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An old Einstein - Eddington generalized gravity and modern ideas on branes and cosmology.

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Abstract

We briefly discuss new models of an 'affine' theory of gravity in multidimensional space-times with *symmetric connections*. We use and generalize Einstein's proposal to specify the space-time geometry by use of the Hamilton principle to determine the connection coefficients from a *geometric Lagrangian* that is an arbitrary function of the generalized Ricci curvature tensor and of other fundamental tensors. Such a theory supplements the standard Einstein gravity with dark energy (the cosmological constant, in the first approximation), a neutral massive (or tachyonic) vector field (*vecton*), and massive (or tachyonic) *scalar fields*. These fields couple only to gravity and can generate dark matter and/or inflation. The concrete choice of the geometric Lagrangian determines further details of the theory. The most natural geometric models look similar to recently proposed brane models of cosmology usually derived from string theory.

The history of science teaches us not to completely forget beautiful and logically consistent papers of the past that were not understood in time. Even though not recognized by the contemporaries (and, often, by the authors) some of them happen to become of interest many years after their publishing. One can recall sufficiently many examples of such work and here we discuss a misunderstood and forgotten model based on work of three eminent scientists (Weyl, Eddington and Einstein) that was reinterpreted and generalized in [1]-[3].

By the end of 1922, Einstein deeply studied and seriously reconsidered attempts of Weyl and Eddington (see [4] - [6]) to construct an affine modification[1] of his general relativity. In 1923 he published three beautiful and concise papers [7] later summarized in [8] and soon forgotten (but see brief discussions in [9], [10]). The most clear exposition of Einstein's approach is given in [8] while the most beautiful model was proposed in the first paper of the series [7]. Here we only briefly summarize general principles, which can more or less naturally restrict possible choice of the physical models. Then a simple model satisfying these principles and generalizing Einstein's first paper, which we call the Einstein - Eddington model, will be introduced and compared to some recently discussed cosmologies based on string theory.

The most important properties of the affine theory are the following. **1.** It predicts the existence of one or more vector fields with real or imaginary mass. **2.** Its D-dimensional

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[1] In 1918, Weyl introduced a special symmetric non-Riemannian connection depending on a metric tensor and on a vector field ('Weyl's connection'), which he attempted to identify with the electromagnetic potential. His theory was severely criticized and is mostly remembered because in it he first introduced a fairly general concept of the gauge symmetry.

generalization predicts (after the simplest dimensional reduction) $(D-4)$ scalar fields with the same mass. **3.** Both the vector and scalar fields couple to gravity only (being the part of the generalized gravitation). **4.** The most natural effective ('physical') Lagrangian contains Eddington - Einstein terms (nowadays often called Dirac - Born - Infeld terms).

Einstein's key idea was to derive the concrete form of the affine connection by applying the Hamilton principle to a generic Lagrangian depending on the generalized Ricci curvature. This assumption completely fixes a geometry, which does not coincide with Weyl's geometry, but belongs to the same simple class of connections introduced and discussed in [1] - [3]. Einstein's unusual result was difficult to comprehend in the first half of the last century and it remains somewhat puzzling even these days. From the modern mathematics viewpoint, its origin could be ascribed to a sort of a mismatch between the affine connection geometry and the Lagrangian 'geometry'. At the moment, it is difficult to find a more detailed explanation. Possibly, this is an interesting mathematical problem.

In Refs.[1] - [3] we follow Einstein's approach and first construct a *geometric Lagrangian density* having the dimension L^{-D} (in units $c=1$). Then we show that, without a metric, one can use scalar densities of weight two constructed of pure geometric fields (see [1] - [3]), the square roots of which give the desirable scalar densities of weight one. The effective *physical Lagrangians* are derived from the geometric ones. A more detailed presentation of the main steps briefly discussed here can be found in [1] - [3].

Here we first outline basic **geometrical facts** and then concentrate on a physical model that looks most interesting for applications to cosmology.

A general exposition of the theory of non-Riemannian spaces equipped with a *symmetric connection* can be found in [11], [9], and in our previous papers. In general, the connection coefficients can be expressed in terms of a Riemannian connection Γ^i_{jk} and of an arbitrary third rank *tensor* a^i_{jk} that is symmetric in the lower indices

$$\gamma^i_{jk} \,=\, \Gamma^i_{jk}[g] + a^i_{jk}\,. \tag{1}$$

Here g_{ij} is and arbitrary symmetric tensor and $\Gamma^i_{jk}[g]$ is its Christoffel symbol. More precisely, for any symmetric connection γ^i_{jk}, there exists a symmetric tensor g_{ij} and a tensor $a^i_{jk} = a^i_{kj}$ such that (1) is satisfied.

The curvature tensor r^i_{jkl} can be defined in terms of γ^i_{jk} by the standard general expression not using any metric,

$$r^i_{jkl} = -\gamma^i_{jk,l} + \gamma^i_{mk}\gamma^m_{jl} + \gamma^i_{jl,k} - \gamma^i_{ml}\gamma^m_{jk}\,. \tag{2}$$

Then, the Ricci-like (but *non-symmetric*) curvature tensor can be defined by contracting the indices i, l:

$$r_{jk} \equiv r^i_{jki} = -\gamma^i_{jk,i} + \gamma^i_{mk}\gamma^i_{ji} + \gamma^i_{ji,k} - \gamma^i_{mi}\gamma^m_{jk} \tag{3}$$

(we again stress that $\gamma^i_{jk} = \gamma^i_{kj}$ but $r_{jk} \neq r_{kj}$). Using only these tensors and the completely antisymmetric tensor density of the rank D, we can construct a quite rich geometric structure.

The *antisymmetric part* of the Ricci curvature r_{ij} can be expressed in terms of the vector field $\gamma_i \equiv \gamma^m_{im}$ or in terms of $a_i \equiv a^m_{im}$, which differ by the gradient term $\partial_i \ln\sqrt{|g|}$ ($g \equiv \det(g_{ij})$):

$$a_{ij} \equiv \frac{1}{2}(r_{ij} - r_{ji}) \equiv -\frac{1}{2}(a_{i,j} - a_{j,i}) \equiv -\frac{1}{2}(\gamma_{i,j} - \gamma_{j,i})\,. \tag{4}$$

We call this field *vecton* and will see that it can be massive or tachyonic depending on a choice of the connection. This definition of the vecton is independent of the division of the connection (1) into the metric and non-metric parts. By the way, $r^m_{mij} = 2a_{ij}$.

Introducing the covariant derivative ∇^γ_i (with respect to the connection γ) we can write the symmetric part of the curvature as

$$s_{ij} \equiv \frac{1}{2}(r_{ij} + r_{j\,i}) = -\nabla^\gamma_m \gamma^m_{ij} + \frac{1}{2}(\nabla^\gamma_i \gamma_j + \nabla^\gamma_j \gamma_i) - \gamma^m_{ni}\gamma^n_{mj} + \gamma^n_{ij}\gamma_n\,. \tag{5}$$

Using the 'metric' covariant derivative $\nabla^g_i \equiv \nabla_i$ we can rewrite s_{ij} in the form

$$s_{ij} = R_{ij}[g] - \nabla_m a^m_{ij} + \frac{1}{2}(\nabla_i\, a_j + \nabla_j\, a_i) + a^m_{ni}a^n_{mj} - a^m_{ij}a_m\,, \tag{6}$$

where $R_{ij}[g]$ is the standard Ricci tensor of a Riemannian space with the metric g_{ij}.

Now, suppose that $\mathbf{c}^{ij} \equiv \sqrt{-g}\, c_{ij}$ is an arbitrary tensor density. Then, its covariant derivative with respect to connection (1) is defined by

$$\nabla^\gamma_i\, \mathbf{c}^{kl} \;=\; \partial_i\, \mathbf{c}^{kl} + \gamma^k_{im}\, \mathbf{c}^{ml} + \gamma^l_{im}\, \mathbf{c}^{km} - \gamma^m_{im}\, \mathbf{c}^{kl}\,. \tag{7}$$

For any *antisymmetric* density, $\mathbf{c}^{ij} \equiv \mathbf{f}^{ij} = -\mathbf{f}^{ji}$, it follows that

$$\nabla^\gamma_i\, \mathbf{f}^{ik} \;=\; \nabla^g_i\, \mathbf{f}^{ik} \;=\; \partial_i \mathbf{f}^{ik}\,. \tag{8}$$

The *symmetric* tensor density $\mathbf{g}^{ij} \equiv \sqrt{-g}\, g^{ij}$ obviously satisfies the equations

$$\nabla^\gamma_i\, \mathbf{g}^{ik} \;=\; \sqrt{-g}\, a^k_{im}\, g^{im}\,, \qquad \nabla^g_i\, \mathbf{g}^{ik} \;=\; 0. \tag{9}$$

Eqs.(5) - (9) will be used in what follows.

For a general symmetric connection one can introduce the concept of the geodesic curve (path), the tangent vector to which is parallel to itself at every point of the curve. The equations for the geodesic curves of any symmetric connection γ^i_{jk} can be written as

$$\ddot{x}^{\,i} + \gamma^i_{jk}\, \dot{x}^j\, \dot{x}^k = 0\,, \tag{10}$$

where the dot denotes differentiating with respect to the so called 'affine' parameter τ of the curve $x^i(\tau)$. Using the affine parameter we can compare the distances between points on the same curve.

For a particular geodesics, the affine parameter is unique up to an affine transformation $\tau \mapsto \tau' = a\tau + b$. Each connection define the unique set of paths, but all symmetric connections (with an arbitrary vector $\hat{a}_k$)

$$\hat{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j\, \hat{a}_k + \delta^i_k\, \hat{a}_j\,, \tag{11}$$

define the same paths. The Weyl (conformal) tensor W^i_{jkl} of connection (11) is independent of $\hat{a}_k$ while the Ricci tensor and its symmetric and antisymmetric parts are $\hat{a}_i$-dependent (see [11] for more details).

Therefore, an interesting class of connections is

$$\hat{\gamma}^i_{jk} = \Gamma^i_{jk}[g] + \delta^i_j\, \hat{a}_k + \delta^i_k\, \hat{a}_j\,, \tag{12}$$

where $\Gamma^i_{jk}[g]$ is a Riemannian connection (the Christoffel symbol of a symmetric tensor g_{ij}). The paths of the connection $\hat{\gamma}^i_{jk}$ coincide with the geodesics of $\Gamma^i_{jk}[g]$, but the Ricci tensor of $\hat{\gamma}$ is symmetric *if and only if* $\hat{a}_i = \partial_i \hat{a}$ with some scalar $\hat{a}$. We see that connection (12) is maximally close to the Riemannian connection $\Gamma^i_{jk}[g]$ and may be called a *geodesically Riemannian* ('g-Riemannian') connection. Weyl and Einstein studied more general connections that belong to the following class introduced in [1], [2]:

$$\gamma^i_{jk} = \Gamma^i_{jk}[g] + \alpha(\delta^i_j\, \hat{a}_k + \delta^i_k\, \hat{a}_j) - (\alpha - 2\beta) g_{jk}\, \hat{a}^i\,, \tag{13}$$

where $\hat{a}^i = g^{im}\hat{a}_m$. The Weyl connection corresponds to $\beta = 0$ and the g-Riemannian connection, to $\alpha = 2\beta$. Einstein derived the connection for the space-time dimension $D = 4$, his result is $\alpha = -\beta = \frac{1}{6}$ (it was generalized to any dimension in [3]).

Using (6) it is easy to calculate the physically important expression for the symmetric part of the Ricci curvature. The terms linear in A are equal to

$$(\alpha + \beta)(\nabla_i \hat{a}_j + \nabla_j \hat{a}_i) + (\alpha - 2\beta)\, g_{ij} \nabla_m \hat{a}^m\,, \tag{14}$$

and the quadratic terms are

$$\hat{a}_i \hat{a}_j\, [(\alpha - 2\beta)^2 - 3\alpha^2] + 2\, g_{ij} \hat{a}^2 (\alpha - 2\beta)(\alpha + \beta)\,. \tag{15}$$

As we shall soon see the presence of the $\hat{a}_i \hat{a}_k$ term in the expression for s_{ij} signals that the vector field a_i has in general a nonzero mass and that the sign of the first term in (15) can be positive or negative (the second term in (15) and the linear terms in (6) in general do not vanish). In particular, for the Weyl, Einstein and g-Riemannian connections the quadratic terms are, respectively:

$$\text{W:}\ \ -\frac{1}{2}[\hat{a}_i \hat{a}_k - 2 g_{ik} \hat{a}^2]\,, \qquad \text{E:}\ \frac{1}{6}\hat{a}_i \hat{a}_k\,, \qquad \text{g-R:}\ \ -\frac{3}{4}\hat{a}_i \hat{a}_k\,. \tag{16}$$

Before we leave pure mathematics and turn to more physical problems, we should mention one of the characteristic properties of symmetric connections. For applications of geometry to gravity, it is very important that at every point of the affine-connected space-time manifold there must exist a geodesic coordinate system, such that the connection coefficients are zero at this point. Using the above formulas it is easy to prove that such a coordinate system exists if and only if the connection is symmetric. For symmetric connections, the Fermi theorem about the existence of geodesic coordinates along the curves also holds (for the precise definitions and proofs see [11]).

Let us turn to **dynamics**. Weyl's approach to constructing a physical theory based on the affine geometry is direct (if we discard his ideas on 'linear metric' and on lengths calibrating): he first chooses a particular geometry (13) with $\beta = 0$ and then constructs tensor equations that should generalize the Einstein equations. He tries to identify the antisymmetric part of the curvature with the electromagnetic field tensor but the $\hat{a}_i \hat{a}_k$ terms spoil this interpretation. He eventually guessed a Lagrangian which is similar to the Einstein theory coupled to a vector field with the mass term that cannot be removed and with the cosmological constant that he introduces 'by hand'. Eddington tried to find a generalization of Einstein's theory by considering the most general nonsymmetric affine connection. In a discussion of possible scalar densities he suggests the simplest one (we call it *Eddington's scalar density*),

$$\mathcal{L} \equiv \sqrt{-\det(r_{ij})}\ \equiv\ \sqrt{-r}\,, \tag{17}$$

This resembles the fundamental scalar density of the Riemannian geometry, $\sqrt{-\det(g_{ij})} \equiv \sqrt{-g}$, and Eddington tried to directly identify s_{ij} with the metric. If we in addition identify a_{ij} with the electromagnetic field we get a Born-Infeld Lagrangian. However, Eddington did not succeed in constructing consistent equations.

A consistent *Lagrangian formulation* of the generalized theory was found by Einstein. His approach is conceptually different both from Weyl's and Eddington's ones and consists of two stages. In the first stage, he assumed that the general symmetric connection should be restricted by the Hamilton principle for a general Lagrangian density depending either on r_{ij} (see the second paper[2].) or on s_{ij} and a_{ij} separately (in the third paper). He gave no motivation for this assumption, but it is easy to see that the resulting theory in the limit $a_{ij} = 0$ is consistent with the standard general relativity supplemented with a cosmological term. In this stage, Einstein succeeded in deriving the remarkable expression for the connection (see (13) with $\alpha = -\beta = \frac{1}{6}$) and the general expression for s_{ij} depending on a massive (tachyonic) vector field and the metric tensor density $\mathbf{g}^{ij}$.

In the next stage, a concrete Lagrangian density $\mathcal{L}(s_{ij}\,, a_{ij})$ should be chosen. Einstein did not formulate any principle for selecting a Lagrangian, and both from geometric and physical standpoint his concrete choice seems sufficiently arbitrary, especially in the third paper where he essentially reproduced one of the Weyl results. We believe that his best choice was made in the first two papers and, indeed, very similar effective Lagrangians are considered in modern applications of the superstring theory to cosmology. We may try to formulate some properties of possible geometric Lagrangian densities $\mathcal{L}$ that are consistent with the Eddington-Einstein Lagrangian but allow for a more general class of them (with different mass terms, different dependence on s_{ij}, a_{ij}, a_i, in different space-time dimensions).

Naturally, the Lagrangian must depend on *tensor variables having a direct geometric meaning*. It is desirable that in the next stage they will acquire a *natural physical interpretation*. As soon as we do not wish to fix the division of the connection into the metric and non-metric parts by Eq.(1), we can take the vector γ_i (not a_i!), second-rank tensors, s_{ij}, a_{ij}, $\gamma_{ij} \equiv \gamma_i \gamma_j$ (if we used representation (1) for γ^k_{ij} we could add to this list $\bar{\gamma}_{ij} \equiv \gamma_k a^k_{ij}$, but this tensor implicitly depends on the metric and we must not use it at the first stage). We can construct higher-rank tensors, but the tensors of the second rank (especially, the first three) look more fundamental from the physics point of view.

Consider the *second-rank tensors* as building blocks of the 'geometric' Lagrangian. They all have the dimension L^{-2} (in the units $c = 1$) and we can use as Lagrangian densities some homogeneous functions of the degree $D/2$ and dimension L^{-D} that are independent on any dimensional constants. After integrating over D-dimensional volume element $dx^0 \bigwedge ... \bigwedge dx^{D-1}$ we then get a dimensionless quantity playing a role of a *geometric action*. The simplest Lagrangian density then depends on three second-rank tensors,

$$\mathcal{L} = \mathcal{L}(s_{ij}, a_{ij}, \gamma_{ij})\,, \tag{18}$$

and a density having the correct dimension L^{-D} can easily be written:

$$\mathcal{L}_g = \sqrt{-\det(s_{ij} + \nu a_{ij} + \nu_1 \gamma_{ij})}\,. \tag{19}$$

Here we take the minus sign because $\det(s_{ij}) < 0$ (due to the local Lorentz invariance) and we naturally assume that the same is true for $\det(s_{ij} + \nu a_{ij} + \nu_1 \gamma_{ij})$ (to reproduce Einstein's

[2] In the first paper Einstein uses as the Lagrangian the Eddington density but later he realized that in the first stage it is sufficient to suppose that the Lagrangian is an arbitrary scalar density depending on r_{ij}

general relativity in the limit $\nu, \nu_1 \to 0$). The ν-parameters are dimensionless, we mainly introduce them to disentangle the scale of the mass parameter of the vector field from the cosmological constant. If we take the original Eddington - Einstein Lagrangian (17), the mass squared will be of the order of the cosmological constant Λ (see [2]). Lagrangian (19) with $\nu_1 = 0$ was proposed and studied in some detail in [2]. The general Lagrangian (19) was first considered in Ref.[3], where we also discussed a more general construction that allows to write other Lagrangians having the desired properties. Unfortunately, these generalized Lagrangians are more complicated both technically and conceptually, and we do not discuss them here.

We emphasize that the Lagrangians (17) and (19) are written in the form independent of D, although the analytic expressions for the dependence of the determinants on s_{ij} and a_{ij} essentially depend on D. Accordingly, the physical equations depend on the space-time dimension as we will shortly demonstrate.

The starting point for Einstein (in his first paper of the series [7]) was the action principle with Lagrangian density (17) depending on 40 connection functions γ^i_{kl}. Varying the action with respect to these functions, he derived 40 equations that allowed him to find the expression for γ^i_{kl} given by (13) with $\alpha = -\beta = \frac{1}{6}$ (in the four-dimensional space-time).

The main steps of his proof were reproduced in [2]. Here we somewhat generalize the derivation to an arbitrary dimension D and assume that the geometric Lagrangian depends also on $\gamma_i \equiv \gamma^m_{im}$. We define the new tensor densities[3].

$$\frac{\partial \mathcal{L}}{\partial s_{ij}} \equiv \mathbf{g}^{ij}\,, \qquad \frac{\partial \mathcal{L}}{\partial a_{ij}} \equiv \mathbf{f}^{ij}\,, \qquad \frac{\partial \mathcal{L}}{\partial \gamma_i} \equiv \mathbf{b}^{i}\,, \tag{20}$$

and introduce a conjugate Lagrangian density $\mathcal{L}^* = \mathcal{L}^*(\mathbf{g}^{ij}, \mathbf{f}^{ij}, \mathbf{b}^i)$ by a Legendre transformation,

$$s_{ij} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{g}^{ij}}\,, \qquad a_{ij} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{f}^{ij}}\,, \qquad \gamma_i = \frac{\partial \mathcal{L}^*}{\partial \mathbf{b}^{i}}\,. \tag{21}$$

By varying $\mathcal{L}$ in γ^i_{kl} and using the above definitions, we can then show that the conditions $\delta\mathcal{L}/\delta\gamma^i_{kl} = 0$ are equivalent to the following 40 equations

$$2\nabla^{\gamma}_i\,\mathbf{g}^{kl} \;=\; \delta^l_i\,[\nabla^{\gamma}_m\,(\mathbf{g}^{km} + \mathbf{f}^{km}) - \mathbf{b}^k] + \delta^k_i\,[\nabla^{\gamma}_m\,(\mathbf{g}^{lm} + \mathbf{f}^{lm}) - \mathbf{b}^l]\,, \tag{22}$$

where ∇^{γ}_i is the covariant derivative with respect to the affine connection γ^i_{jk}. Remembering (8) we define the vector density $\hat{\mathbf{a}}^k$ by

$$\partial_i\,\mathbf{f}^{ki} \,-\, \mathbf{b}^k \;\equiv\; \hat{\mathbf{a}}^k, \tag{23}$$

and then easily find that

$$\nabla^{\gamma}_i\,\mathbf{g}^{ik} \;=\; -\frac{D+1}{D-1}\hat{\mathbf{a}}^k\,, \tag{24}$$

Now it is easy to find the equations from which the connection coefficients can be derived (as in the Riemannian case):

$$\nabla^{\gamma}_i\,\mathbf{g}^{kl} \;=\; -\frac{1}{D-1}(\delta^k_i\hat{\mathbf{a}}^l + \delta^l_i\hat{\mathbf{a}}^k). \tag{25}$$

[3] Following Eddington's notation, we let boldface Latin letters denote tensor densities. The derivatives in (20) and (21) must be properly symmetrized, which is easy in concrete calculations. We tacitly assume that geometry has only a single dimensional constant, e.g., the cosmological constant Λ with the dimension L^{-2}. To restore the correct dimension in (20) and (21), we must then multiply the densities by $\Lambda^{(D-2)/2}$.

Defining the Riemann metric tensor g_{ij} by the equations

$$g^{kl}\sqrt{-g} \,=\, \mathbf{g}^{kl}\,, \quad g_{kl}\, g^{lm} \,=\, \delta_k^m\,, \tag{26}$$

we can then define the corresponding Riemannian covariant derivative ∇_i, for which

$$\nabla_i\, g_{kl} = 0, \quad \nabla_i\, g^{kl} = 0. \tag{27}$$

Taking the above into account, we can now use (25) to derive the expression for γ^i_{jk} in terms of the metric tensor g_{ij} and of the vector $\hat{a}^k \equiv \hat{\mathbf{a}}^k/\sqrt{-g}$,

$$\gamma^i_{jk} \,=\, \Gamma^i_{jk}[g] \,+\, \alpha_D\,[\,\delta^i_j\,\hat{a}_k + \delta^i_k\,\hat{a}_j - (D-1)\,g_{jk}\,\hat{a}^i]\,, \tag{28}$$

which corresponds to $\alpha = \alpha_D$ and $\beta = \beta_D$ in (13), with

$$\alpha_D \equiv [(D-1)(D-2)]^{-1}\,, \qquad \beta_D \equiv -[2(D-1)]^{-1}\,. \tag{29}$$

For $D = 4$, this coincides with Einstein's result for the connection. If we add γ_{ij} as an independent variable, the connection remains the same. Note also that the added variables remain non-dynamical and attempting to make them dynamical in the second stage 'by hand' destroys the beauty of the original Einstein construction.

We cannot go deeper into discussions of further relations between geometry of affine connections and dynamical principles. But the above results show that these relations are rather complex and we do not yet understand their nature. Indeed, we tried to add new natural variables into the geometric Lagrangian, but the class of connections obtained as an output of Einstein's approach did not change at all. It can be argued that there are many other, not yet explored options, but in reality, we do not even know how to obtain Weyl's or g-Riemannian connections following Einstein's approach.

One of the possibilities could be to abandon some of Einstein's assumptions. The most serious drawback (or virtue, depending on a viewpoint) of his approach is that two pairs of the basic variables of the theory, (s_{ij}, $\mathbf{g}^{ij}$) and (a_{ij}, $\mathbf{f}^{ij}$), having very different geometrical and physical meaning, are treated symmetrically. Definition (20) looks quite natural for the metric density because Einstein's Lagrangian for the pure gravity theory is simply $\mathbf{g}^{ij}R_{ij}$. But Einstein's definition of $\mathbf{f}^{ij}$ tacitly (and, as we see, wisely!) assumes that the geometric Lagrangian is independent of γ_i or γ_{ij}. This may look rather paradoxical, but, as we have seen, the mass term is dictated by the geometry because its germ, the term $\sim a_i a_j$, is already present in the expression for s_{ij}.[4] Its interpretation as the physical mass comes when we write an effective physical Lagrangian. Then the geometric Lagrangian generates only the kinetic terms and is in fact the Lagrangian of a brane.

There are many other questions, which should be carefully discussed, but we postpone the discussion to future publications. Here, we present a simple example demonstrating how to eventually pass **from geometry to physics** and to demonstrate a relation of the Einstein approach to the present-day concerns. Our discussion suggests that the geometric Lagrangian (19) with $\nu_1 = 0$ is better motivated by geometry and physics than other ones. This Lagrangian is most natural and gives the effective physical Lagrangian belonging to a class widely discussed in relation to modern problems of cosmology. We only briefly describe this model which is, possibly, the simplest generalization of Einstein's general relativity.

[4] Therefore, it would be more natural to identify the field tensor of the massive vector field directly with a_{ij}, up to a necessary dimensional multiplier.

Pure geometry gives us equations (4) and (6). With a^i_{jk} given by (13), their right-hand sides are given by $(a_{i,j} - a_{j,i})/2$, where $a_i = (D\alpha + 2\beta)\,\hat{a}_i$, and by the sum of R_{ij} with expressions (14), (15). To derive s_{ij} and a_{ij} in terms of the 'physical' variables g_{ij} and f_{ij} we must choose a Lagrangian (e.g., (19)) and then solve equations (20) with respect to the geometric variables s_{ij} and a_{ij}. Alternatively, if we know the conjugate Lagrangian $\mathcal{L}^*(\mathbf{g}^{ij}, \mathbf{f}^{ij})$, we can directly calculate them using (21).

In [2], we reproduced Einstein's result of Refs.[7], [5] (in which it was not written explicitly but could easily be derived):

$$\mathcal{L} \equiv \sqrt{-\det(r_{ij})} \;=\; 4\sqrt{-\det(\mathbf{g}^{ij} + \mathbf{f}^{ij})} \;\equiv\; 4\sqrt{-\det(g_{ij} + f_{ij})} \;=\; \mathcal{L}^*\,. \tag{30}$$

We emphasize that these equations are valid only in the four-dimensional theory. Note that the equality $\mathcal{L}^* = \mathcal{L}$ simply follows from the fact that $\mathcal{L}$ is a homogeneous function of the degree two but, in general, the concrete expression for $\mathcal{L}^*$ must be obtained by a direct calculation. Now we can show that the relation like (30) holds also for Lagrangian (19) with $\nu_1 = 0$ and $\nu \neq 0$, which we rewrite as

$$\mathcal{L}_\nu \equiv \sqrt{-\det(s_{ij} + \nu a_{ij})}\,. \tag{31}$$

This can be done by a direct computation but it is simpler to first dimensionally reduce $\mathcal{L}_\nu$.

Consider the $D = 4$ case and define a *'spherical reduction'* not using any metric. Suppose that s_{ij} and a_{ij} are functions of (x^0, x^1) and that $a_2 = a_3 = 0$ (therefore, only $a_{01} = -a_{10} \neq 0$). We then assume that the symmetric matrix has the following nonzero elements: $s_{ij} = \delta_{ij}\, s_i\,,\; s_{01} = s_{10}$ (our result will not change if also $s_{23} \neq 0$). By explicitly deriving $s_{ij} + \nu a_{ij}$, we can find $\mathbf{g}^{ij}$ and $\mathbf{f}^{ij}$ (using (20)) and hence express $\det(\mathbf{g}^{ij} + \lambda\mathbf{f}^{ij})$ in terms of s_{ij} and a_{ij} :

$$16\det(\mathbf{g}^{ij} + \lambda\mathbf{f}^{ij}) = \det[s_{ij} + (\nu^2\lambda)\, a_{ij}]. \tag{32}$$

It follows that choosing $\lambda = \nu^{-1}$ we have[5]

$$\mathcal{L}_\nu = \sqrt{-\det(s_{ij} + \lambda^{-1}\, a_{ij})} = 4\sqrt{-\det(\mathbf{g}^{ij} + \lambda\mathbf{f}^{ij})} = 4\sqrt{-\det(g_{ij} + \lambda f_{ij})} = \mathcal{L}^*_\lambda\,, \tag{33}$$

where the sign and normalization are arbitrary chosen in relation to the cosmological interpretation. This result is written in the form not implying the spherical reduction, and we suppose it is true in a general four-dimensional theory. In arbitrary dimension ($D \neq 2$) it must be somewhat modified as was first shown in [2].

To similarly treat the **higher dimensional case** we first reduce the D-dimensional Lagrangian to the dimension four. For simplicity, let us consider $D = 5$. Then the field $a_k\,(k = 0, .., 4)$ depends only on $x_i\,(i = 0, .., 3)$, $a_{ij} = \frac{1}{2}(\partial_j a_i - \partial_i a_j)$, and $a_{4i} = 1/2\,(\partial_i\, a_4)$. Therefore the terms containing a^2_{4i} should be interpreted in four dimensions as kinetic terms of the *scalar field* a_4.[6] Applying spherical reduction to the resulting four-dimensional Lagrangian, we can construct a two-dimensional model effectively describing spherically

[5] We ignore the dimensional constants while working mainly with geometrical theory, where presumably exists just one dimensional constant Λ (with $c = 1$). Then emergence of some dimensionless parameters may signal that there exist other dimensional constants (e.g., different scales in symmetric and antisymmetric sectors of geometry may be described by introducing our parameter ν). We eventually restore dimensions in the effective physical Lagrangian.

[6] It can be seen that this scalar field is massive or tachyonic. In the simplest reduction, its mass coincides with that of the vecton.

symmetric solutions of the four-dimensional gravity coupled to the vecton and to the scalar fields. To get the corresponding Lagrangian we derive the determinant of the matrix

$$s_{ij} + \nu a_{ij} \equiv s_i \delta_{ij} + (\delta_{0i}\delta_{1j} + \delta_{0j}\delta_{1i})(s_{ij} + \nu a_{ij}) + (\delta_{i4} + \delta_{j4}) a_{ij} \,, \tag{34}$$

where a_{ij} are defined in terms of a_i, and all the functions in (34) depend on x^0, x^1 (thus $a_{24} = a_{34} = 0$). The determinant is

$$\det(s_{ij} + \nu a_{ij}) = \prod_{i=0}^{4} s_i \, [1 + \tilde{s}_{01}^2 - \nu^2 (\tilde{a}_{01}^2 + \tilde{a}_{04}^2 - \tilde{a}_{14}^2)], \tag{35}$$

where we define $\tilde{m}_{ij} \equiv m_{ij} \, |s_i s_j|^{-1/2}$. The determinant obviously has zeroes and thus its square root is always singular. Therefore, the corresponding two-dimensional dilaton gravity describing spherically symmetric solutions is rather unusual and complex. By further reductions to static or cosmological configurations we can construct corresponding one-dimensional dynamical systems describing static states with horizons as well as cosmological models. The cosmological models look realistic enough because they incorporate a natural sources of the dark energy, inflation, and, possibly, some candidates for the dark matter (for a more detailed discussion see [1], [2]).

Before presenting a simplest cosmological model, we write the general D-dimensional theory. In addition to predicting scalar fields, the higher-dimensional Lagrangians differ from the ones usually considered in modern brane cosmology. In fact, while the square-root Lagrangian $\mathcal{L}$ produces the square-root Lagrangian $\mathcal{L}^*$, which gives the so-called DBI-like term in the effective physical Lagrangian (see many examples in [12] - [22]), our higher-dimensional Lagrangian essentially depends on D:

$$\sqrt{-\det(s_{ij} + \nu a_{ij})} = [-2^D \det(\mathbf{g}^{ij} + \lambda \mathbf{f}^{ij})]^{1/(D-2)} = \sqrt{-g}\, [-2^D \det(\delta_i^j + \lambda f_i^j)]^{1/(D-2)} \,, \tag{36}$$

which coincides with (33) for $D = 4$. Following [2], we may write the corresponding **physical Lagrangian**

$$\mathcal{L}_{eff} = \sqrt{-g} \left[-2\Lambda \, [\det(\delta_i^j + \lambda f_i^j)]^{1/(D-2)} + R(g) + c_a \, g^{ij} a_i a_j \right] , \tag{37}$$

which should be varied with respect to the metric and the vector field; c_a is a parameter depending on D (Einstein's first model is obtained for $D = 4$ and $c_a = 1/6$). When the vecton field is zero, we have the standard Einstein gravity with the cosmological constant. Making the dimensional reduction from $D = 5$ to $D = 4$, we obtain the Lagrangian describing the vecton a_i, $f_{ij} \sim \partial_i a_j - \partial_j a_i$ and $(D-4)$ scalar fields a_k, $k = 4, .., D$.

The theory (37) is very complex, even at the classical level. Its spherically symmetric sector is described by a (1+1)-dimensional dilaton gravity coupled to one massive vector and to several scalar fields. If the mass of the vector field is zero and the scalar fields vanish[7], the *dilaton gravity is classically integrable with a rather general dependence of the Lagrangian on the massless Abelian gauge fields*, $X(\phi, F^2)$, where $F^2 = F_{ij} F^{ij}$ and ϕ is the dilaton field, see [23]-[25]. If $\mu^2 \neq 0$, the theory is certainly not integrable even with vanishing scalar fields. It is also not easy to analytically construct its physically interesting approximate solutions[8].

[7] Such models can be derived by dimensional reductions of some higher-dimensional gravity and supergravity theories, see, e.g, [23] - [26] and references therein.

[8] At first sight, a perturbation theory in μ^2 seems to be a viable alternative to numeric approximations but, when $\mu^2 = 0$, an additional gauge symmetry emerges that makes it difficult to estimate the validity of the approximation, especially, in the physically important asymptotic regions.

Further dimensional reductions to one-dimensional static or cosmological theories also give non-integrable dynamical systems although some approximate solutions can possibly be derived. The naive cosmological reduction of four-dimensional theory (37) can be written using the metric

$$ds_4^2 = e^{2\alpha} dr^2 + e^{2\beta} d\Omega^2(\theta, \phi) - e^{2\gamma} dt^2 \,, \tag{38}$$

where α, β, γ depend on t and $d\Omega^2$ is the metric on the two-dimensional sphere.[9] Now the effective cosmological (one-dimensional) Lagrangian corresponding to theory (37) in the $D = 4$ case is

$$\mathcal{L}_c = -2e^{2\beta} \left[e^{\alpha-\gamma}(\dot{\beta}^2 + 2\dot{\beta}\dot{\alpha}) + \Lambda\sqrt{e^{2(\alpha+\gamma)} - \lambda^2 \dot{A}^2} + \frac{1}{2}\,\mu^2 A^2 e^{-\alpha+\gamma} \right]. \tag{39}$$

As γ is obviously a Lagrange multiplier we can fix the remaining gauge freedom by choosing $\gamma = -\alpha$.[10] Using this gauge and denoting the anisotropy function by $3\sigma \equiv \beta - \alpha$, we have the gauge fixed Lagrangian

$$\mathcal{L}_c = -2e^{2\beta} \left[3e^{2\alpha}(\dot{\rho}^2 - \dot{\sigma}^2) + \Lambda\sqrt{1 - \lambda^2 \dot{A}^2} + \frac{1}{2}\,\mu^2 A^2 e^{-2\alpha} \right], \tag{40}$$

where $\alpha = \rho - 2\sigma$ and $\gamma = \rho + \sigma$.

Up to the dilaton multiplier $e^{2\beta}$, the second term in (40) is the DBI (or, 0-brane) Lagrangian. If we consider constant metric functions α, β, and denote $M_A \equiv 2\lambda^2 \Lambda e^{2\beta}$, we see that the 0-brane term is the relativistic Lagrangian of a particle with the mass M_A (the analog of the velocity of light c is $\lambda^{-1} \equiv \bar{c}$). Introducing the canonical momenta p_ρ, p_σ, p_A we find the Hamiltonian (one should not forget that M_A depends on $\beta(t)$):

$$\mathcal{H} = \bar{c}\sqrt{p_A^2 + M_A^2\,\bar{c}^2} + \mu^2 A^2 e^{2(\beta-\alpha)} + \frac{1}{24}\,e^{2(\beta+\alpha)}\,(p_\sigma^2 - p_\rho^2) \;=\; 0. \tag{41}$$

If $\mu^2 = 0$, the momentum P_A is the integral of motion and we get an integrable 1-dimensional dilaton gravity. (with $\mu^2 \neq 0$, it is not integrable and rather unusual theory). If α and β vary much slower than $A(t)$ this is a more tractable model of a relativistic 'particle' with the slowly varying time dependent mass M_A in a simple potential having time dependent parameters. A simpler effective 'particle' model was used by Gribov for discovering the famous *Gribov copies*. One may hope that a similar interpretation of the theory (41) will help to understand some unusual qualitative features of our generalized gravity.

For small A and slowly varying gravitational fields, one can also use the small-field approximation (see [1], [2]), which is formally equivalent to expanding (37) in powers of λ^2. Keeping only the first-order correction we then obtain a nice-looking field theory:

$$\mathcal{L}_{eff} \;\cong\; \sqrt{-g}\left[R[g] - 2\Lambda - \kappa\Big(\frac{1}{2}F_{ij}F^{ij} + \mu^2 A_i A^i + g^{ij}\partial_i\psi\,\partial_j\psi + m^2\psi^2\Big)\right], \tag{42}$$

where $A_i \sim a_i$, $F_{ij} \sim f_{ij}$, $\kappa \equiv G/c^4$ and we use the CGS dimensions. Note that here we choose the standard normalization of the fields and thus the dimensionless parameters of

[9] The function $\beta(t)$ is the two-dimensional dilaton field and, usually, it is supposed that $\alpha = \beta$ (isotropy condition). With the massive vector field $A_i(t)$, this is not possible because the equations of motion require $A_0 \equiv A_t = 0$ and $A_1 \equiv A_r \neq 0$, which obviously gives an anisotropic configuration, see [1], [2].

[10] The standard gauge fixings are $\gamma = 0$ or $\gamma = \alpha$; in [2] we also used the gauge fixing $\gamma = 3\rho \equiv \alpha + 2\beta$. Varying the Lagrangian multiplier γ gives the *energy constraint*, i.e. vanishing of the Hamiltonian.

the theory (D, λ) are hidden in the masses μ and m. Note also that for Einstein's geometry the masses are imaginary, but we should study the general case when they may also be real.

This simplified theory still keeps traces of its geometric origin: the simplest form of the dark energy (the cosmological constant Λ), massive (or tachyonic) vector and scalar fields, which can describe inflation and/or imitate dark matter. The most popular inflationary models require a few massive scalar particles usually called inflatons (see, e.g., [27] - [31]). Without massive scalar fields, there is no simple inflation mechanism with one massive vecton. However, with the tachyonic vecton (see [32]) or with several massive vector particles, it is probably easier to find more realistic inflation models (see [33] - [37]; some of these papers also discuss a possible role of massive vector particles in dark energy and dark matter mechanisms).

In conclusion, we note that the geometrical and dynamical models discussed in this paper are not well understood, both conceptually and technically. Much work on them should be done before a realistic cosmological model could be constructed. In particular, one should study the relation between the geometry and dynamics discovered by Einstein. Possibly, one shall find behind it some symmetry principles which are not yet understood. One should also study more general theories. For example, why we not add to the geometric Lagrangian the terms quadratic in the curvature tensor that can be constructed not using any metric? Of course, the Eddington - Einstein Lagrangian and its simplest generalizations discussed here are most beautiful and are closely related to the modern theory of branes, but this is not a good enough argument for restricting alternative geometric proposals. The new part of the connection a^i_{jk} is a tensor that can generate some higher spin fields and we must have some serious arguments for excluding this possibility from the very beginning.

Finally, we must clearly state once more that the generalization of gravity considered here has nothing to do with other matter fields. It is not suggesting any unification of gravity with other forces of nature and with the standard matter. The true meaning of it and its unexpected relation to recent discoveries and ideas in cosmology is a real puzzle. Possibly, a role of this theory is to replace the standard gravity inside the string theory which did not yet completely succeed in giving a simple and natural explanation of dark energy, inflation, and dark matter.

It is a great sorrow to dedicate this article to the dear memory of Volodya Gribov and not to hear his sharp critical and highly stimulating remarks on its content. I realize that the ideas treated here might look to Volodya a bit far from physics he liked, but his incredible ability to penetrate deep to the heart of any problem would certainly help to solve a puzzle left to us by three great scientists of the last century.

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