \mathrm{id}_{\mathbb{Q}}$ becomes an isomorphism for finite CW-complexes.

We demand that the Chern character for differential K-theory should display the same properties: be implemented as characteristic class of vector bundles with connection, and then pass to a natural transformation between differential cohomology theories.

Characteristic classes in differential cohomology

Early on, differential cohomology is closely related to characteristic classes of vector bundles with connection. Indeed, $\hat{H}^2(X; \mathbb{Z})$ even is isomorphic to the set of isomorphism classes of complex line bundles with Hermitean connection. This isomorphism is implemented by the differential version of the first Chern class.

Elaborating on this, Cheeger and Simons [32, Section 4] construct differential Chern classes of vector bundles with Hermitean connection with values in integral differential cohomology

$$\hat{c}_k(E, \nabla) \in \hat{H}^{2k}(X; \mathbb{Z})$$

with the following properties:

(1) naturality under pullback with smooth maps;

(2) compatibility with the classical Chern classes:

$$I(\hat{c}_k(E, \nabla)) = c_k(E) \in H^{2k}(X; \mathbb{Z});$$

(3) compatibility with Chern-Weil theory

$$R(\hat{c}_k(E, \nabla)) = C_k(\nabla^2) \in \Omega^{2k}(X)$$

where $C_k(\nabla)$ is the Chern-Weil form of the connection ∇ associated to the invariant polynomial C_k which is (up to a scalar) the elementary symmetric polynomial in the eigenvalues.

Moreover, Cheeger and Simons show that the differential Chern classes are uniquely determined by these requirements. The proof is reminiscent of (and indeed inspired) the proof of uniqueness in Section 3. They use the universal example of $\mathbb{C}^n$-vector bundles with connection —known to exist by [55]. Once the differential Chern classes are chosen for the universal example, they are determined by naturality for every (E, ∇). The integral cohomology of the base space $BU(n)$ of the universal example is concentrated in even degrees. The long exact sequence (2.6) then implies that $I \oplus R\colon \hat{H}^{2k}(BU(n); \mathbb{Z}) \oplus \Omega^{2k}_{d=0}(BU(n))$ is injective, so that (2) and (3) determine the universal $\hat{c}_k$, and they exist by the defining properties of $C_k(\nabla^2)$. One only has to check that the construction is independent of the choice of the universal model, which essentially follows from uniqueness. Note that $BU(n)$ is not itself a finite dimensional manifold so that one has (as usual) to work with finite dimensional approximations.

Differential ordinary cohomology is defined with coefficients in any subring of $\mathbb{R}$, and the inclusion of coefficient rings induces natural maps with all the expected compatibility relations. In particular, we have differential cohomology with coefficients in $\mathbb{Q}$, $\hat{H}^*(\cdot;\mathbb{Q})$, taking values in the category of $\mathbb{Q}$-vector spaces, and commutative diagrams

$$\begin{array}{ccc} \hat{H}(\cdot;\mathbb{Z}) & \longrightarrow & \hat{H}(\cdot;\mathbb{Q}) \\ \downarrow R & & \downarrow R \\ \Omega(\cdot) & \xrightarrow{=} & \Omega(\cdot) \end{array} \quad ; \quad \begin{array}{ccc} \hat{H}(\cdot;\mathbb{Z}) & \longrightarrow & \hat{H}(\cdot;\mathbb{Q}) \\ \downarrow I & & \downarrow I \\ H^*(\cdot;\mathbb{Z}) & \longrightarrow & H^*(\cdot;\mathbb{Q}). \end{array}$$

Expressing the classical Chern character (uniquely) as a rational polynomial in the Chern classes

$$\mathrm{ch}(E) = P_{\mathrm{ch}}(c_1(E), c_2(E), \dots); \quad \text{with } P_{\mathrm{ch}} \in \mathbb{Q}[[x_1, x_2, \dots]],$$

we now define the differential Chern character

$$\hat{\mathrm{ch}}(E,\nabla) := P_{\mathrm{ch}}(\hat{c}_1(E,\nabla), \dots) \in \hat{H}^{2*}(X;\mathbb{Q}). \tag{6.1}$$

By construction, this is a natural characteristic class satisfying

$$I(\hat{\mathrm{ch}}(E,\nabla)) = \mathrm{ch}(E) \in H^{2*}(X;\mathbb{Q})$$
$$R(\hat{\mathrm{ch}}(E,\nabla)) = P_{\mathrm{ch}}(C_1(\nabla^2), \dots) = \mathrm{tr}(-\exp(\nabla^2/2\pi i)) \in \Omega^{2*}(X).$$

Differential Chern character transformation

The second point of view of the differential Chern character is as a natural transformation between differential K-theory and differential cohomology. Indeed, we prove in [26, Section 6] the following theorem.

6.2 Theorem. *There is a unique natural transformation of differential cohomology theories*

$$\hat{\mathrm{ch}}\colon \hat{K}^*(\cdot) \to \hat{H}^*(\cdot;\mathbb{Q}[u,u^{-1}])$$

with the following properties:

(1) Compatibility with the Chern character in ordinary cohomology and with the action of forms, i.e. the following diagram commutes:

$$\begin{array}{ccccc} \Omega(X)/\operatorname{im}(d) & \xrightarrow{a} & \hat{K}(X) & \xrightarrow{I} & K(X) \\ \downarrow = & & \downarrow \hat{\mathrm{ch}} & & \downarrow \mathrm{ch} \\ \Omega(X)/\operatorname{im}(d) & \xrightarrow{a} & \hat{H}(X;\mathbb{Q}[u,u^{-1}]) & \xrightarrow{I} & H(X:\mathbb{Q}[u,u^{-1}]) \end{array} .$$

(2) Compatibility with the curvature homomorphism, i.e. the following diagram commutes

$$\begin{array}{ccc} \hat{K}(X) & \xrightarrow{R} & \Omega_{d=0}(X;\mathbb{R}[u,u^{-1}]) \\ \downarrow \hat{\mathrm{ch}} & & \downarrow = \\ \hat{H}(X;\mathbb{Q}[u,u^{-1}]) & \xrightarrow{R} & \Omega_{d=0}(X;\mathbb{R}[u,u^{-1}]) \end{array} .$$

(3) Compatibility with suspension, i.e. the following diagram commutes

$$\begin{array}{ccc} \hat{K}(S^1 \times X) & \xrightarrow{\hat{\mathrm{ch}}} & \hat{H}(S^1 \times X; \mathbb{Q}[u,u^{-1}]) \\ \downarrow{\scriptstyle \mathrm{pr}_{2!}} & & \downarrow{\scriptstyle \mathrm{pr}_{2!}} \\ \hat{K}(X) & \xrightarrow{\hat{\mathrm{ch}}} & \hat{H}(X; \mathbb{Q}[u,u^{-1}]). \end{array}$$

Here, we use the differential K-orientation of pr_2 *as in Section 5.3.*

Moreover, if $x \in \hat{K}^0(X)$ *is represented (as in 4.1 or 4.3) by the zero-dimensional family* $((E,\nabla),\rho)$ *(with* ρ *the additional form of odd degree), then*

$$\hat{\mathrm{ch}}(x) = \hat{\mathrm{ch}}(E,\nabla) + a(\rho) \in \hat{H}^{2*}(X; \mathbb{Q}[u,u^{-1}]).$$

In addition, $\hat{\mathrm{ch}}$ *is a multiplicative natural transformation and becomes an isomorphism after tensor product with* $\mathbb{Q}$*.*

6.2 Differential Riemann-Roch theorem

We are now in the situation to formulate the Riemann-Roch theorem, which describes the relations between push-forward in differential K-theory and differential cohomology and the differential Chern character. Let $p\colon W \to B$ be a proper submersion with a differential K-orientation o represented by $(g^{T^vp}, T^hp, \tilde{\nabla}, \sigma)$ as in Section 5.1.

The Riemann Roch theorem asserts the commutativity of a diagram

$$\begin{array}{ccc} \hat{K}(W) & \xrightarrow{\hat{\mathrm{ch}}} & \hat{H}(W,\mathbb{Q}) \\ \downarrow{\scriptstyle \hat{p}_!} & & \downarrow{\scriptstyle \hat{p}_!^A} \\ \hat{K}(B) & \xrightarrow{\hat{\mathrm{ch}}} & \hat{H}(B,\mathbb{Q}) \ . \end{array}$$

Here $\hat{p}_!^A$ is the composition of the cup product with a differential rational cohomology class $\hat{\hat{\mathbf{A}}}{}^c(o)$ and the push-forward in differential rational cohomology (uniquely determined by an ordinary orientation of p, in particular by the K-orientation which is already fixed).

We have to define the refinement $\hat{\hat{\mathbf{A}}}(o) \in \hat{H}^{ev}(W,\mathbb{Q})$ of the form $\hat{\mathbf{A}}^c(\tilde{\nabla}) \in \Omega^{ev}(W; R[u,u^{-1}])$. The geometric data of o determines a connection ∇^{T^vp} and hence a geometric bundle $\mathbf{T^vp} := (T^vp, g^{T^vp}, \nabla^{T^vp})$. According to [32] we can define Pontryagin classes

$$\hat{p}_i(\mathbf{T^vp}) \in \hat{H}^{4i}(W,\mathbb{Z}) \ , \quad i \geq 1 \ .$$

The spin^c-structure gives rise to a Hermitean line bundle $L^2 \to W$ with connection ∇^{L^2} (see (5.1)). We set $\mathbf{L}^2 := (L^2, h^{L^2}, \nabla^{L^2})$. Again using [32], we get a class

$$\hat{c}_1(\mathbf{L}^2) \in \hat{H}^2(W,\mathbb{Z}).$$

Inserting the classes $u^{-2i}\hat{p}_i(\mathbf{T^vp})$ into that $\hat{\mathbf{A}}$-series $\hat{\mathbf{A}}(p_1,p_2,\dots) \in \mathbb{Q}[[p_1,p_2\dots]]$ we define

$$\hat{\hat{\mathbf{A}}}(\mathbf{T^vp}) := \hat{\mathbf{A}}(\hat{p}_1(\mathbf{T^vp}), \hat{p}_2(\mathbf{T^vp}), \dots) \in \hat{H}^0(W, \mathbb{Q}[u,u^{-1}]) \ . \tag{6.3}$$

6.4 Definition. We define

$$\hat{\hat{\mathbf{A}}}^c(o) := \hat{\hat{\mathbf{A}}}(\mathbf{T^v p}) \wedge e^{\frac{1}{2u}\hat{c}_1(\mathbf{L}^2)} - a(\sigma) \in \hat{H}^0(W, \mathbb{Q}[u, u^{-1}]) \ .$$

Note that $R(\hat{\hat{\mathbf{A}}}^c(o)) = \hat{\mathbf{A}}^c(\tilde{\nabla})$, with $\hat{\mathbf{A}}^c(\tilde{\nabla})$ of (5.4).

By [26, Lemma 6.17], $\hat{\hat{\mathbf{A}}}^c(o)$ indeed does not depend on the particular representative of o. This follows from the homotopy formula.

We now define

$$\hat{p}_!^A \colon \hat{H}^*(W; \mathbb{Q}[u, u^{-1}]) \to \hat{H}^{*-n}(B; \mathbb{Q}[u, u^{-1}]);\ x \mapsto \hat{p}_!(\hat{\hat{\mathbf{A}}}^c(o) \cup x)$$

The differential Riemann-Roch now reads

6.5 Theorem. *The following square commutes*

$$\begin{array}{ccc} \hat{K}(W) & \xrightarrow{\hat{\mathrm{ch}}} & \hat{H}(W, \mathbb{Q}[u, u^{-1}]) \\ \downarrow{\scriptstyle \hat{p}_!} & & \downarrow{\scriptstyle \hat{p}_!^A} \\ \hat{K}(B) & \xrightarrow{\hat{\mathrm{ch}}} & \hat{H}(B, \mathbb{Q}[u, u^{-1}]). \end{array}$$

This diagram is compatible via the transformations I with the maps of the classical Riemann-Roch theorem.

6.3 Differential Atiyah-Singer index theorem

We want to understand the Atiyah-Singer index theorem as the equality of the analytic and the topological (family) index. To formulate a differential version of this, the first step is to define the differential analytic and topological (family) index.

We start with the proper submersion $p \colon W \to B$ with n-dimensional fibers, and we think of $W \to B$ as the underlying family of manifolds on which the family of elliptic operators shall be given. To be able to define a *differential* index, we have to choose geometry for $p \colon W \to B$, which amounts exactly to the choice of data representing a differential orientation of p as in Section 5.1, namely a tuple $(g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ consisting of vertical metric, horizontal distribution, fiberwise spin^c-structure and compatible spin^c-connection. The form σ could of course be chosen equal to 0.

We follow the definition of the analytic index as given by Freed and Lott in [38]. We start with a vector bundle (E, ∇_E) with connection over W (representing a class in $\hat{K}^0(W)$ using either the model of Section 4.1 or 4.3). We then want to define the analytic index of the family of spin^c-Dirac operators twisted by (E, ∇_E). This is based on Bismut-Quillen superconnections. Indeed, it uses the Bismut-Cheeger eta-form which mediates between the Chern character of the finite dimensional index bundle (giving the naive analytic index) and the Chern character of the Bismut superconnection. For details see below.

The topological index of [38] is modelled closely after the classical definition of the topological index by Atiyah and Singer. One factors $p \colon W \to B$ as a fiberwise embedding $W \to S^N \times B$ and $\mathrm{pr}_2 \colon S^N \times B \to B$. One then uses very explicit formulas for the differential K-theory push-forward of the embedding $W \to S^N \times B$, based on a model of differential K-theory using currents. Finally, a Künneth decomposition of $\hat{K}(S^N \times B)$ gives an explicit push-forward for $\mathrm{pr}_2\colon S^N \times B \to B$. The topological index is defined as a modification of the composition of these two push-forwards. It does not involve spectral analysis nor does it require the solution of differential equations. It does use differential forms, so the term "differential topological index" is indeed quite appropriate. For details again see below.

The main result of [38] is then

6.6 Theorem. *Differential analytic index and differential topological index both define homomorphism*

$$\mathrm{ind}^{an},\ \mathrm{ind}^{top}\colon \hat{K}^0(W) \to \hat{K}^{-n}(B).$$

Moreover, analytic and topological index coincide:

$$\mathrm{ind}^{an} = \mathrm{ind}^{top}$$

Finally, $\mathrm{ind}^{an} = \mathrm{ind}^{top}$ *fits into a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K\mathbb{R}/\mathbb{Z}^{-1}(W) & \longrightarrow & \hat{K}^0(W) & \xrightarrow{R} & \Omega^0_{d=0}(W;\mathbb{R}[u,u^{-1}]) \\ & & \downarrow{\scriptstyle \mathrm{ind}^{an}=\mathrm{ind}^{top}} & & \downarrow{\scriptstyle \mathrm{ind}^{an}=\mathrm{ind}^{top}} & & \downarrow{\scriptstyle \omega\mapsto\int_{W/B}\hat{\mathbf{A}}^c(\tilde{\nabla})\wedge\omega} \\ 0 & \longrightarrow & K\mathbb{R}/\mathbb{Z}^{-1-n}(B) & \longrightarrow & \hat{K}^{-n}(B) & \xrightarrow{R} & \Omega^{-n}_{d=0}(B;\mathbb{R}[u,u^{-1}]). \end{array}$$

In the following, we explain the ingredients of this formula.

(1) On the differential form level, $\hat{\mathbf{A}}^c(\tilde{\nabla})$ of (5.4) coincides with the form $Todd(\hat{\nabla}^W)$ of [38, (2.14)].

(2) For K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$, $\mathrm{ind}^{an}\colon K\mathbb{R}/\mathbb{Z}^{-1}(W) \to K\mathbb{R}/\mathbb{Z}^{-1-n}(B)$ has been constructed in and is the main theme of [51]. In particular, the equality of analytic and topological index in this context is proved there.

6.7 Proposition. *In the end, of course,* $\mathrm{ind}^{an} = \mathrm{ind}^{top}\colon \hat{K}^0(W) \to \hat{K}^{-n}(B)$ *coincide with* $\hat{p}_!\colon \hat{K}^0(W) \to \hat{K}^{-n}(B)$ *of Section 5.2.*

The main point is that ind^{an} is defined as an honest analytic index, whereas $\hat{p}_!$ only formally does so.

In the following, we will assume that the fiber dimension n is even. The case of odd n is easily reduced to this via suspension-desuspension constructions using products with S^1.

Analytic index in differential K-theory

We now define the analytic index of a cycle $(\mathcal{E},\phi) = (E, h_E, \nabla_E, \phi)$ for $\hat{K}^0(W)$ given in terms of a vector bundle with connection and an auxiliary form ϕ as in Section 4.1. To get a cleaner picture, we assume that the family D_E of twisted spin^c-Dirac operators over B, constructed as the geometric Dirac operators provided by the differential K-orientation data twisted with (E,∇_E) has a kernel bundle $\ker(D_E)$ which comes with an induced Hermitean metric and compressed connection $\nabla_{\ker(D_E)}$. The Bismut-Cheeger eta-form in this situation is defined as

$$\tilde{\eta} := u^{-n/2} R_u \int_0^\infty STr\left(u^{-1}\frac{dA_s}{ds} e^{u^{-1}A_s^2}\right)\, ds\ \in \Omega^{-n-1}(B;\mathbb{R}[u,u^{-1}])/\operatorname{im}(d).$$

R_u is introduced to simplify notation, it is induced from the ring homomorphism $\mathbb{R}[u, u^{-1}] \to \mathbb{R}[u, u^{-1}]; u \mapsto (2\pi i)u$. A_s is the Bismut superconnection on the (typically infinite dimensional) bundle $\mathcal{H}$ over B whose fiber over $b \in B$ is the space of sections of the twisted spinor bundle over $W_b = p^{-1}(B)$, the bundle on which D_E acts. More precisely, for $s > 0$

$$A_s = su^{1/2} D_E + \nabla^{\mathcal{H}} - s^{-1} u^{-1/2} c(R)/4.$$

Here, $\nabla^{\mathcal{H}}$ is a canonical connection on $\mathcal{H}$ constructed out of the given connections, and $c(R)$ is Clifford multiplication by the curvature 2-form of $\mathcal{H}$. For details about this construction, see [11, Section 10]. Note that, following Freed-Lott, powers of s instead of powers of $s^{1/2}$ are used in the definition of the superconnection.

The eta-form provides an interpolation between the Chern character form of the kernel bundle and of E as follows (compare [38, (3.11)] and [26, (0.6)]):

$$d\tilde{\eta} = \int_{W/B} \hat{A}^c(\tilde{\nabla}) \wedge \mathrm{ch}(\nabla_E) - \mathrm{ch}(\nabla^{\ker(D_E)}). \tag{6.8}$$

Following [38, Definition 3.12], one now defines for $(\mathcal{E}, \phi) = (E, h_E, \nabla_E, \phi)$

$$\mathrm{ind}^{an}(\mathcal{E}, \phi) := \left((\ker(D_E), h_{\ker(D_E)}, \nabla^{ker(D_E)}), \int_{W/B} \hat{A}^c(\tilde{\nabla}) \wedge \phi + \tilde{\eta} \right), \tag{6.9}$$

given by the zero dimensional family $\ker(D_E)$ with its Hermitean metric and connection, and the differential form part which uses the eta form.

In [38, Theorem 6.2] it is shown that this formula indeed factors through a map ind^{an} on differential K-theory as stated in Theorem 6.6. Alternatively one could use [26, Corollary 5.5] which immediately implies that $\mathrm{ind}^{an}(\mathcal{E}, \phi)$ is equal to the push-forward $\hat{p}^o_!(\mathcal{E}, \phi)$ as defined in (5.9).

The construction of the analytic differential index in the general case, i.e. if the kernels do not form a bundle, is carried out by a perturbation to reduce to the special case treated so far.

Topological index in differential K-theory

The topological index is defined in two steps. One chooses a *fiberwise isometric* fiberwise embedding $i \colon W \to S^N \times B$ of even codimension, where the target is equipped with the product structure. We assume that we have a differential spin^c-structure on the normal bundle ν of the embedding which is compatible with structures on W and on $S^N \times B$ (in the sense of [38, Section 5]).

Given a Hermitean bundle with connection (E, ∇_E) and a differential form ϕ on B, one now has to construct the differential push-forward i_* of $(\mathcal{E}, \phi) = (E, h_E, \nabla_E, \phi)$. This is based on the Thom homomorphism in differential K-theory. More precisely, one has to choose a ($\mathbb{Z}/2\mathbb{Z}$-graded) Hermitean bundle F on $S^N \times B$ with connection together with an endomorphism V such that V is invertible outside $i(W)$ but the kernel of V is isomorphic (as geometric bundle) to the tensor product of E with the spinor bundle, and a suitably defined compression of the covariant derivative of V under this isomorphism becomes Clifford multiplication. In [13, Definition 1.3] in this situation a transgression form γ is defined which interpolated between the Chern form of F and the image image of $\mathrm{ch}(\nabla_E)$ under the differential form spin^c-Thom homomorphism.

We then define

$$\hat{i}_!(\mathcal{E},\phi) := \Big((F,h_F,\nabla_F),\phi\wedge\hat{A}^o(\nabla_\nu)^{-1}\wedge\delta_X-\gamma-C'\Big)\in\hat{K}^{N-n}(S^N\times B) \tag{6.10}$$

where $\hat{A}^o(\nabla_\nu)^{-1}\wedge\delta_X$ is the current representing the spin^c-Thom form and C' is a further correction term which vanishes if the horizontal distribution of $W\to B$ is the restriction of the product horizontal distribution of $S^n\times B$ under the embedding i.

We now define

$$\mathrm{ind}^{top}(\mathcal{E},\phi) := \hat{p}_!(\hat{i}_!(\mathcal{E},\phi))\in\hat{K}^{-n}(B), \tag{6.11}$$

where $p\colon S^N\times B\to B$ is the projection and we use the product structure to define the spin^c-orientation.

Finally, we observe that for the product $S^N\times B$ we can use the Künneth type formula to obtain an explicit formula for $\hat{p}_!(x)$. More precisely, use the Künneth formula in ordinary K-theory to write the underlying K-theory class $I(X)=p^*a+\mathrm{pr}_1^*(t)\cdot p^*b$ where t is a second additive generator (besides 1) of $K^*(S^N)$. We lift all the classes a,b,t to classes A,B,T in differential K-theory and then write $X=p^*A+pr_1^*(T)\cdot p^*B+a(\alpha)$ with a suitable differential form α. One then obtains by [38, (5.31)]

$$\hat{p}_!(X)=B+a(\int_{S^N\times B/B}\hat{A}^o\wedge\alpha) \tag{6.12}$$

where $\hat{A}^o\in\Omega^0(S^N\times B;\mathbb{R}[u,u^{-1}])$ is given by the product structure on $p\colon S^N\times B\to B$. By [38, Corollary 7.36] this construction indeed factors through a homomorphism ind^{top} on differential K-theory.

Proof of the differential index theorem

It remains to prove that for any $(\mathcal{E},\phi)=(E,h_E,\nabla_E,\phi)$ as above, $\mathrm{ind}^{top}(\mathcal{E})-\mathrm{ind}^{an}(\mathcal{E})=0$. The constructions are carried out in such a way that the desired identity holds for the images under R. Therefore $\mathrm{ind}^{top}(\mathcal{E})-\mathrm{ind}^{an}(\mathcal{E})=:T\in K^{-n-1}(B;\mathbb{R}/\mathbb{Z})$ is a flat differential K-theory class. Now Freed and Lott in [38] follow the method of proof of the $\mathbb{R}/\mathbb{Z}$-index theorem of [51]. One has to show that the pairing of T with $K_{-n-1}(B)$ (the ordinary K-homology of B) vanishes. These pairings are given by reduced η-invariants (provided we have a kernel bundle). In the end, one has to establish certain identities for η-invariants on W, on B and on $S^N\times B$. These follow from adiabatic limit considerations and —to deal with the embedding $i\colon W\to S^N\times B$ also the main result [13, Theorem 2.2].

For the general case, one has to follow the effect of the perturbations which reduce to the case of a kernel bundle.

Related work

The thesis [48] discusses a model of even differential K-theory using vector bundles with connection and push-forward maps in this model. The work culminates in the special case of the index theorem for differential K-theory if B is the point.

7 Twisted differential K-theory

Usual cohomology theories often have severe limitations when dealing with situations in which orientations are required, but not present. This happens in particular when one wants to describe the cohomological properties of a fiber bundle which is not oriented for the cohomology theory one wants to study. Closely related is the non-existence of integration maps for non-orientable bundles or more generally non-orientable maps.

This problem is solved by using cohomology with twisted coefficients. For ordinary cohomology, this is just described by a local coefficient system, which one can easily implement e.g. in a Čech description of cohomology, compare e.g. [14, Chapter 10]. Twists have been introduced for generalized cohomology theories and successfully used. We refer to [53] for the approach to twisted cohomology via parametrized spectra and to [1] for a construction using infinity categories.

In particular, twisted K-theory has been studied extensively, motivated by by the classification of D-brane charges in the presence of a background B-field as discussed in Section 1.1.

7.1 Twists for ordinary K-theory

The most general twists for a multiplicative generalized cohomology theory represented by an E_∞-ring spectrum E are (up to equivalence) described by degree 0 cohomology classes with coefficients in $bgl_1(E)$, where $gl_1(E)$ is the spectrum of units of E and $bgl_1(E) := gl_1(E)[1]$ is its one-fold deloop. In the case of K-theory, the spectrum $bgl_1(K)$ contains the summand $H\mathbb{Z}[3]$. In other words, there is a subgroup of the isomorphism classes of twists for K-theory on a space B given by $H^3(B;\mathbb{Z})$. Most authors concentrate on these twists.

However, it is inappropriate to think only in terms of isomorphism classes of twists. The twists always form a pointed groupoid (with a trivial object). Technically, one can take the path groupoid of the mapping space

$$\mathrm{map}_{Spectra}(\Sigma^\infty X_+, bgl_1(E)) \ . \tag{7.1}$$

The twisted cohomology theory is more appropriately understood as a functor from this groupoid to graded abelian groups. This path groupoid is the truncation of the ∞-groupoid given by the mapping space itself which should really be considered as right object. In the present paper we prefer the truncation since it is used in most applications and the generalization to the differential case.

In particular, a given twist usually will have non-trivial automorphisms, and these automorphisms act non-trivially on the twisted cohomology. In our case, the automorphisms of the trivial K-theory twist on B are given by $H^2(B;\mathbb{Z})$. Because of the non-trivial automorphisms, for an isomorphism class of twists, e.g. $c \in H^3(B;\mathbb{Z})$ it does in general not make sense to talk about "‘the"’ twisted K-theory group $K^c(B)$. Only the isomorphism class of this group is well defined, but this is not sufficient e.g. if one wants to discuss functorial properties.

As indicated above one usually does not work with the most general kind of twists determined homotopy theoretically by $gl_1(E)$ but with a more explicit class closely tied to the relevant geometric situation. We therefore collect, following [37, Section 2, Section 3.1], the standard properties of a twisted extension of the cohomology theory E in an axiomatic manner.

7.2 Definition. Let E be a generalized cohomology theory. An extension of E to cohomology with twists consists of the following data:

- For every space X of a (pointed) groupoid $\mathfrak{Twist}_X$.
- For every continuous map $f\colon Y \to X$ a functor $f^*\colon \mathfrak{Twist}_X \to \mathfrak{Twist}_Y$ which is (weakly) functorial in f. Even more: the association $X \to \mathfrak{Twist}_X$ should become a weak presheaf of groupoids. In most examples this presheaf of groupoids satifies descend for open coverings and therefore forms a stack in topological spaces.
- Define then $\mathfrak{Twist}$ as the Grothendieck construction of the presheaf above, i.e. the category with objects (X,τ) where X is a space and $\tau \in \mathfrak{Twist}_X$ and morphisms from (X,τ_X) to (Y,τ_Y) consisting of a map $f\colon X \to Y$ together with an isomorphism $\tau_X \to f^*\tau_Y$. Define the category $\mathfrak{Twist}^2$ of *pairs in twists* with objects (X,A,τ) as before, but where $A \subset X$ and τ is a twist on X.
- The twisted version of E is then a contravariant functor from $\mathfrak{Twist}^2$ to graded abelian groups,

  $$(X,A,\tau) \mapsto E^{\tau+n}(X,A)$$

  together with natural transformation

  $$\delta\colon E^{\tau+n+1}(X,A) \to E^{\tau+n}(A,\emptyset).$$

  These have to satisfy the following properties:
- homotopy invariance: here, a homotopy between $f,g\colon (X,\tau_X) \to (Y,\tau_Y)$ is a morphism $h\colon (X\times[0,1],\mathrm{pr}^*\tau_X) \to (Y,\tau_Y)$ such that $i_0 \circ h = f$ and $i_1 \circ h = g$. The morphism of pairs $i_k\colon (X,\tau_X) \to (X\times[0,1],\mathrm{pr}^*\tau_X)$ uses the identity morphism o twists $i_k^*\,\mathrm{pr}^*\tau_X \to \tau_X$.
- long exact sequence of the pair:

  $$\to E^{\tau+n}(X,A) \to E^{\tau+n}(X) \to E^{\tau+n}(A) \xrightarrow{\delta} E^{\tau+n-1}(X,A) \to$$
- excision isomorphism
- wedge axiom: if $(X,A,\tau) = \amalg_{i\in I}(X_i,A_i,\tau_i)$ then the natural map

  $$E^{\tau+n}(X,A) \to \prod_{i\in I} E^{\tau+n}(X_i,A_i)$$

  is an isomorphism.
- For the base point $0 \in \mathfrak{Twist}_X$, we require that $E^{0+n}(X,A) = E^n(X,A)$ with the given definition of E^n.

Often, one will require additional structure, in particular a monoidal structure on $\mathfrak{Twist}_X$ which one typically writes additively. Then one requires a natural bilinear product

$$E^{\tau_1+n}(X,A) \otimes E^{\tau_2+m}(X,A) \to E^{\tau_1+\tau_2+n+m}(X,A)$$

which should be associative and graded commutative up to the natural isomorphism of twistings coming from the monoidal structure.

In this situation, one would also require a functorial and compatible push-forward for a proper map between smooth manifolds $f\colon X \to Y$

$$f_!\colon E^{f^*\tau+o(f)+*}(X) \to E^{\tau+*-(\dim X-\dim Y)}(Y)$$

where $o(f)$ is an orientation twist associated to the map f. An E-orientation of the map f will give rise to a trivialization $o(f) \to 0$, so that for an oriented map one has a push-forward

$$f_!\colon E^{f^*\tau+*}(X) \to E^{\tau+*-(\dim X-\dim Y)}(Y).$$

As usual with cohomology theories, there are variants, depending on which category of spaces and pairs of spaces one considers, and for which situations precisely one requires excision.

In the approach of [1] these axioms can easily be realized.

7.3 Example. As mentioned in Section 1.1, one can twist de Rham cohomology, defined on the category of smooth manifolds (possibly with boundary), as follows. Let N be a graded vector space, e.g. $N = \mathbb{R}[u, u^{-1}]$.

- $\mathfrak{Twist}_X := \Omega^1_{d=0}(X; N)$, the closed N-valued forms of total degree 1, with pullback the usual pullback. The base point is the form $\Omega = 0$.
- The morphisms from Ω_1 to Ω_2 are given by all forms $\eta \in \Omega^0(X; N)$ with $\Omega_2 = \Omega_1 + d\eta$. Composition is defined as the sum of differential forms.
- $H^{\Omega+n+ev}_{dR}(X, A) := \ker(d^\Omega|_{\bigoplus_{k\in\mathbb{Z}} \Omega^{n+2k}}) / \operatorname{im}(d^\Omega)$, with differential $d^\Omega(\omega) := d\omega + \Omega$. Note that the fact that Ω is closed implies that this is indeed a differential.
- Given a morphism $\eta \in \Omega^0(X; N)$ from Ω_1 to $\Omega_1 + d\eta$, define the induced isomorphism of twisted de Rham groups
$$\eta^*\colon H^{\Omega_1+n}_{dR}(X; N) \to H^{\Omega_1+d\eta+n}_{dR}(X; N); [\omega] \mapsto [\omega \cup \exp(-\eta)]$$
- The sum of forms defines a strictly symmetric monoidal structure on $\mathfrak{Twist}$ and the cup product of differential forms induces an associative and graded commutative product structure on twisted de Rham cohomology.

7.4 Remark. A variant of Example 7.3 uses as twists only the differential forms $\mathfrak{Twist}'_X := \Omega^3_{d=0}(X) \cdot u \subset \Omega^1_{d=0}(X; N)$. This is particularly relevant for the comparison with K-theory.

Note that the for the isomorphism classes of twists one obtains $\pi_0(\mathfrak{Twist}'_X) = H^3_{dR}(X)$.

In the case of K-theory, there are many different models for the groupoid of twists we consider. A particularly simple version is the following: Let U be the unitary group of an infinite dimensional separable Hilbert space H (with norm topology) and $PU := U/S^1$ where S^1 is the center, the multiples of the identity. Because of Kuiper's theorem, U is contractible and PU has the homotopy type of the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. Let K be the C^*-algebra of compact operators on H. Conjugation defines an action of PU on K by C^*-algebra automorphism.

7.5 Example. Assume that B is compact. $\mathfrak{Twist}_B$, the groupoid of twists for K-theory on B is now defined as the category of principal PU-bundles over B, with morphisms the homotopy classes of bundle isomorphisms. If τ is such a bundle, we can form the associated bundle of C^*-algebras $\tau \times_{PU} K$. The sections of this bundle form themselves a C^*-algebra. Now set $K^{\tau+*}(B) := K^*(\Gamma(\tau \times_{PU} K))$. A isomorphism $\beta\colon \tau' \to \tau$ of PU-principal bundles induces an isomorphism of associated K-bundles and therefore also of the C^*-algebras of sections, which finally induces a (functorial) isomorphism $\beta^*\colon K^{*+\tau'}(B) \to K^{*+\tau}(B)$.

Algebras of sections of K-bundles like $\tau \times_{BU} K$ are called "'continuous trace algebras"' and are an important object of study in operator algebras, compare e.g. [58]. This point of view of twisted K-theory —using continuous trace algebras— is exploited and developed e.g. in [52, 57, 61, 62].

Homotopy invariance of K-theory of C^*-algebras implies that β^* depends only on the homotopy class of β. The trivial twist is the trivial bundle $PU \times B$. Its automorphisms are given by maps from B to PU. As $PU = K(\mathbb{Z}, 2)$, the set of homotopy classes of such maps is $H^2(B;\mathbb{Z})$. For the special case of torsion classes in $H^3(B;\mathbb{Z})$, this model has first been considered in [33]. More precisely, this paper uses bundles of finite dimensional matrix algebras over B instead of K-bundles which is exactly the reason why only torsion twists occur. The general case is studied in [62, 61]. Another, closely related model for the twists is given by $U(1)$-bundle gerbes. This point of view is studied e.g. in [15].

Note that PU also acts by conjugation on the space Fred of Fredholm operators on H. The latter is a model for the zeroth space of the K-theory spectrum. Given a twist τ, we can form the associated bundle $\tau \times_{PU}$ Fred. We can then define $K^{0+\tau}(B)$ alternatively as the homotopy classes of sections of $\tau \times_{PU}$ Fred. This model is used e.g. in [5, 6]. One can define $K^{1+\tau}(B)$ by using an appropriate classifying space for K^1 instead of Fred which is a classifying space for K^0.

Obviously, a more refined version of this construction uses bundle of spectra (also called parametrized spectra) instead of bundles of spaces. A very precise version of such a model, with a satisfactory description of a product structure, of orientation and of the natural transformation from twisted spin^c-cobordism to twisted K-theory corresponding to the Atiyah orientation has been worked out in [63]. When dealing with bundles, it is necessary to deal with objects and maps on the nose, and not only up to homotopy.

Our description suggests a further "'categorification"' of the concept of (twisted) generalized cohomology theory. In the same way as twists have to be considered as a groupoid, one should also think of (twisted) generalized cohomology as a groupoid. The objects of this groupoid are the cocycles, and a cochain c of shifted degree (modulo boundaries) is a morphism from x to $x' = x + dc$. This would require a two-groupoid of twists, e.g. a two-truncation of the mapping space (7.1).

More explicitely, in in our example we might think of $H^3(X;\mathbb{Z})$ as the groupoid whose objects are isomorphism classes of principal PU-bundles over X, morphisms are PU-bundle maps and 2-morphism are homotopies of PU-bundle maps. Similarly, we might think of $K^{0+\tau}(B)$ as the groupoid whose objects are sections of $\tau \times_{PU}$ Fred and with morphisms homotopies of sections. If one likes ∞-categories then one could consider twisted cohomology as an ∞-functor which associates to a twist $\tau \in \mathrm{map}_{Spectra}(\Sigma^\infty X_+, gl_1(E))$ the spectrum of sections of the associated bundle E_τ of spectra. This can be made precise using [1]. In this picture it is easy to implement additional structures like multiplication or push-forward.

In the truncated groupoid picture most of this has been carried out in the even more elaborate equivariant situation in [37, Section 3]. The model there is based on the construction of twisted K-theory spectra.

7.2 Twisted differential K-theory

To define the concept of a twisted differential generalized cohomology theory, one has to combine the concept of twist with the concept of differential extension (which is *not* a cohomology theory, but there the deviation is well under control). One does need groupoids of *differential twists* which contain differential form information. Along the way, one will need an appropriate Chern character to twisted de Rham cohomology.

The following definition, what in general a twisted differential cohomology theory should be, follows essentially [44, Appendix A.3].

7.6 Definition. A differential extension of a twisted cohomology theory as in Definition 7.2 consists of the following data:

- for each smooth manifold X a groupoid $\mathfrak{Twist}_{\hat{E},X}$, together with (weakly) functorial pullback along smooth maps. They form a weak presheaf of groupoids. As above, we can "combine" all these groupoids to the category $\mathfrak{Twist}_{\hat{E}}$.
- natural functors
$$F\colon \mathfrak{Twist}_{\hat{E},X} \to \mathfrak{Twist}_X$$
$$\mathrm{Curv}\colon \mathfrak{Twist}_{\hat{E},X} \to \Omega^1_{d=0}(X;N).$$
Here and in the following, we lift the action of $\Omega^0(X;N)$ on twisted de Rham cohomology with the same formula to the twisted de Rham complexes.
- To each $\tau \in \mathfrak{Twist}_{\hat{E},X}$ we assign a Chern character
$$\mathrm{ch}^\tau\colon E^{F(\tau)+*}(X) \to H_{dR}^{\mathrm{Curv}(\tau)+*}(X;N),$$
natural with respect to pullback.
- The differential twisted extension of E is then a functor from the category of differential twists $\mathfrak{Twist}_{\hat{E}}$ to graded abelian groups:
$$(X,\tau) \mapsto \hat{E}^{\tau+*}(X).$$
Note that this includes functoriality with respect to pullback along smooth maps and along isomorphism of twists.
- There are natural (for pullback along smooth maps and along isomorphism of twists) transformations
$$I\colon \hat{E}^{\tau+*}(X) \to E^{F(\tau)+*}(X)$$
$$R\colon \hat{E}^{\tau+*}(X) \to \Omega^*_{d^{\mathrm{Curv}(\tau)}=0}(X;N)$$
$$a\colon \Omega^{*-1}(X;N)/\operatorname{im}(d^{\mathrm{Curv}(\tau)}) \to \hat{E}^{\tau+*}(X).$$

- They satisfy

$$R \circ a = d^{\mathrm{Curv}(\tau)} \qquad \text{and} \qquad \mathrm{ch}^\tau \circ I = \mathrm{pr} \circ R$$

  where $\mathrm{pr}\colon \Omega^*_{d^{\mathrm{Curv}(\tau)}=0}(X;N) \to H^{\mathrm{Curv}(\tau)}_{dR}(X;N)$ is the canonical projection.

- Using these, we get exact sequences

$$E^{F(\tau)+*-1}(X) \to \Omega^{*-1}(X;N)/\operatorname{im}(d^{\mathrm{Curv}(\tau)}) \xrightarrow{a} \hat{E}^{\tau+*}(X) \xrightarrow{I} E^{F(\tau)+*}(X) \to 0$$

Additionally, one would typically like to have a compatible product structure, as in Definition 7.2, with an adopted rule for the compatibility of the transformation a.

Finally, if one has a product structure, one would like to have a push-forward along smooth maps (or at least proper submersions) $f\colon X \to Y$ of the form

$$f_!\colon \hat{E}^{f^*\tau+o(f)+*}(X) \to \hat{E}^{\tau+*+(\dim X-\dim Y)}(Y),$$

where $o(f)$ is a differential orientation twist associated to f. A differential E-orientation should induce a trivialization $o(f) \to 0$ so that in this case one gets a push-forward

$$f_!\colon \hat{E}^{f^*\tau+*}(X) \to \hat{E}^{\tau+*+(\dim X-\dim Y)}(Y).$$

A first attempt toward a definition and description of twisted differential K-theory is given in [31], although not exactly in the setting of Definition 7.2. The main problems are of course:

(1) construction of the groupoid of differential twists

(2) construction of the differential cohomology groups

(3) construction of the push-forward.

[31] works with $U(1)$-banded bundle gerbes with connection and curving as objects of the groupoid of twists, and the curvature 3-form of this connection and curving is the transformation Curv (on objects). Given such a twist, they construct a principal $PU(H)$-bundle. Their twisted differential K-theory is then based on sections of associated bundles of Fredholm operators and explicitly constructed locally defined vector bundles with connection. For the rather elaborate precise definition, we refer to [31, Section 3].

7.7 Definition. The twists for X used in [44, Definition A.1] are *geometric central extensions*. Such a geometric central extension is

(1) a groupoid (P_0, P_1) with a local equivalence to the trivial groupoid (X, X),

(2) a central $U(1)$-extension of groupoids $L \to P_1$

(3) in particular, $L \to P_1$ is a $U(1)$-principal bundle, and another part of the data is a connection ∇ on this principal bundle,

(4) moreover, $L \to P_1$ being a central extension means one has over $P_2 = P_1 \times_{P_0} P_1$ an isomorphism of line bundles $\lambda\colon \operatorname{pr}_1^* L \otimes \operatorname{pr}_2^* L \to \circ^* L$ using the two projections and the composition of arrows $\operatorname{pr}_1, \operatorname{pr}_2, \circ\colon P_2 \to P_1$. λ should satisfy the cocycle condition, i.e. the two different ways to map $L_h \otimes L_g \otimes L_f$ to $L_{h\circ g\circ f}$ on $P_3 = P_1 \times_{P_0} P_1 \times_{P_0} P_1$ coincide.

(5) a 2-form $\omega \in \Omega^2(P_0)$.

These ingredients have to satisfy certain compatibility conditions explained in [44, Definition A.1]. In particular, $p_1^*\omega - p_0^*\omega = \frac{\sqrt{-1}}{2\pi}\Omega^\nabla$, and λ is an isomorphism of line bundles with connection.

In [44] it is observed that no construction of twisted differential K-theory with their twists (the geometric central extensions) is available yet, but one certainly expects that such a construction is possible.

7.3 T-duality

Motivated from string theory, T-duality is expected to be an equivalence of low energy limits of type IIA/B superstring theories on T-dual pairs. In particular, as D-brane charges are classified by twisted K-theory, T-duality predicts a canonical isomorphism between appropriate twisted K-theory groups of the underlying topological spaces of the T-dual pairs. This prediction has been made mathematically rigorous under the term "topological T-duality". It is investigated e.g. in [16, 23, 22, 29].

We briefly introduce into the mathematical setup as proposed in [22, Section 2], compare also [29, Section 4].

7.8 Definition. We let $T^n = U(1)^n$ be the n-dimensional real torus, considered as Lie group. Let B be a topological space (often with some restrictions, e.g. to be a compact CW-complex).

A T-duality triple consists of two T^n-principal bundles E, E' over the common base space B, and twists $\tau \in \mathfrak{Twist}_E$, $\tau' \in \mathfrak{Twist}_{E'}$ for K-theory. The third ingredient of a T-duality triple is an isomorphism of twists $u\colon p^*\tau \to (p')^*\tau'$ over the *correspondence space* $E \times_B E'$ with the two canonical projections $p\colon E \times_B E' \to E$ and $p'\colon E \times_B E' \to E'$.

The twists and the isomorphism u of twists have to satisfy certain conditions. These are most transparent if $n = 1$. In this case, they simply say that

$$\int_{E/B} [\tau] = c_1(E'); \qquad \int_{E'/B} [\tau'] = c_1(E),$$

where $[\tau] \in H^3(E;\mathbb{Z})$ is the characteristic class determined and determining the isomorphism class of the twist τ. Moreover, restricted to a point each $x \in B$, $\tau|_{E_x}$ and $\tau'_{E'_x}$ are canonically trivialized (because $E_x \cong U(1) \cong E'_x$ have vanishing H^2 and H^3). Consequently, using the induced trivializations, the restriction of u to the fiber over x becomes an automorphism of the trivial twist and therefore is classified by an element in $H^2(E_x \times_x E'_x) \cong H^2(U(1) \times U(1);\mathbb{Z}) \cong \mathbb{Z}$. We require that this element is the canonical generator.

For the details for general n, we refer to [30, Section 2], where this is again treated using cohomology, or [29, Defintion 4.1.3] where the language of stacks is used.

Of course, in this setting we first have to choose appropriate data for a twisted extension of K-theory, e.g. the model where twists are PU-principal bundles or $U(1)$-banded gerbes.

7.9 Definition. Given a T-duality triple $((E,\tau),(E',\tau'),u)$ as in Definition 7.8, we define the T-duality transformation of twisted K-theory

$$T := p'_! u^* p^* \colon K^{*+\tau}(E) \to K^{*-n+\tau'}(E'). \tag{7.10}$$

It is defined as the composition of pull-back to the correspondence space, using u to map τ-twisted K-theory to τ'-twisted K-theory and finally integration along p', where we use the fact that T^n-principal bundles are canonically oriented for any cohomology theory, in particular for K-theory.

The main results of [22] and [29] concern

(1) the classification of T-duality triples: there is e.g. a universal T-duality triple over a classifying space $\mathbf{R}_n$ of such triples whose homotopy type is computed: it is the homotopy fiber in the sequence

$$\mathbf{R}_N \to K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2) \xrightarrow{\cup} K(\mathbb{Z},4) \tag{7.11}$$

where the map $\cup$ is the composition of the map associated to the usual cup-product and the standard scalar product of the coefficient group $\mathbb{Z}^n \otimes \mathbb{Z}^n \to \mathbb{Z}$.

Note, therefore, that up to equivalence all the information of a T-duality triple is given by two T^n-principal bundles P, P' with Chern classes $c_1,\dots,c_n$, $c'_1,\dots,c'_n$ with $\sum c_i c'_i = 0$, together with an explicit trivialization of the cycle representing this product (e.g. a lift of its classifying map to the homotopy fiber $\mathbf{R}_n$).

In [22], we also discuss, which pairs (E,τ) can be part of a T-duality triple and in how many ways. For $n=1$, this is always the case, even in a unique way (up to equivalence). For $n>1$, both these assertions are wrong in general.

(2) The T-duality transform T of (7.10) is always an isomorphism (compare [22, Theorem 6.2]).

7.4 T-duality and differential K-theory

Alex Kahle and Alessandro Valentino [44] study the effect of T-duality in differential K-theory. They follow the approach of Section 7.3, i.e. they first make a precise definition of a differential T-duality triple [44, Definition 2.1] (there called "pair").

However, they make very efficient use of the higher structures of differential cohomology alluded to above.

7.12 Definition. Fix a base space B. A differential T-duality triple on B according to Kahle and Valentino [44, Definition 2.1] first consists of two objects $\mathcal{P} = (P,\nabla), \mathcal{P}' = (P',\nabla')$ in the groupoid of cycles for differential cohomology with coefficients in $\mathbb{Z}^n$, given by two T^n-bundles with connection over B, $\pi\colon P \to B$, $\pi'\colon P' \to B$.

Secondly, using a product on the level of the groupoid of cycles, form $\mathcal{P}\cdot\mathcal{P}'$, a cycle for $\hat{H}^4(B;\mathbb{Z})$ (on the level of the coefficients $\mathbb{Z}^n$, for this multiplication we use the standard inner product). The last ingredient for a T-duality triple is an isomorphism in the groupoid of cycles for differential H^4:

$$\sigma\colon 0 \to \mathcal{P}\cdot\mathcal{P}'.$$

Note that the existence of such a trivialization is a strong condition on $\mathcal{P},\mathcal{P}'$.

Observe that this description is very much in line with the description of the homotopy type of the classifying space for topological T-duality triples (7.11) and the resulting description of T-duality triples.

To obtain the twists for differential K-theory one expects for a differential T-duality triple, Kahle and Valentino argue as follows:

The pullback of $\mathcal{P}$ to the total space P of the underlying bundle has a canonical trivialization, and similarly for $\mathcal{P}'$. This trivialization can be multiplied with $\pi^*\mathcal{P}'$ to give a trivialization of $\pi^*\mathcal{P}\cdot\pi^*\mathcal{P}'$. The composition of $\pi^*\sigma$ with the inverse of this is an automorphism of the trivial object 0 and therefore defines a cycle $\hat{\tau}$ for the third differential cohomology of P. Similarly, we obtain a cycle $\hat{\tau}'$ for the third differential cohomology of P'. Finally, in [44, Lemma 2.2], it is shown how the canonical trivializations of $\pi^*\mathcal{P}$ and $(\pi')^*\mathcal{P}'$ give rise to a morphism $\hat{u}$ in the groupoid of cycles for $\hat{H}^3(P\times_B P')$ from $p^*\hat{\tau}$ to $(p')^*\hat{\tau}'$, where $p\colon P\times_B P'\to P$, $p'\colon P\times_B P'\to P'$ are the canonical projections.

The crucial points assumed by Kahle-Valentino is

(1) to have a groupoid cycle model for differential cohomology where cycles for $\hat{H}^2$ are principal $U(1)$-bundles with connection and where one has a multiplication with good properties on the level of cycles

(2) to have a model for a twisted extension of differential K-theory, where the groupoid of cycles for $\hat{H}^3$ is exactly the groupoid of twists.

Let us repeat that, at the moment, no complete construction of twisted differential K-theory satisfying these requirements seems to be available.

With these assumptions, it is now immediate how to define a T-duality transformation in twisted differential K-theory (assuming that "integration along T^n-principal bundles with connection" is also defined for the twisted differential K-theory at hand):

$$\hat{T} := \hat{p}'_! \circ \hat{u}^* \circ p^* \colon \hat{K}^{*+\hat{\tau}}(P) \to \hat{K}^{*-n+\hat{\tau}'}(P'). \tag{7.13}$$

Here $\hat{u}^*$ is the isomorphism induced by the isomorphism of differential twists $\hat{u}$ of [44, Lemma 2.2] as above.

However, there is one observation to be made: upon application of the curvature transformation, simple calculations show that $\hat{T}$ can never be surjective, as the forms in the image of its differential form analog have a very specific invariance property under the action of T^n.

7.14 Definition. For a T^n-principal bundle like P, let $\hat{\tau}$ be a cycle for differential cohomology and twist for differential K-theory as above and assume that the differential form $R(\hat{\tau})$ is T^n-invariant. Define the *geometrically invariant subgroup* of twisted differential K-theory as

$$\hat{K}^{*+\hat{\tau}}(P)^{T^n} := \{x\in \hat{K}^{*+\hat{\tau}}(P) \mid g^*R(x) = R(x)\ \forall g\in T^n\}.$$

With this notion, Kahle and Valentino prove their main result [44, Theorem 2.4].

7.15 Theorem. *The differential T-duality transform $\hat{T}$ of* (7.13) *preserves the geometrically invariant subgroups and defines an isomorphism*

$$\hat{T}\colon \hat{K}^{*+\hat{\tau}}(P)^{T^n} \to \hat{K}^{*-n+\hat{\tau}'}(P')$$

The main point of the proof of this theorem is the construction of the transformation in such a way that it is compatible with all the transformations given for differential K-theory. One then has to check/use that the transformation is an isomorphism for geometrically invariant forms (the image under the curvature homomorphism of geometrically invariant differential K-theory) and for topological twisted K-theory. The proof then concludes using the five lemma.

8 Applications of differential K-theory

Differential K-theory is a natural home for many well known, and hopefully some new, typically secondary invariants. In this section, we want to present some examples of this kind. To be able to do this, we start with a couple of elementary calculations.

8.1 Lemma.

$$\hat{K}^1(*) = \mathbb{R}/\mathbb{Z}; \qquad \hat{K}^0(*) = \mathbb{Z}, \qquad \hat{K}^1_{flat}(*) = \mathbb{R}/\mathbb{Z}; \qquad \hat{K}^0_{flat}(*) = \{0\}$$

as follows directly from the short exact sequence (2.4).

8.1 Holonomy

Let (V, ∇) be a Hermitian vector bundle of rank n over S^1 with unitary connection and with holonomy ϕ (well defined modulo conjugation in $U(n)$). (V, ∇) defines a geometric family $\mathcal{V}$ and therefore an element in differential K-theory $[\mathcal{V}, 0]$. By [26, Lemma 5.3]

8.2 Lemma.

$$[\mathcal{V}, 0] = a(\frac{1}{2\pi i}\det(\phi)).$$

8.2 $\mathbb{Z}/k\mathbb{Z}$-invariants

Recall that a $\mathbb{Z}/k\mathbb{Z}$-manifold is a manifold W with boundary together with a manifold X together with a diffeomorphism $f\colon \partial W \to \underbrace{X \amalg \ldots \amalg X}_{n \text{ copies}}$.

We now associate to a family of $\mathbb{Z}/k\mathbb{Z}$-manifolds over B a class in $\hat{K}_{flat}(B)$.

8.3 Definition. A *geometric family of* $\mathbb{Z}/k\mathbb{Z}$*-manifolds* is a triple $(\mathcal{W}, \mathcal{E}, \phi)$, where $\mathcal{W}$ is a geometric family with boundary, $\mathcal{E}$ is a geometric family without boundary, and $\phi\colon \partial\mathcal{W} \xrightarrow{\sim} k\mathcal{E}$ is an isomorphism of the boundary of $\mathcal{W}$ with k copies of $\mathcal{E}$.

We define $u(\mathcal{W}, \mathcal{E}, \phi) := [\mathcal{E}, -\frac{1}{k}\Omega(\mathcal{W})] \in \hat{K}(B)$.

8.4 Lemma. *We have $u(\mathcal{W},\mathcal{E},\phi)\in \hat{K}_{flat}(B)$. This class is a k-torsion class. It only depends on the underlying differential-topological data.*

8.5 Theorem. *Let $B=*$ and $\dim(\mathcal{W})$ be even. Then $u(\mathcal{W},\mathcal{E},\phi)\in \hat{K}^1_{flat}(*)\cong \mathbb{R}/\mathbb{Z}$. Let $i_k\colon \mathbb{Z}/k\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ the embedding which sends $1+k\mathbb{Z}$ to $\frac{1}{k}$. Then*

$$i_k(\mathrm{ind}_a(\bar{W})) = u(\mathcal{W},\mathcal{E},\phi) \ ,$$

where $i_k(\mathrm{ind}_a(\bar{W}))\in\mathbb{Z}/k\mathbb{Z}$ is the index of the $\mathbb{Z}/k\mathbb{Z}$-manifold $\bar{W}$, and where we use the notation of [39].

8.3 Reduced eta-invariants

Let π be a finite group. We construct a transformation

$$\phi\colon \Omega_d^{\mathrm{spin^c}}(BU(n)\times B\pi)\to \hat{K}^{-d}_{flat}(*) \ .$$

Let $f\colon M\to BU(n)\times B\pi$ represent $[M,f]\in\Omega^{Spin^c}(BU(n)\times B\pi)$. This map determines a π-covering $p\colon \tilde{M}\to M$ and an n-dimensional complex vector bundle $V\to M$. We choose a Riemannian metric g^{TM} and a spin^c-connection $\tilde{\nabla}$. These structures determine a differential K-orientation of $t\colon M\to *$. We further fix a metric h^V and a connection ∇^V in order to define a geometric bundle $\mathbf{V}:=(V,h^V,\nabla^V)$ and the associated geometric family $\mathcal{V}$. The pull back of g^{TM} and $\tilde{\nabla}$ via $\tilde{M}\to M$ fixes a differential K-orientation of $\tilde{t}\colon\tilde{M}\to *$.

Then we set

$$\phi([M,f]) := [\tilde{t}_!(p^*\mathcal{V})\sqcup_* |\pi| t_!\mathcal{V}^{op}, 0]\in \hat{K}_{flat}(*) \ .$$

In [26, Section 5.10] it is shown that this class only depends on the bordism class of $[M,f]$.

More generally, without even the assumption that π is finite, choose two finite dimensional representations ρ_1,ρ_2 of π of the same dimension, with associated flat bundles F_1,F_2. Replace in the above $|\pi|\, t_!\mathcal{V}$ by $t_!(\mathcal{V}\otimes F_2)$ and $\tilde{t}_!(p^*\mathcal{V})$ by $t_!(\mathcal{V}\otimes F_1)$.

Note that this boils down to the previous case if ρ_1 is the regular representation of the finite group π, and ρ_2 is $\mathbb{C}^{|\pi|}$, where $\mathbb{C}$ stands for the trivial representation.

8.6 Proposition. *This construction defines a homomorphism*

$$\phi_{\rho_1,\rho_2}\colon \Omega_d^{\mathrm{spin^c}}(BU(n)\times B\pi)\to \hat{K}_{flat}(-d)(*).$$

If d is even, the target group is trivial. If d is odd, $\hat{K}^{-d}_{flat}(*)\cong\mathbb{R}/\mathbb{Z}$. In this case, ϕ_{ρ_1,ρ_2} coincides with the reduced rho-invariant of Atiyah-Patodi-Singer.

The construction immediately generalizes to a parameterized version: to a smooth family of d-dimensional spin^c-manifolds parameterized by B, with a family of $\mathbb{C}^n$-vector bundles and also of π-coverings one associates in the same way a class in $\hat{K}^{-d}_{flat}(B)$.

For details of all of this, compare [26, Section 5.10].

8.4 e-invariant

A framed manifold is a manifold M together with a trivialization of its tangent bundle.

[26, Proposition 5.22] states that a bundle of framed n-manifolds $\pi\colon E \to B$ has a canonical differential K-orientation, given by the fiberwise spin^c-structure which comes from the trivialization, and the spin^c-connection which again comes from the trivial connection (form part 0). We then define

$$e([\pi\colon E \to B]) := \hat{\pi}_!(1) \in \hat{K}^{-n}_{flat}(B).$$

The push-down is with respect to the canonical $\hat{K}$-orientation of π, and the flatness of the connection of this differential K-orientation in the end implies that $R(\hat{\pi}_1(1)) = 0$.

8.7 Proposition. *If $B = *$ and n is odd, $e([B]) \in K^{-1}_{flat}(*) = \mathbb{R}/\mathbb{Z}$.*

This class coincides with the Adam's classical e-invariant for the stable homotopy groups, identified with the framed bordism groups.

For details, compare [26, Section 5.11].

8.5 Secondary invariants for string bordism

In [19], using spectral invariants of Dirac operators, Bunke and Naumann construct a secondary Witten genus, a bordism invariant of string manifolds. They use differential cohomology to facilitate some of their calculations, compare e.g. [19, Lemma 2.2].

9 Equivariant differential K-theory and orbifold differential K-theory

As explained in Section 1.1, one of the motivations for the study of differential K-theory comes from physics, where fields in abelian gauge theories are suggested to be modelled by cocycles for differential K-theory and where some of the main features are captured by the properties of differential cohomology theories. In Section 7.4 we have seen how this is successfully applied to T-duality, another important subject motivated by string theory.

In particular for the latter, however, mathematical physics also requires the study of singular spaces. Such singular spaces often arise as quotients of smooth manifolds by the action of a group, which is one reason why equivariant situations are important. We would therefore like to study differential K-theory for singular spaces and equivariant differential K-theory. Unfortunately, these theories are not well understood yet.

In [25], we construct differential K-theory of representable smooth orbifolds, i.e. global quotients of a manifold by a compact group. The construction is based on equivariant local index theory in the spirit of Section 4.3. The relevant Chern character takes values in delocalized de Rham cohomology of the orbifold. In case of a global quotient by a finite group, this is defined in terms of the de Rham complexes of the fixed point sets. We obtain a ring valued functor with the usual properties of a differential extension of a cohomology theory.

For proper submersions (with smooth fibers) we construct a push-forward map in differential orbifold K-theory. Finally, we construct a non-degenerate intersection pairing with values in $\mathbb{C}/\mathbb{Z}$ for the subclass of smooth orbifolds which can be written as global quotients by a finite group action. We construct a real subfunctor of our theory, where the pairing restricts to a non-degenerate $\mathbb{R}/\mathbb{Z}$-valued pairing. Indeed, we use in that paper the language of (differentiable étale) stacks which turns out to be particularly convenient.

In [60, Section 5.4],a model for equivariant differential K-theory in the spirit of Section 4.2 is constructed. It uses the fact that there are very nice models for the classifying space for equivariant K-theory. As target for the Chern character on equivariant K-theory [60] uses Bredon cohomology with coefficients in the representation ring tensored with $\mathbb{R}$. Via a de Rham isomorphism, this is canonically isomorphic to delocalized de Rham cohomology. As in the non-equivariant case, this model is not so well suited to the construction of a product structure and of push-forwards, which are therefore not discussed in [60]. However, in [60, Section 6] is described how equivariant differential K-theory can be used to described Ramond-Ramond fields and their flux quantization condition in orbifolds of type II superstring theory.

The preprint [56] gives yet another construction of differential equivariant K-theory for finite group actions along the lines of [43], i.e. of Section 4.2. Moreover, it constructs a product and push-forward to a point. The constructions are mainly homotopy theoretical.Ortiz in [56] raises the interesting question [56, Conjecture 6.1] of identifying his push-forward in analytic terms. In the model of [25], in view of the geometric construction of the push-forward and the analytic nature of the relations, the conjectured relation is essentially a tautology. See [26, Corollary 5.5] for a more general statement in the non-equivariant case. [56, Conjecture 6.1] would be an immediate consequence of a theorem stating that any two models of equivariant differential K-theory for finite group actions are canonically isomorphic in a way compatible with integration. It seems to be plausible that the method of [27] extends to the equivariant case. In [56], the conjectured equivariant index formula is proved in a number of special cases, e.g. if Γ is trivial, or in case the G-manifold is a G- boundary.

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