Capacities of Grassmann channels

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A new class of quantum channels called Grassmann channels is introduced and their classical and quantum capacity is calculated. The channel class appears in a study of the two-mode squeezing operator constructed from operators satisfying the fermionic algebra. We compare Grassmann channels with the channels induced by the bosonic two-mode squeezing operator. Among other results, we challenge the relevance of calculating entanglement measures to assess or compare the ability of bosonic and fermionic states to send quantum information to uniformly accelerated frames.

I. INTRODUCTION

The notion of quantum channel capacity is central to quantum Shannon theory. Early development in the seventies [1] was a starting point to an impressive amount of knowledge that has been acquired in the last two decades [2, 3]. Two of the most investigated areas is the classical [4] and quantum capacity [5] of a quantum channel. The classical/quantum capacity informs us about the ability of a quantum channel to transmit classical or quantum correlations. More precisely, consider a sender who has, in principle, at his disposal an optimal encoder producing a classical or quantum code, and a receiver able to process the channel output and recover the transmitted information (that is, to decode) with an arbitrarily high precision. In this way, the information can be transmitted at the rate given by the capacity and cannot be improved by any other choice of encoding. Various additional conditions or restrictions might be added, for instance if privacy is required [6] or some sort of assistance in terms of other quantum or classical resources available to the communicating parties [7, 8]. This leads to a large number of important capacity definitions relevant under given circumstances and one might even try to characterize the whole capacity multi-dimensional regions in which the axes correspond to various available resources [9–11].

Due to the presence of regularization (see below) the classical or quantum capacity is not efficiently computable. There are, however, particular examples of channels for which the classical or quantum capacity is easy to calculate. In the case of the classical capacity, every such example must be cherished since the proof usually involves some nontrivial manipulations [12–17]. For the quantum capacity almost all known non-trivial examples fall in the class of degradable channels [18, 23, 25]. Among these examples are exceptional cases for which both capacities are known, to our knowledge there are only two examples: (i) a qubit erasure channel [18] and (ii) Hadamard channels [14]. There are also trivial examples of such channels with zero quantum capacity: entanglement-breaking channels [15] and anti-degradable channels. In this paper we add another member into the elite group of non-trivial examples: the Grassmann channels. In order to find the quantum capacity we show that the Grassmann channels are degradable while to find the classical capacity we make use of the fact that the Grassmann channels are of a direct

sum form. We show that the channels which form the structure of the Grassmann channels are new members of the surprisingly broad family of channels studied in [17] for which the classical capacity is efficiently calculable. The lowest-dimensional example of the Grassmann channels turns out to be a qubit erasure channel. In general, higher-dimensional Grassmann channels have certain traits in common with a qubit erasure channel making them interesting in the light of some recent capacity results [3, 19]. We further investigate physical implications of the capacities calculations and conclude, in contrary to the common opinion, that (i) it makes absolutely no qualitative difference for the study of the Unruh effect whether the states are composed of bosons or fermions (at least for the quantum information transmission purposes) and (ii) naïve calculations of certain entanglement measures of states shared between inertial and uniformly accelerated observers do not provide much insight about the channel's ability to reliably send quantum information. Another physical conclusion is an intriguing connection of the Grassmann channels and their bosonic relatives to the well studied family of transpose-depolarizing channels.

The paper is organized as follows. In Sec. II we recall some basic notions from quantum Shannon theory together with the fermionic algebra and briefly discuss various relevant physical situations. It is followed by Sec. III in which we describe the construction of the qudit Grassmann channels and study some of their properties. Sections IV and V contain the calculation of the quantum and classical capacity of the whole (infinite-dimensional) class of Grassmann channels. In Sec. VI we discuss several physical properties of the Grassmann channels.

II. DEFINITIONS

Fermions and their correlated pairs

For a set of modes that are specified by quantum numbers compactly denoted by κ, the exchange characteristics of indistinguishable fermions that may occupy those modes are reflected in the anticommutation relations obeyed by the corresponding creation (annihilation) operators a_κ $(a_\kappa^\dagger)$:

$$\{a_\kappa, a_{\kappa'}^\dagger\} = \delta(\kappa - \kappa'), \tag{1}$$

$$\{a_\kappa^\dagger, a_{\kappa'}^\dagger\} = \{a_\kappa, a_{\kappa'}\} = 0. \tag{2}$$

An interesting transformation that preserves the anticommutation relations results from the following algorithm. Within the available modes two subsets are chosen. Each of the modes from the subset of lower cardinality is related through an injective function $f(\kappa) = \kappa'$ to the elements of the other subset. In this way, the modes are paired and so are the creation-annihilation operators $a_\kappa, a_\kappa^\dagger$ and $a_{f(\kappa)}, a_{f(\kappa)}^\dagger$. In order to make simpler the notation, the operators $a_{f(\kappa)}, a_{f(\kappa)}^\dagger$ from now on will be denoted by a different letter, for instance, $a_{f(\kappa)}, a_{f(\kappa)}^\dagger \to c_\kappa, c_\kappa^\dagger$. It can be directly shown that the so called Bogoliubov transformations

$$b_\kappa = \cos r a_\kappa - e^{-i\phi} \sin r c_\kappa^\dagger \tag{3a}$$

$$b_\kappa^\dagger = \cos r a_\kappa^\dagger - e^{i\phi} \sin r c_\kappa \tag{3b}$$

$$d_\kappa = \cos r c_\kappa + e^{-i\phi} \sin r a_\kappa^\dagger \tag{3c}$$

$$d_\kappa^\dagger = \cos r c_\kappa^\dagger + e^{i\phi} \sin r a_\kappa. \tag{3d}$$

preserve the anticommutation relations Eq. (1). These transformations were first introduced for the generation of the Bardeen-Cooper-Schreiffer (BCS) states, which are an excellent approximation to the ground state of a weakly interacting superconductor [27]. In that case, the

modes are thought to describe electrons with the quantum numbers usually taken as the vector wave number $\vec{k}$ and the projection of the spin s. The function f is such that $f(\vec{k}, s) = (-\vec{k}, -s)$. The Cooper pairs that are described using the operators b_κ, d_κ, as a consequence, have opposite momenta and spin.

Bogoliubov transformations have been very useful in the description of strongly correlated fermions in diverse scenarios like condensed matter [28], ultracold degenerate atomic Fermi gases [29] or quantum field theory of particles on spacetime with a nontrivial metric [30, 31, 33, 34]. To illustrate the impact of these transformations in the latter area, consider negligible interacting massive fermions. For a uniformly accelerated observer, the adequate spacetime coordinates are the Rindler ones; for which two wedges, that can not be causally connected, are identified, we shall denote them by right R and left L wedges. The right (left) Rindler modes are only supported on the right (left) wedge. An important question concerns to the energy spectra seen by a Rindler observer in connection to the Unruh effect [32]. The most clear treatments of the problem rely on comparisons between the so-called Unruh modes and the Rindler modes. They are naturally carried out in terms of generalized Bogoliubov transformations [30] that take into account that each Unruh mode needs for its representation an infinite superposition of Rindler modes. In this kind of scenario, perhaps the simplest realization of a Bogoliubov transformation corresponds to the case of a spin one-half fermion moving in a one dimensional space. Then, the modes of each fermion can be described in terms of a Grassmann field and a basic transformation is

$$
\begin{pmatrix} a_{k,s}^{(R)} \\ \bar{a}_{-k,-s}^{(L)^\dagger} \end{pmatrix} = \begin{pmatrix} \cos r & -e^{-i\phi}\sin r \\ e^{i\phi}\sin r & \cos r \end{pmatrix} \begin{pmatrix} b_{k,s} \\ \bar{b}_{-k,-s}^\dagger \end{pmatrix}.
\tag{4}
$$

where $a, a^\dagger$ stand for particles and $\bar{a}, \bar{a}^\dagger$ for antiparticles, k denotes the wave number and s its spin projection. The real parameter r can be chosen to depend on the acceleration α and the rest mass m of the Rindler observer as $\tan r = e^{-\frac{\pi c^2}{\alpha}\sqrt{k^2+(mc/\hbar)^2}}$ [33]. Note that Eq. (4) is Eqs. (3a) and (3d) written in the matrix form and with the notation adjusted to the relevant physical situation.

Classical and quantum capacity

Let $\mathcal{K} : A' \mapsto A$ be a quantum channel mapping density operator from an input Hilbert space $\mathcal{H}_{A'}$ to an output Hilbert space $\mathcal{H}_A$. We define $V_\mathcal{K} : A' \mapsto AC$ to be its isometric extension. Equivalently, by embedding the isometry into a higher-dimensional Hilbert space we write $U_\mathcal{K} : A'C' \mapsto AC$, where $U_\mathcal{K}$ is a unitary operator. Using the isometry picture the complementary channel of $\mathcal{K}$ is defined as $\mathcal{K}^c(\sigma) = \mathrm{Tr}_A[V_\mathcal{K}\sigma V_\mathcal{K}^\dagger] \equiv \mathrm{Tr}_A \circ V_\mathcal{K} \circ \sigma$. The second equality is how we will occasionally abbreviate similar expressions. When there is no chance of confusion we will omit the mode index A or C for states. The von Neumann entropy $H(\varrho_A) = -\mathrm{Tr}[\varrho_A \log \varrho_A]$ will be written in the economic way as $H(A)_\varrho$ which is mainly suitable for dealing with the parts of multipartite states $\varrho_A = \mathrm{Tr}_{BC\ldots}[\varrho_{ABC\ldots}]$. This convention will also be used for other entropic quantities.

A classical ensemble can be written in the form of a classical-quantum state $\sigma_{XA'} = \sum_x p(x)|x\rangle\langle x|_X \otimes \sigma_{x,A'}$ with a probability distribution function $p(x)$. Then, for a quantum channel $\mathcal{K}$ the capacity is given by the regularized expression [4]

$$
C(\mathcal{K}) = \lim_{n\to\infty} \frac{1}{n} C_{\mathrm{Hol}}(\mathcal{K}^{\otimes n}).
\tag{5}
$$

$C_{\mathrm{Hol}}(\mathcal{K})$ is the Holevo quantity

$$
C_{\mathrm{Hol}}(\mathcal{K}) = \sup_{\{p(x),\sigma_{x,A'}\}} I(X:A)_\varsigma,
\tag{6}
$$

where $\varsigma_{XA} = \mathcal{K}(\sigma_{XA'})$ and $I(X : A) = H(X) + H(A) - H(XA)$ is the mutual information. For the optimization task in Eq. (6) it is sufficient to consider $\sigma_{x,A'}$ to be pure states.

The expression for the quantum capacity contains regularization as well [5]

$$Q(\mathcal{K}) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(\mathcal{K}^{\otimes n}). \tag{7}$$

$Q^{(1)}(\mathcal{K})$ is the optimized coherent information

$$Q^{(1)}(\mathcal{K}) = \sup_{\psi} [H(A)_\tau - H(C)_\tau], \tag{8}$$

where $\tau_{AC} = V_{\mathcal{K}} \circ \psi_{A'}$.

There exists a generalization of the von Neumann entropy known as the α-Rényi entropy $H^\alpha(\rho) \stackrel{\text{df}}{=} 1/(1 - \alpha) \log [\text{Tr} \, \rho^\alpha]$, where the von Neumann entropy is recovered for $\alpha \to 1$. A quantum channel $\mathcal{D}$ is called degradable [23, 24] if there exists another channel $\mathcal{M}$ such that $\mathcal{M} \circ \mathcal{D} = \mathcal{D}^c$ holds. The channel $\mathcal{M}$ is called a degrading map. Finally, ln is the natural logarithm and log denotes the base two logarithm unless stated otherwise.

III. GRASSMANN CHANNELS

Definition 1. *We define a d-mode fermionic state as*

$$|F\rangle = \prod_{i=1}^{d} (a_i^\dagger)^{n_i} |vac\rangle = |n_1 \dots n_d\rangle \equiv |\vec{n}\rangle,$$

where $n_i \in \{0, 1\}$. Following the properties of the fermionic operators we see that (i) each mode is occupied by at most one particle and (ii) the state $|F\rangle$ is completely antisymmetric. For $0 \le k \le d$ there is $\binom{d}{k}$ possible fermionic states for which $\sum_{i=1}^{d} n_i = k$ holds.

The operator related to a Bogoliubov transformation Eq. (4) over the i-th fermionic modes a and c reads

$$U_{A_i C_i} = \exp \left[r(a_i^\dagger c_i^\dagger e^{-i\phi} - c_i a_i e^{i\phi}) \right], \tag{9}$$

where $r, \phi \in \mathbb{R}$. The operator exponent might be factorized according to the following identity:

Theorem 2 ([21]). *Let J_+, J_- and J_3 be operators satisfying the commutation relations*

$$\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2J_3. \end{aligned} \tag{10}$$

Then the following identity holds

$$e^{\lambda_+ J_+ + \lambda_3 J_3 + \lambda_- J_-} = e^{\Lambda_+ J_+} e^{\ln \Lambda_3 J_3} e^{\Lambda_- J_-}, \tag{11}$$

where

$$\begin{aligned} \Lambda_\pm &= \frac{2\lambda_\pm \sinh f}{2f \cosh f - \lambda_3 \sinh f} \\ \Lambda_3 &= \left(\cosh f - \frac{\lambda_3}{2f} \sinh f \right)^{-2} \\ f &= \left((\lambda_3/2)^2 + \lambda_- \lambda_+ \right)^{1/2}. \end{aligned} \tag{12}$$

In our case we have

$$J_+ = a^\dagger c^\dagger$$
$$J_- = ca = -ac$$
$$J_3 = \frac{1}{2}\left(a^\dagger a + c^\dagger c - 1\right). \tag{13}$$

This choice satisfies the commutation relations in Eqs. (10). We get from Eq. (9) $\lambda_\pm = \pm r e^{\mp i\phi}$ and $\lambda_3 = 0$ leading to

$$U_{A_i C_i} = \cos r \exp\left[a_i^\dagger c_i^\dagger e^{-i\phi} \tan r\right] \exp\left[(a_i^\dagger a_i + c_i^\dagger c_i) \ln \cos^{-1} r\right] \exp\left[-c_i a_i e^{i\phi} \tan r\right]. \tag{14}$$

We might safely set $\phi = 0$ since we will later see that it has no relevance in this work.

We collect the majority of identities used in the course of the paper in the following lemma.

Lemma 3. *Let a and c be operators obeying the canonical anticommutation relations Eq. (1). Then the following identities hold*

$$[a_i c_i, a_j^\dagger c_j^\dagger] = 2\delta_{ij} J_3 \tag{15a}$$

$$[a_i^\dagger a_i, a_j^\dagger c_j^\dagger] = \delta_{ij} a_j^\dagger c_j^\dagger \tag{15b}$$

$$[a_i^\dagger a_i, a_j^\dagger] = \delta_{ij} a_j^\dagger \tag{15c}$$

$$[a_j^\dagger c_j^\dagger, a_i^\dagger] = 0 \tag{15d}$$

$$\left[\prod_{j=1}^{k} a_j c_j, a_i^\dagger\right] = 0 \tag{15e}$$

$$\left[\prod_{j=1}^{k} a_j^\dagger c_j^\dagger, c_i^\dagger\right] = 0 \tag{15f}$$

$$(-)^{\Delta_{k-1}} \prod_{j=1}^{k} a_j c_j = \prod_{j=1}^{k} a_j \prod_{j=1}^{k} c_j. \tag{15g}$$

The first equation is a generalization of the second row in Eq. (10), Eq. (15c) simplifies to $a_i^\dagger a_i a_i^\dagger = a_i^\dagger$ for $i = j$, Eq. (15e) holds for $i \neq j$ and $\Delta_{k-1} = k(k-1)/2$.

Proof. Eqs. (15a)-(15f) directly follow from Eq. (1). In Eq. (15g) we move all a_j on the left of all c_j. So all a_j on the LHS of Eq. (15g) for even j 'jump over' an odd number of c_j operators. Similarly every odd a_j switches its position with an even number of c_j operators. The total acquired phase is

$$\prod_{j=1}^{k}(-)^{(k-j)} = (-)^{\sum_{j=1}^{k}(k-j)} = (-)^{k(k-1)/2} \equiv (-)^{\Delta_{k-1}}.$$

■

Taking into account Eq. (15g) the action of d copies of the fermionic unitary operator results in

$$|\Psi\rangle_{AC} = \bigotimes_{i=1}^{d} U_{A_i C_i} |vac\rangle = \cos^d r \sum_{k=0}^{d} (-)^{\Delta_{k-1}} \tan^k r \sum_{n_1,\dots,n_d}^{\binom{d}{k}} |\vec{n}\rangle_A |\vec{n}\rangle_C. \tag{16}$$

If we wanted to see how $U_{AC} = \bigotimes_{i=1}^{d} U_{A_i C_i}$ transforms a fermionic qudit written in the multi-rail basis of d modes

$$|\psi\rangle_{A'C'} = \sum_{i=1}^{d} \beta_i a_i^\dagger |vac\rangle = \sum_{i=1}^{d} \beta_i |i\rangle_{A'} |vac\rangle_{C'}$$

we might just calculate $|\Phi\rangle_{AC} = U_{AC} |\psi\rangle_{A'C'}$. In order to simplify this complicated calculation we first observe

$$U_{AC} = \cos^d r \exp\left[\tan r \sum_{i=1}^{d} a_i^\dagger c_i^\dagger\right] \exp\left[\sum_{i=1}^{d}(a_i^\dagger a_i + c_i^\dagger c_i)\ln\cos^{-1} r\right]\exp\left[\tan r \sum_{i=1}^{d} a_i c_i\right], \quad (17)$$

where Eqs. (15a) and (15b) for $i \neq j$ were utilized. To proceed we make use of the main advantage of the multi-rail encoding. Due to the presence of c_i annihilating the vacuum state we observe

$$\exp\left[\tan r \sum_{i=1}^{d} a_i c_i\right]\left(\sum_{i=1}^{d} \beta_i a_i^\dagger\right)|vac\rangle = \sum_{i=1}^{d} \beta_i a_i^\dagger |vac\rangle.$$

We may then write

$$\begin{aligned}
|\Phi\rangle_{AC} &= \cos^d r \exp\left[\tan r \sum_{i=1}^{d} a_i^\dagger c_i^\dagger\right] \exp\left[\sum_{i=1}^{d}(a_i^\dagger a_i + c_i^\dagger c_i)\ln\cos^{-1} r\right]\left(\sum_{i=1}^{d} \beta_i a_i^\dagger\right)|vac\rangle \\
&= \cos^{d-1} r \exp\left[\tan r \sum_{i=1}^{d} a_i^\dagger c_i^\dagger\right]\left(\sum_{i=1}^{d} \beta_i a_i^\dagger\right)|vac\rangle \qquad (18) \\
&= \cos^{d-1} r \left(\sum_{i=1}^{d} \beta_i a_i^\dagger\right)\exp\left[\tan r \sum_{i=1}^{d} a_i^\dagger c_i^\dagger\right]|vac\rangle.
\end{aligned}$$

The second row follows from Eq. (15c) (see why $i = j$ does not spoil the commutator). The last equality is possible due to Eq. (15d).

The action of the unitary U_{AB} leads to

$$|\Phi\rangle_{AC} = U_{AC} \sum_{i=1}^{d} \beta_i |i\rangle_{A'} |vac\rangle_{C'} = \cos^{d-1} r \left(\sum_{i=1}^{d} \beta_i a_i^\dagger\right)\exp\left[\tan r \sum_{j=1}^{d} a_j^\dagger c_j^\dagger\right]|vac\rangle \qquad (19a)$$

$$= \cos^{d-1} r \left(\sum_{i=1}^{d} \beta_i a_i^\dagger\right) \sum_{k=1}^{d+1} \tan^{k-1} r (-)^{\Delta_{k-2}} \sum_{N_k} |\ldots n_j \ldots\rangle_A |\ldots n_j \ldots\rangle_C \qquad (19b)$$

$$= \cos^{d-1} r \sum_{k=1}^{d} \tan^{k-1} r (-)^{\Delta_{k-2}} \left[\sum_{N_k} \sum_{i\in I}^{d-k+1} (\pm)_i \beta_i |\ldots n_j + 1_i \ldots\rangle_A |\ldots n_j \ldots\rangle_C\right], \quad (19c)$$

where we have defined the set $N_k = \{(n_1, \ldots, n_d)| \sum_{j=1}^{d} n_j = k - 1\}$ to be the sum over all possible states with $k - 1$ fermions. The additional fermion (with respect to the C subsystem) in Eq. (19c) occupies the i-th position. There is $\binom{d}{k-1}$ states $|\ldots n_j \ldots\rangle_C$ in the middle sum of Eq. (19c) for a given N_k ($k = 1\ldots d$). The rightmost sum of Eq. (19c) sums over $i \in I$, for a fixed N_k, such that $I = \{i \mid n_i = 0\}$ and thus contains $d-k+1$ terms, justifying the upper limit in the sum. Taking, for example, $\beta_0 = 1$ we verify $\cos^{2(d-1)} r \sum_{k=1}^{d} \tan^{2(k-1)} r \binom{d-1}{k-1} = 1$. The

multiplicative sign $(\pm)_i$ in Eq. (19c) is dependent on the fermionic state of the A subsystem. It appears following the operator rules specific to the fermionic algebra. Let us recall some basic properties of the algebra

$$a_i^\dagger \left| \ldots n_j \ldots \right\rangle = (1 - n_i)(-)^{\sum_{j=1}^{i-1} n_j} \left| \ldots n_j + 1_i \ldots \right\rangle$$

$$a_i \left| \ldots n_j \ldots \right\rangle = n_i(-)^{\sum_{j=1}^{i-1} n_j} \left| \ldots n_j - 1_i \ldots \right\rangle. \tag{20}$$

Isometry output Eq. (19) gives rise to a new class of channels we nicknamed *Grassmann channels*.

Definition 4. *The d-dimensional Grassmann channel $\mathcal{G}_d$ is a quantum channel defined by the action of its isometry $V_{\mathcal{G}_d}$ as $\mathcal{G}_d(\psi_{A'}) = \mathrm{Tr}_C \circ V_{\mathcal{G}_d} \circ \psi_{A'}$. The action of the channel on an input qudit is given by*

$$\mathcal{G}_d : \psi_{A'} \mapsto \gamma_A^{(d)} = \cos^{2(d-1)} r \bigoplus_{k=1}^{d} \tan^{2(k-1)} r \binom{d-1}{k-1} \chi_k^{(d)} = \bigoplus_{k=1}^{d} p_k \mathcal{G}_{d,k}(\psi_{A'}), \tag{21}$$

where $p_k = \cos^{2(d-1)} r \tan^{2(k-1)} r \binom{d-1}{k-1}$. $\mathcal{G}_{d,k}$ are channels constituting the Grassmann channel $\mathcal{G}_d$ and $\chi_k^{(d)} = \mathcal{G}_{d,k}(\psi_{A'})$ having $\mathrm{Tr}\left[\chi_k^{(d)}\right] = 1$. The output block dimension $\dim \chi_k^{(d)} = \binom{d}{k}$ for $k = 1, \ldots, d$ is known as a Grassmann number, hence the name Grassmann channels.

Remark. The index k in Eq. (21) has two roles in $\chi_k^{(d)}$. It labels the state but it also indicates how many fermions the basis is composed of. We thus know in which basis $\chi_k^{(d)}$ is written.

Remark. The first Grassmann channel $\mathcal{G}_1$ is a trivial trace map. The second Grassmann channel $\mathcal{G}_2$ is a channel recently playing an important role in quantum Shannon theory - a qubit erasure channel [3, 18–20]. We now look at $\mathcal{G}_2$ and $\mathcal{G}_3$ cases in more detail.

Example. For $d = 2$ we get from Eq. (16)

$$\begin{aligned}
\left| \Psi \right\rangle_{AC} &= \cos^2 r \exp\left[\tan r(a_1^\dagger c_1^\dagger + a_2^\dagger c_2^\dagger) \right] \left| vac \right\rangle \\
&= \cos^2 r \left[1 + \tan r(a_1^\dagger c_1^\dagger + a_2^\dagger c_2^\dagger) + \frac{\tan^2 r}{2}(a_1^\dagger c_1^\dagger + a_2^\dagger c_2^\dagger)^2 \right] \left| vac \right\rangle \\
&= \cos^2 r \left(\left| vac \right\rangle + \tan r \left(\left| 10 \right\rangle_A \left| 10 \right\rangle_C + \left| 01 \right\rangle_A \left| 01 \right\rangle_C \right) - \tan^2 r \left| 11 \right\rangle_A \left| 11 \right\rangle_C \right).
\end{aligned} \tag{22}$$

Following Eq. (19c) we obtain

$$\left| \Phi \right\rangle_{AC} = \cos r \Big((\beta_1 \left| 10 \right\rangle + \beta_2 \left| 01 \right\rangle) \left| vac \right\rangle + \tan r \left(\beta_1 \left| 11 \right\rangle \left| 01 \right\rangle - \beta_2 \left| 11 \right\rangle \left| 10 \right\rangle \right) \Big) \tag{23a}$$

$$= \sqrt{1-p}\big(\beta_1 \left| 1 \right\rangle + \beta_2 \left| 2 \right\rangle \big)_A \left| 0 \right\rangle_C + \sqrt{p} \left| 3 \right\rangle_A \big(\beta_1 \left| 2 \right\rangle - \beta_2 \left| 1 \right\rangle \big)_C. \tag{23b}$$

The last equation is Eq. (23a) rewritten using a reparametrization $\sqrt{1-p} = \cos r$, $\sqrt{p} = \sin r$ and a logical ket notation $(\left| 0 \right\rangle, \ldots, \left| 3 \right\rangle)$ to facilitate the comparison with an isometry output for a qubit erasure channel

$$\left| \Phi \right\rangle_{AC}^{erase} = \sqrt{1-p}\big(\beta_1 \left| 1 \right\rangle + \beta_2 \left| 2 \right\rangle \big)_A \left| f \right\rangle_C + \sqrt{p} \left| f \right\rangle_A \big(\beta_1 \left| 1 \right\rangle + \beta_2 \left| 2 \right\rangle \big)_C. \tag{24}$$

$\left| f \right\rangle$ is a flag state orthogonal to both $\left| 1 \right\rangle$ and $\left| 2 \right\rangle$. The most notable difference between Eq. (23b) and Eq. (24) is the dimension of the output Hilbert space. For the latter the Hilbert space is three-dimensional. But if we trace over the A or C subsystems and compare them we immediately see that they indeed induce the same channel. The Grassmann channel $\mathcal{G}_2$ is just embedded in a higher-dimensional space than the corresponding erasure channel and unitarily rotated (note the difference in the second brackets of Eqs. (23b) and (24)).

Example. For $\mathcal{G}_3$ we get from Eq. (19c)

$$|\Phi\rangle_{AC} = \cos^2 r \Big((\beta_1 |100\rangle + \beta_2 |010\rangle + \beta_3 |001\rangle) |vac\rangle \tag{25a}$$

$$+ \tan r \left[(-\beta_1 |101\rangle - \beta_2 |011\rangle) |001\rangle - (\beta_1 |110\rangle - \beta_3 |011\rangle) |010\rangle + (\beta_2 |110\rangle + \beta_3 |101\rangle) |100\rangle \right] \tag{25b}$$

$$- \tan^2 r |111\rangle (\beta_1 |011\rangle - \beta_2 |101\rangle + \beta_3 |110\rangle) \Big). \tag{25c}$$

The trace over the C subsystem gives us the output of $\mathcal{G}_3$

$$\mathcal{G}_3 : \psi_{A'} \mapsto \gamma_A^{(3)} = p_1 \chi_1^{(3)} \oplus p_2 \chi_2^{(3)} \oplus p_3 \chi_3^{(3)}, \tag{26}$$

where $p_1 = \cos^4 r, p_2 = 2 \tan^2 r \cos^4 r, p_3 = \tan^4 r \cos^4 r$. $\chi_1^{(3)}$ is the input state itself and $\chi_3^{(3)}$ is a flag state $|111\rangle$. We are interested in the form of $\chi_2^{(3)}$

$$\chi_2^{(3)} = \frac{1}{2} \begin{pmatrix} |\beta_2|^2 + |\beta_3|^2 & \beta_2 \bar{\beta}_1 & -\beta_3 \bar{\beta}_1 \\ \bar{\beta}_2 \beta_1 & |\beta_1|^2 + |\beta_3|^2 & \beta_3 \bar{\beta}_2 \\ -\bar{\beta}_3 \beta_1 & \bar{\beta}_3 \beta_2 & |\beta_1|^2 + |\beta_2|^2 \end{pmatrix} \tag{27}$$

where the two-fermionic basis is ordered as $\{|011\rangle, |101\rangle, |110\rangle\}$. We easily verify that the Kraus operators

$$K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, K_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{28}$$

yields $\chi_2^{(3)}$.

In the proof of the next theorem we will extensively use the geometric picture from the representation theory of the $\mathsf{sl}(d, \mathbb{C})$ Lie algebras. We summarized some basic facts in the appendix.

Theorem 5. *Let the first block of $\gamma_A^{(d)}$ in Eq. (21) be written as*

$$\chi_1^{(d)} = \frac{1}{d} \Big(\mathbb{1} + \sum_{\alpha=1}^{L} n_\alpha \lambda_\alpha^{(1)} \Big), \tag{29}$$

where $\lambda_\alpha^{(1)}$ are generators of the dual representation to the fundamental representation of the $\mathsf{sl}(d, \mathbb{C})$ algebra, $L = 2d(d-1)$ and n_α are functions of $\beta_i \bar{\beta}_j$. Then the remaining blocks in Eq. (21) can be expanded with the same coefficients n_α

$$\chi_k^{(d)} = \frac{1}{\binom{d}{k}} \Big(\mathbb{1} + \sum_{\alpha=1}^{L} n_\alpha \lambda_\alpha^{(k)} \Big), \tag{30}$$

where $\lambda_\alpha^{(k)}$ are generators of the k-th completely antisymmetric representation of the $\mathsf{sl}(d, \mathbb{C})$ algebra.

Remark. The number L in Eqs. (29) and (30) comes from the use of a redundant number of the $\mathsf{sl}(2, \mathbb{C})$ subalgebras as discussed in the appendix. Namely, there is $d(d-1)$ diagonal generators H_n, $d(d-1)/2$ off-diagonal generators E_{ij} (recall that $i < j$) and $d(d-1)/2$ of their Hermitian conjugates $E_{ij}^\dagger$.

Remark. Since we use a linearly dependent set of algebra generators the expansion coefficients n_α are not unambiguously determined. For us, however, it is sufficient to show that at least for one specific construction (the one presented here) the coefficients can be chosen to stay preserved when switching to a higher-dimensional representation of $\mathsf{sl}(d, \mathbb{C})$.

The upcoming proof will be divided into two parts. We will separately prove Eq. (30) for diagonal and off-diagonal generators. To better follow the proof it might be helpful to watch the example of Eq. (19c) for $d = 4$. In the course of the proof there is a remark illustrating results on this case.

Example. For $d = 4$ we get from Eq. (19c)

$$
\begin{aligned}
|\Phi\rangle_{AC} = \cos^3 r \Bigg(& \Big(\beta_1 |1000\rangle + \beta_2 |0100\rangle + \beta_3 |0010\rangle + \beta_4 |0001\rangle \Big) |vac\rangle \\
& + \tan r \Big[\Big(\beta_1 |1001\rangle + \beta_2 |0101\rangle + \beta_3 |0011\rangle \Big) |0001\rangle \\
& \quad + \Big(\beta_1 |1010\rangle + \beta_2 |0110\rangle - \beta_4 |0011\rangle \Big) |0010\rangle \\
& \quad + \Big(\beta_1 |1100\rangle - \beta_3 |0110\rangle - \beta_4 |0101\rangle \Big) |0100\rangle \\
& \quad - \Big(\beta_2 |1100\rangle + \beta_3 |1010\rangle + \beta_4 |1001\rangle \Big) |1000\rangle \Big] \\
& - \tan^2 r \Big[\Big(\beta_1 |1011\rangle + \beta_2 |0111\rangle \Big) |0011\rangle + \Big(\beta_1 |1101\rangle - \beta_3 |0111\rangle \Big) |0101\rangle \\
& \quad - \Big(\beta_2 |1101\rangle + \beta_3 |1011\rangle \Big) |1001\rangle + \Big(\beta_1 |1110\rangle + \beta_4 |0111\rangle \Big) |0110\rangle \\
& \quad + \Big(- \beta_2 |1110\rangle + \beta_4 |1011\rangle \Big) |1010\rangle + \Big(\beta_3 |1110\rangle + \beta_4 |1101\rangle \Big) |1100\rangle \Big] \\
& - \tan^3 r \, |1111\rangle \Big(- \beta_1 |0111\rangle + \beta_2 |1011\rangle - \beta_3 |1101\rangle + \beta_4 |1110\rangle \Big) \Bigg).
\end{aligned}
\tag{31}
$$

We do not indicate the subsystems but in a product of two kets the first one is the A subsystem and the second one is the C subsystem.

We will employ the fermionic representation of the $\mathsf{sl}(2, \mathbb{C})$ algebra [35] (see also Eqs. (A3))

$$
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a_m^\dagger a_m - a_l^\dagger a_l
\tag{32a}
$$

$$
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = a_m^\dagger a_l
\tag{32b}
$$

$$
\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = a_l^\dagger a_m,
\tag{32c}
$$

where $m \neq l$ are mode labels.

Proof. (i) Let us first find the coefficients n_α from Eq. (29) and then show that they remain preserved in Eq. (30). Since we have

$$
\frac{|\beta_1|^2}{d} \Big(\sum_{i=1}^d a_i^\dagger a_i + \sum_{j=1}^{d-1} (a_1^\dagger a_1 - a_{1+j}^\dagger a_{1+j}) \Big) = |\beta_1|^2 a_1^\dagger a_1 \equiv |\beta_1|^2 |1 \ldots 0\rangle\langle 1 \ldots 0|
\tag{33}
$$

and so

$$
\frac{|\beta_i|^2}{d} \Big(\sum_{i=1}^d a_i^\dagger a_i + \sum_{j=1}^{d-1} (a_i^\dagger a_i - a_{i\oplus j}^\dagger a_{i\oplus j}) \Big) = |\beta_i|^2 a_i^\dagger a_i \equiv |\beta_i|^2 |0 \ldots 1_i \ldots 0\rangle\langle 0 \ldots 1_i \ldots 0|
\tag{34}
$$

there is $(d-1)$ coefficients n_α that equal $|\beta_i|^2$ for $i = 1 \ldots d$. Note that $i \oplus j \equiv i + j \mod (d+1)$.

For $k > 1$ we first notice in Eq. (19c) the presence of the factors $(\pm)_i$. They are irrelevant for now since we investigate the diagonal generators. The important fact is that the numerical coefficients of all β_i's are the same in the absolute value, that is, equal to one. We trace over C and the diagonal part of the A subsystem for a given k reads

$$\text{diag}\,[\chi_k^{(d)}] = \frac{1}{\binom{d}{k}}\left[\sum_{N_k}^{} \sum_{i \in I}^{d-k+1} |\beta_i|^2 |\ldots n_j + 1_i \ldots\rangle\langle\ldots n_j + 1_i \ldots|\right]. \tag{35}$$

Each diagonal element $\sum_{i \in I}^{d-k+1} |\beta_i|^2 |\ldots n_j + 1_i \ldots\rangle\langle\ldots n_j + 1_i \ldots|$ corresponding to a given n_j can be factorized

$$\sum_{i \in I}^{d-k+1} |\beta_i|^2 |\ldots n_j + 1_i \ldots\rangle\langle\ldots n_j + 1_i \ldots|$$
$$= \left(\sum_{i \in I}^{d-k+1} |\beta_i|^2 |0\ldots 1_i \ldots 0\rangle\langle 0 \ldots 1_i \ldots 0|\right)|\ldots n_j \ldots\rangle\langle\ldots n_j \ldots|, \tag{36}$$

where the expression in the parenthesis in the second line can be rewritten using the fermionic representation of the $\mathsf{sl}(2, \mathbb{C})$ algebra from Eq. (34). The rest serves as a label (recall that there is $\binom{d}{k-1}$ orthogonal states for which $\sum n_j = k - 1$). However, because of the factorization in Eq. (36) the dimension of the first expression will become $d' = d - \sum_{j=1}^{d} n_j$ giving us

$$\sum_{i \in I}^{d-k+1} |\beta_i|^2 |\ldots n_j + 1_i \ldots\rangle\langle\ldots n_j + 1_i \ldots|$$
$$= \left(\sum_{i \in I}^{d-k+1} |\beta_i|^2 \Big(\sum_{i=1}^{d'} a_i^\dagger a_i + \sum_{j=1}^{d'-1}(a_i^\dagger a_i - a_{i \oplus j}^\dagger a_{i \oplus j})\Big)\right)|\ldots n_j \ldots\rangle\langle\ldots n_j \ldots|. \tag{37}$$

So the diagonal elements for $k > 1$ can also be expressed using only the $\mathsf{sl}(2, \mathbb{C})$ diagonal algebra generators. This is what we expected following the discussion in the appendix.

Now we have to make sure that there is the right number of summands when constructing the direct sum $\mathsf{sl}(2, \mathbb{C})$ subalgebra representations also discussed in the appendix in the last paragraph. This is indeed the case. The orthogonal 'label states' factorized out in Eq. (37) label the subspace in which the $\mathsf{sl}(2, \mathbb{C})$ subalgebra in the parenthesis lives. We want to count how often this situation happens. Since it is a $\mathsf{sl}(2, \mathbb{C})$ subalgebra we have two spots out of d in each ket occupied by two fermions (by choosing, for example, β_1 and β_2 or any other pair) and the remaining number of spots can be occupied by $\sum_{j=1}^{d} n_j$ fermions. The number of possibilities is

$$\#[orthogonal\ subspaces] = \binom{d-2}{\sum_{j=1}^{d} n_j} \equiv \binom{d-2}{k-1}.$$

We thus reproduced the result found in the appendix for all completely antisymmetric representations of the $\mathsf{sl}(d, \mathbb{C})$ algebras.

Remark. We might verify the above considerations on Eq. (31). For $k = 2$ (lines 2 to 5) tracing over C and taking the diagonal part gives us unnormalized Eq. (35) for which $d - k + 1 = 3$. There is $\binom{4}{1} = 4$ states where $\sum_{j=1}^{d} n_j = 1$ holds and they can be factorized out as in Eq. (37). There are therefore four blocks with mutually orthogonal flags attached to them but only $\binom{2}{1} = 2$

of them have a common pair of coefficients (e.g., β_1 and β_2). This is the case illustrated in Fig. 5b. All edges of the octahedron are $\mathsf{sl}(2,\mathbb{C})$ algebras. The pairs of parallel lines form the generators of a completely antisymmetric representation of $\mathsf{sl}(4,\mathbb{C})$ in the form of a direct sum. They span a 4-dimensional space but none of the spaces is orthogonal to any other.

(ii) Proving Eq. (30) for off-diagonal generators is considerably simpler. Looking at the rightmost sum of Eq. (19c) we see that every state $|\ldots n_j \ldots\rangle_C$ is multiplied by $\sum_j \beta_j(\pm)_j |\ldots n_j + 1_i \ldots\rangle_A$ containing one fermion more in the i-th mode compared to the C subsystem. Tracing over C and taking the off-diagonal part gives us expressions of the following type

$$\sum_{l,m} \beta_l \bar{\beta}_m (\pm)_l (\pm)_m |\ldots n_i + 1_l \ldots\rangle\langle\ldots n_j + 1_m \ldots|. \tag{38}$$

We immediately recognize the off-diagonal matrices forming Eq. (38) to be proportional to the step operators pertaining to the $\mathsf{sl}(2,\mathbb{C})$ subalgebra from Eq (32). The situation is only complicated by the presence of a function $(\pm)_l(\pm)_m$ responsible for the change of a sign. The function comes from the fermionic relations Eq. (20). The change of sign therefore appears if we annihilate the fermion in the l-th mode and create it in the m-th mode

$$(\pm)_l(\pm)_m |\ldots n_j + 1_m \ldots\rangle_A = a_m^\dagger a_l |\ldots n_j + 1_l \ldots\rangle.$$

But this is precisely the action of the operator from Eq. (32b) representing the $\mathsf{sl}(2,\mathbb{C})$ subalgebra. We again take a direct sum of $\binom{d-2}{k-1}$ off-diagonal generators. They form the off-diagonal generators of the k-th lowest completely antisymmetric representation of $\mathsf{sl}(d,\mathbb{C})$. $\blacksquare$

Corollary 6. The Grassmann channels $\mathcal{G}_d$ are $SU(d)$ covariant.

Proof. The proof is identical to Corollary 17 for completely symmetric representations of $\mathsf{sl}(d,\mathbb{C})$ in [22] showing that the Unruh channel is $SU(d)$ covariant. $\blacksquare$

IV. QUANTUM CAPACITY

There is an interesting symmetry between the A and C output subsystem captured in the following lemmas and later necessary for the proof of degradability of the Grassmann channels.

Lemma 7. *Labeling the Pauli matrices σ_Z and σ_X by Z and X and a d-dimensional identity by I_d we first introduce an infinite product $Z \otimes I_2 \otimes Z \otimes I_2 \otimes Z \otimes I_2 \otimes \ldots$. A d-mode unitary operator Ξ_d is defined as the first d unitaries of the product. Then the following identity holds*

$$\mathcal{O}_{0\to(d-1)} \circ X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ (-)^d \Big(\sum_{i=1}^d \beta_i a_i^\dagger\Big) |vac\rangle_{AC} = \tan^{d-1} r \Big(\sum_{i=1}^d \beta_i c_i^\dagger\Big) |\vec{n}\rangle_A |\vec{n}\rangle_C, \tag{39}$$

where $\sum_{i=1}^d n_i = d-1$ and $\mathcal{O}_{(k-1)\to(d-k)} : \tan^{k-1} r \to \tan^{d-k} r$ is a non-physical operation solely acting on the trigonometric function in Eq. (19b) (or Eq. (19c)). For $k = 1$ we understand $\mathcal{O}_{0\to(d-1)}(1) = \tan^{d-1} r$ in Eq. (39).

Proof. We prove the identity by a direct calculation. Using properties of the fermionic algebra,

the following hold:

$$X^{\otimes 2d} \circ (\Xi_d \otimes I_d)\Big(\sum_{i=1}^{d} \beta_i a_i^\dagger \ket{vac}_{AC}\Big) = X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ \Big(\sum_{i=1}^{d} \beta_i \ket{0\ldots 1_i \ldots 0}_A\Big)\ket{vac}_C$$
(40a)

$$= (-)^i X^{\otimes 2d} \circ \Big(\sum_{i=1}^{d} \beta_i \ket{0\ldots 1_i \ldots 0}_A\Big)\ket{vac}_C \qquad (40b)$$

$$= (-)^i \Big(\sum_{i=1}^{d} \beta_i \ket{1\ldots 0_i \ldots 1}_A\Big)\ket{1\ldots 1}_C. \qquad (40c)$$

On the other hand we have

$$\tan^{d-1} r \sum_{i=1}^{d} \beta_i c_i^\dagger \ket{\vec{n}}_A \ket{\vec{n}}_C = \tan^{d-1} r \Big(\sum_{i=1}^{d} (-)^{d-1+i-1} \beta_i \ket{1\ldots 0_i \ldots 1}_A\Big) \ket{1\ldots 1}_C. \qquad (41)$$

By multiplying Eq. (40a) by $(-)^{d-2}$ and exchanging the trigonometric function (represented by the action of $\mathcal{O}_{0\to(d-1)}$) we get the RHS of Eq. (41) and therefore also Eq. (39). ∎

The purpose of the next lemma is to present two extensively used identities.

Lemma 8. *Let a be a fermionic operator. Then, the following identities hold*

$$Za^\dagger = -a^\dagger Z \qquad (42a)$$

$$Xa^\dagger = aX. \qquad (42b)$$

Proof. Directly follows from Eq. (1) once we realize that $X = a + a^\dagger$ and $Z = a^\dagger a - aa^\dagger = 2a^\dagger a - 1 = 1 - 2aa^\dagger$. ∎

Lemma 9. *The following relation holds*

$$\mathcal{O}_{(k-1)\to(d-k)} \circ X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ \Big(\sum_{i=1}^{d} \beta_i a_i^\dagger\Big) \ket{\Psi}_{AC} = \Big(\sum_{i=1}^{d} \beta_i c_i^\dagger\Big) \ket{(\pm)_{\vec{m},k,d}\Psi}_{AC}, \qquad (43)$$

where $\ket{(\pm)_{\vec{m},k,d}\Psi}_{AC}$ is $\ket{\Psi}_{AC}$ from Eq. (16) with some of the summands having a negative sign.

Remark. The indices of the phase function will be described and the function explicitly written.

Proof. To make the derivation smoother we will first prove a specific part of Eq. (43) in which $k = 2$

$$\mathcal{O}_{1\to(d-2)}(\tan r) \circ X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ \Big(\sum_{\substack{i=1 \\ i\neq j}}^{d} \beta_i a_i^\dagger\Big) a_j^\dagger c_j^\dagger \ket{vac}_{AC}$$

$$= \tan^{d-2} r (-)^{j-1} \Big(\sum_{\substack{i=1 \\ i\neq j}}^{d} \beta_i c_i^\dagger\Big) \ket{\vec{m}^j}_A \ket{\vec{m}^j}_C. \qquad (44)$$

where $\sum_{i=1}^{d} m_i^j = d - 2$ and $m_j^j = 0$. Let us ignore for a while the function $\mathcal{O}_{1\to(d-2)}$. Using identities (15d), (42a),(42b) and Lemma 7 (note the missing j-th summand from both sides of Eq. (39)) the LHS of Eq. (44) reads

$$(-)^j a_j c_j \circ X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ \Big(\sum_{\substack{i=1 \\ i\neq j}}^{d} \beta_i a_i^\dagger \Big) |vac\rangle_{AC} = (-)^j (-)^d a_j c_j \Big(\sum_{\substack{i=1 \\ i\neq j}}^{d} \beta_i c_i^\dagger \Big) |\vec{n}\rangle_A |\vec{n}\rangle_C \quad (45a)$$

$$= (-)^j (-)^d (-)^{d-1} \Upsilon \Big(\sum_{\substack{i=1 \\ i\neq j}}^{d} \beta_i c_i^\dagger \Big) |\vec{m}^j\rangle_A |\vec{m}^j\rangle_C, \quad (45b)$$

where $\Upsilon = (-)^{j-2}(-)^{j-2} = (-)^{j-1}(-)^{j-1} \equiv 1$ depending on whether $i < j$ or $j < i$. Thus we recovered the phase on the RHS of Eq. (44). It is time to justify the presence of $\mathcal{O}_{1\to(d-2)}$ in Eq. (44). Eq. (19c) indicates that the exponent of $\tan^{k-1} r$ coincides with the number of excitations of the C subsystem and is one less than the number of excitations in the A subsystem. From reasons that will become clear later we would like the exponent of the trigonometric function $\tan^{d-2} r$ on the RHS to correspond to the number of excitations of the A subsystem which is indeed equal to $d - 2$.

To prove Eq. (43) we first rewrite Eq. (16) as

$$\sum_{k=1}^{d+1} \tan^{k-1} r (-)^{\Delta_{k-2}} \sum_{N_k} |\vec{n}^J\rangle_A |\vec{n}^J\rangle_C = \sum_{k=1}^{d+1} \tan^{k-1} r \sum_{N_k} (a_j^\dagger c_j^\dagger)^{n_j} |vac\rangle_{AC} = \frac{1}{\cos^d r} |\Psi\rangle_{AC},$$

where we have used the notation introduced in Eq. (19c). We have also introduced the set J, for a fixed N_k, as $J = \{j | 1 \leq j \leq d, \ n_j = 1\}$. Let us investigate the expression for a chosen k:

$$\mathcal{O}_{(k-1)\to(d-k)}(\tan^{k-1} r) \circ X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ \Big(\sum_{\substack{i=1 \\ i\notin J}}^{d} \beta_i a_i^\dagger \Big) \sum_{N_k} (a_j^\dagger c_j^\dagger)^{n_j} |vac\rangle_{AC} \quad (46a)$$

$$= \mathcal{O}_{(k-1)\to(d-k)}(\tan^{k-1} r) \circ \sum_{N_k} ((-1)^j a_j c_j)^{n_j} \circ X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ \Big(\sum_{\substack{i=1 \\ i\notin J}}^{d} \beta_i a_i^\dagger \Big) |vac\rangle_{AC} \quad (46b)$$

$$= \tan^{d-k} r (-)^d \sum_{N_k} ((-1)^j a_j c_j)^{n_j} \Big(\sum_{\substack{i=1 \\ i\notin J}}^{d} \beta_i c_i^\dagger \Big) |\vec{n}'\rangle_A |\vec{n}'\rangle_C \quad (46c)$$

$$= \tan^{d-k} r (-)^d \sum_{N_k} (-)^{jn_j} (-)^{(k-1)(d-1)} \Upsilon^{k-1} (-)^{\Delta_{k-2}} \Big(\sum_{\substack{i=1 \\ i\notin J}}^{d} \beta_i c_i^\dagger \Big) |\vec{m}^J\rangle_A |\vec{m}^J\rangle_C \quad (46d)$$

$$= \tan^{d-k} r \Big(\sum_{\substack{i=1 \\ i\notin J}}^{d} \beta_i c_i^\dagger \Big) \sum_{N_k} (\pm)_{\vec{m},k,d} |\vec{m}^J\rangle_A |\vec{m}^J\rangle_C, \quad (46e)$$

In the first equality of Eq. (46) we used Eqs. (15f), (42a) and (42b), and in the second equality Lemma 7 was used with the summands corresponding to the set J removed from both sides of Eq. (39). Here $\sum_i n_i' = d - 1$ holds. In the third equality, Eqs. (15e), (15g) and the properties of the fermionic algebra leading to Υ in Eq. (45b) were used and we introduced a vector $|\vec{m}^J\rangle$

such that $\sum_{i=1}^{d} m_i^J = d - k$ and $m_j^J = 0$ if $j \in J$. Finally in the last equation we collected all phases into a single function $(\pm)_{\vec{m},k,d}$.

To prove the lemma we realize that the LHS of Eq. (43) is Eq. (46a) summed over k

$$\cos^d r \sum_{k=1}^{d} \left[\mathcal{O}_{(k-1)\to(d-k)}(\tan^{k-1} r) \circ X^{\otimes 2d} \circ (\Xi_d \otimes I_d) \circ \left(\sum_{\substack{i=1 \\ i \notin J}}^{d} \beta_i a_i^\dagger \right) \sum_{N_k} (a_j^\dagger c_j^\dagger)^{n_j} \right] |vac\rangle_{AC} \quad (47)$$

(note that we sum only to d since the $(d+1)$-th summand disappeared). The RHS of Eq. (43) is just a sum of Eq. (46e) over k

$$\cos^d r \sum_{k=1}^{d} \tan^{d-k} r \Big(\sum_{\substack{i=1 \\ i \notin J}}^{d} \beta_i c_i^\dagger \Big) \sum_{N_k} (\pm)_{\vec{m},k,d} \left|\vec{m}^J\right\rangle_A \left|\vec{m}^J\right\rangle_C = \Big(\sum_{i=1}^{d} \beta_i c_i^\dagger \Big) |(\pm)_{\vec{m},k,d}\Psi\rangle_{AC} \quad (48)$$

Notice that if we exchange $(d-k)$ for $(k-1)$ in $\tan^{d-k} r, N_k, \left|\vec{m}^J\right\rangle_A$ and $\left|\vec{m}^J\right\rangle_C$ such that $\sum_{i=1}^{d} m_i^J = k - 1$ we indeed get $\Big(\sum_{i=1}^{d} \beta_i c_i^\dagger \Big) |\Psi\rangle_{AC}$ up to the phase function. $\blacksquare$

Remark. The important fact about Eq. (46e) is that the sign depends on chosen d, k and N_k so it is in general different for each summand over N_k but common for each sum over i.

Lemma 10. *Following the notation of Definition 4 we label $\gamma_C^{(d)} = \mathrm{Tr}_A \circ \Phi_{AC}$. Then we have*

$$\mathcal{O}_{(k-1)\to(d-k)} \circ W \circ \gamma_A^{(d)} = \gamma_C^{(d)}, \quad (49)$$

where W is a unitary transformation.

Proof. Lemma 9 is powerful since it allows us to compare the outputs of Grassmann channels with their complementary outputs $\gamma_C^{(d)}$. Indeed, from the form of $|\Phi\rangle_{AC}$ we can deduce the explicit form of the block matrices from which $\gamma_A^{(d)}$ is composed of (see Eq. (21)). However, if we wanted to compare $\gamma_A^{(d)}$ with $\gamma_C^{(d)}$ given by unitary U_{AC} in Eqs. (19) it would be a difficult task. The advantage of Lemma 9 is that we don't even need to know $\gamma_A^{(d)}$ or $\gamma_C^{(d)}$ explicitly to find the relation between them.

Let us rewrite the previous lemma result in the following way

$$\Big(\sum_{i=1}^{d} \beta_i a_i^\dagger \Big) |\Psi\rangle_{AC} = \mathcal{O}_{(k-1)\to(d-k)} \circ (\Xi_d \otimes I_d) \circ X^{\otimes 2d} \circ \Big(\sum_{i=1}^{d} \beta_i c_i^\dagger \Big) |(\pm)_{\vec{m},k,d}\Psi\rangle_{AC}. \quad (50)$$

Recall that $\mathcal{O}_{(k-1)\to(d-k)} \circ (\Xi_d \otimes I_d) \circ X^{\otimes 2d}$ is an involution and $\mathcal{O}_{(k-1)\to(d-k)}$ is a scalar function. Note that $\Xi_d \otimes I_d$ commutes or anticommutes with $X^{\otimes 2d}$ depending on the specific form of Ξ_d. We incorporate this sign change into $(\pm)_{\vec{m},k,d}$. Tracing out the A subsystem we have

$$\gamma_C^{(d)} = \mathrm{Tr}_A \circ \Big(\sum_{i=1}^{d} \beta_i a_i^\dagger \Big) |\Psi\rangle_{AC} = \mathcal{O}_{(k-1)\to(d-k)} \circ W \circ \mathrm{Tr}_A \circ \Big(\sum_{i=1}^{d} \beta_i c_i^\dagger \Big) |(\pm)_{\vec{m},k,d}\Psi\rangle_{AC} \quad (51a)$$

$$= \mathcal{O}_{(k-1)\to(d-k)} \circ W \circ \mathrm{Tr}_A \circ \Big(\sum_{i=1}^{d} \beta_i c_i^\dagger \Big) |\Psi\rangle_{AC} \quad (51b)$$

$$= \mathcal{O}_{(k-1)\to(d-k)} \circ W \circ \mathrm{Tr}_C \circ \Big(\sum_{i=1}^{d} \beta_i a_i^\dagger \Big) |\Psi\rangle_{AC} \quad (51c)$$

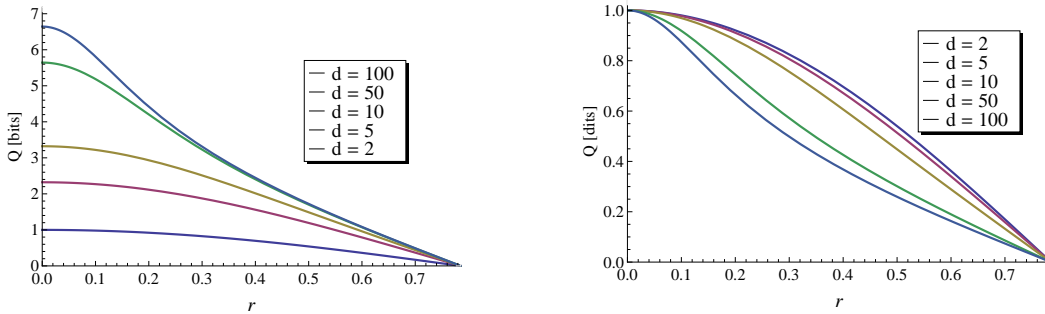$$= \mathcal{O}_{(k-1)\to(d-k)} \circ W \circ \gamma_A^{(d)}. \quad (51d)$$

FIG. 1: Quantum capacity for Grassmann channels for $d = 2, 5, 10, 50$ and 100. In the left plot the capacity is calculated using the base two logarithm and in the right plot using the base d logarithm. The curves follow the order in the legend.

The first equality holds since all the unitaries in Eq. (50) act locally and $\mathcal{O}_{(k-1)\rightarrow(d-k)}$ is again a scalar function (the unitary W appears as the result of tracing over unitarily-locally transformed state $|\Phi\rangle_{AC}$). The sign ambivalence in the second equality is irrelevant because tracing over A means creating of a convex sum of states belonging to the C subsystem so the phase disappears, the third equality is a simple permutation argument due to the symmetry of $|\Psi\rangle_{AC}$ between the modes A and C and in Eq. (51d) we invoke the definition of $\gamma_A^{(d)}$. ∎

Let us formally define the complementary Grassmann channel based on Lemma 10 and Definition 4:

Definition 11. *The d-dimensional complementary Grassmann channel $\mathcal{G}_d^c$ is the quantum channel defined by the isometry $V_{\mathcal{G}_d}$ as $\mathcal{G}_d^c(\psi_{A'}) = \mathrm{Tr}_A \circ V_{\mathcal{G}_d} \circ \psi_{A'}$. The action of the channel on an input qudit is given by*

$$\mathcal{G}_d^c : \psi_{A'} \mapsto \gamma_C^{(d)} = \cos^{2(d-1)} r \bigoplus_{k=1}^{d} \tan^{2(d-k)} r \binom{d-1}{k-1} W \circ \chi_k^{(d)}, \tag{52}$$

where $\chi_k^{(d)}$ has been introduced in Eq. (21).

For the purpose of proving degradability, W is a harmless unitary matrix independent on k so we may just ignore it. But $\mathcal{O}_{(k-1)\rightarrow(d-k)}$ is not a completely positive map so degradability remains to be proven.

Theorem 12. *All Grassmann channels $\mathcal{G}_d$ from Eq. (21) are degradable for $r \in [0, \pi/4]$.*

Remark. Note that the degradability of a qubit erasure channel on the whole interval is recovered for $d = 2$. This corresponds to $p \in [0, 1/2]$ in Eq. (23b) as is valid for a 'standard' qubit erasure channel Eq. (24) [18].

Proof. We will prove the theorem by a direct construction of the degrading map. Rewriting Eqs. (21) and (52) we get

$$\gamma_A^{(d)} = \bigoplus_{k=1}^{d} p_k \chi_k^{(d)} \equiv \bigoplus_{k=1}^{d} p_k \mathcal{G}_{d,k}(\psi_{A'}) \tag{53}$$

$$\tilde{\gamma}_C^{(d)} = W \gamma_C^{(d)} W^\dagger = \bigoplus_{k=1}^{d} \tilde{p}_k \chi_k^{(d)}, \tag{54}$$

where $p_k = \cos^{2(d-1)} r \tan^{2(k-1)} r \binom{d-1}{k-1}$ and $\tilde{p}_k = \cos^{2(d-1)} r \tan^{2(d-k)} r \binom{d-1}{k-1}$. Recall that for the purpose of the degrading map construction we may work with $\tilde{\gamma}_C^{(d)}$ instead of $\gamma_C^{(d)}$. We assume the existence of the following degrading map

$$\mathcal{M} : \gamma_A^{(d)} \mapsto q_1 \gamma_A^{(d)} + \sum_{k=2}^d q_k \chi_k^{(d)} = \sum_{k=1}^d \tilde{p}_k \chi_k^{(d)} \equiv \tilde{\gamma}_C^{(d)}. \tag{55}$$

In order for $\mathcal{M}$ to be a completely positive map we have to show that $0 \le q_k$ and $\sum_{k=1}^d q_k = 1$ for $r \in [0, \pi/4]$. For all d we get from Eq. (55) the following set of d linear equations

$$\begin{aligned} q_1 p_1 &= \tilde{p}_1 \\ q_1 p_2 + q_2 &= \tilde{p}_2 \\ &\vdots \\ q_1 p_d + q_d &= \tilde{p}_d. \end{aligned} \tag{56}$$

The set is easily solvable. The first equation gives us $q_1 = \tan^{2(d-1)} r$ which we plug into the remaining equations. We get

$$q_k = \binom{d-1}{k-1} \cos^{2(d-1)} r \left(\tan^{2(d-k)} r - \tan^{2(d+k-2)} r \right) \tag{57}$$

for $k = 2 \ldots d$. Since for $r \in [0, \pi/4]$ the tangent function is monotonously increasing and $0 \le \tan r \le 1$ holds as well we may conclude that for $k = 1 \ldots d$ all q_k are positive. Finally, summing the left and right side of the equation set and using $\sum_{k=1}^d p_k = \sum_{k=1}^d \tilde{p}_k = 1$ we find that $\sum_{k=1}^d q_k = 1$. ∎

We might proceed to the calculation of the quantum capacity of the Grassmann channels. Due to the degradability the quantum capacity formula Eq. (7) reduces to the optimized coherent information Eq. (8). Furthermore, according to Theorem 5 the Grassmann channels are covariant. Therefore, the supremum in Eq. (8) is achieved for a maximally mixed input qudit and the quantum capacity formula reads

$$Q(\mathcal{G}_d) = \frac{1}{d} \cos^{2(d-1)} r \sum_{k=1}^d k \binom{d}{k} \log k \left(\tan^{2(d-k)} r - \tan^{2(k-1)} r \right). \tag{58}$$

The plots for various d can be found in Fig. 1.

V. CLASSICAL CAPACITY

We first present a simple generalization of the characterization theorem derived in [17] (Theorem 1).

Theorem 13 ([17]). *The quantum channel $T : \mathcal{H}(\mathbb{C}^d) \to \mathcal{H}(\mathbb{C}^{d'})$ is of the form*

$$T(\rho) = \frac{I_{d'} - mM(\rho)}{d' - m},$$

where M is a positive, linear and trace-preserving map such that there exists a state ρ_0 where $mM(\rho_0)$ is a projection of rank m if and only if the α-Rényi minimal output entropy $H_{min}^\alpha(T) = \min_\rho H^\alpha(T(\rho))$ is α-independent.

Remark. The (almost trivial) generalization lies in setting $d' \neq d$.

We now show that the blocks from which all the Grassmann channels are composed fulfill the required conditions.

Lemma 14. *Every block of the qudit Grassmann channel $\mathcal{G}_d : \mathcal{H}(\mathbb{C}^d) \rightarrow \mathcal{H}(\mathbb{C}^{2^d - 1})$ has the following form:*

$$\mathcal{G}_{d,k}(\rho) = \frac{I_{d'} - mM_{d,k}(\rho)}{d' - m} \tag{59}$$

where $d' = \binom{d}{k}$ is the dimension of the output space. Moreover, for all input pure states ρ_0, $mM_{d,k}(\rho_0)$ is a projection of rank m

Proof. Consider the k-th block of the qudit Grassmann channel with input of the form $|1\rangle\langle 1|$. Then the output state will have the following form:

$$\phi_A^{(k)} = \frac{1}{\binom{d-1}{k-1}} \sum_{s_i} |1\, s_2 \dots s_d\rangle\langle 1\, s_2 \dots s_d|_A \tag{60}$$

where we are using the convention $\sum_{i=2}^d s_i = k - 1$ and normalized the block. Now suppose we let $m = \binom{d-1}{k}$ and we know the dimension of the output space for the k-th block is equal to $\binom{d}{k}$, then $d' - m = \binom{d}{k} - \binom{d-1}{k} = \binom{d-1}{k-1}$. Therefore,

$$I_{d'} - (d' - m)\mathcal{G}_{d,k}(|1\rangle\langle 1|) = I_{d'} - \sum_{s_i} |1\, s_2 \dots s_d\rangle\langle 1\, s_2 \dots s_d| \tag{61}$$

$$= \sum_{t_i} |0\, t_2 \dots t_d\rangle\langle 0\, t_2 \dots t_d|. \tag{62}$$

with the convention now being $\sum_{i=2}^d t_i = k$. The matrix in Eq. (62) has rank $m = \binom{d-1}{k}$. Thus with an input of the form $\rho_0 = |1\rangle\langle 1|$, we have found a matrix $M_{d,k}(\rho_0)$ that satisfies the condition in Eq. (59).

Now, in order to generalize this result to arbitrary pure inputs, one can use the $SU(d)$ covariance of the Grassmann channel presented in Corollary 6. Assume $R_{d,r}$ to be an r-dimensional unitary representation of $SU(d)$. Then due to covariance the following holds:

$$\mathcal{G}_{d,k}(\rho) = \mathcal{G}_{d,k}(R_{d,d} \circ \rho_0) = R_{d,d'} \circ \mathcal{G}_{d,k}(\rho_0). \tag{63}$$

Thus the following chain of equalities hold:

$$R_{d,d'} \circ \big(mM_{d,k}(\rho_0)\big) = R_{d,d'} \circ \big((I_{d'} - (d' - m)\mathcal{G}_{d,k}(\rho_0)\big) \tag{64a}$$

$$= I_{d'} - (d' - m)R_{d,d'} \circ \mathcal{G}_{d,k}(\rho_0) \tag{64b}$$

$$= I_{d'} - (d' - m)\mathcal{G}_{d,k}(R_{d,d} \circ \rho_0) \tag{64c}$$

$$= mM_{d,k}(R_{d,d} \circ \rho_0). \tag{64d}$$

The last equality follows from defining $mM_{d,k}(R_{d,d} \circ \rho_0)$ in this way, since it has rank m since it is just equal to a rotated state of rank m. Thus every block, after normalization, has the form of Eq. (59). ∎

Corollary 15. Grassmann channels have a direct sum form Eq. (21). Hence from Lemma 3 in [13] it follows that all Grassmann channels have the Holevo capacity additive. From Eqs. (5) and (6) we get

$$C(\mathcal{G}_d) = \sup_{\{p(x), \sigma_{x,A'}\}} \left[H(A)_{\mathcal{G}_d(\sigma_{XA'})} - \sum_x p(x)H(A)_{\mathcal{G}_d(\sigma_{x,A'})} \right]. \tag{65}$$
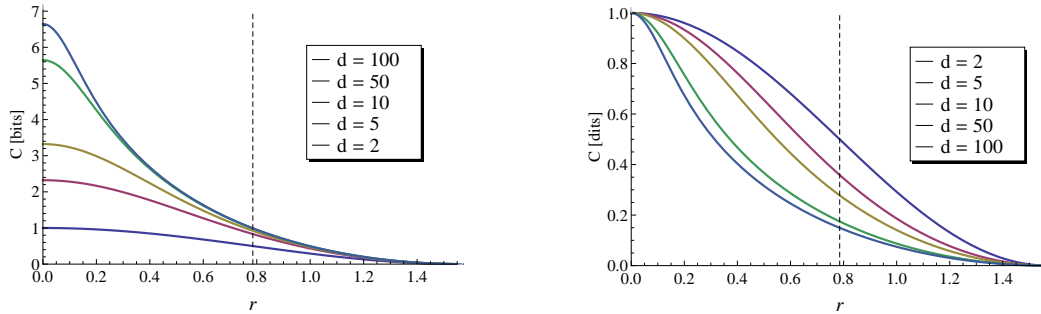
FIG. 2: Classical capacity for Grassmann channels for $d = 2, 5, 10, 50$ and 100. The vertical dashed lines at $r = \pi/4$ label the point where the quantum capacity in Fig. 1 equals zero. In the left plot the capacity is calculated using the base two logarithm and in the right plot using the base d logarithm. The curves follow the order in the legend.

Consequently, exploiting the result from [14, 26] for covariant channels, the classical capacity is given by

$$
\begin{aligned}
C(\mathcal{G}_d) &= H(\{p_k\}) + \sum_{k=1}^{d} p_k \log \binom{d}{k} - H(A)_{\mathcal{G}_d(\rho_0)} \\
&= \log d - \cos^{2(d-1)} r \sum_{k=1}^{d} \tan^{2(k-1)} r \binom{d-1}{k-1} \log k.
\end{aligned}
\tag{66}
$$

VI. PHYSICAL IMPLICATIONS

Bosons, fermions and non-inertial observers

It is often claimed that there is a fundamental difference between the entanglement behavior of maximally entangled states built upon fermions and bosons in a relativistic setting [33]. Namely, provided that a uniformly accelerated observer has one half of an initially maximally entangled state it is found that in the infinite acceleration limit the bosonic entanglement disappears while the fermionic entanglement partially persists. Based on this naïve approach it is concluded that in the infinite limit entangled fermionic states might be useful for various quantum-informational protocols where a certain amount of shared entanglement is usually needed.

However, our capacity results suggest something different at least for the purpose of quantum communication between an inertial and noninertial observer. For the purpose of sending quantum messages the ultimate measure of a channel's capability to transmit the information is its quantum capacity which has nothing to do with any particular entanglement measure. We found that there is no qualitative difference between the behavior of the quantum capacity for the qudit Grassmann channel (fermions) and its bosonic equivalent introduced in Ref. [22] where we study the bosonic version of transformation Eq. (9). This led to the definition of the qudit Unruh channels as the bosonic counterpart of the qudit Grassmann channel. One of the resolved problems is the quantum capacity of the qudit Unruh channels which we can compare with the Grassmann channels. As a result we find that quantum capacities for both channels converge to zero as we approach the infinite acceleration limit. So from the viewpoint of quan-
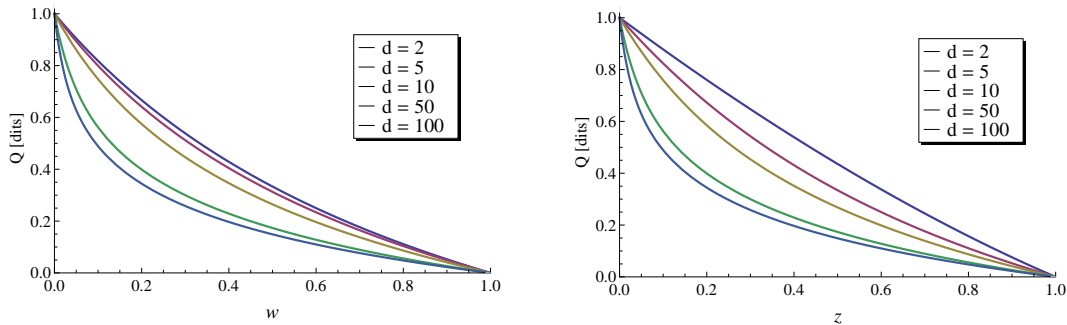
FIG. 3: Quantum capacity for the Grassmann channel (on the left) and the Unruh channel (on the right) taken from [22]. The parameters $w = \tan^2 r$ and $z = \tanh^2 r$ are chosen to compare the capacities for the same proper acceleration. The capacity of the Unruh channel is in general higher but importantly for the infinite acceleration limit ($w = z = 1$) both capacities are equal zero. The curves follow the order in the legend.

tum Shannon theory entangled resources based on bosons or fermions are equally useful. The only difference is in the capacity value for a finite acceleration. To fairly compare the capacities we rewrite Eq. (58) as a function of $w = \tan^2 r$. Using $\cos^2 r = 1/(1 + \tan^2 r)$ we get

$$Q(\mathcal{G}_d) = \frac{1}{d}\left(\frac{1}{1+w}\right)^{d-1}\sum_{k=1}^{d} k\binom{k}{d}\log k\left(w^{d-k} - w^{k-1}\right). \tag{67}$$

This expression can be directly compared to the one we get for the qudit Unruh channel $\mathcal{U}_d$ [22]

$$Q(\mathcal{U}_d) = \frac{1}{d}(1-z)^{d+1}\sum_{k=1}^{\infty} k\binom{d+k-1}{k}\log\frac{d+k-1}{k}z^{k-1}. \tag{68}$$

The Unruh channel outperforms the Grassmann channel for low-dimensional inputs as we can see in Fig. 3. Due to the local similarities between the Rindler and Schwarzschild spacetime (explicitly spelled out, for example, in [34]) the similar conclusion regarding the quantum capacities also holds in the black hole scenario.

Grassmann channels and transpose-depolarizing channels

Let us define the family of qudit transpose-depolarizing channels [16].

Definition 16. *Let $\mathcal{T} : \mathcal{H}(\mathbb{C}^d) \to \mathcal{H}(\mathbb{C}^d)$ be a map defined as*

$$\mathcal{T}(\sigma) = t\bar{\sigma} + (1 - t)\mathbb{1}/d \tag{69}$$

acting on normalized density matrices σ. The bar denotes complex conjugation in a given basis and the map is a quantum channel in the following interval of t:

$$-\frac{1}{d-1} \leq t \leq \frac{1}{d+1}. \tag{70}$$

Remark. The transpose-depolarizing channel for $t = -1/(d-1)$ is known as the d-dimensional Werner-Holevo channel [36].

Lemma 17. *The complementary channels of the $\mathcal{G}_{d,2}$ channels presented in Definition 4 are the d-dimensional Werner-Holevo channels.*

Proof. We take the corresponding part of the isometry output Eq. (19c) for $k = 2$ and rewrite it as

$$
\Phi_{AC}^{(2)} = \cos^{d-1} r \tan r \sum_{j=1}^{d} \sum_{\substack{i=1 \\ i\neq j}}^{d-1} (\pm)_i \beta_i \left| 0 \ldots 1_j \ldots 1_i \ldots 0 \right\rangle_A \left| 0 \ldots 1_j \ldots 0 \right\rangle_C
$$

$$
\equiv \sum_{j=1}^{d} \sum_{\substack{i=1 \\ i\neq j}}^{d-1} (\pm)_i \beta_i \left| [ji] \right\rangle_A \left| j \right\rangle_C .
\tag{71}
$$

Note that $\left| [ji] \right\rangle = - \left| [ij] \right\rangle$ so we can further write (leaving out the irrelevant trigonometric functions)

$$
\tilde{\Phi}_{AC}^{(2)} = \sum_{j=1}^{d} \sum_{i>j}^{d-1} \left| [ji] \right\rangle_A \left(-\beta_i \left| j \right\rangle + \beta_j \left| i \right\rangle \right)_C .
\tag{72}
$$

Considering an input pure state $\left| \psi \right\rangle_{A'} = \sum_{l=1}^{d} \beta_l \left| l \right\rangle_{A'}$ the isometry of $\mathcal{G}_{d,2}$ reads

$$
V_{\mathcal{G}_{d,2}} = \sum_{j=1}^{d} \sum_{i>j}^{d-1} \left(-\delta_{il} \left| [ji] \right\rangle_A \left| j \right\rangle\!\langle l |_{CA'} + \delta_{jl} \left| [ji] \right\rangle_A \left| i \right\rangle\!\langle l |_{CA'} \right).
\tag{73}
$$

Tracing over A followed by normalizing by $\sqrt{1/\binom{d-1}{k-1}} = \sqrt{1/(d-1)}$ and changing the overall sign leads to $\binom{d}{2}$ Kraus operators of the form

$$
\mathcal{K}_{ij} = \sqrt{\frac{1}{d-1}} \left(\left| j \right\rangle\!\langle i | - \left| i \right\rangle\!\langle j | \right),
\tag{74}
$$

where $1 \leq i < j \leq d$. These are well known as the Kraus operators for the d-dimensional Werner-Holevo channels [36]. ∎

This result brings us two interesting points. As mentioned earlier, in [10] we study the capacity region of the bosonic version of the transformation from Eq. (9) known as the qudit Unruh channel [22]. One of the results is the characterization of the complementary channels of the Unruh channel. Their structure is also block-diagonal as in the present case and the first nontrivial complementary block for each d is the transpose-depolarizing channel Eq. (69) for $t = 1/(d+1)$. This is a peculiar observation and raises a number of questions. First of all, why for fermions we get the transpose-depolarizing channel from one end of the allowed interval Eq. (70) and for bosons from the other end? Also, does some sort of intermediate statistics interpolating between bosons and fermions correspond to the whole interval? One of the obvious possibilities are anyons whose appearance is not limited just to the two-dimensional world [37].

The identification of the d-dimensional Werner-Holevo channel also implies that the Grassmann channels do not belong to the class of Hadamard channels [10, 11] – the channels whose complementary channel is entanglement-breaking. The reason is that the Werner-Holevo channels are known not to be entanglement-breaking. The transpose-depolarizing channels are entanglement-breaking for $-1/(d^2-1) \leq t$ [16]. Henceforth, at least one of the complementary blocks of all Grassmann channels has negative partial transpose.

VII.   CONCLUSIONS

We have introduced a new class of quantum channels to the group with computable classical and quantum capacities, the Grassmann channels. Such channels are rare in quantum Shannon theory since the calculation of the classical and quantum capacities requires an optimization over an infinite number of channel uses. The Grassmann channels' isometric extension is physically well motivated and stems from the Bogoliubov transformation which preserve the canonical anticommutation relations corresponding to the Fermi-Dirac statistics.

In order to determine the quantum capacity of the Grassmann channels, we have shown that these channels are degradable and have explicitly calculated the degrading map. Combining this with the result that the Grassmann channels are covariant we were able to give a closed form for the quantum capacity. A different technique was used in order to calculate the classical capacity of the set of Grassmann channels. We showed that each block, in the block diagonal form of the matrix, had a particular form [17] which enabled the calculation of their minimum output entropy. Exploiting this result allowed us to calculate the classical capacity of the Grassmann channels.

To appreciate the capacity results from the physical point of view we compare the Grassmann channels with the Unruh channels studied elsewhere [11, 22]. They share a close analog to the Grassmann channels in the sense that their isometric extension appears formally identical. The difference is that for the Unruh channels the isometry is built upon the operators obeying the canonical commutation relations (relevant to the Bose-Einstein statistics) inducing a completely different class of channels. In particular, the Unruh channels belong to the set of Hadamard channels - the channels whose complementary channels are entanglement breaking. The set of Grassmann channels do not fall in this class. This is the first example to our knowledge of a set of channels that do not belong to the Hadamard class that have a computable and at the same time nonzero classical and quantum capacity.

The main physical consequences also come from the comparison between the fermionic and bosonic case. One of the discussed physical motivations for investigating this type of channel is that both the fermionic and bosonic Bogoliubov transformation occurs in the study of particle production in uniformly accelerated frames. Fermionic entanglement between two parties who originally share a maximally entangled state exists to an extent even in the limit of infinite acceleration of one of the participants. Contrary to the fermionic case, bosonic entanglement vanishes in the infinite acceleration limit. Based on this observation it is believed that there is a difference between these types of resources. The result of our work suggests that at least for quantum communication purposes there is no difference whatsoever. The Unruh channel does demonstrate a greater quantum capacity than the Grassmann channel when the acceleration parameter is small, however in the infinite acceleration limit, both quantum capacities tend to zero.

There exists another connection between the Unruh and Grassmann channels. Both channels are direct sums of other quantum channels and so are their complementary channels. The first non-trivial block of the complementary channel for a given dimension d belongs to the family of qudit transpose-depolarizing channels which is a single-parameter family of quantum channels. Interestingly, this holds both for the Grassmann channels and Unruh channels. The difference is that the complementary block of the Unruh channel corresponds to the transpose-depolarizing channel with the parameter at the upper limit of the allowed parameter interval, while the first nontrivial complementary block of the Grassmann channel corresponds to the lower limit of the allowed interval (the Werner-Holevo channel). This immediately raises the question: Why does the fermionic case occupy one end of the interval while the bosonic case occupy the other? Perhaps and even more interestingly, could there be intermediate statistics model (like anyons, for example) that would explain intermediate values of the interval? Another direction in which future research could be done is to consider the case of coupled fermions with a

more active role of the spin variable and one can ask how the additional spin variable alters the Grassmann channels and whether the classical and quantum capacity is still calculable. Finally, the implications of the results here obtained to quantum information protocols inspired by Cooper pairs in solid state or atomic physics scenarios deserve an independent detailed study.

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Appendix A: Geometric picture of the $\mathsf{sl}(d,\mathbb{C})$ Lie algebra representations

In the $\mathsf{sl}(d,\mathbb{C})$ case there may be several mutually commuting operators. A maximal linearly-independent commuting set of operators of a (semi-simple) Lie algebra is called a Cartan subalgebra. Once a Cartan subalgebra has been chosen we can use the common eigenvectors to label the basis vectors of an irreducible representation.

Definition 18. *An r-tuple $\alpha = (\alpha_1, \ldots, \alpha_r)$ of complex numbers is called a root if: (i) not all the α_i are zero, (ii) there is an element E_α of $\mathsf{sl}(d,\mathbb{C})$ such that*

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{A1}$$

The set $\{H_i, E_\alpha\}$ is called the Cartan-Weyl basis [35]. The Cartan subalgebra of $\mathsf{sl}(d,\mathbb{C})$ has dimension $r = d - 1$; we say that the rank of the Lie algebra is r. We write $\mathbf{H} = (H_1, \ldots, H_r)$ for the Cartan subalgebra generated by the elements $\{H_1, \ldots, H_r\}$ of the Lie algebra; these elements are assumed to be independent. Among all the roots there is a class of special roots called simple roots.

Definition 19. *A root is called a simple root if it cannot be written as a linear combination of other positive roots.*

Definition 20. *If ρ (where $\rho : \mathsf{sl}(d,\mathbb{C}) \to GL(V)$ for some V) is a representation of $\mathsf{sl}(d,\mathbb{C})$ then a r-tuple $\mu = (\mu_1, \ldots, \mu_r)$ of complex numbers is a weight for ρ if there is a nonzero vector $\psi \in V$ such that ψ is an eigenvector of each H_i with eigenvalue μ_i.*

If μ is a weight and ψ a weight vector for ρ and α is a root then

$$\rho(H_i)\rho(E)\psi = (\mu_i + \alpha_i)\rho(E)\psi. \tag{A2}$$

In short, E changes all the eigenvalues of the Cartan operators and it creates a new weight vector (or kills the weight vector). The root is a vector in the weight space that points in the direction in which the weights are changing. The E operators are called shift or raising and lowering operators. Roughly speaking, the positive roots correspond to raising operators while the negative roots to lowering operators. We classify the irreducible representations by the *highest possible value of the weight.* For general semi-simple Lie algebras we do exactly the same thing once we have a suitable order on the weights in order to define the right notion of highest weight.

A special example of the Cartan-Weyl basis is the Chevalley-Serre basis [35]. Two aspects make this basis special. (i) The step operators are associated to simple roots and (ii) the normalization is chosen such that the roots are integers. Unless explicitly stated we work in this basis due to its accessible geometric interpretation.

To give the operators a geometric interpretation we will work with a specific matrix representation of the $\mathsf{sl}(d,\mathbb{C})$ algebra. Let us define E_{ij}, where $1 \le i \ne j \le d$, as the matrix having one where the i-th row and the j-th column intersect and the rest are zeros. Furthermore we define a diagonal matrix H_i in which the i-th diagonal entry is 1, the $(i+1)$-th diagonal entry is -1 and the rest are zeros. If we assume $j = i+1$ the following set $\{H_i, E_{ij}, E_{ij}^\dagger\}$ forms the Chevalley-Serre basis for $\mathsf{sl}(d,\mathbb{C})$. More explicitly, for $d = 2$ we get

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A3a}$$

$$E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{A3b}$$

$$E_{12}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{A3c}$$

Therefore in this basis we have

$$[H_i, E_{ij}] = 2E_{ij} \tag{A4a}$$

$$[H_i, E_{ij}^\dagger] = -2E_{ij}^\dagger. \tag{A4b}$$

The simple roots are elements of a vector space dual to the one spanned by elements of the Cartan subalgebra.

This structure opens the door to an insightful geometric picture in terms of the so-called root space diagram. The dual space will be called the space of roots. For the $\mathsf{sl}(d,\mathbb{C})$ Lie algebra the space is $(d-1)$-dimensional and the simple root vectors defined as an r-tuple of simple roots form a non-orthogonal basis. Let us call the basis spanning the space of roots the root basis. We are aware of the overuse of the expressions root and root vectors. The terminology is not stabilized and differ in various textbooks. Also, the root space diagrams are sometimes called weight diagrams. It can be shown that each consecutive simple root vectors subtend the angle $2\pi/3$. We rewrite Eq. (A1) as

$$[H_i, E_{ij}] = (\mu_i^{(i)} - \mu_i^{(j)})E_{ij}, \tag{A5}$$

where $\mu_i^{(j)}$ are called fundamental weights. The word fundamental reflects the fact that we are dealing only with simple roots. We explicitly rewrite Eq. (A2) as the spectral decomposition

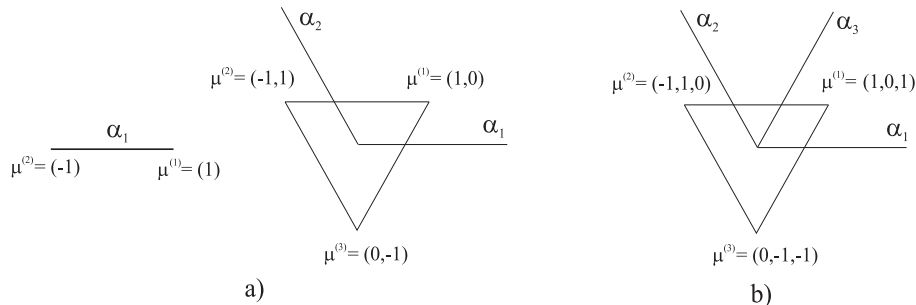$$H_i = \sum_{j=1}^{d} \mu_i^{(j)} |\psi_j\rangle\langle\psi_j|,$$

FIG. 4: (a) We illustrate the spaces of roots for $\mathsf{sl}(2,\mathbb{C})$ and $\mathsf{sl}(3,\mathbb{C})$. The basis vectors are simple roots and are indicated by α_i. For the weights (vertices) we explicitly write down their coordinates in this basis. We can read off the generators of the Cartan subalgebra from the weights. (b) It is sometimes helpful to introduce an overcomplete basis. The position of α_3 reflects the relation $H_3 = H_1 + H_2$ for H_3 in Eq. (A6) and it is not a simple root.

where j labels the j-th component of a d-component spinor $|\psi\rangle$. The important role played by the (fundamental) weights is that they are coordinates of the eigenvectors in the space of roots. Fig. 4a illustrates the situation for $d = 2$ and $d = 3$. Hence for each fundamental representation of $\mathsf{sl}(d,\mathbb{C})$ there is d points each representing an eigenvector ψ_j.

What about the role played by the shift operators? They have a precise geometric interpretation as well. All points of the $\mathsf{sl}(d,\mathbb{C})$ fundamental representations are interconnected. The operator responsible for a transition from site $|\psi_i\rangle$ to $|\psi_j\rangle$ is the operator E_{ij} or $E_{ij}^\dagger$ for the opposite direction. It also follows from Eqs. (A4) that each segment connecting two neighboring spinors has length two.

The fundamental representations of $\mathsf{sl}(d,\mathbb{C})$ algebra contain $r = d - 1$ independent $\mathsf{sl}(2,\mathbb{C})$ subalgebras each satisfying Eqs. (A3). However, since the root space diagram is a complete graph there are in total $\binom{d}{2}$ linearly dependent $\mathsf{sl}(2,\mathbb{C})$ subalgebras corresponding to the number of edges. The diagonal generators of the 'additional' $\mathsf{sl}(2,\mathbb{C})$ subalgebras are constructed similarly to the H_i's above. The only difference is that 1 and -1 on the diagonal are separated by one or more zeros. As an example $(d = 3)$, the remaining diagonal generator is

$$H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{A6}$$

Therefore, the rank of the weight vectors equals three and it correspond to introducing an overcomplete root basis, see Fig. 4b. Note that H_3 does not correspond to a simple root. Indeed, the axis α_3 in Fig. 4b can be obtained by a linear combination of α_1 and α_2 which are both positive root vectors.

Completely antisymmetric representations of the $\mathsf{sl}(d,\mathbb{C})$ algebra

For the purpose of this article we are interested in particular higher-dimensional representations of $\mathsf{sl}(d,\mathbb{C})$ - the completely antisymmetric representations. The lowest-dimensional antisymmetric representation is formed as the dual of the fundamental representation

$$\psi^{j_1} = \varepsilon^{j_1 \ldots j_d} \psi_{j_2} \ldots \psi_{j_d}, \tag{A7}$$

where $\varepsilon^{j_1\ldots j_d}$ is a completely antisymmetric tensor and $1 \leq j_1,\ldots,j_d \leq d$. The dual representation (of the fundamental representation only!) coincides with its complex conjugate representation. We get the dual representation from the fundamental representation by $G_{\mathsf{sl}(d,\mathbb{C})} \mapsto -\bar{G}_{\mathsf{sl}(d,\mathbb{C})}$ where $G_{\mathsf{sl}(d,\mathbb{C})}$ are all generators of $\mathsf{sl}(d,\mathbb{C})$. The eigenvalues remain the same and the weight vectors just change the sign. Hence, the root space diagrams are the same and they are just, vaguely speaking, pointing in the opposite direction. The interpretation of the edges and points follows the fundamental case. The difference lies in the fact that the antisymmetric representations of $\mathsf{sl}(d,\mathbb{C})$ are carried by d-component antisymmetrized spinors.

Higher-dimensional completely antisymmetric representations of the $\mathsf{sl}(d,\mathbb{C})$ algebra are formed in an intuitive way. First of all, the dimension of the spaces of roots remains the same as well as the number of algebra generators. The generators clearly satisfy the same commutation relations since it is just a different representation of the same $\mathsf{sl}(d,\mathbb{C})$ algebra. The root diagram 'is grown' in the direction of roots but this process cannot go on forever. The spinors carrying the higher-dimensional representation are completely antisymmetrized and so there are only $\binom{d}{k}$ states in the k-th completely antisymmetric representation of $\mathsf{sl}(d,\mathbb{C})$ where $1 \leq k \leq d$ (formally, we should also include the trivial representation, $k = 0$). Fig. 5 illustrates
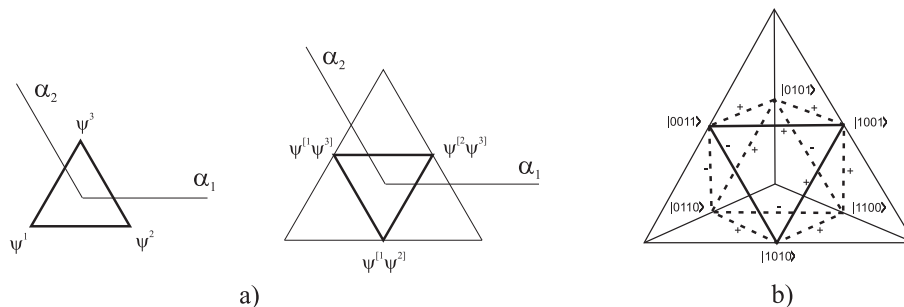

a)   b)

FIG. 5: (a) The left plot is the dual representation to the fundamental representation of $\mathsf{sl}(3,\mathbb{C})$ where ψ^i are completely antisymmetric spinors from Eq. (A7). The middle plot is the second-lowest ($k = 2$) completely antisymmetric representation of $\mathsf{sl}(3,\mathbb{C})$. Due to the antisymmetrization procedure (indicated by the square brackets) the vertices do not correspond to any state. (b) The second lowest ($k = 2$) completely antisymmetric representation of $\mathsf{sl}(4,\mathbb{C})$. The antisymmetrization procedure leaves us with a six-dimensional space (the inner octahedron) with the spanning basis indicated. The spinors are written in the Fock basis following Definition 1. The sign attached to an edge indicates the sign induced by the (fermionic) $\mathsf{sl}(2,\mathbb{C})$ shift operator when applied on spinors the edge connects.

the growth of these representations for $d = 3$ and $d = 4$. The connection to the fermionic states brought in Definition 1 is straightforward: The state $|F\rangle = |n_1 \ldots n_d\rangle$ for which $\sum_{i=1}^{d} n_i = k$ holds is an antisymmetric spinor carrying the k-th completely antisymmetric representation of $\mathsf{sl}(d,\mathbb{C})$.

We find the matrix form of the generators of all higher-dimensional completely antisymmetric $\mathsf{sl}(d,\mathbb{C})$ algebra representations. Because of the complete antisymmetry condition there are only segments connecting two neighboring points. Therefore only the lowest-dimensional representation of the $\mathsf{sl}(2,\mathbb{C})$ subalgebra appears in the construction. However, because for a given edge there might be more segments parallel to it, the $\mathsf{sl}(d,\mathbb{C})$ subalgebra generators are formed by a direct sum of the $\mathsf{sl}(2,\mathbb{C})$ subalgebras. The reason for a direct sum is that they, by construction, act on mutually orthogonal subspaces. Finally, for every d there is only $r = d - 1$ linearly independent directions (or, said otherwise, only $r = d - 1$ independent sets of parallel lines) so the number of generators equals the number of independent $\mathsf{sl}(2,\mathbb{C})$ subalgebras and they manifestly satisfy the commutation relations for $\mathsf{sl}(d,\mathbb{C})$. Note that it does NOT mean that the completely antisymmetric representations of $\mathsf{sl}(d,\mathbb{C})$ are direct sum representations.

As a matter of fact, they are irreducible. The direct sum subalgebras we have created do not themselves span mutually orthogonal subspaces. For illustration see Fig. 5b where any pair of parallel segments 'share' spinors with some other pair of parallel segments.

The only ambiguity lies in the sign of the shift operators. We can see from Eq. (A3) that switching their sign does not spoil the commutation relations. Let us illustrate the ambiguity on an example from Fig. 5b. The transition from state $|0011\rangle$ to $|0101\rangle$ is provided by $a_2^\dagger a_3 |0011\rangle = |0101\rangle$ meanwhile to get from $|0011\rangle$ to $|0110\rangle$ we acquire a minus sign $a_2^\dagger a_4 |0011\rangle = - |0110\rangle$. Not accidentally, the operators $a_i^\dagger a_j, a_j^\dagger a_i$ are the fermionic representation of the shift operators of the $\mathsf{sl}(2, \mathbb{C})$ algebra [35]. They play an important role in the proof of Theorem 5 where the fermionic representation is properly introduced.

So far we talked about a direct sum of several $\mathsf{sl}(2, \mathbb{C})$ subalgebras but for the sake of proof of Theorem 5 we need to specify how many summands there actually is. This transforms into the question how many different $\mathsf{sl}(2, \mathbb{C})$ subalgebras corresponding to a chosen direction exist. Let the direction be chosen by the step operators $\{a_2^\dagger a_1, a_1^\dagger a_2\}$. Then for a given d and k there is $\binom{d-2}{k-1}$ parallel segments in this particular direction for the k-th lowest completely antisymmetric representation of $\mathsf{sl}(d, \mathbb{C})$. We get this number by a combinatorial argument: We have a spinor with d positions where the first two slots are occupied. For the lowest-dimensional representation ($k = 1$) the rest is occupied by zeros and therefore for the k-th lowest dimensional representation there is ($k - 1$) ones to occupy the remaining ($d - 2$) free spaces. By a simple permutation argument we can see that this holds for any of the ($d - 1$) linearly independent directions. For the example in Fig. 5b we indeed get two parallel segments for each of the three independent directions.