The quantum H_4 integrable system

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Abstract

The quantum H_4 integrable system is a $4D$ system with rational potential related to the non-crystallographic root system H_4 with 600-cell symmetry. It is shown that the gauge-rotated H_4 Hamiltonian as well as one of the integrals, when written in terms of the invariants of the Coxeter group H_4, is in algebraic form: it has polynomial coefficients in front of derivatives. Any eigenfunctions is a polynomial multiplied by ground-state function (factorization property). Spectra corresponds to one of the anisotropic harmonic oscillator. The Hamiltonian has infinitely-many finite-dimensional invariant subspaces in polynomials, they form the infinite flag with the characteristic vector $\vec{\alpha} = (1, 5, 8, 12)$.

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I. THE HAMILTONIAN

The quantum H_4 system is a four-dimensional system related to the non-crystallographic root system H_4 [1]. The Hamiltonian of this model is invariant with respect to the H_4 Coxeter group, which is the full symmetry group of the 600-cell polytope. The H_4 Coxeter group is discrete subgroup of $O(4)$ and its dimension is 14400. In Cartesian coordinates the H_4 rational Hamiltonian has the form (see [1])

$$\begin{aligned}\mathcal{H}_{H_4} = \frac{1}{2}\sum_{k=1}^{4}\left[-\frac{\partial^2}{\partial x_k^2}+\omega^2 x_k^2+\frac{g}{x_k^2}\right]+\sum_{\mu_{2,3,4}=0,1}\frac{2g}{[x_1+(-1)^{\mu_2}x_2+(-1)^{\mu_3}x_3+(-1)^{\mu_4}x_4]^2}\\ +\sum_{\{i,j,k,l\}}\sum_{\mu_{1,2}=0,1}\frac{2g}{[x_i+(-1)^{\mu_1}\varphi_+ x_j+(-1)^{\mu_2}\varphi_- x_k+0\cdot x_l]^2}\ ,\end{aligned} \tag{1}$$

where $\{i,j,k,l\}=\{1,2,3,4\}$ and its even permutations. Here $g=\nu(\nu-1)>-1/4$ is the coupling constant, $\varphi_\pm=(1\pm\sqrt{5})/2$ the *golden ratio* and its algebraic conjugate. The configuration space is the subspace of $\mathbf{R}^4$ where the condition $(\alpha\cdot x)>0$ holds for any positive root α of H_4. It is an analogue of the principal Weyl chamber in the case of crystallographic root systems.

The ground state eigenfunction and its eigenvalue are

$$\Psi_0(x)=\Delta_1^\nu\Delta_2^\nu\Delta_3^\nu\exp\left(-\frac{\omega}{2}\sum_{k=1}^{4}x_k^2\right)\ ,\qquad E_0=2\omega(1+30\nu)\ , \tag{2}$$

where

$$\Delta_1 = \prod_{k=1}^{4}x_k, \tag{3}$$

$$\Delta_2 = \prod_{\mu_{2,3,4}=0,1}[x_1+(-1)^{\mu_2}x_2+(-1)^{\mu_3}x_3+(-1)^{\mu_4}x_4], \tag{4}$$

$$\Delta_3 = \prod_{\{i,j,k,l\}}\prod_{\mu_{1,2}=0,1}[x_i+(-1)^{\mu_1}\varphi_+ x_j+(-1)^{\mu_2}\varphi_- x_k+0\cdot x_l]\,. \tag{5}$$

The ground state eigenfunction (2) does not vanish in the configuration space.

The main object of our study is the gauge-rotated Hamiltonian (1) with the ground state eigenfunction (2) taken as a factor,

$$h_{\mathrm{H}_4} = -2(\Psi_0)^{-1}(\mathcal{H}_{\mathrm{H}_4}-E_0)(\Psi_0)\ , \tag{6}$$

where E_0 is the ground state energy given by (2). The gauge rotated operator (6) is the second-order differential operator without free term. By construction its lowest eigenfunction is a constant and the lowest eigenvalue is equal to zero.

Now let us define new variables in (6). The H_4 root space is characterized by four fundamental weights $w_c,\ c = 1,2,3,4$ (see e.g. [2]). Taking action of all group elements on fundamental weight ω_c we generate orbit $\Omega(w_c)$ of a certain length (length $\equiv$ #elements of the orbit). The results are summarized as

weight	orbit length
$w_1 = (0,\ 0,\ 0,\ 2\varphi_+)$	120
$w_2 = (1,\ \varphi_+^2,\ 0,\ \varphi_+^4)$	600
$w_3 = (0,\ \varphi_+,\ 1,\ \varphi_+^4 - 1)$	720
$w_4 = (0,\ 2\varphi_+,\ 0,\ 2\varphi_+^3)$	1200

Now let us find H_4-invariants. In order to do it we choose for simplicity the shortest orbit $\Omega(w_1)$ and make averaging,

$$t_a^{(\Omega)}(x) = \sum_{w\in\ \Omega(w_1)} (w\cdot x)^a\ , \tag{7}$$

where $a = 2, 12, 20, 30$ are the degrees of the H_4 group. It is worth noting that these invariants are defined ambiguously, up to a non-linear combination of the invariants of the lower degrees

$$\begin{aligned}
&t_2^{(\Omega)} \mapsto t_2^{(\Omega)}\ ,\\
&t_{12}^{(\Omega)} \mapsto t_{12}^{(\Omega)} + A_1\ (t_2^{(\Omega)})^6\ ,\\
&t_{20}^{(\Omega)} \mapsto t_{20}^{(\Omega)} + A_2\ (t_2^{(\Omega)})^4 t_{12}^{(\Omega)} + A_3\ (t_2^{(\Omega)})^{10}\ ,\\
&t_{30}^{(\Omega)} \mapsto t_{30}^{(\Omega)} + A_4\ (t_2^{(\Omega)})^5 t_{20}^{(\Omega)} + A_5\ (t_2^{(\Omega)})^3 (t_{12}^{(\Omega)})^2 + A_6\ (t_2^{(\Omega)})^9 (t_{12}^{(\Omega)}) + A_7\ (t_2^{(\Omega)})^{15}\ ,
\end{aligned} \tag{8}$$

where $\{A\}$ are parameters. Canonical invariant basis for the H_4 was found only recently by Mehta [4] (see also [5]). Now we use H_4 invariants as new coordinates in (6).

Now we can make a change of variables in the gauge-rotated Hamiltonian (6):

$$(x_1, x_2, x_3, x_4) \to (t_2^{(\Omega)}, t_{12}^{(\Omega)}, t_{20}^{(\Omega)}, t_{30}^{(\Omega)})\ .$$

The first observation is that the transformed Hamiltonian $h_{\mathrm{H}_4}(t)$ (6) takes on an algebraic form for any value of the parameters $\{A\}$ in variables t's (8). The second observation is that for any value of the parameters $\{A\}$ the operator $h_{\mathrm{H}_4}(t)$ has infinitely-many finite-dimensional invariant subspaces in polynomials which form infinite flag. Our goal is to find the parameters for which $h_{\mathrm{H}_4}(t)$ preserves a minimal flag (for a discussion see e.g. [3]). After some analysis such a set of parameters is found

$$
\begin{aligned}
A_1 &= -1\,, \quad A_2 = -\frac{43510}{1809}\,, \quad A_3 = \frac{41701}{1809}\,, \quad A_4 = -\frac{17583778485}{146142376}\,, \\
A_5 &= -\frac{313009515}{15383408}\,, \quad A_6 = \frac{22081114965}{7691704}\,, \quad A_7 = -\frac{798259915667}{292284752}\,.
\end{aligned} \tag{9}
$$

Hereafter we call the t-variables for such values of parameters as τ-variables.

In order to write down explicit expressions for variables τ it is useful to exploit the notation for multivariate polynomials introduced by Iwasaki et al [5]. For given partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$, let denote as M_λ the associated monomial symmetric polynomial of the variables $(x_1^2, x_2^2, x_3^2, x_4^2)$,

$$
M_\lambda = \sum x_1^{2\mu_1} x_2^{2\mu_2} x_3^{2\mu_3} x_4^{2\mu_4}\ ,
$$

where the sum is taken over all permutations $(\mu_1, \mu_2, \mu_3, \mu_4)$ of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. If λ consists of mutually distinct numbers $p_1 > \cdots > p_m$ with p_j appearing k_j times in λ, then we denote the polynomial as

$$
M_\lambda = [p_1^{k_1}|\cdots|p_m^{k_m}]\ .
$$

Let introduce also Δ_4 as the fundamental alternating polynomial of $(x_1^2, x_2^2, x_3^2, x_4^2)$,

$$
\Delta_4 = \prod_{1\leq i<j\leq 4} (x_i^2 - x_j^2)\ .
$$

Then, the τ-variables (7)-(9)are written in these notations as

$$
\begin{aligned}
\tau_1 &= [1|0^3] \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2\ , \\
\tau_2 &= 14[3^2|0^2] - 6[4|2|0^2] + 2[5|1|0^2] - 270[2^2|1^2] + 30[2^3|0] \\
&\quad -12[4|1^2|0] + 348[3|1^3] + 9[3|2|1|0] + 33\sqrt{5}\Delta_4\ , \\
\tau_3 &= 2[8|2|0^2] + 4[8|1^2|0] - 10[7|3|0^2] - 45[7|2|1|0] + 60[7|1^3] \\
&\quad +22[6|4|0^2] + 157[6|3|1|0] + 270[6|2^2|0] - 150[6|2|1^2] \\
&\quad -22[5^2|0^2] - 131[5|4|1|0] - 733[5|3|2|0] - 2156[5|3|1^2] \\
&\quad +4050[5|2^2|1] + 1320[4^2|2|0] + 4650[4^2|1^2] + 6[4|3^2|0] \\
&\quad -2175[4|3|2|1] - 19050[4|2^3] + 10800[3^2|2^2] + 3336[3^3|1] \\
&\quad +3\sqrt{5}\Delta_4\{\ 5[4|0^3] - 18[3|1|0^2] + 49[2^2|0^2] + 3[2|1^2|0] \\
&\quad +1146[1^4]\ \}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\tau_4 \ = \ & 65742[15|0^3] - 504[13|2|0^2] + 830[13|1^2|0] + 61690[12|3|0^2] \\
& -5130[12|2|1|0] - 9495[12|1^3] + 18795[11|4|0^2] \\
& +28560[11|3|1|0] - 43500[11|2^2|0] - 53070[11|2|1^2] \\
& -156330[10|5|0^2] + 59130[10|4|1|0] + 26415[10|3|2|0] \\
& +405255[10|3|1^2] + 1350[10|2^2|1] + 19710[9|6|0^2] \\
& -20[9|4|2|0] - 8663355[9|4|1^2] - 120[9|3^2|0] + 450[9|3|2|1] \\
& -962715[9|2^3] + 13860[8|7|0^2] - 94530[8|6|1|0] \\
& -353160[8|5|2|0] - 1452060[8|5|1^2] + 5557050[8|4|3|0] \\
& +590580[8|4|2|1] - 198270[8|3^2|1] + 389250[7^2|1|0] \\
& +2897820[7|6|2|0] - 5227920[7|6|1^2] + 1134540[7|5|3|0] \qquad (10) \\
& -4041270[7|5|2|1] - 591330[7|4^2|0] + 23417850[7|4|3|1] \\
& -22770[7|4|2^2] - 23528790[7|3^2|2] + 29647380[6^2|3|0] \\
& +36597510[6^2|2|1] - 1649925[6|5|4|0] + 150[6|5|3|1] \\
& +40935[6|5|2^2] - 510[6|4^2|1] - 60[6|4|3|2] + 242505[6|3^3] \\
& +270060[5^3|0] - 528270[5^2|4|1] - 36255[5|4^2|2] + 825[5|4|3^2] \\
& +707085[4^3|3] + 45\sqrt{5}\Delta_4\{\ -27040[9|0^3] - 5[8|1|0^2] \\
& -1914[7|1^2|0] + 23[6|3|0^2] + 91[6|2|1|0] - 44[6|1^3] \\
& -352[5|4|0^2] + 8[5|3|1|0] + 1085[5|2^2|0] + 6875[5|2|1^2] \\
& -5168[4^2|1|0] - 934[4|3|2|0] - 568[4|3|1^2] + 1773[4|2^2|1] \\
& +20911[3^3|0] + 15915[3^2|2|1] + 573[3|2^3]\ \}\ .
\end{aligned}
$$

Thus, the variables $\tau_{1,2,3,4}$ are homogeneous polynomials in x^2's of the degrees 1,3,10,15 , respectively.

Finally, the gauge-rotated Hamiltonian (6) in the τ-coordinates (10) written as

$$
h_{\mathrm{H}_4} \ = \ \sum_{i,j=1}^{4} A_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=1}^{4} B_i(\tau)\frac{\partial}{\partial\tau_i} \quad , \quad A_{ij} = A_{ji}\ , \qquad (11)
$$

takes amazingly simple form with the coefficient functions

$$
A_{11} \ = \ 4\ \tau_1\ ,\ A_{12} \ = \ 24\ \tau_2\ ,
$$

$$
A_{13} \ = \ 40\ \tau_3\ ,\ A_{14} \ = \ 60\ \tau_4\ ,
$$

$$
\begin{aligned}
A_{22} &= 88\ \tau_1\tau_3 + 8\ \tau_1^5\tau_2\ , \\
A_{23} &= -4\ \tau_1^3\tau_2^2 + 24\ \tau_1^5\tau_3 - 8\ \tau_4\ , \\
A_{24} &= 10\ \tau_1^2\tau_2^3 + 60\ \tau_1^4\tau_2\tau_3 + 40\ \tau_1^5\tau_4 - 600\ \tau_3^2\ , \\
A_{33} &= -\frac{38}{3}\ \tau_1\tau_2^3 + 28\ \tau_1^3\tau_2\tau_3 - \frac{8}{3}\ \tau_1^4\tau_4\ , \\
A_{34} &= 210\ \tau_1^2\tau_2^2\tau_3 + 60\ \tau_1^3\tau_2\tau_4 - 180\ \tau_1^4\tau_3^2 + 30\ \tau_2^4\ , \\
A_{44} &= -2175\ \tau_1\tau_2^3\tau_3 - 450\tau_1^2\tau_2^2\tau_4 - 1350\ \tau_1^3\tau_2\tau_3^2 - 600\ \tau_1^4\tau_3\tau_4\ ,
\end{aligned}
\qquad (12)
$$

$$
\begin{aligned}
B_1 &= 8(1+30\nu)\ - 4\omega\tau_1\ , \\
B_2 &= 12(1+10\nu)\ \tau_1^5\ -\ 24\omega\tau_2\ , \\
B_3 &= 20(1+6\nu)\ \tau_1^3\tau_2\ -\ 40\omega\tau_3\ , \\
B_4 &= 15(1-30\nu)\ \tau_1^2\tau_2^2\ -\ 450(1+2\nu)\ \tau_1^4\tau_3\ -\ 60\omega\tau_4\ .
\end{aligned}
$$

It can be easily checked that the operator (11) is triangular with respect to action on monomials $\tau_1^{p_1}\tau_2^{p_2}\tau_3^{p_3}\tau_4^{p_4}$. One can find the spectrum of (11) $h_{\mathrm{H}_4}\varphi = -2\epsilon\varphi$ explicitly

$$
\epsilon_{n_1,n_2,n_3,n_4} = 2\omega(n_1 + 6n_2 + 10n_3 + 15n_4)\,, \qquad (13)
$$

where $n_i = 0, 1, 2, \ldots$. Degeneracy of the spectrum is related to the number of solutions of the equation $n_1 + 6n_2 + 10n_3 + 15n_4 = n$ for $n = 0, 1, 2\ldots$ in non-negative numbers $n_{1,2,3,4}$. The spectrum ϵ does not depend on the coupling constant g and it is equidistant. It coincides to the spectrum of $4D$ anisotropic harmonic oscillator with frequencies $(2\omega, 12\omega, 20\omega, 30\omega)$. The energies of the original rational H_4 Hamiltonian (1) are $E = E_0 + \epsilon$. It is worth noting that the Hamiltonian (11) has infinite family of eigenfunctions $\phi_{n_1,0,0,0}$ depending on single variable τ_1. They are given by the Laguerre polynomials and the eigenvalues are linear in quantum number (cf.(13))

$$
\phi_{n_1,0,0,0}(\tau_1)\ =\ L_{n_1}^{(1+60\nu)}(\omega\tau_1)\ , \quad \epsilon_{n_1,0,0,0} = 2\omega n_1\ , \quad n_1 = 0, 1, 2, \ldots\ . \qquad (14)
$$

The boundary of the configuration space of the rational H_4 model (1) in the τ variables is determined by the zeros of the ground state eigenfunction, hence, by pre-exponential factor in (2). It is the algebraic surface of degree 120 in Cartesian coordinates being a product of monomials. In τ-coordinates (10) it can be written as

$$64\ \tau_1^{15}\tau_4^3 + 1440\ \tau_1^{14}\tau_2\tau_3\tau_4^2 + 10800\ \tau_1^{13}\tau_2^2\tau_3^2\tau_4 + 27000\ \tau_1^{12}\tau_2^3\tau_3^3 - 240\ \tau_1^{12}\tau_2^3\tau_4^2$$
$$- 3600\ \tau_1^{11}\tau_2^4\tau_3\tau_4 - 13500\ \tau_1^{10}\tau_2^5\tau_3^2 + 34992\ \tau_1^{10}\tau_3^5 - 1440\ \tau_1^{10}\tau_3^2\tau_4^2 + 300\ \tau_1^9\tau_2^6\tau_4$$
$$- 2160\ \tau_1^9\tau_2\tau_3^3\tau_4 - 1440\tau_1^9\tau_2\tau_4^3 + 2250\ \tau_1^8\tau_2^7\tau_3 - 22680\ \tau_1^8\tau_2^2\tau_3^4 - 28080\ \tau_1^8\tau_2^2\tau_3\tau_4^2$$
$$- 203760\ \tau_1^7\tau_2^3\tau_3^2\tau_4 - 125\ \tau_1^6\tau_2^9 - 493020\ \tau_1^6\tau_2^4\tau_3^3 + 3600\ \tau_1^6\tau_2^4\tau_4^2 + 57780\ \tau_1^5\tau_2^5\tau_3\tau_4$$
$$- 8640\ \tau_1^5\tau_3^4\tau_4 + 4320\ \tau_1^5\tau_3\tau_4^3 + 221310\ \tau_1^4\tau_2^6\tau_3^2 - 648000\ \tau_1^4\tau_2\tau_3^5 + 116640\ \tau_1^4\tau_2\tau_3^2\tau_4^2$$
$$- 4680\ \tau_1^3\tau_2^7\tau_4 + 712800\ \tau_1^3\tau_2^2\tau_3^3\tau_4 + 6480\ \tau_1^3\tau_2^2\tau_4^3 - 35640\ \tau_1^2\tau_2^8\tau_3 + 2052000\ \tau_1^2\tau_2^3\tau_3^4$$
$$\begin{aligned} &+ 62640\ \tau_1^2\tau_2^3\tau_3\tau_4^2 + 259200\ \tau_1\tau_2^4\tau_3^2\tau_4 + 1944\ \tau_2^{10} + 129600\ \tau_2^5\tau_3^3 + 2592\ \tau_2^5\tau_4^2 \\ &+ 2160000\ \tau_3^6 - 86400\ \tau_3^3\tau_4^2 + 864\ \tau_4^4\ =\ 0\ , \end{aligned} \tag{15}$$

which is the algebraic surface of degree 18 being given by a polynomial of degree 15 in τ_1, of the degree 10 in τ_2, of the degree 6 in τ_3 and of the degree 4 in τ_4. It is worth mentioning that l.h.s. of (15) is proportional to the square of Jacobian, $J^2(\frac{\partial\tau}{\partial x})$.

The Hamiltonian $h_{\mathrm{H}_4}(\tau)$ has infinitely-many finite-dimensional invariant subspaces

$$\mathcal{P}_n^{(1,5,8,12)} = \langle \tau_1^{p_1}\tau_2^{p_2}\tau_3^{p_3}\tau_4^{p_4}\,|\, 0 \le p_1 + 5p_2 + 8p_3 + 12p_4 \le n\rangle\ , \quad n = 0, 1, 2, \dots\ , \tag{16}$$

which form the (minimal) infinite flag. Its characteristic vector is

$$\vec{\alpha}_{min}\ =\ (1, 5, 8, 12)\ . \tag{17}$$

It is worth noting that each particular space $\mathcal{P}_n^{(1,5,8,12)}$ (16) as well as the whole flag are invariant with respect to a weighted projective transformation

$$\begin{aligned} \tau_1\ &\to\ \tau_1 + a\,, \\ \tau_2\ &\to\ \tau_2 + b_1\,\tau_1^5 + b_2\,\tau_1^4 + b_3\,\tau_1^3 + b_4\,\tau_1^2 + b_5\,\tau_1 + b_6\,, \\ \tau_3\ &\to\ \tau_3 + c_1\,\tau_1^3\tau_2 + c_2\,\tau_1^2\tau_2 + c_3\,\tau_1\tau_2 + c_4\,\tau_2 + c_5\,\tau_1^8 + c_6\,\tau_1^7 \\ &\qquad + c_7\,\tau_1^6 + c_8\,\tau_1^5 + c_9\,\tau_1^4 + c_{10}\,\tau_1^3 + c_{11}\,\tau_1^2 + c_{12}\,\tau_1 + c_{13}\,, \\ \tau_4\ &\to\ \tau_4 + d_1\,\tau_1^4\tau_3 + d_2\,\tau_1^3\tau_3 + d_3\,\tau_1^2\tau_3 + d_4\,\tau_1\tau_3 + d_5\,\tau_3 \\ &\qquad + d_6\,\tau_1^7\tau_2 + d_7\,\tau_1^6\tau_2 + d_8\,\tau_1^5\tau_2 + d_9\,\tau_1^4\tau_2 + d_{10}\,\tau_1^3\tau_2 \\ &\qquad + d_{11}\,\tau_1^2\tau_2 + d_{12}\,\tau_1\tau_2 + d_{13}\,\tau_2 + d_{14}\,\tau_1^{12} + d_{15}\,\tau_1^{11} \\ &\qquad + d_{16}\,\tau_1^{10} + d_{17}\,\tau_1^9 + d_{18}\,\tau_1^8 + d_{19}\,\tau_1^7 + d_{20}\,\tau_1^6 + d_{21}\,\tau_1^5 \\ &\qquad + d_{22}\,\tau_1^4 + d_{23}\,\tau_1^3 + d_{24}\,\tau_1^2 + d_{25}\,\tau_1 + d_{26}\,, \end{aligned} \tag{18}$$

where $\{a, b, c, d\}$ are parameters. It manifests a hidden invariance of the Hamiltonian (1) preserving its algebraic form. A meaning of this invariance is unclear.

II. INTEGRAL

The Hamiltonian (1) can be written in *hyperspherical coordinates*

$$\begin{aligned} x_1 &= r \sin\psi \sin\theta \cos\phi \ , \\ x_2 &= r \sin\psi \sin\theta \sin\phi \ , \\ x_3 &= r \sin\psi \cos\theta \ , \\ x_4 &= r \cos\psi \ , \end{aligned} \tag{19}$$

where it takes the form

$$\mathcal{H}_{H_4} = -\frac{1}{2}\Delta^{(4)} + \frac{1}{2}\omega^2 r^2 + \frac{W(\psi,\theta,\phi)}{r^2} \ . \tag{20}$$

Here $\Delta^{(4)}$ is the $4D$ Laplacian and the angular function

$$\begin{aligned} W(\psi,\theta,\phi) = &\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi + \varphi_+ s_\psi s_\theta s_\phi + \varphi_- s_\psi c_\theta)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi - \varphi_+ s_\psi s_\theta s_\phi + \varphi_- s_\psi c_\theta)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi + \varphi_+ s_\psi s_\theta s_\phi - \varphi_- s_\psi c_\theta)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi - \varphi_+ s_\psi s_\theta s_\phi - \varphi_- s_\psi c_\theta)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi + \varphi_+ s_\psi c_\theta + \varphi_- c_\psi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi - \varphi_+ s_\psi c_\theta + \varphi_- c_\psi)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi + \varphi_+ s_\psi c_\theta - \varphi_- c_\psi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi - \varphi_+ s_\psi c_\theta - \varphi_- c_\psi)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi + \varphi_+ c_\psi + \varphi_- s_\psi s_\theta s_\phi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi - \varphi_- c_\psi + \varphi_- s_\psi s_\theta s_\phi)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi + \varphi_+ c_\psi - \varphi_- s_\psi s_\theta s_\phi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi - \varphi_+ c_\psi - \varphi_- s_\psi s_\theta s_\phi)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi + \varphi_+ s_\psi s_\theta c_\phi + \varphi_- c_\psi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi - \varphi_+ s_\psi s_\theta c_\phi + \varphi_- c_\psi)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi + \varphi_+ s_\psi s_\theta c_\phi - \varphi_- c_\psi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi - \varphi_+ s_\psi s_\theta c_\phi - \varphi_- c_\psi)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi + \varphi_+ s_\psi c_\theta + \varphi_- s_\psi s_\theta c_\phi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi - \varphi_+ s_\psi c_\theta + \varphi_- s_\psi s_\theta c_\phi)^2} \\ &+ \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi + \varphi_+ s_\psi c_\theta - \varphi_- s_\psi s_\theta c_\phi)^2} + \frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi - \varphi_- s_\psi c_\theta - \varphi_- s_\psi s_\theta c_\phi)^2} \end{aligned}$$

$$
\begin{aligned}
&+\frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi+\varphi_+c_\psi+\varphi_-s_\psi c_\theta)^2}+\frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi-\varphi_+c_\psi+\varphi_-s_\psi c_\theta)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi+\varphi_+c_\psi-\varphi_-s_\psi c_\theta)^2}+\frac{2\nu(\nu-1)}{(s_\psi s_\theta s_\phi-\varphi_+c_\psi-\varphi_-s_\psi c_\theta)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_+s_\psi s_\theta c_\phi+\varphi_-s_\psi s_\theta s_\phi)^2}+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_-s_\psi s_\theta c_\phi+\varphi_-s_\psi s_\theta s_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_+s_\psi s_\theta c_\phi-\varphi_-s_\psi s_\theta s_\phi)^2}+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_-s_\psi s_\theta c_\phi-\varphi_-s_\psi s_\theta s_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_+c_\psi+\varphi_-s_\psi s_\theta c_\phi)^2}+\frac{2\nu(\nu-1)}{(s_\psi c_\theta-\varphi_+c_\psi+\varphi_-s_\psi s_\theta c_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_+c_\psi-\varphi_-s_\psi s_\theta c_\phi)^2}+\frac{2\nu(\nu-1)}{(s_\psi c_\theta-\varphi_+c_\psi-\varphi_-s_\psi s_\theta c_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_+s_\psi s_\theta s_\phi+\varphi_-c_\psi)^2}+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_-s_\psi s_\theta s_\phi+\varphi_-c_\psi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_+s_\psi s_\theta s_\phi-\varphi_-c_\psi)^2}+\frac{2\nu(\nu-1)}{(s_\psi c_\theta+\varphi_-s_\psi s_\theta s_\phi-\varphi_-c_\psi)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_+s_\psi s_\theta c_\phi+\varphi_-s_\psi c_\theta)^2}+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_-s_\psi s_\theta c_\phi+\varphi_-s_\psi c_\theta)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_+s_\psi s_\theta c_\phi-\varphi_-s_\psi c_\theta)^2}+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_-s_\psi s_\theta c_\phi-\varphi_-s_\psi c_\theta)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_+s_\psi s_\theta s_\phi+\varphi_-s_\psi s_\theta c_\phi)^2}+\frac{2\nu(\nu-1)}{(c_\psi-\varphi_+s_\psi s_\theta s_\phi+\varphi_-s_\psi s_\theta c_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_+s_\psi s_\theta s_\phi-\varphi_-s_\psi s_\theta c_\phi)^2}+\frac{2\nu(\nu-1)}{(c_\psi-\varphi_+s_\psi s_\theta s_\phi-\varphi_-s_\psi s_\theta c_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_+s_\psi c_\theta+\varphi_-s_\psi s_\theta s_\phi)^2}+\frac{2\nu(\nu-1)}{(c_\psi-\varphi_+s_\psi c_\theta+\varphi_-s_\psi s_\theta s_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(c_\psi+\varphi_+s_\psi c_\theta-\varphi_-s_\psi s_\theta s_\phi)^2}+\frac{2\nu(\nu-1)}{(c_\psi-\varphi_+s_\psi c_\theta-\varphi_-s_\psi s_\theta s_\phi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi+s_\psi s_\theta s_\phi+s_\psi c_\theta+c_\psi)^2}+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi-s_\psi s_\theta s_\phi+s_\psi c_\theta+c_\psi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi+s_\psi s_\theta s_\phi-s_\psi c_\theta+c_\psi)^2}+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi+s_\psi s_\theta s_\phi+s_\psi c_\theta-c_\psi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi-s_\psi s_\theta s_\phi-s_\psi c_\theta+c_\psi)^2}+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi-s_\psi s_\theta s_\phi+s_\psi c_\theta-c_\psi)^2}\\
&+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi+s_\psi s_\theta s_\phi-s_\psi c_\theta-c_\psi)^2}+\frac{2\nu(\nu-1)}{(s_\psi s_\theta c_\phi-s_\psi s_\theta s_\phi-s_\psi c_\theta-c_\psi)^2}\\
&+\frac{\nu(\nu+1)}{2s_\psi^2s_\theta^2c_\phi^2}+\frac{\nu(\nu+1)}{2s_\psi^2s_\theta^2s_\phi^2}+\frac{\nu(\nu+1)}{2s_\psi^2c_\theta^2}+\frac{\nu(\nu+1)}{2c_\psi^2}\ .
\end{aligned}
\tag{21}
$$

Here, for the sake of simplicity we denoted $c_\psi\equiv\cos\psi$, $s_\psi\equiv\sin\psi$, $c_\theta\equiv\cos\theta$, $s_\theta\equiv\sin\theta$ and $c_\phi\equiv\cos\phi$, $s_\phi\equiv\sin\phi$. It is seen immediately, that the Schroedinger equation (20) admits a separation of radial variable r: any solution can be written in factorized form

$$\Psi(r,\psi,\theta,\phi) = R(r)Q(\psi,\theta,\phi)\ . \tag{22}$$

Functions R and Q are the solutions of the equations

$$\left[-\frac{1}{2r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right)+\frac{1}{2}\omega^2 r^2+\frac{\gamma}{r^2}\right]R(r) \;=\; ER(r)\ , \tag{23}$$

$$\mathcal{F}\ Q(\psi,\theta,\phi) = \gamma\ Q(\psi,\theta,\phi)\ , \tag{24}$$

respectively, while γ is the constant of separation. The operator $\mathcal{F}$ has the form

$$\mathcal{F} \;=\; -\frac{1}{2\sin^2\psi}\left[\frac{\partial}{\partial\psi}\left(\sin^2\psi\frac{\partial}{\partial\psi}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]+W(\psi,\theta,\phi)\ , \tag{25}$$

It can be immediately checked that the Hamiltonian $\mathcal{H}_{H_4}$ and $\mathcal{F}$ commute,

$$[\mathcal{H}_{H_4},\mathcal{F}] \;=\; 0\ . \tag{26}$$

Hence, $\mathcal{F}$ is an integral of motion. Thus, it has common eigenfunctions with the Hamiltonian $\mathcal{H}_{H_4}$.

Let us make a gauge rotation of the operator $\mathcal{F}$ (25) with the ground state function Ψ_0 as a gauge factor,

$$f \;=\; (\Psi_0)^{-1}(\mathcal{F}-\gamma_0)\Psi_0\ , \qquad \gamma_0 = 60\nu(1+30\nu)\ , \tag{27}$$

where γ_0 is the lowest eigenvalue of $\mathcal{F}$. Then make a change of variables to the τ-variables (10). The operator f gets an algebraic form,

$$f \;=\; \sum_{i,j=1}^{4} F_{ij}\frac{\partial^2}{\partial\tau_i\partial\tau_j}+\sum_{j=1}^{4} G_j\frac{\partial}{\partial\tau_j}\ , \quad F_{ij}=F_{ji} \tag{28}$$

where

$$F_{11} \;=\; 0\ ,\ \ F_{12} \;=\; 0\ ,$$

$$F_{13} \;=\; 0\ ,\ \ F_{14} \;=\; 0\ ,$$

$$F_{22} \;=\; -4\tau_1^6\tau_2-44\tau_1^2\tau_3+72\tau_2^2\ ,$$

$$F_{23} \;=\; -12\tau_1^6\tau_3+2\tau_1^4\tau_2^2+4\tau_1\tau_4+120\tau_2\tau_3\ ,$$

$$F_{24} \;=\; -20\tau_1^6\tau_4-30\tau_1^5\tau_2\tau_3-5\tau_1^3\tau_2^3+300\tau_1\tau_3^2+180\tau_2\tau_4\ ,$$

$$
\begin{aligned}
F_{33} &= \frac{4}{3}\tau_1^5\tau_4 - 14\tau_1^4\tau_2\tau_3 + \frac{19}{3}\tau_1^2\tau_2^3 + 200\tau_3^2\,,\\
F_{34} &= 90\tau_1^5\tau_3^2 - 30\tau_1^4\tau_2\tau_4 - 105\tau_1^3\tau_2^2\tau_3 - 15\tau_1\tau_2^4 + 300\tau_3\tau_4\,,\\
F_{44} &= 300\tau_1^5\tau_3\tau_4 + 675\tau_1^4\tau_2\tau_3^2 + 225\tau_1^3\tau_2^2\tau_4 + \frac{2175}{2}\tau_1^2\tau_2^3\tau_3 + 450\tau_4^2\,,\\
G_1 &= 0\,,\\
G_2 &= -6(1+10\nu)\tau_1^6 + 12(7+60\nu)\tau_2\,,\\
G_3 &= -10(1+6\nu)\tau_1^4\tau_2 + 20(11+60\nu)\tau_3\,,\\
G_4 &= 225(1+2\nu)\tau_1^5\tau_3 - \frac{15}{2}(1-30\nu)\tau_1^3\tau_2^2 + 40(12+45\nu)\tau_4\,.
\end{aligned}
\tag{29}
$$

It is worth noting that in the operator f the variable τ_1 appears as a parameter. It implies that any eigenfunction of the Hamiltonian h_{H_4}, which depends on τ_1 only, is an eigenfunction of the integral f with zero eigenvalue.

It can be also shown that the operator f has infinitely many finite-dimensional invariant subspaces in polynomials

$$
\mathcal{P}_n^{(1,6,10,15)} \;=\; \langle \tau_1^{p_1}\tau_2^{p_2}\tau_3^{p_3}\tau_4^{p_4}\,|\, 0 \le p_1 + 6p_2 + 10p_3 + 15p_4 \le n\rangle\ ,\ n = 0,1,2,\dots\ , \tag{30}
$$

which form a flag with characteristic vector $(1,6,10,15)$. The spectrum of the integral $\mathcal{F}\Psi = \Gamma\Psi$ can be found in a closed form,

$$
\Gamma_{0,k_2,k_3,k_4} \;\equiv\; \gamma_{0,k_2,k_3,k_4} + \gamma_0 \;=
$$

$$
72k_2^2+200k_3^2+450k_4^2+120k_2k_3+180k_2k_4+300k_3k_4+2(1+60\nu)(6k_2+10k_3+15k_4)+\gamma_0\ , \tag{31}
$$

where $k_2, k_3, k_4 = 0,1,2,\dots$ and γ_0 is given by (27).

It can be shown that the Hamiltonian h_{H_4} has a certain degeneracy – it preserves two different flags: one with (minimal) characteristic vector (1,5,8,12) and another one with characteristic vector (1,6,10,15). The fact that the operator h_{H_4} with coefficients (12) commutes with f given by (28) implies that common eigenfunctions of the operators h_{H_4} and f are elements of the flag of spaces $\mathcal{P}^{(1,6,10,15)}$.

Let us denote $\phi_{n,i}$ the eigenfunctions of h_{H_4} which are elements of the invariant space $P_n^{(1,5,8,12)}$ and their respectful eigenvalues $\epsilon_{n,i}$. The index i numerates these eigenfunctions for given n starting from 0. It is evident that an eigenfunction with $n < 5$ depends on τ_1 only and its γ is equal to zero. The eigenfunctions with $4 < n < 8$ depend on $\tau_{1,2}$ only while the dependence on the $\tau_{1,2,3}$ occurs for the eigenfunctions with $7 < n < 12$. The eigenfunctions with $n \ge 12$ depend on all four variables $\tau_{1,2,3,4}$.

The function $\phi_{n,i}$ is related to the eigenfunction of the Hamiltonian $\mathcal{H}_{H_4}$ (1) (and the integral $\mathcal{F}$) through $\Psi_{n,i} = \Psi_0 \phi_{n,i}$. Thus, the eigenfunctions $\{\phi\}$ are orthogonal with the weight factor $|\Psi_0|^2$. As an illustration let us give explicit expressions for several eigenfunctions $\phi_{n,i}$ and their respectful eigenvalues,

- $n = 0$

$$\phi_{0,0} = 1 \ , \quad \epsilon_{0,0} = 0 \ ,$$

- $n = 1$

$$\phi_{1,0} = \omega\tau_1 - 2(1+30\nu) \ , \quad \epsilon_{1,0} = 2\omega \ ,$$

- $n = 2$

$$\phi_{2,0} = \omega^2\tau_1^2 - 6\omega(1+20\nu)\tau_1 + 6(1+20\nu)(1+30\nu) \ , \quad \epsilon_{2,0} = 4\omega \ ,$$

- $n = 3$

$$\phi_{3,0} \ = \ \omega^3\tau_1^3 - 12\omega^2(1+15\nu)\tau_1^2 + 36\omega(1+15\nu)(1+20\nu)\tau_1 - 24(1+15\nu)(1+20\nu)(1+30\nu) \ ,$$

$$\epsilon_{3,0} \ = \ 6\omega \ ,$$

- $n = 4$

$$\phi_{4,0} = L_4^{(1+60\nu)}(\omega\tau_1) \ , \ \epsilon_{4,0} = 8\omega \ ,$$

- $n = 5$

$$\phi_{5,0} = L_5^{(1+60\nu)}(\omega\tau_1) \ , \ \epsilon_{5,0} = 10\omega \ ,$$

$$\phi_{5,1} \ = \ \omega^6\tau_2 - 3(1+10\nu)\omega^5\tau_1^5 + 45\omega^4(1+10\nu)^2\tau_1^4 - 300\omega^3(1+12\nu)(1+10\nu)^2\tau_1^4 +$$
$$900\omega^2(1+15\nu)(1+12\nu)(1+10\nu)^2\tau_1^2 - 1080\omega(1+20\nu)(1+15\nu)(1+12\nu)(1+10\nu)^2\tau_1 +$$
$$360(1+30\nu)(1+20\nu)(1+15\nu)(1+12\nu)(1+10\nu)^2 \ ,$$

$$\epsilon_{5,1} = 12\omega \ .$$

Let us denote $\tilde{\phi}_{n,i}$ the common eigenfunctions of h_{H_4} and f which are elements of the invariant space $P_n^{(1,6,10,15)}$ and their respectful eigenvalues $\tilde{\epsilon}_{n,i}, \gamma_{n,i}$. The index i numerates these eigenfunctions for given n starting from 0. It is evident that an eigenfunction with $n < 6$ depends on τ_1 only and its γ is equal to zero. The eigenfunctions with $6 \le n < 10$ can depend on $\tau_{1,2}$ and the dependence on the $\tau_{1,2,3}$ occurs for the eigenfunctions with $10 \le n < 15$. It is evident that all eigenstates (n,i) at fixed n and different i are degenerate: their eigenvalues are equal to $2\omega n$. We give some eigenstates from $P_n^{(1,6,10,15)}$ explicitly,

- $n=0$

$$\tilde{\phi}_{0,0} = 1\ , \quad \tilde{\epsilon}_{0,0} = 0\ , \quad \gamma_{0,0} = 0\ .$$

- $n=1$

$$\tilde{\phi}_{1,0} = \omega\tau_1 - 2(1+30\nu)\ , \quad \tilde{\epsilon}_{1,0} = 2\omega\ , \quad \gamma_{1,0} = 0$$

- $n=2$

$$\tilde{\phi}_{2,0} = \omega^2\tau_1^2 - 6\omega(1+20\nu)\tau_1 + 6(1+20\nu)(1+30\nu)\ , \quad \tilde{\epsilon}_{2,0} = 4\omega\ , \quad \gamma_{2,0} = 0$$

- $n=3$

$$\tilde{\phi}_{3,0} \;=\; \omega^3\tau_1^3 - 12\omega^2(1+15\nu)\tau_1^2 + 36\omega(1+15\nu)(1+20\nu)\tau_1 - 24(1+15\nu)(1+20\nu)(1+30\nu)\,,$$

$$\tilde{\epsilon}_{3,0} = 6\omega\ , \ \gamma_{3,0} = 0$$

- $n=4$

$$\tilde{\phi}_{4,0} = L_4^{(1+60\nu)}(\omega\tau_1)\ , \ \tilde{\epsilon}_{4,0} = 8\omega\ , \ \gamma_{4,0} = 0$$

- $n=5$

$$\tilde{\phi}_{5,0} = L_5^{(1+60\nu)}(\omega\tau_1)\ , \ \tilde{\epsilon}_{5,0} = 10\omega\ , \ \gamma_{5,0} = 0$$

- $n=6$

$$\tilde{\phi}_{6,0} = L_6^{(1+60\nu)}(\omega\tau_1)\ , \ \tilde{\epsilon}_{6,0} = 12\omega\ , \ \gamma_{6,0} = 0$$

$$\tilde{\phi}_{6,1} \;=\; \tau_2 - \frac{1+10\nu}{4(7+60\nu)}\tau_1^6\ , \ \tilde{\epsilon}_{6,1} = 12\omega\ , \ \gamma_{6,1} = 12(7+60\nu)\ .$$

It is worth noting that $\phi_{5,1} = \omega^6\tilde{\phi}_{6,1} + A\tilde{\phi}_{6,0}$ where A is a parameter. Eigenfunctions $\phi_{n,0} = \tilde{\phi}_{n,0}$ at $n \le 6$.

III. CONCLUSIONS

We have shown that the H_4 rational system related to the non-crystallographic root system H_4 is exactly solvable with the characteristic vector $(1, 5, 8, 12)$. This work complements the previous studies of the rational (and trigonometric) models, related with crystallographic root systems (e.g. [6] - [9], [3]) and non-crystallographic root systems $I_2(k)$ [10] and H_3 [11]. A certain significance of exploration of the H_4 rational system is due to a fact that this model is defined in four-dimensional Euclidian space. There are very few known exactly-solvable systems in this space – five-body Calogero-Sutherland (A_4) and BC_4 rational-trigonometric models among them. All of them are completely-integrable.

Taking Coxeter invariants of H_4 as coordinates provided us a way to reduce the rational H_4 Hamiltonian to algebraic form. It gave us a chance to find the eigenfunctions of the rational H_4 Hamiltonian which are proportional to polynomials in these invariant coordinates. It seems correct that these eigenfunctions exhaust all eigenfunctions in the Hilbert space. It is worth noting that the matrix $A_{ij}(\tau)$ which appears in front of the second derivatives after changing variables in Laplacian from Cartesian to the H_4 Coxeter invariant coordinates (see Eqs. (12)) has polynomial entries corresponding to flat space metric, hence the Riemann tensor vanishes.

It should be stressed that it was stated in Lax pair formalism that the Hamiltonian of the H_4 rational system (1) is completely integrable [12]. This implies the existence of three mutually-commuted operators (the 'higher Hamiltonians') which commute with the Hamiltonian forming a commutative algebra. It is known (see [1]) for the crystallographic systems that these higher Hamiltonians are the differential operators of the degrees which coincide to the minimal degrees of the root space (the Lie algebra) or their doubles for the A_N case. It may suggest that for the H_4 rational system the commuting integrals might be differential operators of the orders 12, 20 and 30. Their explicit forms are not known so far. It seems evident that these commuting operators should take on an algebraic form after a gauge rotation (with the ground state function as a gauge factor), and a change of variables from Cartesian coordinates to the Coxeter invariant variables τ's. Interesting open question is about a flag of invariant subspaces: would it be one of these two flags preserved by h_{H_4}? Following the experience with different integrable systems, it seems the integral(s) related with separation of variables do not enter to the commutative algebra. Therefore, the integral $\mathcal{F}$ lays out of the commutative algebra of integrals. It might serve as an indication to a superintegrability of the H_4 rational system.

It should be pointed out that unlike the rational models of the crystallographic root spaces it is not possible to construct integrable (and exactly-solvable) trigonometric systems related to the non-crystallographic root spaces as a natural generalization of the Hamiltonian Reduction Method [1].

The existence of algebraic form of the H_4 rational Olshanetsky-Perelomov Hamiltonian makes possible the study of their polynomial perturbations which are invariant wrt the H_4 Coxeter group by purely algebraic means: one can develop a perturbation theory in which all corrections are found by linear algebra methods [13]. In particular, it gives a chance to calculate the H_4 Coxeter-invariant, polynomial correlation functions by algebraic means.

Another important property of the existence of algebraic form of the H_4 rational Hamiltonian is a chance to perform a canonical, Lie-algebraic discretization to uniform ([14]) and exponential [15] lattices, or mixed uniform-exponential lattices. In the case of all three lattices such a discretization preserves a property of integrability, polynomiality of the eigenfunctions remains and it is isospectral. Making the weighted projective transformation (18) of the H_4 algebraic form (11) we arrive at different algebraic form of the H_4 Hamiltonian. Making then the Lie-algebraic discretization we arrive at a discrete model related to an original discrete model via change of variables. It can be considered as a definition of a polynomial change of variables for discrete operators.

One can find the $sl(2)$-quasi-exactly-solvable generalization [16] of the H_4 model which remains integrable. This is one of the first examples of quasi-exact-solvability related to non-crystallographic root systems. It complements the results obtained previously for all rational models related to crystallographic systems (see [17]), for the $I_2(k)$ rational model [10] and for H_3 [11] – each of these models admit a certain $sl(2)$-QES generalization in a form of sixth degree polynomial potential.

Owing to the explicit knowledge of the ground state function (2) supersymmetric H_4 model can be constructed following a procedure realized in [18] for A_N rational model, in [19] for the BC_N rational model and in [20] for the $I_2(k)$ rational model. It can be done elsewhere.

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