Finite Casimir Energies of Intersecting Objects

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Finite Casimir energies for bosonic fields interacting locally with N objects that do not all have a common intersection are defined by geometrical subtractions. The perturbative expansion for $\beta \sim 0$ of the corresponding subtracted spectral function $\tilde{\phi}^{(N)}(\beta)$ vanishes to all orders. These subtracted spectral functions and their associated Casimir energies can be efficiently computed numerically without regularization or implicit knowledge of the spectrum. They are analytic in the parameters describing the shape and position of the individual objects and remain finite when some, but not all, objects overlap. In the case of $(d+1)$, $(d-1)$-dimensional intersecting hyper-planes embedded in $\mathcal{R}^d$, the $(d+1)$-body Casimir energy gives the work required to move the last hyperplane into position from infinity. With Dirichlet boundary conditions on objects embedded in Euclidean space, the subtracted spectral function is the probability that a standard Brownian process touches all N objects before returning to the starting point after time β. The many-body Casimir energy in this scalar case is positive for an odd, and negative for an even, number of objects. An explicitly finite multiple scattering representation of the general 3-body Casimir energy is given.

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The Casimir interaction energy between two disjoint bodies is finite and may be estimated[1–5]. In principle, it can be numerically computed to arbitrary precision[6–8]. For disjoint bodies, the multiple scattering representation of the interaction energy[9–12] solves most problems encountered in technological applications[6, 13]. We here develop an extension of this formalism that defines finite Casimir energies of more than two bodies that are not necessarily all disjoint. The analysis improves our understanding of the physical origin and interpretation of (finite) parts of zero-point energies. It may also provide a conceptual foundation for exploring gravitational effects arising from vacuum energies[14] and give a systematic approach to Casimir self-stresses for arbitrarily shaped bodies.

Generalizing the case of two disjoint bodies, we here extract a finite part of the vacuum energy for a system consisting of N objects whose common intersection vanishes. For $N > 2$ some of the objects may overlap with others. Divergences that might arise due to intersections of fewer objects are geometrically subtracted and this N-body part of the vacuum energy is explicitly finite. For clarity of presentation we assume that the objects are all embedded in a, large but finite, connected Euclidean space $\mathfrak{D}_\emptyset$ of Euclidean dimension d. The thermodynamic limit $\mathfrak{D}_\emptyset \to \mathcal{R}^d$ may be taken at the end. Following the geometrical subtraction procedure of ref.[15], we relate the finite N-body Casimir energy $\tilde{\mathcal{E}}^{(N)}$ to a subtracted N-body spectral function $\tilde{\phi}^{(N)}(\beta)$,

$$\tilde{\mathcal{E}}^{(N)} = -\frac{\hbar c}{\sqrt{8\pi}} \int_0^\infty \tilde{\phi}^{(N)}(\beta) \frac{d\beta}{\beta^{3/2}} \ . \tag{1}$$

This subtracted N-body spectral function $\tilde{\phi}^{(N)}(\beta)$ is constructed as follows. Let $\mathfrak{D}_s$ represent the domain $\mathfrak{D}_\emptyset$ with objects $\{O_j; j \in s\}$ embedded, $\mathfrak{D}_{1\ldots N}$ being the finite domain $\mathfrak{D}_\emptyset$ with all N objects included and denote with $\mathfrak{P}(s)$ the power set of the elements of a set s of finite cardinality $|s| \leq N$. To simplify some formulae we write $\mathfrak{P}_N$ for $\mathfrak{P}(\{1\ldots N\})$. Let $\phi_s(\beta)$ further denote the spectral function, or trace of the heat kernel $\mathfrak{K}_{\mathfrak{D}_s}$, for the domain $\mathfrak{D}_s$,

$$\phi_s(\beta) = \mathrm{Tr}\mathfrak{K}_{\mathfrak{D}_s}(\beta) = \sum_{n\in\mathbb{N}} e^{-\beta\lambda_n(\mathfrak{D}_s)/2}. \tag{2}$$

Here $\{\lambda_n(\mathfrak{D}_s) > 0, n \in \mathbb{N}$ is the spectrum of a bosonic field that vanishes on the boundary of $\mathfrak{D}_\emptyset$ and whose interactions with the objects in $\mathfrak{D}_s$ are *local*. We assume the action of the field with the objects may be described by positive local potentials or take the form of (compatible) local boundary conditions.

The subtracted spectral function $\tilde{\phi}^{(N)}(\beta)$ of Eq.(1) is the alternating sum of spectral functions $\phi_s(\beta)$ for the individual domains D_s,

$$\tilde{\phi}^{(N)}(\beta) := \sum_{s\in\mathfrak{P}_N} (-1)^{N-|s|} \phi_s(\beta) \ . \tag{3}$$

Fig. 1 gives a pictorial description of Eq.(3) for four line segments as objects in a bounded 2-dimensional Euclidean space.

To facilitate proving that the β-integral in Eq.(1) is finite, we demand that the individual heat kernels are uniformly bounded by the free heat kernel of $\mathcal{R}^d$,

$$0 < \mathfrak{K}_{\mathfrak{D}_s}(\mathbf{x},\mathbf{y};\beta) \leq K(2\pi\beta)^{-d/2} e^{-(\mathbf{x}-\mathbf{y})^2/(2\beta)} \ , \tag{4}$$

for some finite $K > 0$. For physical local interactions with positive potentials this is the case. The bound also appears to hold when objects are represented by local boundary conditions. It in fact is sufficient to demand that for any *finite* separation $(\mathbf{x}-\mathbf{y})^2 > \delta^2 > 0$ the correlation functions vanish faster than any power of β as $\beta \to 0$. One therefore may be able to relax the uniform bound of Eq.(4) considerably.

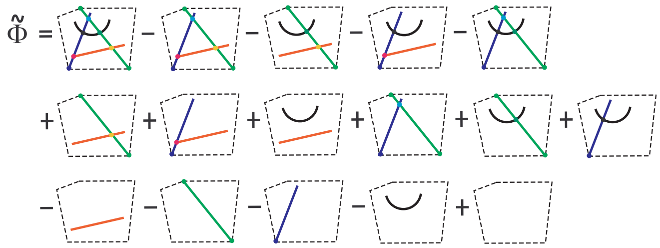

FIG. 1: (color online) The subtracted spectral function $\tilde{\phi}^{(N)}(\beta)$ defined in Eq.(3) for a bounded two-dimensional domain $\mathfrak{D}_\emptyset$ with four intersecting line segments as objects. Each pictograph represents the spectral function of the corresponding domain with the indicated sign. Various local features that contribute to the asymptotic expansion of each spectral function at high temperatures (small β) have been highlighted: lines of different color correspond to possibly different, but compatible, boundary conditions or local potentials. Since the intersections of line segments generally differ, each vertex is shown in a different color. Note that the contribution to the asymptotic expansion from any particular local feature vanishes: the total signed number of times any particular line segment contributes is zero, as is the total signed number of times any particular vertex occurs.

We assume that the spectrum is discrete and positive semi-definite. $\phi_s(\beta)$ may be interpreted as a bosonic single particle partition function at inverse temperature β and a positive spectrum is equivalent to the absence of tachyons in a causal local theory. The spectral functions $\phi_s(\beta > 0)$ of Eq.(3) in this case are positive and monotonically decreasing, approaching at most a finite positive constant for $\beta \sim \infty$. Although we are mainly concerned with a scalar bosonic system, the following also holds for electro-magnetic fields in the absence of free charges.

In local field theories, the asymptotic expansion of $\phi_s(\beta)$ for small β has the general form[16–19],

$$\phi_s(\beta \sim 0) \sim \sum_{\nu=-D}^{\infty} (2\pi\beta)^{\nu/2} A_s^{(\nu)} + \mathcal{O}(e^{-\ell^2_{\rm min}/(2\beta)}) \ , \quad (5)$$

where the Hadamard-Minakshisundaram-DeWitt-Seeley coefficients $A_s^{(\nu)}$ for the domain $\mathfrak{D}_s$ have length-dimension $(-\nu)$. Note that if Eq.(4) holds, exponentially suppressed terms are associated with the presence of classical periodic paths of finite length ℓ_{min}. We decompose the heat kernel coefficients $A_s^{(\nu)}$ further into parts arising from local features of the individual objects and their overlaps,

$$A_s^{(\nu)} = \sum_{\tau \in \mathfrak{P}(s)} a_\tau^{(\nu)} \ , \quad (6)$$

where the sum extends over all ($|s|!$) sets in the power set $\mathfrak{P}(s)$ of the set s. Eq.(6) recursively defines local heat kernel coefficients $a_\tau^{(\nu)}$: the $a_\emptyset^{(\nu)}$ are the heat kernel coefficients associated with the Euclidean domain $\mathfrak{D}_\emptyset$; the $a_{\{j\}}^{(\nu)}$ give their *change* when object j is inserted; $a_{\{jk\}}^{(\nu)}$ accounting for further changes in the asymptotic heat kernel coefficients due to local overlaps of objects j and k. Note that $a_{\{jk\}}^{(\nu)} = 0$ for disjoint objects j and k, if we assume that asymptotic correlations over any finite distance $\delta > 0$ vanish faster than any power in β as implied by Eq.(4). In this case, the power series in the asymptotic high temperature behavior of Eq.(5) arises from correlations over arbitrary short distances only and these do not change if objects are disjoint.

The argument may be extended to show that for local interactions the correction,

$$a_{\{1\ldots N\}}^{(\nu)} = 0, \text{ if } O_1 \cap \cdots \cap O_N = \emptyset \ . \quad (7)$$

It then is a combinatoric problem to show that the contribution of any non-zero $a_\tau^{(\nu)}$ to the asymptotic expansion of $\tilde{\phi}^{(N)}(\beta)$ in Eq.(3) vanishes. Because the other $|s|-|\tau|$ objects may be picked from the remaining $N - |\tau|$ in any order, the number of times the set τ occurs as a subset of the sets in $\mathfrak{P}_N$ with cardinality $|s| \geq |\tau|$ is $\frac{(N-|\tau|)!}{(N-|s|)!(|s|-|\tau|)!} = \binom{N-|\tau|}{N-|s|}$. For $N > |\tau|$ the contribution to the asymptotic expansion of $\tilde{\phi}^{(N)}(\beta)$ in Eq.(3) proportional to $a_\tau^{(\nu)}$ therefore is,

$$(2\pi\beta)^{\nu/2} a_\tau^{(\nu)} \sum_{|s|=|\tau|}^{N} (-1)^{N-|s|} \binom{N-|\tau|}{N-|s|} = 0. \quad (8)$$

When N objects have no common intersection, the asymptotic expansion of $\tilde{\phi}^{(N)}(\beta)$ thus is of the form,

$$\tilde{\phi}^{(N)}(\beta \sim 0) \sim \mathcal{O}(e^{-\ell^2/(2\beta)}) \ , \quad (9)$$

and vanishes faster than any power of β. Together with the fact that the spectral functions $\phi_s(\beta)$ decay monotonically and remain bounded for large β, the asymptotic behavior of Eq.(9) implies that the Casimir energy given by the integral in Eq.(1) is *finite*.

The geometric subtraction procedure allows one to formally interpret $\tilde{\mathcal{E}}^{(N)}$ as the alternating sum of vacuum energies $\mathcal{E}_s$ associated with the domains $\mathfrak{D}_s$,

$$\tilde{\mathcal{E}}^{(N)} = \sum_{s \in \mathfrak{P}_N} (-1)^{N-|s|} \mathcal{E}_s \ . \quad (10)$$

The sum on the right-hand side of Eq.(10) requires some kind of regularization to be meaningful, but as long as this procedure is generic and does not depend on the specific domain $\mathfrak{D}_s$ (for instance a lower bound for the proper time β), the previous considerations show that $\tilde{\mathcal{E}}^{(N)}$ remains well defined when the regularization is removed. The absence of a power series in the asymptotic expansion of $\tilde{\phi}^{(N)}(\beta \sim 0)$ also explains why a semi-classical approach based on classical periodic orbits tends to approximate Casimir energies fairly well[1–3, 23–25], because this approximation reproduces the leading exponentially suppressed terms of the asymptotic expansion.

The subtraction procedure we have outlined becomes particularly transparent for a massless scalar field in a bounded Euclidean space $\mathfrak{D}_\emptyset$ with objects represented by Dirichlet boundary conditions on their surfaces. The Feynman-Kac theorem[20, 21] in this case is the statement that,

$$\phi_s(\beta) = \int_{\mathfrak{D}_\emptyset} \frac{d\mathbf{x}}{(2\pi\beta)^{d/2}} \mathcal{P}_{\mathfrak{D}_s}[\ell_\beta(\mathbf{x})] \ , \tag{11}$$

where $\mathcal{P}_{\mathfrak{D}_s}[\ell_\beta(\mathbf{x})]$ denotes the probability for a standard Brownian bridge[29], $\ell_\beta(\mathbf{x})$, that starts at $\mathbf{x}$ and returns to $\mathbf{x}$ after time β, to *not* encounter any Dirichlet boundary in $\mathfrak{D}_s$ and not exit $\mathfrak{D}_\emptyset$.

A particular loop $\ell_\beta^{(\tau)}(\mathbf{x})$ that remains within $\mathfrak{D}_\emptyset$ and encounters all objects of $\tau \subset \{1 \ldots N\}$ but no others, contributes equally to all $\phi_s(\beta)$ with $s \cap \tau = \emptyset$. For $N > |\tau|$ the contribution of such a loop to $\tilde{\phi}^{(N)}(\beta)$ is proportional to,

$$\sum_{\substack{s\in\mathfrak{P}_N \\ s\cap\tau=\emptyset}} (-1)^{N-|s|} = \sum_{s=0}^{N-|\tau|} (-1)^{N-s} \binom{N-|\tau|}{s} = 0 \ . \tag{12}$$

Only loops that touch all N objects contribute to the alternating sum in Eq.(3) and we have that,

$$\tilde{\phi}^{(N)}(\beta) = (-1)^N \int_{\mathfrak{D}_\emptyset} \frac{d\mathbf{x}}{(2\pi\beta)^{d/2}} \tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta] \ , \tag{13}$$

where $\tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta]$ is the probability that a standard Brownian bridge that starts at $\mathbf{x}$ and returns to $\mathbf{x}$ after time β encounters *all* N objects and does not exit $\mathfrak{D}_\emptyset$. The $(-1)^N$ factor determining the sign of $\tilde{\phi}^{(N)}(\beta)$ arises because such loops contribute to $\phi_\emptyset(\beta)$ only. For a scalar field satisfying Dirichlet boundary conditions on N objects without common intersection we thus have that,

$$(-1)^N \tilde{\mathcal{E}}^{(N)} < 0. \tag{14}$$

For a Dirichlet scalar the sign of $\tilde{\mathcal{E}}^{(N)}$ depends only on the number of objects and not on their shape or position. The construction by geometric subtraction clearly exhibits the finite part of the vacuum energy that is being computed (see Eq.(10)). It is important to correctly interpret this energy. The finite N-body Casimir energy defined here is the N-body *correction* to the vacuum energy that remains when *all* the M-body vacuum energies for $0 \leq M < N$ have been accounted for. The latter may themselves be finite, but very often are not and the sign of $\tilde{\mathcal{E}}^{(N)}$ given by Eq.(14) is that of the N-body correction only. It in general does not coincide with the sign of the overall work required for assembly of all N objects.

Eq.(13) implies that for a Dirichlet scalar $\tilde{\phi}^{(N)}(\beta)$ is the probability of a random walk to fulfill certain geometric conditions. Since they have to touch N objects without common intersection, the Brownian bridges that contribute in Eq.(13) all have *finite* length. The probability $\tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta]$ thus is bounded below by the shortest closed classical path of length $\ell_{\rm min}$ that achieves this,

$$0 \leq \tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta] \leq e^{-\ell^2_{min}/(2\beta)} \ . \tag{15}$$

For a domain $\mathfrak{D}_\emptyset$ of finite volume, the bound of Eq.(15) implies that the asymptotic power series in β of $\tilde{\phi}(\beta \sim 0)$ vanishes to all orders, as we have already argued in a more general setting.

Consider the example of a scalar field in $\mathcal{R}^d$ satisfying Dirichlet boundary conditions on $d+1$, intersecting $d-1$-dimensional hyper-planes. In this case $\tilde{\mathcal{E}}^{(d+1)}$ indeed is the work required to adiabatically move the last hyperplane into position: $\tilde{\mathcal{E}}^{(d+1)}$ vanishes as the volume enclosed by the hyper-planes becomes infinite (if none of them are parallel) and depends continuously on their position. These are consequences of the probability of Brownian bridges to cross all of them. The $d+1$ $d-1$-dimensional hyper-planes forming a simplex, such as a triangle($d=2$) or a tetrahedron($d=3$), thus tend to repel (triangle) for even d and to attract for odd d (tetrahedron). The subtracted Casimir energies of domains with group symmetries have been computed analytically[22, 26, 27], but the world-line method[7, 15, 23] outlined above could provide fairly accurate numerical estimates for N-body Casimir energies of a scalar field satisfying Dirichlet boundary conditions on any set of N intersecting hyper-planes.

An analytically tractable example is provided by 2^d pairwise parallel $(d-1)$-dimensional hyper-planes forming a multi-dimensional tic-tac-toe-like pattern in $\mathcal{R}^d$ that encloses an inner hyper-rectangle with dimensions $\ell_1 \times \cdots \times \ell_d$. The corresponding 2^d-body Casimir energy $\tilde{\mathcal{E}}^{2^d}_{\rm rect.}$ for a scalar field satisfying Dirichlet conditions on all the hyper-planes and vanishing at spatial infinity is found to be[28],

$$\tilde{\mathcal{E}}^{2^d}_{\rm rect.} = -\frac{\hbar c \Gamma[(d+1)/2]}{4\pi^{(d+1)/2}} \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{V_{\rm rect.}}{L^{d+1}(\mathbf{n})}, \tag{16}$$

where $V_{\rm rect.} = \prod_{j=1}^d \ell_j$ is the volume of the hyper-rectangle and $L(\mathbf{n}) = \sqrt{\sum_{j=1}^d n_j^2 \ell_j^2}$ is half the length of a classical periodic orbit in its *interior* that reflects n_j times off the jth parallel pair of hyper-planes. Only classical periodic orbits that touch all hyper-planes contribute to $\tilde{\mathcal{E}}^{2^d}_{\rm rect.}$ and there is no sign of any exterior diffractive orbits (whose lengths are multiples of a cycle). Note that $\tilde{\mathcal{E}}^{2^d}_{\rm rect.}$ remains finite in the limit in which one of the dimensions of the rectangle vanishes and a pair of hyper-planes coincides. As mentioned previously this analyticity in the shape and dimensions of the objects is expected in the world-line description and is one of the more fundamental characteristics of the N-body Casimir energies defined by Eq.(1) and Eq.(3).

For electromagnetic applications one seeks to represent N-body Casimir energies in terms of one-body scattering matrices[12]. A representation of the 3-body energy in terms of one-body T-matrices may be obtained as follows. In the notation of [13], the subtracted 3-body Casimir energy $\tilde{\mathcal{E}}^{(3)}$ of the geometric subtraction scheme expressed in terms of the free-, one-, two- and 3-body Greens functions is,

$$\begin{aligned}\tilde{\mathcal{E}}^{(3)} &= \tfrac{i}{2\tau}\mathrm{Tr}(\ln G_{123} - \ln G_{12} - \ln G_{23} - \ln G_{13} \\ &\quad + \ln G_1 + \ln G_2 + \ln G_3 - \ln G_\emptyset) \qquad (17)\\ &= \tfrac{-i}{2\tau}\mathrm{Tr}(\ln \tilde{G}_1\tilde{G}_{123}^{-1}\tilde{G}_{23} - \ln \tilde{G}_1\tilde{G}_{12}^{-1}\tilde{G}_2 - \ln \tilde{G}_1\tilde{G}_{13}^{-1}\tilde{G}_3)\ ,\end{aligned}$$

where $G_\alpha = G_\emptyset \tilde{G}_\alpha$ is the Greens function for the domain $\mathfrak{D}_\alpha$. The trace is over space and time, with τ here being the temporal extent. Using $\tilde{G}_{ij}^{-1} = \tilde{G}_i^{-1} + \tilde{G}_j^{-1} - \mathbb{1}$ and $\tilde{G}_{123}^{-1} = \tilde{G}_1^{-1} + \tilde{G}_{23}^{-1} - \mathbb{1}$ with $\tilde{G}_i = \mathbb{1} - \tilde{T}_i$, the subtracted 3-body Casimir energy of Eq.(17) in terms of one-body scattering matrices T_i becomes,

$$\begin{aligned}\tilde{\mathcal{E}}^{(3)} &= \tfrac{-i}{2\tau}\mathrm{Tr}(\ln[\mathbb{1} - \tilde{T}_1(\mathbb{1} - \tilde{G}_{23})] - \ln[1 - \tilde{T}_1\tilde{T}_2] \\ &\qquad - \ln[\mathbb{1} - \tilde{T}_1\tilde{T}_3]) \qquad (18)\\ &= \tfrac{-i}{2\tau}\mathrm{Tr}\ln\Big[\mathbb{1} - \tilde{T}_1\tfrac{\mathbb{1}}{\mathbb{1}-\tilde{T}_2\tilde{T}_1}\Big(\tilde{T}_2\tilde{T}_1\tilde{T}_3 + \\ &\quad +(\tilde{T}_2 - \mathbb{1})\tfrac{\tilde{T}_3\tilde{T}_2}{\mathbb{1}-\tilde{T}_3\tilde{T}_2} + (\tilde{T}_3 - \mathbb{1})\tfrac{\tilde{T}_2\tilde{T}_3}{\mathbb{1}-\tilde{T}_2\tilde{T}_3}\Big)\tfrac{\mathbb{1}}{\mathbb{1}-\tilde{T}_1\tilde{T}_3}\Big]\ .\end{aligned}$$

Here $\tilde{T}_i = T_i G_\emptyset = (\mathbb{1} - \tilde{G}_i)$, with $G_\emptyset$ the Green-function for the domain $\mathfrak{D}_\emptyset$ with no objects inserted. The expression in Eq.(18) differs from that given in [12] only in that all (three) two-body interactions have been subtracted. Our previous considerations show that the 3-body vacuum interaction energy given by Eq.(18) is continuous in the position of the three objects and remains finite when some of them overlap. Every term in Eq.(18) requires scattering off all three objects and is individually finite. Explicit calculation[28] shows that this 3-body correction of Eq.(18) to the Casimir energy of three parallel plates is as expected symmetric under the exchange symmetry and remains finite when any two of the three plates coincide. The advantage of writing finite N-body Casimir energies in terms of scattering matrices is that this representation unambiguously defines them for any local field theory and in particular for the physically interesting electromagnetic case. Some previously unresolved conceptual issues, such as whether exterior modes contribute to the Casimir energy of various polytopes[26, 27], have been reduced to purely computational[28] ones.

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