

Entropy of extremal black holes from entropy of quasiblack holes

José P. S. Lemos^a, Oleg B. Zaslavskii^b

^a*Centro Multidisciplinar de Astrofísica – CENTRA, Departamento de Física, Instituto Superior Técnico - IST, Universidade Técnica de Lisboa - UTL, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal,*

^b*Institute of Theoretical Physics - ITP, Freie Universität Berlin, Arnimallee 14 D-14195 Berlin, Germany.*

^b*Astronomical Institute of Kharkov V.N. Karazin National University, 35 Sumskaya St., Kharkov, 61022, Ukraine.*

Abstract

The entropy of extremal black holes (BHs) is obtained using a continuity argument from extremal quasiblack holes (QBHs). It is shown that there exists a smooth limiting transition in which (i) the system boundary approaches the extremal Reissner-Nordström (RN) horizon, (ii) the temperature at infinity tends to zero and quantum backreaction remains bounded on the horizon, and (iii) the first law of thermodynamics is satisfied. The conclusion is that the entropy S of extremal QBHs and of extremal BHs can take any non-negative value, only in particular cases it coincides with $S = A/4$. The choice $S = 0$ with non-zero temperature at infinity is rejected as physically unsatisfactory.

Keywords: quasiblack holes, black holes, extremal horizon, entropy, thermodynamics

1. Introduction

The issue of black hole (BH) entropy is one of the most intriguing in BH physics. For non-extremal BHs the entropy S is given in terms of the horizon area A by the Bekenstein-Hawking formula $S = \frac{A}{4}$ [1], a puzzle not yet resolved in fundamental micro-level terms. Surprisingly, the issue becomes even more intriguing in what concerns extremal BHs, as there are two mutually inconsistent results. There is the prescription $S = 0$ obtained from the fact that for extremal BHs the period of the Euclidean time is not fixed in a classical calculation of the action [2], and there is the usual $S = \frac{A}{4}$ value obtained from string theory [3]. There have been some interesting proposals to further understand the issue, see [4] for a thermodynamical treatment and [5, 6] for a semiclassical approach, but the situation remains contradictory up to now, see [7] for the latest comments. Here, we suggest a resolution of this problem on the basis of pure thermodynamic arguments. In doing so, we exploit the quasiblack hole (QBH) approach.

What is the QBH approach? A QBH is a system whose boundary approaches the would-be horizon as nearly as one likes, and yet the system does not collapse; a horizon is almost formed but never does [8]. The approach consists in finding the limiting properties of the system when the boundary tends to its quasihorizon [9]. Properties of such systems are then compared with pure BH properties. It has been found that, though worked out through totally different methods, QBHs and pure BHs share for outside observers the same properties, such as the mass formula and many others [9], although the interior of both systems is totally different, interior made of matter for QBHs, vacuum interior for pure BHs.

In the work [10] important developments on QBH properties were advanced. The entropy of non-extremal QBH systems was found thermodynamically and shown to be equal to the BH entropy $S = \frac{A}{4}$. This was achieved by using on one hand QBH procedures, and on the other hand the formalism for gravitating systems, such as BHs, of Brown and York [11] for the definition of quasilocal energy and other quasilocal thermodynamic quantities. Now, QBHs can be obtained from a quite

Email addresses: joselemos@ist.utl.pt (José P. S. Lemos), ozaslav@kharkov.ua (Oleg B. Zaslavskii)

generic class of systems, but a simple realization of them is provided through thin charged shells when these are brought to their own horizon radius. In the works [12] these systems were thermodynamically studied but the analysis fell short of letting the shell approach the horizon. In [4] a specially arranged shell system was imagined in order to analyze its behavior when lowering quasistatically the shell into its own horizon, with the shell being immersed in an appropriate quantum vacuum. It was found that $S = \frac{A}{4}$, as well. Our QBH approach for non-extremal systems [10] is generic and thus has none of the drawbacks of specialization to simple thin shell systems.

Since many properties, in particular the entropy S of non-extremal QBHs, can be found and match the corresponding quantities of pure BHs, we continue our pursue and want to shed light on the entropy of extremal BHs by studying the entropy of QBHs, more specifically, extremal QBHs. One can then consider a sequence of stars made of some sort of usual matter, each member of the sequence with lesser radius say, in which the last member of the sequence is an extremal QBH. By using stars made of matter one is enabled to consider more prosaic systems, i.e., systems that do not possess a horizon, and study them through the usual textbook formalism of thermodynamics. Only for the last members of the sequence of stars one takes the limit to the transition to the QBH state. At this very last stage of the sequence, in order to have a well defined thermodynamic system, one has to use results from quantum field theory in curved background [13]. The sequential procedure is of immense importance, as due to the continuity of the calculation process, the QBH approach enables to evade difficulties connected with the often invoked potential discontinuity between non-extremal and extremal BHs. Using the QBH, we follow the procedure developed in [10] (see also [4]), i.e., we calculate S of the material system when the would-be horizon is approached. Since for an external observer a QBH and a BH cannot be distinguished [9], one expects that the entropy of such a system without a true horizon tends to the entropy of the corresponding BH. In this sense our calculations not only give an answer for the QBH entropy but elucidate the value of the entropy of a BH to which the exterior of a QBH tends due to the continuity of the process in the latter case.

We show that our consistent thermodynamic treatment rejects definitely the choice $S = 0$ but does not give an unambiguous universal result for S . The entropy depends on the properties of the working material and, moreover, on the manner the temperature approaches the zero value. In particular $S = \frac{A}{4}$ is not singled out beforehand for the extremal BH entropy.

2. Basic formulas

The study of extremal QBHs has one advantage over the study of non-extremal ones. While non-extremal QBHs show a sort of singular behavior at the quasihorizon, such as a singular stress-energy tensor, extremal QBHs are nonsingular well behaved systems throughout [9]. In order to make the problem tractable we stick to spherically symmetric systems.

Consider a spherical symmetric compact body with boundary at $r = R$ such that $r < R$ defines the inner region, $r > R$ the outer one, and its total charge q is equal to its ADM mass m . The generic space-time line element in the usual coordinates (t, r, θ, ϕ) is then

$$ds^2 = -V \exp(2\psi) dt^2 + \frac{dr^2}{V} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where V and ψ are functions of r . In general one also needs an expression for the electric potential $\phi(r)$. For $r \geq R$ the space-time is described by the extremal Reissner-Nordström (RN) metric, in which case $\psi(r) = 0$, and by the Coulomb electric potential,

$$V(r) = \left(1 - \frac{r_+}{r}\right)^2, \quad \phi(r) = \frac{r_+}{r} + \text{constant}, \quad (2)$$

where $r_+ = m = q$, r_+ being the gravitational radius of the body, i.e., the radius of the would-be horizon, and the constant can be chosen in convenient terms. $R \geq r_+$ always holds here and at $R = r_+$ a QBH forms.

The whole system, compact body plus spacetime, is assumed to be in thermodynamic equilibrium at some non-zero temperature. The entropy of the system is calculated from integrating the first law when it undergoes a reversible process. In general, the integration requires knowledge of the matter equation

of state, but we show that, when one deals with systems on the threshold of forming an extremal horizon, deep conclusions can be drawn without that knowledge. The first law of thermodynamics for our system can be written as [11]

$$TdS = dE + \lambda dA - \varphi de. \quad (3)$$

We now go through TdS , dE , λdA , and $-\varphi de$ carefully.

The local temperature T on the boundary R is related to the temperature T_0 at infinity by the Tolman formula

$$T = \frac{T_0}{\sqrt{V(R)}} = \frac{T_0}{1 - \frac{r_+}{R}}. \quad (4)$$

Since S is the entropy of the system, dS is the change of the entropy upon changing the other quantities.

The quasilocal energy E is given by [11], $E = R \left(1 - \sqrt{V(R)}\right)$. It is seen from this and (1) that for our extremal system $E = r_+$ does not depend on R . Thus

$$dE = dr_+. \quad (5)$$

The gravitational pressure λ is found from the inner region. For the inner region $r \leq R$ the metric is given as in Eq. (1). Then the gravitational pressure λ at the boundary at $r = R$ equals to [11] $8\pi\lambda = \frac{1}{R} \left(\sqrt{V(R)} - 1\right) + \left(\frac{1}{2} \frac{1}{\sqrt{V(r)}} \frac{dV(r)}{dr} + \sqrt{V(r)} \frac{d\psi(r)}{dr}\right)_{r=R_-}$, where $r = R_-$ means that the derivatives should be taken from the inner region. It follows from the tt and rr Einstein equations for the inner region and the boundary condition $\psi(R) = 0$ (mandatory for a smooth matching with the outer region) that $\psi = 4\pi \int_R^r d\bar{r} \frac{\bar{r}(p_r(\bar{r}) + \rho(\bar{r}))}{V(\bar{r})}$. Here p_r is the radial pressure and ρ is the energy density, both include contributions from the matter and the electromagnetic field, i.e., $p_r = p_r^{\text{matter}} + p_r^{\text{em}}$ and $\rho = \rho^{\text{matter}} + \rho^{\text{em}}$. It also follows from the tt Einstein equation that $V(r) = 1 - \frac{2m(r)}{r}$, with $m(r) = 4\pi \int_0^r d\bar{r} \bar{r}^2 \rho(\bar{r})$. Then, for our concrete system, using (2) and the equation for λ , one finds after some manipulations, $8\pi\lambda R = 4\pi p_r^{\text{matter}} R^2 / (1 - r_+/R)$. Now, to make progress we have to understand the system at the threshold of being a QBH. We have to take into account that on the quasihorizon $p_r^{\text{matter}}(r_+) = 0$

according to our general results on pressure in [9]. When matter is absent in the inner region, as in a thin shell, this condition is exact. When there is matter, one can write quite generally $p_r^{\text{matter}}(r) = \frac{b(r_+, R)}{4\pi R^2} \left(1 - \frac{r_+}{R}\right)$, valid near $R = r_+$ and with the function $b(r_+, R)$ model-dependent. Note that we do not need to impose that $p_r(R) = 0$ for $R > r_+$, the surface can move due to thermal motion or something else, or even be a cold star in which case $p_r(R) = 0$. The point is that if the body is sufficiently compressed it follows that $p_r^{\text{matter}}(r_+) = 0$ [9]. Thus, finally,

$$\lambda = \frac{1}{8\pi} \frac{b(r_+, R)}{R}. \quad (6)$$

On the other hand, the area A is defined as $A = 4\pi R^2$, so that

$$dA = 8\pi R dR. \quad (7)$$

The electric potential φ represents the difference in electrostatic potential between a reference point with potential ϕ_0 and the boundary R with potential $\phi(R) = q/R$, blue-shifted from infinity to R through the factor $1/\sqrt{V}$, where V is the time component of the static metric. Thus,

$$\varphi = \frac{\phi_0 - \phi(R)}{\sqrt{V(R)}}. \quad (8)$$

For de , since at the quasihorizon limit $e = r_+$, one has

$$de = dr_+. \quad (9)$$

We are now ready to analyze the entropy of quasiblack holes.

3. Entropy of quasiblack holes and entropy of extremal black holes

First, let us consider the simplest case: a charged shell with a flat space-time inside and an extremal RN metric outside. Then, $b = 0$ since there is no matter inside. Also, as the potential is everywhere constant inside, one has $\varphi = 0$. Then, we obtain the first law in the form,

$$TdS = dr_+. \quad (10)$$

It is instructive to recall that in the case of uncharged shells, as treated in [12], the integrability

condition of the first law yields $T_0 = T_0(r_+)$, so T_0 is not a function of R in such a case. Now our case is an extremal charged shell rather than an uncharged one. In this case the integrability condition for Eq. (10) is $T = T(r_+)$, i.e., the local temperature is a function of r_+ alone. On the other hand the temperature at infinity has thus the form,

$$T_0 = T(r_+) \left(1 - \frac{r_+}{R}\right), \quad (11)$$

It contains a dependence on R , but, as usual, it does not depend on r . With these remarks we can now integrate Eq. (10) and obtain

$$S = S(r_+) = \int_0^{r_+} d\bar{r}_+ \frac{1}{T(\bar{r}_+)}, \quad (12)$$

where the constant of integration ensures that $S \rightarrow 0$ when the system shrinks to nothing. To be sure, Eq. (12) is valid for any $R \geq r_+$.

Second, we consider a more general configuration, with the inside having some type or another of distribution of matter other than vacuum. Clearly, one has to assume that the integrability conditions for the system are valid, otherwise there is no thermodynamic system. Then, since S is a total differential one can integrate along any path. Choose the path $R = r_+(1 + \delta)$ with δ constant and small, so that $dS = (\text{something}) dr_+$. Then one can integrate this equation to obtain S . Taking then at once the limit $R \rightarrow r_+$, we obtain instead of (12) the following equation,

$$S = S(r_+) = \int_0^{r_+} d\bar{r}_+ \frac{D(\bar{r}_+)}{T(\bar{r}_+)}, \quad (13)$$

where,

$$D(r_+) = 1 + b_+ - \varphi_+, \quad (14)$$

$b_+ = b(r_+, R = r_+)$ and $\varphi_+ = \varphi_+(r_+, R = r_+)$. In general, we only require $1 + b_+ - \varphi_+ > 0$ to ensure the positivity of the entropy. Note that if the density of matter inside vanishes at $r = R$, we return to the thin shell situation, since $b_+ \rightarrow 0$, $\varphi_+ \rightarrow 0$, and so $D(r_+) = 1$.

Thus, we can state the following. For QBHs, for any finite generic $T(r_+)$, one obtains a well-defined positive entropy, $S > 0$, from (13), as well as a vanishing temperature at infinity, $T_0 \rightarrow 0$, from (11). In addition one can consider the case in which $T(r_+)$ is not finite, $T(r_+) \rightarrow \infty$ as $T(r_+) = (T_0 / (1 - \frac{r_+}{R}))|_{R \rightarrow r_+}$. In this particular instance

one obtains from (13) that for QBHs $S = 0$ and from (11) that T_0 is positive and finite, not equal to zero, $T_0 > 0$. This latter particular case of QBH behavior is equivalent to the prescription given in [2] for extremal pure BHs.

We now argue that for extremal pure BHs the prescription $T_0 \neq 0$ of [2] is unsatisfactory. As pointed in [13], the prescription that $T_0 \neq 0$ is an arbitrary finite quantity, is inconsistent with quantum backreaction. Indeed, the corresponding quantum stress-energy tensor is of the form $T_{\mu}^{\text{quant}\nu} = T^4 f_{\mu}^{\nu} + h_{\mu}^{\nu}$ where h_{μ}^{ν} is a term finite everywhere. Near the horizon the first term of $T_{\mu}^{\text{quant}\nu}$ diverges as the local temperature T diverges due to the redshift factor. This unstabilizes the system and is physically inappropriate at the semi-classical level [13]. Actually, if this were true, the temperature of the quantum fields and that of the BH itself would not coincide, making thermal equilibrium impossible. $T_0 \neq 0$ and $S = 0$ cannot be a solution. One is left with vanishing T_0 (T finite) and $S \geq 0$ undetermined for extremal BHs.

Our QBH approach gives consistency to this solution of the thermodynamic extremal BH problem. Indeed, the result provided by Eqs. (11) and (13) is free of difficulties. As the local temperature $T(r_+)$ remains finite when $R \rightarrow r_+$, the quantum stress-energy tensor $T_{\mu}^{\text{quant}\nu}$ on the quasihorizon remains finite or even negligible. Moreover, the first law of thermodynamics is also satisfied with the choice (13). Thus, thermal equilibrium is kept in the system, the temperature tends to the Hawking value with a suitable rate, given by (11), and $S \geq 0$ is somehow undetermined. Can we nevertheless say something more definite about the form of the function $S(r_+)$? Eq. (13) tell us that the situation is model-dependent, it depends on $D(r_+)/T(r_+)$, which depends on the properties of the particular system under study. For instance, only for special cases, when the quantity $D(r_+)/T(r_+)$ is given by $D(r_+)/T(r_+) = 2\pi r_+$, can we obtain the Bekenstein-Hawking value $\frac{A}{4}$ where A is the area of the quasihorizon surface. In addition, for a given model, changing the parameter $T(r_+)$, say, one can obtain any desirable value for S , with $S > 0$, $S = 0$ being ruled out.

In deriving that the entropy of extremal BHs is model-dependent we are not alone. We were preceded by the results of [4]. In [4] particular thin

shells as working material were analyzed, and the Gibbs-Duhem relation (which for self-gravitating systems is, in general, not valid) was used, to support the conclusion that extremal BH entropy is model-dependent. Our approach is much more general, makes no use of thin shells neither of the Gibbs-Duhem relation. Moreover, in deriving that the manner in which the temperature T_0 approaches zero is not well fixed, as $T(r_+)$ is a free quantity, we are also not alone. We were preceded by the results of [5] and [6]. Indeed, remarkably, on a totally different setting and actually in a work which raised for the first time problems connected to extremal BHs alone, it was shown in [5] that at the extremal state fluctuations on the temperature grow unbound. Our work shows the appearance of unusual features in the thermal description even without considering such fluctuations. This problem is a quite separate non-trivial issue needing further consideration. In addition, [6] has concluded that the notion of zero temperature is ill-defined for extremal BHs, whereas we defined it but in a rather delicate way (see Eq. (11)), so it changes when we go through the referred sequence of configurations.

4. Conclusions

As the value of S depends on T , it means that the way T approaches zero leaves its imprint on the entropy. This is different from traditional thermodynamics where the state is characterized by its thermodynamic parameters with no memory on how their values were achieved. Our approach implies either that the entropy of extremal BHs is not a full-fledged unambiguous quantity, in the sense that any desirable value of S can be achieved by tuning T say, or that T is unique and can be found on fundamental grounds in a semiclassical theory. One should verify this hypothesis. In any case, our work shows that near $T_0 = 0$, i.e., near the extremal BH or QBH limit, the usual thermodynamic picture can change drastically and the third law of thermodynamics (the way T_0 approaches zero) is a much more intricate issue than expected.

We stressed the key role played by the QBH concept and have shown how to substantiate the choice for the extremal BH entropy from a thermodynamic stand. The result is not universal, with $S = 0$, $T_0 \neq 0$ being ruled out. We used continuity arguments and so one question we should ask is whether

the limiting configuration in the QBH setup yields an entropy S that can be consider the entropy of an extremal BH. Our approach stems from taking the horizon limit of matter configurations with time-like boundaries, whereas BHs have from the start a lightlike horizon. Can we trust that by continuity from the QBH approach we get the correct entropy of an extremal BH? Non-extremal QBHs yield to continuity arguments [10], but there one knew the result beforehand. However, now the entropy of an extremal BH is unknown, so there is no gauge to compare with. Thus, the situation is more tricky, and though we do not possess a rigorous proof, we can add arguments in its favor. When we change m and q , approaching the extremal RN BH metric from a non-extremal one, jumps in S are not excluded. However, these jumps should be connected with jumps in the temperature. If we take the prescription of [2], T_0 changes from $T_0 \approx 0$ for the near-extremal configuration to finite T_0 . In contrast, in our QBH approach $T_0 \rightarrow 0$ smoothly with no source of discontinuity. Moreover, using the standard approach for the entropy of an extremal BH, there remains the difficulty of its calculation and definition within thermodynamics. If one takes the prescription of [2], T_0 is finite and arbitrary, but backreaction destroys the horizon. If, instead, one puts T_0 to zero in accordance with its Hawking value, it is not quite clear how to obtain an entropy by differentiating the system's free energy with respect to a fixed zero temperature. On the other hand, the QBH approach evades these problems since a horizon is absent and at each stage it has a well-defined small non-zero T_0 . It seems appropriate to consider the limiting entropy of the sequence of the QBH configurations precisely as a definition of extremal BH entropy, analogously to the operational definition substantiated in [4].

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