YANG–BAXTER OPERATORS FROM $(\mathbb{G}, \theta)$-LIE ALGEBRAS

F. F. Nichita [1] and Bogdan Popovici [2]
[1] Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O. Box 1-764, RO-014700 Bucharest, Romania
E-mail: Florin.Nichita@imar.ro
[2] Horia Hulubei National Institute for Physics and Nuclear Engineering,
P.O.Box MG-6, Bucharest-Magurele, Romania
E-mail: popobog@theory.nipne.ro

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Abstract

The $(\mathbb{G}, \theta)$-Lie algebras are structures which unify the Lie algebras and Lie superalgebras. We use them to produce solutions for the quantum Yang–Baxter equation. The constant and the spectral-parameter Yang-Baxter equations and Yang-Baxter systems are also studied.

1 Introduction

The $(\mathbb{G}, \theta)$-Lie algebras are structures which unify the Lie algebras and Lie superalgebras, and, in this paper, they will be used to produce solutions for the celebrated quantum Yang–Baxter equation (QYBE). The theory of integrable Hamiltonian systems makes great use of the solutions of the one-parameter form of the QYBE (which are related to the two-parameter form of the QYBE), since coefficients of the power series expansion of such a solution give rise to commuting integrals of motion. *Yang–Baxter systems* emerged from the study of quantum integrable systems, as generalizations of the QYBE related to nonultralocal models.

This paper presents some of the latest results on Yang-Baxter operators from algebra structures and related topics, such as enhanced versions of Theorem 1 (from [17]). Also, we study Yang-Baxter operators from Lie superalgebras and from $(\mathbb{G}, \theta)$-Lie algebras. The following authors constructed Yang-Baxter operators from Lie (co)algebras and Lie superalgebras before: [12], [1], [14], etc. We extend some of the above results to Yang-Baxter systems and spectral-parameter dependent Yang-Baxter equations.

2 The constant QYBE

Throughout this paper k is a field. All tensor products appearing in this paper are defined over k. For V a k-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I : V \to V$ the identity map of the space V.

We use the following notations concerning the Yang-Baxter equation.

If $R : V \otimes V \to V \otimes V$ is a k-linear map, then $R^{12} = R \otimes I, R^{23} = I \otimes R, R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 2.1. An invertible k-linear map $R : V \otimes V \to V \otimes V$ is called a Yang-Baxter operator if it satisfies the equation

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{2.1}$$

Remark 2.2. The equation (2.1) is usually called the braid equation. It is a well-known fact that the operator R satisfies (2.1) if and only if $R \circ \tau$ satisfies the constant QYBE (if and only if $\tau \circ R$ satisfies the constant QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{2.2}$$

Remark 2.3. (i) $\tau : V \otimes V \to V \otimes V$ is an example of a Yang-Baxter operator.

(ii) An exhaustive list of invertible solutions for (2.2) in dimension 2 is given in [7] and in the appendix of [9].

(iii) Finding all Yang-Baxter operators in dimension greater than 2 is an unsolved problem.

Let A be a (unitary) associative k-algebra, and $\alpha, \beta, \gamma \in k$. We define the k-linear map: $R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A$, $R^A_{\alpha,\beta,\gamma}(a \otimes b) = \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b$.

Theorem 2.4. *(S. Dăscălescu and F. F. Nichita, [3]) Let A be an associative k-algebra with $\dim A \geq 2$, and $\alpha, \beta, \gamma \in k$. Then $R^A_{\alpha,\beta,\gamma}$ is a Yang-Baxter operator if and only if one of the following holds:*

(i) $\alpha = \gamma \neq 0, \quad \beta \neq 0$;

(ii) $\beta = \gamma \neq 0, \quad \alpha \neq 0$;

(iii) $\alpha = \beta = 0, \quad \gamma \neq 0$.

If so, we have $(R^A_{\alpha,\beta,\gamma})^{-1} = R^A_{\frac{1}{\beta},\frac{1}{\alpha},\frac{1}{\gamma}}$ in cases (i) and (ii), and $(R^A_{0,0,\gamma})^{-1} = R^A_{0,0,\frac{1}{\gamma}}$ in case (iii).

Remark 2.5. The Yang–Baxter equation plays an important role in knot theory. Turaev has described a general scheme to derive an invariant of oriented links from a Yang–Baxter operator, provided this one can be "enhanced". In [13], we considered the problem of applying Turaev's method to the Yang–Baxter operators derived from algebra structures presented in the above theorem. We concluded that Turaev's procedure invariably produces from any of those enhancements the Alexander polynomial of knots.

Remark 2.6. Let us observe that $R' = R^A_{\alpha,\beta,\alpha} \circ \tau$ is a solution for the equation (2.2). In dimension two, after getting rid of the auxiliary parameters, we obtain the simplest form of R':

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-q & q & 0 \\ \eta & 0 & 0 & -q \end{pmatrix} \tag{2.3}$$

where $\eta \in \{0,\ 1\}$, and $q \in k - \{0\}$. The matrix form (2.3) was obtained as a consequence of the fact that isomorphic algebras produce isomorphic Yang-Baxter operators, and it is compatible with Remark 2.3 (ii).

3 The two-parameter form of the QYBE

Formally, a colored Yang-Baxter operator is defined as a function $R : X \times X \to \mathrm{End}_k(V \otimes V)$, where X is a set and V is a finite dimensional vector space over a field k. Thus, for any $u, v \in X$, $R(u,v) : V \otimes V \to V \otimes V$ is a linear operator. We consider three operators acting on a triple tensor product $V \otimes V \otimes V$, $R^{12}(u,v) = R(u,v) \otimes I$, $R^{23}(v,w) = I \otimes R(v,w)$, and similarly $R^{13}(u,w)$ as an operator that acts non-trivially on the first and third factor in $V \otimes V \otimes V$.

R satisfies the two-parameter form of the QYBE if:

$$R^{12}(u,v)R^{13}(u,w)R^{23}(v,w) = R^{23}(v,w)R^{13}(u,w)R^{12}(u,v) \tag{3.4}$$

$\forall\ u, v, w \in X$.

Theorem 3.1. *(F. F. Nichita and D. Parashar, [15]) Let A be an associative k-algebra with $\dim A \geq 2$, and $X \subset k$. Then, for any two parameters $p, q \in k$, the function $R : X \times X \to \mathrm{End}_k(A \otimes A)$ defined by*

$$R(u,v)(a \otimes b) = p(u-v)1 \otimes ab + q(u-v)ab \otimes 1 - (pu - qv)b \otimes a, \tag{3.5}$$

satisfies the colored QYBE (3.4).

Remark 3.2. If $pu \neq qv$ and $qu \neq pv$ then the operator (3.5) is invertible. Moreover, the following formula holds: $R^{-1}(u,v)(a \otimes b) = \frac{p(u-v)}{(qu-pv)(pu-qv)} ba \otimes 1 + \frac{q(u-v)}{(qu-pv)(pu-qv)} 1 \otimes ba - \frac{1}{(pu-qv)} b \otimes a$.

Algebraic manipulations of the previous theorem lead to the following result.

Theorem 3.3. *Let A be an associative k-algebra with* $\dim A \geq 2$ *and* $q \in k$. *Then the operator*

$$S(\lambda)(a \otimes b) = (e^{\lambda} - 1)1 \otimes ab + q(e^{\lambda} - 1)ab \otimes 1 - (e^{\lambda} - q)b \otimes a \tag{3.6}$$

satisfies the one-parameter form of the Yang-Baxter equation:

$$S^{12}(\lambda_1 - \lambda_2)S^{13}(\lambda_1 - \lambda_3)S^{23}(\lambda_2 - \lambda_3) =$$

$$= S^{23}(\lambda_2 - \lambda_3)S^{13}(\lambda_1 - \lambda_2)S^{12}(\lambda_1 - \lambda_2). \tag{3.7}$$

If $e^{\lambda} \neq q, \ \frac{1}{q},$ *then the operator (3.6) is invertible.*

Moreover, the following formula holds:

$$S^{-1}(\lambda)(a \otimes b) = \frac{e^{\lambda}-1}{(qe^{\lambda}-1)(e^{\lambda}-q)} ba \otimes 1 + \frac{q(e^{\lambda}-1)}{(qe^{\lambda}-1)(e^{\lambda}-q)} 1 \otimes ba - \frac{1}{e^{\lambda}-q} b \otimes a.$$

Remark 3.4. The operator from Theorem 3.3 can be obtained from Theorem 2.4 and the **Baxterization** procedure from [5] (page 22).

Hint: Consider the operator $R^{A}_{q,\frac{1}{q},\frac{1}{q}} : A\otimes A \to A\otimes A, \quad a\otimes b \mapsto qab\otimes 1 + \frac{1}{q}\otimes ab - \frac{1}{q}a\otimes b$ and its inverse, $R^{A}_{q,\frac{1}{q},q}$.

4 Yang-Baxter systems

From the physical point of view the above relations are used to study a certain class of quantum integrable systems, the ultralocal models [6, 10]. However, interesting physical models which have nonultralocal interactions appear, and they require the study of extensions of the QYBE [8, 9]. In the following we describe the Yang-Baxter systems in terms of the Yang-Baxter commutators.

Let V, V', V'' be finite dimensional vector spaces over the field k, and let $R : V\otimes V' \to V\otimes V'$, $S : V\otimes V'' \to V\otimes V''$ and $T : V'\otimes V'' \to V'\otimes V''$ be three linear maps. The *Yang–Baxter commutator* is a map $[R, S, T] : V\otimes V'\otimes V'' \to V\otimes V'\otimes V''$ defined by

$$[R, S, T] := R^{12}S^{13}T^{23} - T^{23}S^{13}R^{12}. \tag{4.8}$$

Note that $[R, R, R] = 0$ is just a short-hand notation for writing the constant QYBE (2.2).

A system of linear maps $W : V\otimes V \to V\otimes V, \quad Z : V'\otimes V' \to V'\otimes V', \quad X : V\otimes V' \to V\otimes V'$, is called a WXZ–system if the following conditions hold:

$$[W, W, W] = 0 \qquad [Z, Z, Z] = 0 \qquad [W, X, X] = 0 \qquad [X, X, Z] = 0 \tag{4.9}$$

Remark 4.1. It was observed that WXZ–systems with invertible W, X and Z can be used to construct dually paired bialgebras of the FRT type leading to quantum doubles. The above is one type of a constant Yang–Baxter system that has recently been studied in [15] and also shown to be closely related to entwining structures [2].

Theorem 4.2. *(F. F. Nichita and D. Parashar, [15]) Let A be a k-algebra, and $\lambda, \mu \in k$. The following is a WXZ–system:*
$W : A\otimes A \to A\otimes A, \quad W(a\otimes b) = ab\otimes 1 + \lambda 1\otimes ab - b\otimes a,$
$Z : A\otimes A \to A\otimes A, \quad Z(a\otimes b) = \mu ab\otimes 1 + 1\otimes ab - b\otimes a,$
$X : A\otimes A \to A\otimes A, \quad X(a\otimes b) = ab\otimes 1 + 1\otimes ab - b\otimes a.$

Remark 4.3. Let R be a solution for the two-parameter form of the QYBE, i.e. $R^{12}(u,v)R^{13}(u,w)R^{23}(v,w) = R^{23}(v,w)R^{13}(u,w)R^{12}(u,v) \;\; \forall\, u,v,w \in X$.

Then, if we fix $s, t \in X$, we obtain the following WXZ–system:
$W = R(s,s)$, $X = R(s,t)$ and $Z = R(t,t)$.

5 Lie superalgebras

Using some of the above techniques we now present enhanced versions of Theorem 1 (from [17]).

Theorem 5.1. *(F. F. Nichita and B. P. Popovici, [16]) Let $V = W \oplus kc$ be a k-space, and $f, g : V\otimes V \to V$ k-linear maps such that $f, g = 0$ on $V\otimes c + c\otimes V$. Then, $R : V\otimes V \to V\otimes V$, $R(v\otimes w) = f(v\otimes w)\otimes c + c\otimes g(v\otimes w)$ is a solution for QYBE (2.2).*

Definition 5.2. A Lie superalgebra is a (nonassociative) $\mathbb{Z}_2$-graded algebra, or superalgebra, over a field k with the Lie superbracket, satisfying the two conditions:

$$[x,y] = -(-1)^{|x||y|}[y,x]$$

$$(-1)^{|z||x|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0$$

where x, y and z are pure in the $\mathbb{Z}_2$-grading. Here, $|x|$ denotes the degree of x (either 0 or 1). The degree of $[x,y]$ is the sum of degree of x and y modulo 2.

Let $(L, [,])$ be a Lie superalgebra over k, and $Z(L) = \{z \in L : [z,x] = 0 \;\; \forall\, x \in L\}$.

For $z \in Z(L)$, $|z| = 0$ and $\alpha \in k$ we define:

$$\phi_\alpha^L \;:\; L\otimes L \quad\longrightarrow\quad L\otimes L$$

$$x\otimes y \mapsto \alpha[x,y]\otimes z + (-1)^{|x||y|} y\otimes x\ .$$

Its inverse is:

$$\phi_\alpha^{L^{-1}} \;:\; L\otimes L \quad\longrightarrow\quad L\otimes L$$

$$x\otimes y \mapsto \alpha z\otimes [x,y] + (-1)^{|x||y|} y\otimes x$$

Theorem 5.3. *Let $(L,[,])$ be a Lie superalgebra and $z\in Z(L), |z|=0$, and $\alpha\in k$. Then: ϕ_α^L is a YB operator.*

Proof. The verification of the Yang-Baxter equation follows below:

$$\phi_\alpha^{L_{23}}\phi_\alpha^{L_{12}}\phi_\alpha^{L_{23}}(a\otimes b\otimes c) = \phi_\alpha^{L_{12}}\phi_\alpha^{L_{23}}\phi_\alpha^{L_{12}}(a\otimes b\otimes c),$$

$$\begin{gathered}\phi_\alpha^{L_{23}}\phi_\alpha^{L_{12}}\phi_\alpha^{L_{23}}(a\otimes b\otimes c) = \phi_\alpha^{L_{23}}\phi_\alpha^{L_{12}}(\alpha a\otimes[b,c]\otimes z + (-1)^{|b||c|}a\otimes c\otimes b) = \\ \phi_\alpha^{L_{23}}(\alpha^2[a,[b,c]]\otimes z\otimes z + (-1)^{|a||[b,c]|}\alpha[b,c]\otimes a\otimes z + (-1)^{|b||c|}\alpha[a,c]\otimes z\otimes b + \\ (-1)^{|b||c|}(-1)^{|a||c|}c\otimes a\otimes b) = \alpha^2[a,[b,c]]\otimes z\otimes z + (-1)^{|a||[b,c]|}\alpha[b,c]\otimes z\otimes a + \\ (-1)^{|b||c|}\alpha[a,c]\otimes b\otimes z + (-1)^{|b||c|}(-1)^{|a||c|}\alpha c\otimes\ [a,b]\otimes z + \\ (-1)^{|b||c|}(-1)^{|a||c|}(-1)^{|a||b|}c\otimes b\otimes a\end{gathered} \tag{5.10}$$

$$\begin{gathered}\phi_\alpha^{L_{12}}\phi_\alpha^{L_{23}}\phi_\alpha^{L_{12}}(a\otimes b\otimes c) = \phi_\alpha^{L_{12}}\phi_\alpha^{L_{23}}(\alpha[a,b]\otimes z\otimes c + (-1)^{|a||b|}b\otimes a\otimes c) = \\ \phi_\alpha^{L_{12}}(\alpha[a,b]\otimes c\otimes z + (-1)^{|a||b|}\alpha b\otimes[a,c]\otimes z + (-1)^{|a||b|}(-1)^{|a||c|}b\otimes c\otimes a) = \\ \alpha^2[[a,b],c]\otimes z\otimes z + (-1)^{|[a,b]||c|}\alpha c\otimes[a,b]\otimes z + (-1)^{|a||b|}\alpha^2[b,[a,c]]\otimes z\otimes z + \\ (-1)^{|a||b|}(-1)^{|[a,c]||b|}\alpha[a,c]\otimes\ b\otimes z + (-1)^{|a||b|}(-1)^{|a||c|}\alpha[b,c]\otimes z\otimes a \\ +(-1)^{|a||b|}(-1)^{|a||c|}(-1)^{|b||c|}c\otimes b\otimes a\end{gathered} \tag{5.11}$$

i.e.

$$\begin{gathered}\alpha^2[a,[b,c]]\otimes z\otimes z + (-1)^{|a||[b,c]|}\alpha[b,c]\otimes z\otimes a + (-1)^{|b||c|}\alpha[a,c]\otimes b\otimes z + \\ (-1)^{|b||c|+|a||c|}\alpha c\otimes\ [a,b]\otimes z = \alpha^2[[a,b],c]\otimes z\otimes z + (-1)^{|[a,b]||c|}\alpha c\otimes[a,b]\otimes z + \\ (-1)^{|a||b|}\alpha^2[b,[a,c]]\otimes z\otimes z + (-1)^{|a||b|}(-1)^{|[a,c]||b|}\alpha[a,c]\otimes\ b\otimes z + \\ (-1)^{|a||b|}(-1)^{|a||c|}[b,c]\otimes z\otimes a\end{gathered} \tag{5.12}$$

and the two terms are equal given the choice of z and the Jacobi relations for the superalgebra:

$$\alpha^2[a,[b,c]]\otimes z\otimes z = \alpha^2[[a,b],c]\otimes z\otimes z + (-1)^{|a||b|}\alpha^2[b,[a,c]]\otimes z\otimes z \tag{5.13}$$

It is easily checked that the two operators are inverse to each other.

$$\phi_\alpha^L \phi_\alpha^{L-1}(x\otimes y) = \phi_\alpha^L(\alpha z\otimes[x,y] + (-1)^{|x||y|}y\otimes x) = \alpha^2[z,[x,y]]\otimes z + (-1)^{|[x,y]||z|}\alpha[x,y]\otimes z + (-1)^{|x||y|}\alpha[y,x]\otimes z + (-1)^{|x||y|}(-1)^{|x||y|}x\otimes y = x\otimes y \tag{5.14}$$

□

Theorem 5.4. *Let $(L,[,])$ be a Lie superalgebra, $z \in Z(L), |z| = 0$, $X \subset k$, and $\alpha, \beta : X \times X \to k$. Then, $R : X \times X \to \mathrm{End}_k(L \otimes L)$ defined by*

$$R(u,v)(a\otimes b) = \alpha(u,v)[a,b]\otimes z + \beta(u,v)(-1)^{|a||b|}a\otimes b, \tag{5.15}$$

satisfies the colored QYBE (3.4) $\iff \beta(u,w)\alpha(v,w) = \alpha(u,w)\beta(v,w)$.

Proof. Following similar steps as in the previous proof, we need that the next relations are true:

$$\begin{aligned}
&\alpha(u,v)\alpha(v,w)\beta(u,w)[a,[b,c]]\otimes z\otimes z + (-1)^{|b||c|}\alpha(u,v)\alpha(u,w)\beta(v,w)[[a,c],b]\otimes z\otimes z = \\
&\qquad \alpha(u,v)\alpha(u,w)\beta(v,w)[[a,b],c]\otimes z\otimes z \\
&(-1)^{|a||[b,c]|}\alpha(v,w)\beta(u,w)\beta(u,v)a\otimes[b,c]\otimes z = (-1)^{|a||b|+|a||c|}\alpha(v,w)\beta(u,v)\beta(u,w)a\otimes[b,c]\otimes z \\
&(-1)^{|b||c|+|[a,c]||b|}\alpha(u,w)\beta(u,v)\beta(v,w)[a,c]\otimes b\otimes z = (-1)^{|a||b|}\alpha(u,w)\beta(u,v)\beta(v,w)[a,c]\otimes b\otimes z \\
&(-1)^{|b||c|+|a||c|}\alpha(u,v)\beta(u,w)\beta(v,w)[a,b]\otimes z\otimes c = (-1)^{|[a,b]||c|}\alpha(u,v)\beta(u,w)\beta(v,w)[a,b]\otimes z\otimes c
\end{aligned} \tag{5.16}$$

It is easily observed that beside the first one all the relations are automatically fulfilled; as for the first relation, a sufficient condition is:

$$\frac{\alpha(v,w)}{\alpha(u,w)} = \frac{\beta(v,w)}{\beta(u,w)}.$$

For example, $\alpha(u,v) = f(v)$ and $\beta(u,v) = g(v)$ could be chosen. □

Remark 5.5. Letting $u = v$ above, we obtain that:

$$\phi_{\alpha,\beta}^L \; : \; L\otimes L \quad \longrightarrow \quad L\otimes L$$

$$x\otimes y \mapsto \alpha[x,y]\otimes z + (-1)^{|x||y|}\beta y\otimes x \; .$$

and its inverse:

$$\phi_{\alpha,\beta}^{L}{}^{-1} \; : \; L\otimes L \quad \longrightarrow \quad L\otimes L$$

$$x \otimes y \mapsto \frac{\alpha}{\beta^2} z \otimes [x,y] + (-1)^{|x||y|} \frac{1}{\beta} y \otimes x$$

are Yang-Baxter operators.

Remark 5.6. Let us consider the above data and apply it to Remark 4.3. Then, if we let $s, t \in X$, we obtain the following WXZ–system:

$W(a \otimes b) = R(s,s)(a \otimes b) = f(s)[a,b] \otimes z + g(s)(-1)^{|a||b|} a \otimes b$, and

$Z(a \otimes b) = R(t,t)(a \otimes b) = \ X(a \otimes b) = R(s,t)(a \otimes b) = f(t)[a,b] \otimes z + g(t)(-1)^{|a||b|} a \otimes b$.

Remark 5.7. The results presented in this section hold for Lie algebras as well. This is a consequence of the fact that these operators restricted to the first component of a Lie superalgebra have the same properties.

6 $(\mathbb{G}, \theta)$-Lie algebras

We now consider the case of $(\mathbb{G}, \theta)$-Lie algebras as in [18]: a generalization of Lie algebras and Lie superalgebras.

A $(\mathbb{G}, \theta)$-Lie algebras consists of a $\mathbb{G}$-graded vector space L, with $L = \oplus_{g \in \mathbb{G}} L_g$, $\mathbb{G}$ a finite abelian group, a non associative multiplication $\langle .., .. \rangle : L \times L \to L$ respecting the graduation in the sense that $\langle L_a, L_b \rangle \subseteq L_{a+b}, \ \ \forall a, b \in \mathbb{G}$ and a function $\theta : \mathbb{G} \times \mathbb{G} \to C^*$ taking non-zero complex values. The following conditions are imposed:

- θ-braided (G-graded) antisymmetry: $\langle x, y \rangle = -\theta(a,b) \langle y, x \rangle$
- θ-braided (G-graded) Jacobi id: $\theta(c,a)\langle x, \langle y,z \rangle \rangle + \theta(b,c)\langle z, \langle x,y \rangle \rangle + \theta(a,b)\langle y, \langle z,x \rangle \rangle = 0$
- $\theta : G \times G \to C^*$ color function $\begin{cases} \theta(a+b,c) = \theta(a,c)\theta(b,c) \\ \theta(a,b+c) = \theta(a,b)\theta(a,c) \\ \theta(a,b)\theta(b,a) = 1 \end{cases}$

for all homogeneous $x \in L_a, y \in L_b, z \in L_c$ and $\forall a, b, c \in \mathbb{G}$.

Theorem 6.1. *Under the above assumptions,*

$$R(x \otimes y) = \alpha[x,y] \otimes z + \theta(a,b) x \otimes y, \tag{6.17}$$

with $z \in Z(L)$, satisfies the equation (2.2) $\iff$ $\theta(g,a) = \theta(a,g) = \theta(g,g) = 1$, $\forall x \in L_a$ and $z \in L_g$.

The inverse operator reads: $R^{-1}(x \otimes y) = \alpha[y,x] \otimes z + \theta(b,a) x \otimes y$

Proof. If we consider the homogeneous elements $x \in L_a$, $y \in L_b$, $t \in L_c$, as before,

$$R^{12}R^{13}R^{23}(x \otimes y \otimes t) = R^{23}R^{13}R^{12}(x \otimes y \otimes t)$$

is equivalent to

$$\theta(a,g)[x,[y,t]] \otimes z \otimes z + \theta(b,c)[[x,t],y] \otimes z \otimes z = \theta(g,g)[[x,y],c] \otimes z \otimes z \quad (6.18)$$

$$\theta(a,g)\theta(a,b+c)x \otimes [y,t] \otimes z = \theta(a,b)\theta(a,c)x \otimes [y,t] \otimes z \quad (6.19)$$

$$\theta(b,c)\theta(a+c,b)[x,t] \otimes y \otimes z = \theta(a,b)\theta(b,g)[x,t] \otimes y \otimes z \quad (6.20)$$

$$\theta(b,c)\theta(a,c)[x,y] \otimes z \otimes t = \theta(a+b,c)\theta(g,c)[x,y] \otimes z \otimes t \quad (6.21)$$

Due to the conditions $\langle L_a, L_b \rangle \subseteq L_{a+b}$ the above relations are true if $\theta(a,g) = \theta(b,g) = \theta(g,c) = \theta(g,g) = 1$ is assumed. □

7 Conclusions

Motivated by the need to create a better frame for the study of Lie (super)algebras than that presented in [17], this paper extends that construction and makes an analysis on the constructions of solutions for the two-parameter form of the QYBE and Yang-Baxter systems.

Following a series of posters presented at National Conferences on Theoretical Physics, our paper generalizes the constructions from [12] (to (G,θ)-Lie algebras). Somewhat less sophisticated than that of [12] (we do not use Category Theory), our approach is direct and more suitable for applications. [14] considered the constructions of Yang-Baxter operators from Lie (co)algebras, suggesting an extension (to a bigger category with a self-dual functor acting on it) for the duality between the category of finite dimensional Lie algebras and the category of finite dimensional Lie coalgebras. This duality extension was explained in [4].

Finally, some applications of these results could be in constructions of: FRT bialgebras (from the Yang-Baxter operators obtained in this paper), knot invariants (see Remark 2.5), solutions for the classical Yang-Baxter equation (see below), etc.

Theorem 7.1. *Let $(L, [,])$ be a Lie algebra and $z \in Z(L)$. Then:*
$r : L \otimes L \longrightarrow L \otimes L, \quad x \otimes y \mapsto [x,y] \otimes z$
satisfies the classical Yang-Baxter equation:
$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$

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