Path Integrals for Quadratic Lagrangians on p-Adic and Adelic Spaces

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Abstract

Feynman's path integrals in ordinary, p-adic and adelic quantum mechanics are considered. The corresponding probability amplitudes $\mathcal{K}(x'', t''; x', t')$ for two-dimensional systems with quadratic Lagrangians are evaluated analytically and obtained expressions are generalized to any finite-dimensional spaces. These general formulas are presented in the form which is invariant under interchange of the number fields $\mathbb{R} \leftrightarrow \mathbb{Q}_p$ and $\mathbb{Q}_p \leftrightarrow \mathbb{Q}_{p'}$, $p \neq p'$. According to this invariance we have that adelic path integral is a fundamental object in mathematical physics of quantum phenomena.

1 Introduction

To describe dynamics of a particle in classical mechanics, there are Hamiltonian and Lagrangian formalisms which are equivalent. Quantum mechanics is usually related to quantization of a classical Hamiltonian consisting of a particle in an effective field given by a potential.

Starting from the Hamiltonian there are two ways to treat quantum evolution of a physical system: (i) the Heisenberg picture, where time dependence is directly related to the operator of an observable A, i.e.

$$i\hbar \frac{d\hat{A}}{dt} = i\hbar \frac{\partial \hat{A}}{\partial t} + [\hat{A}, \hat{H}], \tag{1}$$

and (ii) the Schrödinger picture, where time evolution is governed by the Schrödinger equation

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = H(\hat{k}, x)\Psi(x,t), \quad \hat{k} = -i\hbar \frac{\partial}{\partial x}. \tag{2}$$

Both approaches are invented in the 1925-26 and shown to be equivalent versions of the same theory called Quantum Mechanics.

Quantum mechanics related to the Lagrangian formalism started in the 1932 by Dirac's observation that the quantum state in a point $q + dq$ at the time $t + dt$ is connected with the state in the point q at t by the transformation function $\exp\left(\frac{i\, L\, dt}{\hbar}\right)$, where $L = L(\dot{q}, q, t)$ is

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the classical Lagrangian. In the 1940's, Feynman developed Dirac's approach and shown that dynamical evolution of the wave function $\Psi(x,t)$ is

$$\Psi(x'',t'') = \int \mathcal{K}(x'',t'';x',t')\Psi(x',t')dx', \tag{3}$$

where

$$\mathcal{K}(x'',t'';x',t') = \int_{x',t'}^{x'',t''} \exp\left(\frac{2\pi i}{h}\int_{t'}^{t''} L(\dot{q},q,t)dt\right)\mathcal{D}q, \tag{4}$$

and $\int_{t'}^{t''} L(\dot{q},q,t)\,dt = S[q]$ is the action for a path $q(t)$ connecting points x' and x''. The integral in (4) is known as the Feynman path integral. In the Feynman definition [1], discretizing the time t into equidistant subintervals, the path integral (4) is the limit of the corresponding multiple integral of N variables $q_i = q(t_i), \quad (i = 1,2,...,N)$, when $N \to \infty$. It is the primary object of the Feynman's path integral approach to quantum mechanics which is related to the classical Lagrangian formalism. Feynman's, Schrödinger's and Heisenberg's approaches to ordinary quantum mechanics are equivalent, but their formalisms are not equally suitable in some generalizations.

$\mathcal{K}(x'',t'';x',t')$ is the kernel of the corresponding unitary integral operator $U(t'',t')$ acting as follows:

$$\Psi(t'') = U(t'',t')\Psi(t'). \tag{5}$$

$\mathcal{K}(x'',t'';x',t')$ is also called the probability amplitude for a quantum particle to pass from a point x' at the time t' to the other point x'' at t''. It is closely related to Green's function and the quantum-mechanical propagator.

Starting from (3) one can easily derive the following three general properties:

$$\int \mathcal{K}(x'',t'';x,t)\mathcal{K}(x,t;x',t')dx = \mathcal{K}(x'',t'';x',t'), \tag{6}$$

$$\int \bar{\mathcal{K}}(x'',t'';x',t')\mathcal{K}(y,t'';x',t')dx' = \delta(x''-y), \tag{7}$$

$$\mathcal{K}(x'',t'';x',t'') = \lim_{t'\to t''}\mathcal{K}(x'',t'';x',t') = \delta(x''-x'), \tag{8}$$

where integration is over all the configuration space.

For all its history, the path integral has been a subject of great interest in theoretical and mathematical physics. It has became, not only in quantum mechanics (see, e.g. [2]) but also in the entire quantum theory, one of its the most profound and suitable approaches to foundations and elaborations. Feynman's path integral construction is also a natural and very successful instrument in formulation and investigation of p-adic [3] and adelic [4, 5] quantum mechanics. Moreover there are no p-adic analogs of the differential equations (1) and (2).

Adelic quantum mechanics contains complex-valued functions of real and all p-adic arguments in the adelic form. There is not the corresponding Schrödinger equation for p-adic dynamics, but Feynman's path integral method is quite appropriate. Feynman's path integral for probability amplitude in p-adic quantum mechanics $\mathcal{K}_p(x'',t'';x',t')$ [3], where $\mathcal{K}_p$ is complex-valued and x'',x',t'',t' are p-adic variables, is a direct p-adic generalization of (4), i.e.

$$\mathcal{K}_p(x'',t'';x',t') = \int_{x',t'}^{x'',t''} \chi_p\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q},q,t)dt\right)\mathcal{D}q, \tag{9}$$

where $\chi_p(a) = \exp 2\pi i\{a\}_p$ is p-adic additive character. The Planck constant h in (4) and (9) is the same rational number. We consider p-adic valued integral $\int_{t'}^{t''} L(\dot{q}, q, t)dt$ as the difference of antiderivative (without pseudoconstants) of $L(\dot{q}, q, t)$ in final (t'') and initial (t') times. In the case of time discretization we have $\mathcal{D}q = \prod_{i=1}^{N} dq(t_i)$, where $dq(t_i)$ is the p-adic additive Haar measure. Thus, p-adic path integral is the limit of the multiple Haar integral when $N \to \infty$. To calculate (9) in this way one has to introduce some ordering in the time $t \in \mathbb{Q}_p$, and it is successfully done in [6]. On previous investigations of p-adic path integrals one can see [7, 8, 9, 10] and references therein. Some mathematical aspects of Feynman's path integral on real space are vastly considered, e.g. see [11, 12, 13]. Path integral on p-adic space with p-adic valued probability amplitude was considered in [14].

Our main task here is an analytic evaluation of the p-adic (9) and the corresponding adelic Feynman path integrals for the general case of Lagrangians $L(\dot{q}, q, t)$, which are quadratic polynomials in $\dot{q}$ and q, without making time discretization. In fact, we will use the general requirements (in particular, (6) and (7)) which any (ordinary, p-adic and adelic) path integral must satisfy. In some parts of this evaluation, there is some similarity with Ref. [11]. Adelic path integral may be viewed as an infinite product of the ordinary one and p-adic path integrals for all primes p. Formal definition of adelic path integral, with some of its basic properties, will be presented in Section 5.

Some of the main motivations to apply p-adic numbers and adeles in quantum physics are: (*i*) the field of rational numbers $\mathbb{Q}$, which contains all observational and experimental numerical data, is the dense subfield not only in the field of real numbers $\mathbb{R}$ but also in the fields of p-adic numbers $\mathbb{Q}_p$, (*ii*) there is well developed analysis (e.g. see [15]) with p-adic valued and complex-valued functions over $\mathbb{Q}_p$ which is suitable in modern mathematical physics, (*iii*) general mathematical methods and fundamental physical laws should be invariant[16] under an interchange of the number fields $\mathbb{R}$ and $\mathbb{Q}_p$, (*iv*) there is a quantum gravity uncertainty Δx of distances around the Planck length ℓ_0, $\Delta x \geq \ell_0 = \sqrt{\hbar G/c^3} \sim 10^{-33}\text{cm}$, which restricts priority of archimedean geometry based on real numbers and gives rise to employment of nonarchimedean geometry related to p-adic numbers [16], and (*v*) it seems to be quite natural to extend path integral on real spaces to adelic one by adding probability amplitudes over the paths on all p-adic spaces.

Since 1987, there have been many publications (for a review, see, e.g. [15, 17, 18, 19]) on possible applications of p-adic numbers and adeles in modern theoretical and mathematical physics. The first successful employment of p-adic numbers was in string theory. In Volovich's article [20], a hypothesis on the existence of nonarchimedean geometry at the Planck scale was proposed and p-adic string theory was initiated. Using p-adic Veneziano amplitude as the Gel'fand-Graev [21] beta function, Freund and Witten obtained [22] an attractive adelic formula, which states that the product of the standard crossing symmetric Veneziano amplitude and all its p-adic counterparts equals a constant. Such approach gives a possibility to consider some ordinary string amplitudes as an infinite product of their inverse p-adic analogs. Many aspects of p-adic string theory have been of the significant interest.

For a systematic investigation of p-adic quantum dynamics, two kinds of p-adic quantum mechanics have been formulated: with complex-valued and p-adic valued wave functions of p-adic variables (for a review, see [3, 15] and [18], respectively). This paper is related to the first kind of quantum mechanics, which can be presented as a triple

$$(L_2(\mathbb{Q}_p), W, U(t)), \tag{10}$$

where $L_2(\mathbb{Q}_p)$ is the Hilbert space on $\mathbb{Q}_p$. W denotes the Weyl quantization procedure and $U(t)$ is the unitary representation of an evolution operator on $L_2(\mathbb{Q}_p)$. In our approach, $U(t)$ is naturally realized by the Feynman path integral method. In order to connect p-adic with standard quantum mechanics, adelic quantum mechanics was formulated [4]. Within adelic quantum mechanics a few basic physical systems [23, 24], including some minisuperspace cosmological models [25], have been successfully considered. As a result of p-adic effects in the adelic approach, a space-time discreteness at the Planck scale is obtained. Adelic path integral plays a central role and provides an extension of contributions from quantum trajectories over real space to probability amplitudes over paths in all p-adic spaces. There have been also investigations on application of p-adic numbers in the spin glasses, Brownian motion, stochastic processes, information systems, hierarchy structures, genetic code, dynamics of proteins and some other phenomena related to very complex dynamical systems (for a review see [15, 18, 26, 27, 19]).

2 p-Adic Numbers, Adeles and Their Functions

In this section we give a brief review of some basic properties of p-adic numbers, adeles, and their functions, which provides a minimum of mathematical background for next sections.

There are physical and mathematical reasons to start with the field of rational numbers $\mathbb{Q}$. From physical point of view, numerical results of all experiments and observations are some rational numbers, i.e. they belong to $\mathbb{Q}$. From algebraic point of view, $\mathbb{Q}$ is the simplest number field of characteristic 0. Recall that any $0 \neq x \in \mathbb{Q}$ can be presented as infinite expansions into the two different forms:

$$x = \sum_{k=n}^{-\infty} a_k 10^k, \quad a_k = 0, 1, \cdots, 9, \quad a_n \neq 0, \tag{11}$$

which is the ordinary one to the base 10, and the other one to the base p (p is any prime number)

$$x = \sum_{k=m}^{+\infty} b_k p^k, \quad b_k = 0, 1, \cdots, p-1, \quad b_m \neq 0, \tag{12}$$

where n and m are some integers which depend on x. The above representations (11) and (12) exhibit the usual repetition of digits, however the expansions are in the mutually opposite directions. The series (11) and (12) are convergent with respect to the metrics induced by the usual absolute value $|\cdot|_\infty$ and p-adic norm $|\cdot|_p$, respectively. Note that these valuations exhaust all possible inequivalent non-trivial norms on $\mathbb{Q}$. Performing completions, i.e. allowing all possible realizations of digits, one obtains standard representation of real and p-adic numbers in the form (11) and (12), respectively. Thus, the field of real numbers $\mathbb{R}$ and the fields of p-adic numbers $\mathbb{Q}_p$ exhaust all number fields which may be obtained by completion of $\mathbb{Q}$, and which contain $\mathbb{Q}$ as a dense subfield. Since p-adic norm of any term in (12) is $|b_k p^k|_p = p^{-k}$ if $b_k \neq 0$, geometry of p-adic numbers is the nonarchimedean one, *i.e.* strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$ holds and $|x|_p = p^{-m}$. $\mathbb{R}$ and $\mathbb{Q}_p$ have many distinct algebraic and geometric properties.

There is no natural ordering on $\mathbb{Q}_p$. However one can introduce a linear order on $\mathbb{Q}_p$ in the following way: $x < y$ if $|x|_p < |y|_p$, or if $|x|_p = |y|_p$ then there exists such index $r \geq 0$ that digits satisfy $x_m = y_m, x_{m+1} = y_{m+1}, \cdots, x_{m+r-1} = y_{m+r-1}, x_{m+r} < y_{m+r}$. Here, x_k and y_k are digits related to x and y in expansion (12). This ordering is very useful in time discretization and calculation of p-adic functional integral as a limit of the N-multiple Haar integral when $N \to \infty$.

There are mainly two kinds of analysis on $\mathbb{Q}_p$ which are of interest for physics, and they are based on two different mappings: $\mathbb{Q}_p \to \mathbb{Q}_p$ and $\mathbb{Q}_p \to \mathbb{C}$, where $\mathbb{C}$ is the field of ordinary complex numbers. We use both of these analyses, in classical and quantum p-adic models, respectively.

Elementary p-adic valued functions and their derivatives are defined by the same series as in the real case, but the regions of convergence of these series are determined by means of p-adic norm. As a definite p-adic valued integral of an analytic function $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ we take difference of the corresponding antiderivative in end points, i.e.

$$\int_a^b f(x) = \sum_{n=0}^{\infty} \frac{f_n}{n+1} \left(b^{n+1} - a^{n+1}\right).$$

Usual complex-valued functions of p-adic variable, which are employed in mathematical physics, are: (i) an additive character $\chi_p(x) = \exp 2\pi i \{x\}_p$, where $\{x\}_p$ is the fractional part of $x \in \mathbb{Q}_p$, (ii) a multiplicative character $\pi_s(x) = |x|_p^s$, where $s \in \mathbb{C}$, and (iii) locally constant functions with compact support, like $\Omega(|x|_p)$, where

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1, \\ 0, & |x|_p > 1. \end{cases} \tag{13}$$

There is well defined Haar measure and integration. So, we have

$$\int_{\mathbb{Q}_p} \chi_p(ayx)\, dx = \delta_p(ay) = |a|_p^{-1}\, \delta_p(y), \quad a \neq 0, \tag{14}$$

$$\int_{\mathbb{Q}_p} \chi_p(\alpha x^2 + \beta x)\, dx = \lambda_p(\alpha)\, |2\alpha|_p^{-\frac{1}{2}}\, \chi_p\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0, \tag{15}$$

where $\delta_p(u)$ is the p-adic Dirac δ-function. The number-theoretic function $\lambda_p(x)$ in (15) is a map $\lambda_p : \mathbb{Q}_p^* \to \mathbb{C}$ defined as follows [28]:

$$\lambda_p(x) = \begin{cases} 1, & m = 2j, & p \neq 2, \\ \sqrt{\left(\frac{-1}{p}\right)} \left(\frac{x_m}{p}\right), & m = 2j+1, & p \neq 2, \end{cases} \tag{16}$$

$$\lambda_2(x) = \begin{cases} \exp\left[\pi i (1/4 + x_{m+1})\right], & m = 2j, \\ \exp\left[\pi i (1/4 + x_{m+1}/2 + x_{m+2})\right], & m = 2j+1, \end{cases} \tag{17}$$

where x is presented in the form (12), $j \in \mathbb{Z}$, $\left(\frac{x_m}{p}\right)$ is the Legendre symbol defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if} \quad a \equiv y^2 (\text{mod } p), \\ -1, & \text{if} \quad a \not\equiv y^2 (\text{mod } p), \end{cases} \tag{18}$$

and $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$. We will also take $\lambda_p(0) = 1$. It is often sufficient to use standard properties:

$$\lambda_p(a^2 x) = \lambda_p(x), \quad \lambda_p(x)\lambda_p(-x) = 1, \quad \lambda_p\left(\frac{xy}{x+y}\right) = \frac{\lambda_p(x)\,\lambda_p(y)}{\lambda_p(x+y)},$$

$$\lambda_p(x)\,\lambda_p(y) = (x,y)_p\,\lambda_p(xy)\,\lambda_p(1), \qquad |\lambda_p(x)|_\infty = 1, \qquad a \neq 0, \tag{19}$$

where $(x,y)_p$ is the Hilbert symbol. Recall that the Hilbert symbol $(a,b)_p\,, \quad a,b \in \mathbb{Q}_p$, is $+1$ or -1 if there exist such $x,y,z \in \mathbb{Q}_p$ that equation $a\,x^2 + b\,y^2 = z^2$ has or has not a nontrivial solution, respectively.

Recall that the real analogs of (14) and (15) have the same form , *i.e.*

$$\int_{\mathbb{Q}_\infty} \chi_\infty(ayx)\,dx = \delta_\infty(ay) = |a|_\infty^{-1}\,\delta_\infty(y)\,, \quad a \neq 0, \tag{20}$$

$$\int_{\mathbb{Q}_\infty} \chi_\infty(\alpha x^2 + \beta x)\,dx = \lambda_\infty(\alpha)\,|2\alpha|_\infty^{-\frac{1}{2}}\,\chi_\infty\left(-\frac{\beta^2}{4\alpha}\right)\,, \quad \alpha \neq 0, \tag{21}$$

where $\mathbb{Q}_\infty \equiv \mathbb{R}$, $\chi_\infty(x) = \exp(-2\pi i x)$ is additive character in the real case and δ_∞ is the ordinary Dirac δ-function. Function $\lambda_\infty(x)$ is defined as

$$\lambda_\infty(x) = \exp\left[-\pi i \frac{\mathrm{sgn}\ x}{4}\right], \quad x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \tag{22}$$

and exhibits the same properties (19), i.e. equalities (19) have place if we replace index p by ∞. In the real case, the Hilbert symbol $(x,y)_\infty$ is equal to -1 if $x < 0$, $y < 0$ and otherwise is $+1$.

Since we are interested in Feynman's path integral on spaces with any finite number of dimensions, generalization of some previous formulas has to be introduced.

Definition 2.1. Let

$$\Lambda_v(x_1\,,x_2\,,\cdots\,,x_n) = \prod_{i=1}^{n} \lambda_v(x_i) \tag{23}$$

be new number-theoretic functions, where subscript $v = \infty, 2, 3, \cdots, p, \cdots$ denotes real as well as any p-adic case.

Proposition 2.2. *The new functions $\Lambda_v(x_1\,,x_2\,,\cdots\,,x_n)$ satisfy the following property:*

$$\Lambda_v(x_1\,,x_2\,,\cdots\,,x_n) = \lambda_v(x_1 x_2 \cdots x_n)\lambda_v^{n-1}(1) \prod_{i<j\leq n} (x_i, x_j)_v\,. \tag{24}$$

Proof. Formula (24) follows from the above property $\lambda_v(a)\,\lambda_v(b) = (a,b)_v\,\lambda_v(1)\,\lambda_v(ab)$ and the properties of the Hilbert symbol: $(a,b)_p = (b,a)_p$ and $(a,bc)_p = (a,b)_p\,(a,c)_p$, see [28]. □

Proposition 2.3. *Let $x = (x_1\,,x_2\,,\cdots,x_n)$, $y = (y_1\,,y_2\,,\cdots,y_n)$ be column vectors, and let $B = (B_{kl})$ be a nonsingular $n \times n$ matrix, where x_k, y_k, $B_{kl} \in \mathbb{Q}_v$. Then*

$$\int_{\mathbb{Q}_v^n} \chi_v(y^T B x)\,d^n x = |\det(B_{kl})|_v^{-1} \prod_{k=1}^{n} \delta_v(y_k)\,, \tag{25}$$

where y^T denotes transpose map of y.

Proof. Let us change variables of integration by $z_k = \sum_{l=1}^{n} B_{kl} x_l$. Then we have $d^n z = |\det(B_{kl})|_v \, d^n x$. The integral (25) can be rewritten as $|\det(B_{kl})|_v^{-1} \prod_{k=1}^{n} \int_{\mathbb{Q}_v} \chi_v(y_k z_k)\, dz_k$. According to (14) and (20), we obtain (25). $\square$

Proposition 2.4. *Let $x = (x_1, x_2, \cdots, x_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ be two column vectors, and let $\alpha = (\alpha_{kl})$ be a nonsingular symmetric $n \times n$ matrix, where x_k, β_k, $\alpha_{kl} \in \mathbb{Q}_v$. Then*

$$\int_{\mathbb{Q}_v^n} \chi_v(x^T \alpha x + \beta^T x)\, d^n x = \Lambda_v(\alpha_1, \alpha_2, \cdots, \alpha_n)\, |\det(2\alpha_{kl})|_v^{-\frac{1}{2}} \chi_v\left(-\frac{1}{4}\beta^T \alpha^{-1} \beta\right), \tag{26}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are eigenvalues of the matrix α.

Proof. Consider first the case $\beta = 0$. Using an orthogonal rotation $n \times n$ matrix A such that $x' = Ax$ and $x^T \alpha x = x'^T \alpha' x'$, where $\alpha' = A \alpha A^T = diag(\alpha_1, \alpha_2, \cdots, \alpha_n)$, one obtains

$$\begin{aligned} \int_{\mathbb{Q}_v^n} \chi_v(x^T \alpha x)\, d^n x &= \prod_{k=1}^{n} \int_{\mathbb{Q}_v} \chi_v(x'_k \alpha_k x'_k)\, dx'_k = \prod_{k=1}^{n} \lambda_v(\alpha_k)\, |2\alpha_k|_v^{-\frac{1}{2}} \\ &= \Lambda(\alpha_1, \alpha_2, \ldots, \alpha_n) |\det(2\alpha_{kl})|_v^{-\frac{1}{2}} . \end{aligned}$$

Employing (15) and (21), as well as (23) and the property that the determinant of a matrix is the product of all its eigenvalues, we gain (26) for $\beta = 0$. The final result follows from the identity

$$x^T \alpha\, x + \beta^T x = (x + \frac{1}{2}\alpha^{-1}\beta)^T \alpha\, (x + \frac{1}{2}\alpha^{-1}\beta) - \frac{1}{4}\beta^T \alpha^{-1} \beta$$

and after shifting the integration variable. $\square$

Remark 2.5. Since the determinant of a matrix is the product of all its eigenvalues, it is worth noting that according to (24) one can express $\Lambda_v(\alpha_1 \alpha_2 \cdots \alpha_n)$ in (26) in the following form:

$$\Lambda_v(\alpha_1\,, \alpha_2\,, \cdots, \alpha_n) = \lambda_v(\det(\alpha_{kl}))\, \lambda_v^{n-1}(1) \prod_{i<j\leq n} (\alpha_i, \alpha_j)\,. \tag{27}$$

For more information on usual properties of p-adic numbers and related analysis one can see [15, 21, 29].

Real and p-adic numbers are unified in the form of adeles. An adele x [21] is an infinite sequence

$$x = (x_\infty, x_2, \cdots, x_p, \cdots), \tag{28}$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set $\mathcal{P}$ of primes p one has $x_p \in \mathbb{Z}_p$, where $\mathbb{Z}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\}$ is the ring of p-adic integers. Componentwise addition and multiplication are natural operations on the ring of adeles $\mathbb{A}$, which can be regarded as

$$\mathbb{A} = \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P}), \qquad \mathbb{A}(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p. \tag{29}$$

$\mathbb{A}$ is a locally compact topological space.

There are also two kinds of analysis over topological ring of adeles $\mathbb{A}$, which are generalizations of the corresponding analyses over $\mathbb{R}$ and $\mathbb{Q}_p$. The first one is related to mapping $\mathbb{A} \to \mathbb{A}$ and the other one to $\mathbb{A} \to \mathbb{C}$. In complex-valued adelic analysis it is worth mentioning an additive character

$$\chi(x) = \chi_\infty(x_\infty) \prod_p \chi_p(x_p), \tag{30}$$

a multiplicative character

$$|x|^s = |x_\infty|_\infty^s \prod_p |x_p|_p^s, \quad s \in \mathbb{C}, \tag{31}$$

and elementary functions of the form

$$\phi(x) = \phi_\infty(x_\infty) \prod_{p \in \mathcal{P}} \phi_p(x_p) \prod_{p \notin \mathcal{P}} \Omega(|x_p|_p), \tag{32}$$

where $\phi_\infty(x_\infty)$ is an infinitely differentiable function on $\mathbb{R}$ such that $|x_\infty|_\infty^n \phi_\infty(x_\infty) \to 0$ as $|x_\infty|_\infty \to \infty$ for any $n \in \{0, 1, 2, \cdots\}$, and $\phi_p(x_p)$ are locally constant functions with compact support. All finite linear combinations of elementary functions (32) make the set $S(\mathbb{A})$ of the Schwartz-Bruhat adelic functions. The Fourier transform of $\phi(x) \in S(\mathbb{A})$, which maps $S(\mathbb{A})$ onto $S(\mathbb{A})$, is

$$\tilde{\phi}(y) = \int_{\mathbb{A}} \phi(x) \chi(xy) dx, \tag{33}$$

where $\chi(xy)$ is defined by (30) and $dx = dx_\infty dx_2 dx_3 \cdots$ is the Haar measure on $\mathbb{A}$.

It is worth mentioning the following adelic products [28]:

$$\chi_\infty(x) \prod_p \chi_p(x) = 1, \quad x \in \mathbb{Q} \tag{34}$$

$$|x|_\infty^s \prod_p |x|_p^s = 1, \quad x \in \mathbb{Q}^*, \; s \in \mathbb{C} \tag{35}$$

$$\lambda_\infty(x) \prod_p \lambda_p(x) = 1, \quad x \in \mathbb{Q}^* \tag{36}$$

$$(x, y)_\infty \prod_p (x, y)_p = 1, \quad x, y \in \mathbb{Q}^*. \tag{37}$$

One can define the Hilbert space on $\mathbb{A}$, which we will denote by $L_2(\mathbb{A})$. It contains infinitely many complex-valued functions of adelic argument (for example, $\Psi_1(x), \Psi_2(x), \cdots$) with scalar product $(\Psi_1, \Psi_2) = \int_{\mathcal{A}} \bar{\Psi}_1(x) \Psi_2(x) dx$ and norm $||\Psi|| = (\Psi, \Psi)^{\frac{1}{2}} < \infty$, where dx is the Haar measure on $\mathbb{A}$. A basis of $L_2(\mathbb{A})$ may be given by the set of orthonormal eigefunctions in spectral problem of the evolution operator $U(t)$, where $t \in \mathbb{A}$. Such eigenfunctions have the form

$$\psi_{\mathcal{P},\alpha}(x, t) = \psi_n^{(\infty)}(x_\infty, t_\infty) \prod_{p \in \mathcal{P}} \psi_{\alpha_p}^{(p)}(x_p, t_p) \prod_{p \notin \mathcal{P}} \Omega(|x_p|_p), \tag{38}$$

where $\psi_n^{(\infty)}$ and $\psi_{\alpha_p}^{(p)}$ are eigenfunctions in ordinary and p-adic cases, respectively. $\Omega(|x_p|_p)$ is defined by (13) and presents a state invariant under transformation of $U_p(t_p)$ evolution operator. Adelic quantum mechanics [4, 5] may be regarded as a triple

$$(L_2(\mathbb{A}), W(z), U(t)),$$

where $W(z)$ and $U(t)$ are unitary representations of the Heisenberg-Weyl group and evolution operator on $L_2(\mathbb{A})$, respectively.

3 Quadratic Lagrangians and Their Actions

A general quadratic Lagrangian can be written in matrix form as follows:

$$L(\dot{q}, q, t) = \frac{1}{2}\,\dot{q}^T A\,\dot{q} + \dot{q}^T B\,q + \frac{1}{2}\,q^T C\,q + D^T\,\dot{q} + E^T\,q + \varepsilon \tag{39}$$

where $A = (\alpha_{kl}(t))$ is a regular symmetric matrix, $C = (\gamma_{kl}(t))$ is a symmetric matrix, $B = (\beta_{kl}(t))$ is a matrix, $D = (\delta_k(t))$, $E = (\eta_k(t))$, $q = (q_k(t))$ and $\dot{q} = (\dot{q_k}(t))$ are vectors in $\mathbb{R}^n$. All matrices are of type $n \times n$ with matrix elements viewed as analytic functions of the time t. In fact, we want to consider the corresponding adelic Lagrangian, i.e. an adelic collection of Lagrangians of the same form (39) which differ only by their valuations $v = \infty, 2, 3, \cdots$. In this section we present some results valid simultaneously for real as well as for p-adic classical mechanics. In adelic case their power series expansions will have the same rational coefficients in the real and all p-adic cases.

The Euler-Lagrange equations of motion are

$$A\,\ddot{q} + (\dot{A} + B - B^T)\,\dot{q} + (\dot{B}\, - C)\,q = E - \dot{D}. \tag{40}$$

Generally, (40) represents a system of n coupled linear inhomogeneous differential equations of the second order. When it is coupled, starting from the homogeneous system and eliminating derivatives of all but one coordinate, one can construct a system of n uncoupled (resolvent) homogeneous linear differential equations of the $2n$ order. Thus a general solution of (40), which describes classical trajectory, can be found by means of solution of the corresponding uncoupled equations. In this way we have

$$q_k = x_k(t) = \sum_{m=1}^{2n} f_{km}(t)\,C_m + \xi_k(t), \quad \text{or} \quad q = x(t) = F(t)\,\mathcal{C} + \xi(t), \tag{41}$$

where $F(t) = [f_{km}(t)] \in M_{n,2n}$ is a solution of the corresponding system of homogeneous differential equations, $\mathcal{C} = [C_m] \in M_{2n,1}$ is the vector of constants, and $\xi(t) = [\xi_k(t)] \in M_{n,1}$ is a particular solution of the complete system of differential equations (40). If we choose $f_{1m}(t)$ as linearly independent solutions for $x_1(t)$ then solutions $f_{km}(t)$ for $x_k(t),\ k \neq 1,$ are determined by the system (40) and they are related to $f_{1m}(t)$.

For the boundary conditions $x'_k = x_k(t')$ and $x''_k = x_k(t'')$, let us introduce the following useful notations:

$$f_i(t) = [f_{i1}(t), \ldots, f_{i\,2n}(t)], \quad f^i(t'', t') = [f_{1i}(t''), \ldots, f_{ni}(t''), f_{1i}(t') \ldots, f_{n\,i}(t')]^T\,, \tag{42}$$

$$f''_i = f_i(t''), \quad f'_i = f_i(t'), \quad \dot{f}''_i = \dot{f}_i(t''), \quad \dot{f}'_i = \dot{f}_i(t'), \tag{43}$$

$$\mathcal{F} = \mathcal{F}(t'',t') = \left[\begin{array}{c} F(t'') \\ F(t') \end{array} \right] = \left[\begin{array}{c} F'' \\ F' \end{array} \right] = \begin{cases} [f_1(t''), \dots, f_n(t''), f_1(t'), \dots, f_n(t')]^T \\ [f^1(t'',t'), f^2(t'',t'), \dots, f^{2n}(t'',t')], \end{cases} \tag{44}$$

where $[f_1(t''),..,f_n(t''),f_1(t'),..,f_n(t')]^T$ is a matrix with rows $f_1(t''),..,f_n(t')$

$$\triangle = \triangle(t'',t') = \det \mathcal{F}, \quad \mathcal{F}_{ij} = (ij)\text{-algebraic complement of } \mathcal{F}, \quad \triangle_{i,j} = \det \mathcal{F}_{ij}, \tag{45}$$

$$F'' = F(t''), \quad F' = F(t'), \quad x\xi = [x_1'' - \xi_1'', \dots, x_n'' - \xi_n'', x_1' - \xi_1', \dots, x_n' - \xi_n']^T \tag{46}$$

$$\triangle_i = \triangle_i(t'',t') = \det[f^1(t'',t'), \dots, f^{i-1}(t'',t'), x\xi, f^{i+1}(t'',t') \dots, f^{2n}(t'',t')], \tag{47}$$

$$\dot{\triangle}_i(f_j')(t'',t') = \dot{\triangle}_i(f_j') = \det[\, f_1'',..,f_{i-1}'', \dot{f}_j', f_{i+1}'',..,f_n'', f_1',..,f_n'\,], i,j = 1,..,n, \tag{48}$$

$$\dot{\triangle}_{i+n}(f_j'')(t'',t') = \dot{\triangle}_{i+n}(f_j'') = \det[\, f_1'',..,f_n'', f_1',..,f_{i-1}', \dot{f}_j'', f_{i+1}',..,f_n'\,], i,j = 1,..,n. \tag{49}$$

Proposition 3.1. *Imposing the boundary conditions $x_k' = x_k(t')$ and $x_k'' = x_k(t'')$, vector of constants of integration $\mathcal{C}$ becomes:*

$$\mathcal{C} = \mathcal{C}(t'',t') = \frac{1}{\triangle(t'',t')} \, [\triangle_1(t'',t'), \triangle_2(t'',t'), \dots, \triangle_{2n}(t'',t')]^T. \tag{50}$$

Proof. It follows after performing relevant computations. □

Note that in the real case for periodic solutions $f_{km}(t+T) = f_{km}(t)$, the determinant $\triangle$ can be singular. To avoid such problem one has then to restrict the time interval $t'' - t'$ to be smaller than the period T.

Taking into account (50), one can rewrite (41) in the following form

$$x_k(t) = \frac{1}{\triangle(t'',t')} \sum_{i=1}^{2n} \triangle_i(t'',t') f_{ki}(t) + \xi_k(t), \qquad k = 1,2,\dots,n.$$

Using the equations of motion (40), the Lagrangian (39) can be rewritten as

$$L(\dot{x},x,t) = \frac{1}{2}\frac{d}{dt}\left[x^T A\,\dot{x} + x^T B\,x + D^T x\right] + \frac{1}{2}\left(D^T \dot{x} + E^T x\right) + \varepsilon \tag{51}$$

where $x(t)$ denotes now the classical trajectory (41).

Using method which was described in [12] in the case $n = 3$ one can find the following very important result.

Theorem 3.2. Let $\{f_{1j}\ , j = 1,2,\dots,2\,n\}$ be any linearly independent solutions of the resolvent equation for $x_1(t)$ in (40), then solutions $f_{km}(t)$ for $x_k(t),\ k \neq 1,$ are determined by the system (40) and the following equality holds

$$\det \left[\begin{array}{c} F(t) \\ \dot{F}(t) \end{array} \right] = \frac{\mathcal{D}}{\det A}, \tag{52}$$

where $\mathcal{D}$ is a non-zero constant, which could be chosen to be equal 1.

Proposition 3.3. *The general form of the action for classical trajectory $x(t)$ of a quadratic Lagrangian, for a particle being in point x' at the time t' and in position x'' at t'', is*

$$\bar{S}(x'',t'';x',t') = \frac{1}{2}\, x''^T \,\bar{A}\, x'' + x''^T \bar{B}\, x' + \frac{1}{2}\, x'^T \bar{C}\, x' + \bar{D}^T x'' + \bar{E}^T\, x' + \bar{\varepsilon}, \tag{53}$$

where $\bar{A} = [\bar{A}_{kl}]$, $\bar{B} = [\bar{B}_{kl}]$, $\bar{C} = [\bar{C}_{kl}]$, $\bar{D} = [\bar{D}_k]$, *and* $\bar{E} = [\bar{E}_k]$.

$$\bar{A}_{kl} = \bar{A}_{kl}(t'',t') = \frac{\partial^2 \bar{S}_0}{\partial x''_k \partial x''_l}, \quad \bar{B}_{kl} = \bar{B}_{kl}(t'',t') = \frac{\partial^2 \bar{S}_0}{\partial x''_k \partial x'_l}, \quad \bar{C}_{kl} = \bar{C}_{kl}(t'',t') = \frac{\partial^2 \bar{S}_0}{\partial x'_k \partial x'_l},$$

$$\bar{D}_k = \bar{D}_k(t'',t') = \frac{\partial \bar{S}_0}{\partial x''}, \quad \bar{E} = \bar{E}_k(t'',t') = \frac{\partial \bar{S}_0}{\partial x'}, \quad \bar{\varepsilon} = \bar{\varepsilon}(t'',t') = \bar{S}_0$$

and subscript $_0$ *in the classical action means that after performing derivatives of the* $\bar{S}(x'',t'';x',t')$ *one has to replace* x'' *and* x' *by* $x'' = x' = 0$.

Proof. From (50) it is clear that constants of integration $C_i(t'',t')$ are linear in x''_k and x'_l. Then the corresponding classical action

$$\begin{aligned} \bar{S}(x'',t'';x',t') &= \int_{t'}^{t''} L(\dot{x},x,t)\,dt = \frac{1}{2}\left[x^T A\,\dot{x} + x^T B\,x + D^T\,x\right]\,\Big|_{t'}^{t''} \\ &+ \frac{1}{2}\int_{t'}^{t''} \left(D^T\dot{x} + E^T x\right)\,dt + \int_{t'}^{t''} \varepsilon(t)\,dt \end{aligned} \tag{54}$$

is quadratic in x''_k and x'_l, where subscripts run again the same values, k, $l = 1, \cdots, n$. □

For our evaluation of the path integrals it is especially important to have explicit dependence of coefficients $\bar{A}_{kl}$, $\bar{B}_{kl}$ and $\bar{C}_{kl}$ on coefficients in the Lagrangian (39) and on ingredients of the classical trajectory (41). Because of that, we can rewrite (54) in the following way:

$$\begin{aligned} \bar{S}(x'',t'';x',t') &= \int_{t'}^{t''} L(\dot{x},x,t)\,dt = \frac{1}{2}\left[\,x^T A\,\dot{x} + x^T B\,x\,\right]\,\Big|_{t'}^{t''} + \mathcal{L}in(x'',x') \\ &= \frac{1}{2}\,[\,(\mathcal{C}^T F(t'')^T + \xi(t'')^T) A(t'')\,(\mathcal{C}\dot{F}(t'') + \dot{\xi}(t'')) + (\mathcal{C}^T F(t'')^T + \xi(t'')^T) \\ &\quad \times B(t'')(\mathcal{C}F(t'') + \xi(t'')) - (\mathcal{C}^T F(t')^T + \xi(t')^T) A(t')\,(\mathcal{C}\dot{F}(t') + \dot{\xi}(t')) \\ &\quad + (\mathcal{C}^T F(t')^T + \xi(t')^T) B(t')(\mathcal{C}F(t') + \xi(t'))] + \mathcal{L}in(x'',x') \\ &= \frac{1}{2}\,[\mathcal{C}^T F(t'')^T A(t'')\,\mathcal{C}\dot{F}(t'') + \mathcal{C}^T F(t'')^T B(t'')\mathcal{C}F(t'') \\ &\quad - \mathcal{C}^T F(t')^T A(t')\mathcal{C}\dot{F}(t') - \mathcal{C}^T F(t')^T B(t')\mathcal{C}F(t')] + \widetilde{\mathcal{L}in}(x'',x'), \end{aligned} \tag{55}$$

where $\mathcal{L}in(x'',x')$ means that this expression is linear in x'' and x'. Since we want to find $\bar{G} \in \{\bar{A},\bar{B},\bar{C}\}$, and e.g.

$$\bar{G}_{kl} = \bar{G}_{kl}(t'',t') = \frac{\partial^2 \bar{S}_0}{\partial x''_k \partial x''_l} = \frac{1}{2}\,\frac{\partial^2\Big(\left[x^T A\,\dot{x} + x^T B\,x\right]\,\Big|_{t'}^{t''}\Big)}{\partial x''_k \partial x''_l}, \tag{56}$$

it is necessary to have the following properties.

Lemma 3.4 *The following relations hold*

$$
\begin{aligned}
\frac{\partial \mathcal{C}^T}{\partial x''_k} &= \frac{1}{\triangle}\,[\,(-1)^{k+1}\triangle_{k,1}, (-1)^{k+2}\triangle_{k,2}, \ldots, (-1)^{k+2n}\triangle_{k,2n}\,]\,,\\
\frac{\partial \mathcal{C}^T}{\partial x'_k} &= \frac{1}{\triangle}\,[\,(-1)^{n+k+1}\triangle_{n+k,1}, (-1)^{n+k+2}\triangle_{n+k,2}, \ldots, (-1)^{n+k+2n}\triangle_{n+k,2n}\,]\,,\\
\frac{\partial (\mathcal{C}^T\, F^T(t''))}{\partial x''_k} &= \frac{\partial \mathcal{C}^T}{\partial x''_k}\; F^T(t'') = [\,0,\ldots,0,\overset{k}{1},0,\ldots,0\,]\,,\\
\frac{\partial (\mathcal{C}^T\, F^T(t'))}{\partial x''_k} &= \frac{\partial \mathcal{C}^T}{\partial x''_k}\; F^T(t') = 0 = \frac{\partial \mathcal{C}^T}{\partial x'_k}\; F^T(t'') = \frac{\partial (\mathcal{C}^T\, F^T(t''))}{\partial x'_k}\,,\\
\frac{\partial (\dot{F}' C)}{\partial x''_k} &= \dot{F}'\frac{\partial \mathcal{C}}{\partial x''_k} = \frac{1}{\triangle}\,[\,\dot{\triangle}_k(f'_1), \dot{\triangle}_k(f'_2), \ldots, \dot{\triangle}_k(f'_n)\,]\,, \qquad (57)\\
\frac{\partial (\dot{F}'' C)}{\partial x''_k} &= \dot{F}''\frac{\partial \mathcal{C}}{\partial x''_k} = \frac{1}{\triangle}\,[\,\dot{\triangle}_k(f''_1), \dot{\triangle}_k(f''_2), \ldots, \dot{\triangle}_k(f''_n)\,]\,,\\
\frac{\partial (\dot{F}' C)}{\partial x'_k} &= \dot{F}'\frac{\partial \mathcal{C}}{\partial x'_k} = \frac{1}{\triangle}\,[\,\dot{\triangle}_{n+k}(f'_1), \dot{\triangle}_{n+k}(f'_2), \ldots, \dot{\triangle}_{n+k}(f'_n)\,]\,,\\
\frac{\partial (\dot{F}'' C)}{\partial x'_k} &= \dot{F}''\frac{\partial \mathcal{C}}{\partial x'_k} = \frac{1}{\triangle}\,[\,\dot{\triangle}_{n+k}(f''_1), \dot{\triangle}_{n+k}(f''_2), \ldots, \dot{\triangle}_{n+k}(f''_n)\,]\,.
\end{aligned}
$$

Let us find now, the matrix elements $\bar{A}_{kl}$, $\bar{B}_{kl}$ and $\bar{C}_{kl}$. Firstly we have

$$
\begin{aligned}
\frac{\partial^2 (x(t'')^T\, A''\, \dot{x}(t''))}{\partial x''_k \partial x''_l} &= \frac{\partial^2 (\mathcal{C}^T F''^{\,T}\, A''\, \dot{F}''\mathcal{C})}{\partial x''_k \partial x''_l} = \frac{\partial}{\partial x''_k}\left(\frac{\partial (\mathcal{C}^T F''^{\,T}\, A''\, \dot{F}''\mathcal{C})}{\partial x''_l}\right)\\
&= \frac{\partial}{\partial x''_k}\left(\frac{\partial \mathcal{C}^T}{\partial x''_l}\; F''^{\,T}\, A''\, \dot{F}''\mathcal{C} + \mathcal{C}^T F''^{\,T}\, A''\, \dot{F}''\frac{\partial \mathcal{C}}{\partial x''_l}\right)\\
\mathcal{C} \text{ is linear in } x''_k \text{ and } x''_l &= \frac{\partial \mathcal{C}^T}{\partial x''_l}\; F''^{\,T}\, A''\, \dot{F}''\frac{\partial \mathcal{C}}{\partial x''_k} + \frac{\partial \mathcal{C}^T}{\partial x''_k} F''^{\,T}\, A''\, \dot{F}''\frac{\partial \mathcal{C}}{\partial x''_l} \overset{(57)}{=}\\
&= \frac{1}{\triangle}\,[0,\ldots,0,\overset{l}{1},0,\ldots,0]\,A''\,[\,\dot{\triangle}_k(f''_1), \dot{\triangle}_k(f''_2), \ldots, \dot{\triangle}_k(f''_n)\,]^T\\
&\quad + \frac{1}{\triangle}\,[0,\ldots,0,\overset{k}{1},0,\ldots,0]\,A''\,[\,\dot{\triangle}_l(f''_1), \dot{\triangle}_l(f''_2), \ldots, \dot{\triangle}_l(f''_n)\,]^T\\
&= \frac{1}{\triangle}\sum_t \left(\alpha''_{lt}\dot{\triangle}_k(f''_t) + \alpha''_{kt}\dot{\triangle}_l(f''_t)\right). \qquad (58)
\end{aligned}
$$

$$
\frac{\partial^2 (x(t')^T\, A'\, \dot{x}(t'))}{\partial x''_k \partial x''_l} = \frac{\partial^2 (\mathcal{C}^T F'^{\,T}\, A'\, \dot{F}'\mathcal{C})}{\partial x''_k \partial x''_l} = \frac{\partial}{\partial x''_k}\left(\frac{\partial (\mathcal{C}^T F'^{\,T}\, A'\, \dot{F}'\mathcal{C})}{\partial x''_l}\right)
$$

$$= \frac{\partial \mathcal{C}^T}{\partial x''_l}\, F'^{\,T} A' \dot{F}' \frac{\partial \mathcal{C}}{\partial x''_k} + \frac{\partial \mathcal{C}^T}{\partial x''_k} F'^{\,T} A' \dot{F}' \frac{\partial \mathcal{C}}{\partial x''_l} \overset{(57)}{=} 0\,. \tag{59}$$

$$\frac{\partial^2 (x(t')^T A' \dot{x}(t'))}{\partial x'_k \partial x'_l} = \frac{\partial^2 (\mathcal{C}^T F'^{\,T} A' \dot{F}' \mathcal{C})}{\partial x'_k \partial x'_l} = \frac{\partial}{\partial x'_k} \left(\frac{\partial (\mathcal{C}^T F'^{\,T} A' \dot{F}' \mathcal{C})}{\partial x'_l} \right)$$
$$= \frac{1}{\triangle} \sum_t \left(\alpha'_{lt} \dot{\triangle}_{n+k}(f'_t) + \alpha'_{kt} \dot{\triangle}_{n+l}(f'_t) \right). \tag{60}$$

$$\frac{\partial^2 (x(t'')^T A'' \dot{x}(t''))}{\partial x'_k \partial x'_l} = \frac{\partial^2 (\mathcal{C}^T F''^{\,T} A'' \dot{F}'' \mathcal{C})}{\partial x'_k \partial x'_l} = \frac{\partial}{\partial x'_k} \left(\frac{\partial (\mathcal{C}^T F''^{\,T} A'' \dot{F}'' \mathcal{C})}{\partial x'_l} \right)$$
$$= \frac{\partial \mathcal{C}^T}{\partial x'_l}\, F''^{\,T} A'' \dot{F}'' \frac{\partial \mathcal{C}}{\partial x'_k} + \frac{\partial \mathcal{C}^T}{\partial x'_k} F''^{\,T} A'' \dot{F}'' \frac{\partial \mathcal{C}}{\partial x''_l} \overset{(57)}{=} 0\,. \tag{61}$$

$$\frac{\partial^2 (x(t'')^T B'' x(t''))}{\partial x''_k \partial x''_l} = \frac{\partial^2 (\mathcal{C}^T F''^{\,T} B'' F'' \mathcal{C})}{\partial x''_k \partial x''_l} = \frac{\partial}{\partial x''_k} \left(\frac{\partial (\mathcal{C}^T F''^{\,T} B'' F'' \mathcal{C})}{\partial x''_l} \right)$$
$$= \frac{\partial \mathcal{C}^T}{\partial x''_l}\, F''^{\,T} B'' F'' \frac{\partial \mathcal{C}}{\partial x''_k} + \frac{\partial \mathcal{C}^T}{\partial x''_k} F''^{\,T} B'' F'' \frac{\partial \mathcal{C}}{\partial x''_l} \overset{(57)}{=}$$
$$= [0, \ldots, 0, \overset{k}{1}, 0, \ldots, 0]\, B''\, [0, \ldots, 0, \overset{l}{1}, 0, \ldots, 0]^T$$
$$+ [0, \ldots, 0, \overset{l}{1}, 0, \ldots, 0]\, B''\, [0, \ldots, 0, \overset{k}{1}, 0, \ldots, 0]^T = \beta''_{lk} + \beta''_{kl}\,, \tag{62}$$

$$\frac{\partial^2 (x(t')^T B' x(t'))}{\partial x'_k \partial x'_l} = \beta'_{lk} + \beta'_{kl}\,, \tag{63}$$

$$\frac{\partial^2 (x(t'')^T B'' x(t''))}{\partial x'_k \partial x'_l} = \frac{\partial^2 (x(t')^T B' x(t'))}{\partial x''_k \partial x''_l} = 0\,. \tag{64}$$

$$\frac{\partial^2 (x(t'')^T A'' \dot{x}(t''))}{\partial x''_k \partial x'_l} = \frac{\partial^2 (\mathcal{C}^T F''^{\,T} A'' \dot{F}'' \mathcal{C})}{\partial x''_k \partial x'_l} = \frac{\partial}{\partial x''_k} \left(\frac{\partial (\mathcal{C}^T F''^{\,T} A'' \dot{F}'' \mathcal{C})}{\partial x'_l} \right)$$
$$= \frac{\partial \mathcal{C}^T}{\partial x'_l}\, F''^{\,T} A'' \dot{F}'' \frac{\partial \mathcal{C}}{\partial x''_k} + \frac{\partial \mathcal{C}^T}{\partial x''_k} F''^{\,T} A'' \dot{F}'' \frac{\partial \mathcal{C}}{\partial x'_l} \overset{(57)}{=}$$
$$= \frac{1}{\triangle} [0, \ldots, 0, \overset{k}{1}, 0, \ldots, 0]\, A''\, [\, \dot{\triangle}_{n+l}(f''_1), \dot{\triangle}_{n+l}(f''_2), \ldots, \dot{\triangle}_{n+l}(f''_n)\,]^T$$
$$= \frac{1}{\triangle} \sum_t \alpha''_{kt} \dot{\triangle}_{n+l}(f''_t)\,, \tag{65}$$

$$\frac{\partial^2 (x(t')^T A' \dot{x}(t'))}{\partial x''_k \partial x'_l} = \frac{\partial^2 (\mathcal{C}^T F'^{\,T} A' \dot{F}' \mathcal{C})}{\partial x''_k \partial x'_l} = \frac{\partial}{\partial x''_k} \left(\frac{\partial (\mathcal{C}^T F'^{\,T} A' \dot{F}' \mathcal{C})}{\partial x'_l} \right)$$

$$= \frac{\partial \mathcal{C}^T}{\partial x'_l} F'^{\,T} A' \dot{F}' \frac{\partial \mathcal{C}}{\partial x''_k} + \frac{\partial \mathcal{C}^T}{\partial x''_k} F'^{\,T} A' \dot{F}' \frac{\partial \mathcal{C}}{\partial x'_l} \overset{(57)}{=}$$

$$= \frac{1}{\triangle} [0, \ldots, 0, \overset{l}{1}, 0, \ldots, 0] A' [\dot{\triangle}_k(f'_1), \dot{\triangle}_k(f'_2), \ldots, \dot{\triangle}_k(f'_n)]^T$$

$$= \frac{1}{\triangle} \sum_t \alpha'_{lt} \dot{\triangle}_k(f'_t) . \tag{66}$$

$$\frac{\partial^2 (x(t'')^T B'' x(t''))}{\partial x''_k \partial x'_l} = \frac{\partial^2 (\mathcal{C}^T F''^{\,T} B'' F'' \mathcal{C})}{\partial x''_k \partial x'_l} = \frac{\partial}{\partial x''_k} \left(\frac{\partial (\mathcal{C}^T F''^{\,T} B'' F'' \mathcal{C})}{\partial x'_l} \right)$$

$$= \frac{\partial \mathcal{C}^T}{\partial x'_l} F''^{\,T} B'' F'' \frac{\partial \mathcal{C}}{\partial x''_k} + \frac{\partial \mathcal{C}^T}{\partial x''_k} F''^{\,T} B'' F'' \frac{\partial \mathcal{C}}{\partial x'_l} \overset{(57)}{=} 0 , \tag{67}$$

$$\frac{\partial^2 (x(t')^T B' x(t'))}{\partial x''_k \partial x'_l} = \frac{\partial^2 (\mathcal{C}^T F'^{\,T} B' F' \mathcal{C})}{\partial x''_k \partial x'_l} = \frac{\partial}{\partial x''_k} \left(\frac{\partial (\mathcal{C}^T F'^{\,T} B' F' \mathcal{C})}{\partial x'_l} \right)$$

$$= \frac{\partial \mathcal{C}^T}{\partial x'_l} F'^{\,T} B' F' \frac{\partial \mathcal{C}}{\partial x''_k} + \frac{\partial \mathcal{C}^T}{\partial x''_k} F'^{\,T} B' F' \frac{\partial \mathcal{C}}{\partial x'_l} \overset{(57)}{=} 0 . \tag{68}$$

Now, using (56), as well as (57)-(68) we have **Theorem 3.5.** *The related coefficients are:*

$$\bar{A}_{kl} = \bar{A}_{kl}(t'', t') = \frac{1}{2\triangle} \sum_{t=1}^{n} \left(\alpha''_{lt} \dot{\triangle}_k(f''_t) + \alpha''_{kt} \dot{\triangle}_l(f''_t) \right) + \frac{\beta''_{lk} + \beta''_{kl}}{2} , \tag{69}$$

$$\bar{B}_{kl} = \bar{B}_{kl}(t'', t') = \frac{1}{2\triangle} \sum_{t=1}^{n} \left(\alpha''_{kt} \dot{\triangle}_{n+l}(f''_t) - \alpha'_{lt} \dot{\triangle}_k(f'_t) \right) \tag{70}$$

$$\bar{C}_{kl} = \bar{C}_{kl}(t'', t') = \frac{-1}{2\triangle} \sum_{t=1}^{n} \left(\alpha'_{lt} \dot{\triangle}_{n+k}(f'_t) + \alpha'_{kt} \dot{\triangle}_{n+l}(f'_t) \right) - \frac{\beta'_{lk} + \beta'_{kl}}{2} . \tag{71}$$

4 Path Integrals on Real and p-Adic Spaces

According to Feynman's path integral approach, discussed in the Introduction, to obtain the complete transition amplitude from (x', t') to (x'', t'') one has to take sum of amplitudes over all possible trajectories $q(t)$ which interpolate between points (x', t') and (x'', t''). Any quantum path may be regarded as a deviation $y(t)$ with respect to the classical one $x(t)$, i.e. $q(t) = x(t) + y(t)$, where $y' = y(t') = 0$ and $y'' = y(t'') = 0$. The corresponding Taylor expansion of the quadratic action functional $S[q]$ around classical path $x(t)$ is

$$S[q] = S[x + y] = S[x] + \delta S[x] + \frac{1}{2!} \delta^2 S[x] \tag{72}$$

$$= S[x] + \frac{1}{2} \int_{t'}^{t''} \left(\dot{y}_k \frac{\partial}{\partial \dot{q}_k} + y_k \frac{\partial}{\partial q_k} \right)^2 L(\dot{q}, q, t) dt.$$

Since our Lagrangian is a polynomial up to quadratic order in $\dot{q}_k$ and q_k, the terms with higher derivatives in (72) are equal zero. Note also that the classical path $x(t)$ gives an extremum of the action and hence we take $\delta S[x] = 0$. According to (4) and (9), for any $v = \infty, 2, 3, \cdots$, we can write

$$\mathcal{K}_v(x'', t''; x', t') = \int \chi_v \left(-\frac{1}{h} S[x + y] \right) \mathcal{D}y, \tag{73}$$

where we replaced $\mathcal{D}q$ by $\mathcal{D}y$, since x is a fixed classical trajectory. Due to (72), the expression (73) gains the more explicit form

$$\begin{aligned} \mathcal{K}_v(x'', t''; x', t') &= \chi_v \left(-\frac{1}{h} \bar{S}(x'', t''; x', t') \right) \\ &\times \int_{y' \to 0, t'}^{y'' \to 0, t''} \chi_v \left(-\frac{1}{2h} \int_{t'}^{t''} \left(\dot{y}_k \frac{\partial}{\partial \dot{q}_k} + y_k \frac{\partial}{\partial q_k} \right)^2 L(\dot{q}, q, t) dt \right) \mathcal{D}y, \end{aligned} \tag{74}$$

where we used $y'' = y' = 0, \quad S[x] = \bar{S}(x'', t''; x', t')$.

Proposition 4.1. *$\mathcal{K}_v(x'', t''; x', t')$ has the form*

$$\mathcal{K}_v(x'', t''; x', t') = N_v(t'', t') \chi_v \left(-\frac{1}{h} \bar{S}(x'', t''; x', t') \right), \tag{75}$$

where $N_v(t'', t')$ does not depend on end points x'' and x'.

Proof. It follows from (74). □

To compute $N_v(t'', t')$, let us note that (74) can be rewritten as

$$\mathcal{K}_v(x'', t''; x', t') = \chi_v \left(-\frac{1}{h} \bar{S}(x'', t''; x', t') \right) K_v(0, t''; 0, t'), \tag{76}$$

where $K_v(0, t''; 0, t') = K_v(y'', t''; y', t')|_{y''=y'=0}$ and

$$K_v(y'', t''; y', t') = \int_{y', t'}^{y'', t''} \chi_v \left(-\frac{1}{h} \int_{t'}^{t''} \left[\frac{1}{2} \dot{y}_k \, \alpha_{kl} \, \dot{y}_l + \dot{y}_k \, \beta_{kl} \, y_l + \frac{1}{2} y_k \, \gamma_{kl} \, y_l \right] \right) \mathcal{D}y. \tag{77}$$

Note that coefficients α_{kl}, β_{kl} and γ_{kl} are those of the initial Lagrangian (39). According to (75) and (76) one has

$$N_v(t'', t') = K_v(y'', t''; y', t')|_{y''=y'=0}, \tag{78}$$

where

$$K_v(y'', t''; y', t') = N_v(t'', t') \chi_v \left(-\frac{1}{h} \left[\frac{1}{2} y''_k \, \bar{A}_{kl} \, y''_l + y''_k \, \bar{B}_{kl} \, y'_l + \frac{1}{2} y'_k \, \bar{C}_{kl} \, y'_l \right] \right). \tag{79}$$

with $\bar{A}_{kl}$, $\bar{B}_{kl}$ and $\bar{C}_{kl}$ given by equations (69)-(71).

To find the corresponding expression for $N_v(t'', t')$ we shall employ conditions (6) and (7). The unitary condition (7) now reads:

$$\int_{\mathbb{Q}_v^n} \bar{K}_v(y'', t''; y', t') K_v(y, t''; y', t') \, d^n y' = \prod_{k=1}^{n} \delta_v(y''_k - y_k). \tag{80}$$

Proposition 4.2. *The absolute value of $N_v(t'',t')$ in (75) is*

$$|N_v(t'',t')|_\infty = \left|\frac{1}{h^n}\det\left[\frac{\partial^2}{\partial x_k''\partial x_l'}\bar{S}_0(x'',t'';x',t')\right]\right|_v^{\frac{1}{2}}. \tag{81}$$

Proof. Substituting $K_v(y'',t'';y',t')$ from (79) to (80), and taking into account that the time t'' is the same in points y'' and y, one obtains

$$|N_v(t'',t')|_\infty^2 \chi_v\left[\frac{1}{2h}\bar{A}_{kl}(t'',t')(y_k''y_l''-y_ky_l)\right]\int_{\mathbb{Q}_v^n}\chi_v\left[\frac{1}{h}(y_k''-y_k)\bar{B}_{kl}(t'',t')y_l'\right]d^ny' = \\ = \prod_{k=1}^n \delta_v(y_k''-y_k). \tag{82}$$

Using (25), one has

$$|N_v(t'',t')|_\infty^2 \chi_v\left[\frac{1}{2h}\bar{A}_{kl}(t'',t')(y_k''y_l''-y_ky_l)\right]\left|\det\left[\frac{1}{h}\bar{B}_{kl}(t'',t')\right]\right|_v^{-1}\prod_{k=1}^n \delta_v(y_k''-y_k) \\ = \prod_{k=1}^n \delta_v(y_k''-y_k). \tag{83}$$

Performing integration in (83) over variable y_k, it follows

$$|N_v(t'',t')|_\infty = \left|\det\left(\frac{1}{h}\bar{B}_{kl}(t'',t')\right)\right|_v^{\frac{1}{2}}. \tag{84}$$

Since $\bar{B}_{kl}(t'',t')$ is the same for $\bar{S}(x'',t'';x',t')$ and $\bar{S}(y'',t'';y',t')$, according to (54) one obtains (81). $\square$

We have now

$$N_v(t'',t') = \left|\det\left(\frac{1}{h}\frac{\partial^2\bar{S}_0(y'',t'';y',t')}{\partial y_k''\partial y_l'}\right)\right|_v^{\frac{1}{2}}\mathcal{A}_v(t'',t'), \tag{85}$$

where $|\mathcal{A}_v(t'',t')|_\infty = 1$ and $\mathcal{A}_v(t'',t')$ remains to be determined explicitly. To this end, we use condition (6), which has now the form

$$\int_{\mathbb{Q}_v^n} K_v(y'',t'';y,t)K_v(y,t;y',t')\,d^ny = K_v(y'',t'';y',t'). \tag{86}$$

Inserting (79) into (86), where $N_v(t'',t')$ has the form (85), we get the following equation:

$$N_v(t'',t')\chi_v\Big(-\frac{1}{h}\big(\frac{1}{2}y''^T\bar{A}(t'',t')y''+y''^T\bar{B}(t'',t')y'+\frac{1}{2}y'^T\bar{C}(t'',t')y'\big)\Big) = N_v(t'',t)N_v(t,t') \\ \times\int_{\mathbb{Q}_v^n}\chi_v\Big(-\frac{1}{h}\big(\frac{1}{2}y''^T\bar{A}(t'',t)y''+y''^T\bar{B}(t'',t)y+y^T\frac{1}{2}\bar{C}(t'',t)y+\frac{1}{2}y^T\bar{A}(t,t')y+y^T\bar{B}(t,t')y' \\ +y'^T\frac{1}{2}\bar{C}(t,t')y'\big)\Big)d^ny = N_v(t'',t)N_v(t,t')\chi_v\Big(-\frac{1}{2h}\big(y''^T\bar{A}(t'',t)y''+y'^T\bar{C}(t,t')y'\big)\Big)$$

$$\times \int_{\mathbb{Q}_v^n} \chi_v\Big(y^T\Big(\frac{\bar{C}(t'',t)+\bar{A}(t,t')}{-2h}\Big)y+\Big(\frac{y''^T\bar{B}(t'',t)+y'^T\bar{B}^T(t,t')}{-h}\Big)y\Big)d^ny = \{\text{Prop. 2.4}\}$$

$$= N_v(t'',t)N_v(t,t')\chi_v\Big(-\frac{1}{2h}\big(y''^T\bar{A}(t'',t)y''+y'^T\bar{C}(t,t')y'\big)\Big)\Lambda_v(\alpha_1,\alpha_2,\ldots,\alpha_n)\,|\det(2H)|_v^{-\frac{1}{2}}$$

$$\times\chi_v\Big(-\frac{1}{4}\left(z^TH^{-1}z\right)\Big),$$

where $\alpha_1,\ldots,\alpha_n$ are eigenvalues of the symmetric matrix

$$H=\frac{\bar{C}(t'',t)+\bar{A}(t,t')}{-2h} \quad \text{and} \quad z=\frac{y''^T\bar{B}(t'',t)+y'^T\bar{B}^T(t,t')}{-h}.$$

Taking into account (85), we can expect the following relations

$$\Big|\det\Big(\frac{1}{h}\frac{\partial^2}{\partial y''\partial y}\bar{S}_0(y'',y)\Big)\Big|_v^{\frac{1}{2}}\Big|\det\Big(\frac{1}{h}\frac{\partial^2}{\partial y\partial y'}\bar{S}_0(y,y')\Big)\Big|_v^{\frac{1}{2}}\,|\det(2H)|_v^{-\frac{1}{2}}$$

$$=\Big|\det\Big(\frac{1}{h}\frac{\partial^2}{\partial y''\partial y'}\bar{S}_0(y'',y')\Big)\Big|_v^{\frac{1}{2}}, \tag{87}$$

$$\chi_v\Big(-\frac{1}{h}\big(\frac{1}{2}y''^T\bar{A}(t'',t')y''+y''^T\bar{B}(t'',t')y'+\frac{1}{2}y'^T\bar{C}(t'',t')y'\big)\Big)$$

$$=\chi_v\Big(-\frac{1}{2h}\big(y''^T\bar{A}(t'',t)y''+y'^T\bar{C}(t,t')y'\big)\Big)\,\chi_v\Big(-\frac{1}{4}\left(z^TH^{-1}z\right)\Big), \tag{88}$$

which implies the third one

$$\mathcal{A}_v(t'',t)\mathcal{A}_v(t,t')\Lambda_v(\alpha_1,\alpha_2,\cdots,\alpha_n)=\mathcal{A}_v(t'',t'). \tag{89}$$

Let us introduce the following notations

$$U=(U_{ij}), \qquad U_{ij}=\frac{1}{2}\,\frac{\dot{\triangle}_i(f_j)(t,t')}{\triangle(t,t')}-\frac{1}{2}\,\frac{\dot{\triangle}_{n+i}(f_j)(t'',t)}{\triangle(t'',t)},$$

$$\mathcal{H}=\bar{A}(t,t')+\bar{C}(t'',t)=A(t)\times U=(w_{ij}), \qquad w_{ij}=\alpha^i\cdot U_j+\alpha^j\cdot U_i, \tag{90}$$

where α^i is $i-$th column of matrix A, and U_j is $j-$th row of matrix U. By the multi-linearity of determinant one reads

$$\det\mathcal{H}=\sum_{\substack{i_1<i_2\dots<i_k\\ j_1<j_2\dots<j_{n-k}}}\det[\alpha^{i_1},\ldots,\alpha^{i_k},U^{j_1},\ldots,U^{j_{n-k}}]\,\det[U^{i_1},\ldots,U^{i_k},\alpha^{j_1},\ldots,\alpha^{j_{n-k}}].$$

Then one can see that above determinant is equal to

$$\det\mathcal{H} = 2^n\det[\,\alpha^1,\alpha^2,\ldots,\alpha^n\,]\,\det[\,U^1,U^2,\ldots,U^n\,]+\mathcal{S}$$

$$=\det A\,\det 2U+\mathcal{S}. \tag{91}$$

Using Euler-Lagrange equations following ideas in [12] one can find that $\mathcal{S}=0$.

For $X=(x_{ij})$, $Y=(y_{ij})$, $Z=(z_{ij})$ and $W=(w_{ij})$ arbitrary matrices $(n,2\,n)$ $(n\in\mathbb{N})$, and let Y_{kl} be obtained from Y replacing $k-$th row of Y by the $l-$th row of $X\,(k,l=1,2\ldots,n)$. Define the (n,n) matrix $T=(t_{kl})$ by

$$t_{kl}=\det\begin{bmatrix}Y_{kl}\\ Z\end{bmatrix}\det\begin{bmatrix}Y\\ W\end{bmatrix}-\det\begin{bmatrix}Y_{kl}\\ W\end{bmatrix}\det\begin{bmatrix}Y\\ Z\end{bmatrix}.$$

Then the following identity holds

$$\det T = \left(\det \begin{bmatrix} Y \\ Z \end{bmatrix}\right)^{n-1} \left(\det \begin{bmatrix} Y \\ W \end{bmatrix}\right)^{n-1} \det \begin{bmatrix} Z \\ W \end{bmatrix} \det \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{92}$$

Using identity (92) one can find the following relation

$$\det 2\,U = (-1)^n \frac{\triangle(t'',t')}{\triangle(t'',t)\,\triangle(t,t')} \det \begin{bmatrix} F(t) \\ \dot F(t) \end{bmatrix}, \tag{93}$$

which combined with the relations (52) (with $\mathcal{D} = 1$), and (91) implies

$$\det \mathcal{H} = (-1)^n \frac{\triangle(t'',t')}{\triangle(t'',t)\,\triangle(t,t')}, \tag{94}$$

and since $\mathcal{H} = -2\,hH$, we have

$$\det 2H = \det(\frac{-1}{h}\,\mathcal{H}) = \frac{(-1)^n}{h^n} \det \mathcal{H} = \frac{1}{h^n} \frac{\triangle(t'',t')}{\triangle(t'',t)\,\triangle(t,t')}. \tag{95}$$

Now, it is clear that to prove (87) is enough to show that

$$\det \bar B(t'',t') = \frac{1}{\triangle(t'',t')}, \tag{96}$$

where $\bar B(t'',t')$ is the matrix $(\bar B_{kl}(t'',t'))$.

In the case $n = 1$, it is shown (see [8]) that (96) holds and relations (87)-(89) are satisfied with

$$\mathcal{A}_v(t'',t') = \lambda_v \left(-\frac{1}{2h} \frac{\partial^2}{\partial y'' \partial y'} \bar S_0(y'',t'';y',t')\right). \tag{97}$$

In the case $n = 2$, using expressions (70) after long calculations, we find

$$\begin{aligned} \det \bar B(t'',t') &= \bar B_{11}\bar B_{22} - \bar B_{12}\bar B_{21} = \frac{\left(\det A(t'') \det \begin{bmatrix} F(t'') \\ \dot F(t'') \end{bmatrix} + \det A(t') \det \begin{bmatrix} F(t') \\ \dot F(t') \end{bmatrix}\right) \triangle(t'',t')}{4\,\triangle(t'',t')^2} \\ &+ \frac{\mathrm{Tr}((A(t') \otimes A(t''))\,\tilde\triangle)}{4\,\triangle(t'',t')^2} = \frac{1}{2\,\triangle(t'',t')} + \frac{\mathrm{Tr}((A(t') \otimes A(t''))\,\tilde\triangle)}{4\,\triangle(t'',t')^2}, \end{aligned} \tag{98}$$

where $\tilde\triangle = \begin{bmatrix} \triangle_{2,1}\triangle_{4,1} & \triangle_{2,1}\triangle_{4,2} & -\triangle_{1,1}\triangle_{4,1} & -\triangle_{1,1}\triangle_{4,2} \\ \triangle_{2,2}\triangle_{4,1} & \triangle_{2,2}\triangle_{4,2} & -\triangle_{1,2}\triangle_{4,1} & -\triangle_{1,2}\triangle_{4,2} \\ -\triangle_{2,1}\triangle_{3,1} & -\triangle_{2,1}\triangle_{3,2} & \triangle_{1,1}\triangle_{3,1} & \triangle_{1,1}\triangle_{3,2} \\ -\triangle_{2,2}\triangle_{3,1} & -\triangle_{2,2}\triangle_{3,2} & \triangle_{1,2}\triangle_{3,1} & \triangle_{1,2}\triangle_{3,2} \end{bmatrix}$ and where $\triangle_{i,j} = \dot\triangle_i(f'_j)$, $i = 1,2$; $j = 1,2$ and $\triangle_{i,j} = \dot\triangle_i(f''_j)$, $i = 3,4$; $j = 1,2$.

From the properties of the function λ_v, we have

$$\lambda_v\Big(\frac{1}{a}\Big) = \lambda_v\Big(a^2\,\frac{1}{a}\Big) = \lambda_v(a),\ \lambda_v\Big(\frac{1}{x} + \frac{1}{y}\Big) = \lambda_v\Big(\frac{x+y}{x\,y}\Big) = \lambda_v\Big(\frac{x\,y}{x+y}\Big) = \frac{\lambda_v(x)\lambda_v(y)}{\lambda_v(x+y)}. \tag{99}$$

Let us introduce the following notations

$$x = \triangle(t'', t) = \frac{1}{\det \bar{B}(t'', t)}, \quad y = \triangle(t, t') = \frac{1}{\det \bar{B}(t, t')},$$

then one can write (as in $n = 1$ case)

$$\lambda_v(\triangle(t'', t) + \triangle(t, t')) = \lambda_v(\triangle(t'', t')) . \tag{100}$$

Note that now

$$\Lambda_v(\alpha_1, \alpha_2, \cdots, \alpha_n) = \lambda_v^{n-1}(1)\, \lambda_v(\det H) \prod_{i<j\leq n} (\alpha_i, \alpha_j)_v = \frac{\mathcal{A}_v(t'', t')}{\mathcal{A}_v(t'', t)\, \mathcal{A}_v(t, t')}. \tag{101}$$

Generally, product of the Hilbert symbols can be $+1$ or -1, and we will take here that it is $+1$, i.e. $\Lambda_v(\alpha_1, \alpha_2, \cdots, \alpha_n) = \lambda_v^{n-1}(1)\, \lambda_v(\det H)$. Then we want to show that $\mathcal{A}_v(t'', t')$ has the form $\mathcal{A}_v(t'', t') = \lambda_v^{1-n}(1)\, \lambda_v(-\xi\, \triangle(t'', t')) = \lambda_v^{1-n}(1)\, \lambda_v(-\xi/\triangle(t'', t'))$, where $\xi = \frac{1}{(2h)^n}$. Since

$$\begin{aligned}
\lambda_v(\det H) &= \lambda_v\Big(\frac{\xi\triangle(t'', t')}{\xi\,\triangle(t'', t)\xi\triangle(t, t')}\Big) = \lambda_v\Big(\frac{\xi\triangle(t'', t)\,\xi\triangle(t, t')}{\xi\triangle(t'', t')}\Big) \\
&= \frac{\lambda_v(\xi\triangle(t'', t))\,\lambda_v(\xi\triangle(t, t'))}{\lambda_v(\xi\triangle(t'', t) + \xi\triangle(t, t'))} = \frac{\lambda_v(\xi\triangle(t'', t))\lambda_v(\xi\triangle(t, t'))}{\lambda_v(\xi\triangle(t'', t'))} \\
&= \frac{\lambda_v(\xi \det \bar{B}(t'', t))\lambda_v(\xi \det \bar{B}(t, t'))}{\lambda_v(\xi \det \bar{B}(t'', t'))} = \frac{\lambda_v(-\xi \det \bar{B}(t'', t'))}{\lambda_v(-\xi \det \bar{B}(t'', t))\lambda_v(-\xi \det \bar{B}(t, t'))}.
\end{aligned} \tag{102}$$

So, if we compare (101) and (102) we see that we obtain one class of solutions for

$$A_v(t'', t') = \lambda_v^{1-n}(1)\, \lambda_v\left(\frac{-1}{(2h)^n} \det \bar{B}(t'', t')\right). \tag{103}$$

In virtue of the above evaluation one can formulate the following

Theorem 4.3. *The v-adic kernel $\mathcal{K}_v(x'', t''; x', t')$ of the unitary evolution operator, defined by (1) and evaluated as the Feynman path integral, for quadratic Lagrangians (39) (and consequently, for quadratic classical actions (51)) has the form*

$$\begin{aligned}
\mathcal{K}_v(x'', t''; x', t') &= \lambda_v^{1-n}(1)\, \lambda_v\left(\frac{-1}{(2h)^n} \det\left(\frac{\partial^2}{\partial x_k'' \partial x_l'} \bar{S}_0(x'', t''; x', t')\right)\right) \left|\det\left(\frac{1}{h}\frac{\partial^2}{\partial x_k'' \partial x_l'} \bar{S}_0(x'', t''; x', t')\right)\right|_v^{\frac{1}{2}} \\
&\quad \times \chi_v\left(-\frac{1}{h}\bar{S}(x'', t''; x', t')\right)
\end{aligned} \tag{104}$$

and satisfies the general properties (6)-(7).

Proof. The formula (104) is a result of the above analytic evaluation, and one has to show that this expression also satisfies explicitly (8). □

Starting from (104) and using definition (22) for λ_∞-function one can rederive well-known result in ordinary quantum mechanics:

$$\mathcal{K}_\infty(x'', t''; x', t') = \sqrt{\frac{1}{(ih)^n} \det\left(-\frac{\partial^2}{\partial x_k'' \partial x_l'} \bar{S}_0(x'', t''; x', t')\right)} \exp\left(\frac{2\pi i}{h} \bar{S}(x'', t''; x', t')\right). \tag{105}$$

5 Adelic Path Integral

Adelic path integral can be introduced as a generalization of ordinary and p-adic path integrals. As adelic analogue of (3) it is related to eigenfunctions in adelic quantum mechanics in the form

$$\psi_{\mathcal{P},\alpha}(x'',t'') = \int_{\mathbb{A}} \mathcal{K}_{\mathbb{A}}(x'',t'';x',t')\psi_{\mathcal{P},\alpha}(x',t')dx', \tag{106}$$

where $\psi_{\mathcal{P},\alpha}(x,t)$ has the form (32). Since the equation (106) must be valid for any set $\mathcal{P}$ of primes p, and adelic eigenstate is an infinite product of real and p-adic eigenfunctions, it is natural to consider adelic probability amplitude in the following form:

$$\mathcal{K}_{\mathbb{A}}(x'',t'';x',t') = \mathcal{K}_{\infty}(x''_{\infty},t''_{\infty};x'_{\infty},t'_{\infty}) \prod_p \mathcal{K}_p(x''_p,t''_p;x'_p,t'_p), \tag{107}$$

where $\mathcal{K}_{\infty}(x''_{\infty},t''_{\infty};x'_{\infty},t'_{\infty})$ and $\mathcal{K}_p(x''_p,t''_p;x'_p,t'_p)$ are probability amplitudes in ordinary and p-adic quantum mechanics, respectively.

From (107), we see that one can introduce adelic path integral as an infinite product of ordinary and p-adic path integrals for all primes p. We consider adelic Feynman's path integral as a path integral on an adelic space. Now we can rewrite (107) in the form

$$\mathcal{K}_{\mathbb{A}}(x'',t'';x',t') = \int_{x',t'}^{x'',t''} \chi_{\mathbb{A}}\left(-\frac{1}{h}S_{\mathbb{A}}[q]\right)\mathcal{D}_{\mathbb{A}}q, \tag{108}$$

where $\chi_{\mathbb{A}}(x)$ is adelic additive character, $S_{\mathbb{A}}[q]$ and $\mathcal{D}_{\mathbb{A}}q$ are adelic action and the Haar measure, respectively. For practical considerations, we define adelic path integral in the form

$$\mathcal{K}_{\mathcal{A}}(x'',t'';x',t') = \prod_v \int_{x'_v,t'_v}^{x''_v,t''_v} \chi_v\left(-\frac{1}{h}\int_{t'_v}^{t''_v} L(\dot{q}_v,q_v,t_v)dt_v\right)\mathcal{D}q_v. \tag{109}$$

Adelic Lagrangian is the infinite sequence

$$L_{\mathbb{A}}(\dot{q},q,t) = (L(\dot{q}_\infty,q_\infty,t_\infty), L(\dot{q}_2,q_2,t_2), L(\dot{q}_3,q_3,t_3),\cdots,L(\dot{q}_p,q_p,t_p),\cdots), \tag{110}$$

where $|L(\dot{q}_p,q_p,t_p)|_p \leq 1$ for all primes p but a finite set $\mathcal{P}$ of them. Consequently, an adelic quadratic Lagrangian looks like (110), where each element $L(\dot{q}_v,q_v,t_v)$ has the same form (39).

Taking into account results obtained in the previous sections, we can write adelic path integral for n-dimensional quadratic Lagrangians (and consequently, quadratic classical actions) as

$$\mathcal{K}_{\mathbb{A}}(x'',t'';x',t') = \prod_v \lambda_v^{1-n}(1)\ \lambda_v\left[-\frac{1}{(2h)^n}\det\left(\frac{\partial^2}{\partial x''_{(v)k}\partial x'_{(v)l}}\bar{S}_0(x''_v,t''_v;x'_v,t'_v)\right)\right]$$

$$\times\left|\det\left(\frac{1}{h}\frac{\partial^2}{\partial x''_{(v)k}\partial x'_{(v)l}}\bar{S}_0(x''_v,t''_v;x'_v,t'_v)\right)\right|_v^{\frac{1}{2}} \chi_v\left(-\frac{1}{h}\bar{S}(x''_v,t''_v;x'_v,t'_v)\right). \tag{111}$$

Note that vacuum state $\Omega(|x_p|_p)$ transforms as

$$\Omega(|x''_p|_p) = \int_{\mathbb{Q}_p} \mathcal{K}_p(x''_p,t''_p;x'_p,t'_p)\Omega(|x'_p|_p)dx'_p = \int_{\mathbb{Z}_p} \mathcal{K}_p(x''_p,t''_p;x'_p,t'_p)dx'_p. \tag{112}$$

As a consequence of (112) one has

$$\int_{\mathbb{Z}_p} \mathcal{K}_p(x''_p, t''_p; x_p, t_p)\mathcal{K}_p(x_p, t_p; x'_p, t'_p)dx_p = \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p), \tag{113}$$

which may be viewed as an additional condition on p-adic path integrals in adelic quantum mechanics for all but a finite number of primes p. Conditions (112) and (113) impose a restriction on a dynamical system to be adelic. It is practically a restriction on time t_p to have consistent adelic time t.

6 Concluding Remarks

Evaluating path integrals simultaneously on real and p-adic n-dimensinal spaces, in the previous sections we derived some general expressions related to probability amplitudes $\mathcal{K}(x'', t''; x', t')$ in ordinary, p-adic and adelic quantum mechanics. It has been done for Lagrangians $L(\dot{q}, q, t)$ which are polynomials at most the second degree in dynamical variables $\dot{q}_k$ and q_k, where $k = 1, 2, \cdots, n$.

It is worth pointing out that the formalism of ordinary and p-adic path integrals can be regarded as the same at different levels of evaluation, and the obtained results have the same form. In fact, this property of number field invariance has to be natural for general mathematical methods in theoretical physics and fundamental physical laws (*cf.* [16]).

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