Chiral Resonant Solitons in Broer-Kaup Type New Hydrodynamic Systems

Jyh-Hao Lee and Oktay K. Pashaev
Institute of Mathematics, Academia Sinica Taipei, Taiwan
(e-mail: leejh@ math.sinica.edu.tw)
Department of Mathematics, Izmir Institute of Technology
Urla-Izmir, 35430, Turkey (e-mail: oktaypashaev@iyte.edu.tr)

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Abstract

New Broer-Kaup type systems of hydrodynamic equations are derived from the derivative reaction-diffusion systems arising in SL(2,R) Kaup-Newell hierarchy, represented in the non-Madelung hydrodynamic form. A relation with the problem of chiral solitons in quantum potential as a dimensional reduction of 2+1 dimensional Chern-Simons theory for anyons is shown. By the Hirota bilinear method, soliton solutions are constructed and the resonant character of soliton interaction is found.

1 Introduction

Recently, a modification of the nonlinear Schrödinger (NLS) equation by a quantum potential has been studied in several problems arising in low dimensional gravity, [1], plasma physics, [3], the capillary wave, and information theory, [8]. Subsequently, the influence of this potential on anyons in 2+1 dimensions has been studied [4], and the Abelian Chern-Simons gauge field interacting with NLS has been represented as a planar Madelung fluid [6] , where the Chern-Simons Gauss law has the simple physical meaning of creation of the local vorticity for the flow. For the static flow when the velocity of the center-of-mass motion is equal to the quantum velocity, the fluid admits an N-vortex solution. It turns out that in this theory the Chern-Simons coupling constant and the quantum potential strength are quantized.

Reduction of problem to 1+1 dimensions leads to JNLS and some versions of Derivative NLS in quantum potential. Hence the chiral solitons appear as solutions of the derivative NLS with quantum potential [5] . The last one by the Madelung transform is represented as the derivative Reaction-Diffusion (DRD) system, arising in SL(2,R) Kaup-Newell hierarchy, and giving rise to the resonant soliton phenomena [7]..

In the present paper by using new, the non-Madelung representation, we formulate the problem in terms of novel hydrodynamic systems of the Broer-Kaup type. Then by Hirota's bilinear method we construct chiral solitons for the system and show the resonance character of their interaction.

2 Dimensional reduction of Chern-Simons theory

We consider the Chern-Simons gauged Nonlinear Schrödinger model (the Jackiw-Pi model) with nonlinear quantum potential term of strength s [4]:

$$L = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{i}{2}(\bar{\psi}D_0\psi - \psi\bar{D}_0\bar{\psi}) - \bar{\mathbf{D}}\bar{\psi}\mathbf{D}\psi + s\nabla|\psi|\nabla|\psi| + V(\bar{\psi}\psi), \quad (1)$$

where $D_\mu = \partial_\mu + ieA_\mu$, $(\mu = 0, 1, 2)$. Classical equations of motion are

$$iD_0\psi + \mathbf{D}^2\psi + V'\psi = s\frac{\Delta|\psi|}{|\psi|}\psi, \quad (2)$$

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa}\bar{\psi}\psi, \quad (3)$$

$$\partial_0 A_j - \partial_j A_0 = -\frac{e}{\kappa}i\epsilon_{jk}(\bar{\psi}D_k\psi - \psi\bar{D}_k\bar{\psi}), \; (j, k = 1, 2). \quad (4)$$

We consider dimensional reduction of this model when all field are independent of x_2 space variable, so that $\partial_2 = 0$. Then in terms of $\tilde{A}_0 \equiv A_0 + eA_2^2$, and $B \equiv A_2$, we obtain

$$i(\partial_0 + ieA_0)\psi + (\partial_1 + ieA_1)^2\psi + V'\psi = s\frac{\partial_1^2|\psi|}{|\psi|}\psi, \quad (5)$$

$$\partial_0 A_1 - \partial_1 A_0 = 0 \quad (6)$$

$$\partial_1 B = \frac{e}{\kappa}\bar{\psi}\psi \quad (7)$$

$$\partial_0 B = i\frac{e}{\kappa}\left[\bar{\psi}(\partial_1 + ieA_1)\psi - \psi(\partial_1 - ieA_1)\bar{\psi}\right] \quad (8)$$

Here and below we skip the tilde sign for A_0. The last two equations are compatible due to (5) and the corresponding continuity equation

$$\partial_0(\bar{\psi}\psi) = i\partial_1\left[\bar{\psi}(\partial_1 + ieA_1)\psi - \psi(\partial_1 - ieA_1)\bar{\psi}\right] \quad (9)$$

implies compatibility of equations (7),(8). Integrating these equations we find B in terms of density $\bar{\psi}\psi$

$$B = \frac{e}{\kappa}\int^x \bar{\psi}\psi\, dx'. \quad (10)$$

From another side, flatness of connection (6) implies $A_0 = \partial_0 \phi$, $A_1 = \partial_1 \phi$, and these potentials can be removed by the gauge transformation, $\psi = e^{-e\phi}\Psi$. As a result we obtain the Schrödinger equation with self-interacting nonlinear potential $V(\bar{\psi}\psi)$, and the quantum potential

$$i\partial_0 \Psi + \partial_1^2 \Psi + V'\Psi = s\frac{\partial_1^2 |\Psi|}{|\Psi|}\Psi \tag{11}$$

2.1 Madelung representation and RNLS

If we substitute the Madelung Ansatz $\Psi = \sqrt{\rho}\, e^{-iS}$ to wave function in (11) then we get the coupled system

$$\partial_0 S - (\partial_1 S)^2 + V'(\rho) + (1-s)\frac{\partial_1^2 \sqrt{\rho}}{\sqrt{\rho}} = 0 \tag{12}$$

$$\partial_0 \rho - \partial_1 (2\rho\, \partial_1 S) = 0 \tag{13}$$

For velocity field $v = -2\partial_1 S$ it implies the hydrodynamical system

$$\partial_0 v + v\partial_1 v = 2\partial_1 \left(V'(\rho) + (1-s)\frac{\partial_1^2 \sqrt{\rho}}{\sqrt{\rho}} \right) \tag{14}$$

$$\partial_0 \rho + \partial_1 (\rho\, v) = 0 \tag{15}$$

which is the Madelung fluid representation of (11).

First we consider the under-critical case, when the strength of the quantum potential $s < 1$. Then introducing rescaled the time and the phase $\tilde{t} = t\sqrt{1-s}$, $\tilde{S} = \frac{S}{\sqrt{1-s}}$, for new wave function $\tilde{\Psi} = \sqrt{\rho}\, e^{-i\tilde{S}}$ we get

$$i\partial_{\tilde{0}} \tilde{\Psi} + \partial_1^2 \tilde{\Psi} + \frac{V'}{1-s}\tilde{\Psi} = 0 \tag{16}$$

In the over-critical case when $s > 1$, we can't reduce (11) to (16). However, if we introduce pair of real functions

$$e^{(+)}(x,t) = \sqrt{\rho} e^{\tilde{S}}, \quad e^{(-)}(x,t) = \sqrt{\rho} e^{-\tilde{S}} \tag{17}$$

instead of one complex wave function, then we get the time reversal pair of reaction-diffusion equations

$$\partial_{\tilde{0}} e^{(+)} + \partial_1^2 e^{(+)} - \frac{V'}{s-1} e^{(+)} = 0 \tag{18}$$

$$-\partial_{\tilde{0}} e^{(-)} + \partial_1^2 e^{(-)} - \frac{V'}{s-1} e^{(-)} = 0 \tag{19}$$

where $\tilde{t} = t\sqrt{s-1}$, $\tilde{S} = \frac{S}{\sqrt{s-1}}$, $\rho = e^{(+)}e^{(-)}$, $V = V(e^{(+)}e^{(-)})$.

If interaction between material particles is the delta function pair form then potential $V(\rho) = g\rho^2/2$ and equation (11) becomes

$$i\partial_0\Psi + \partial_1^2\Psi + g|\Psi|^2\Psi = s\frac{\partial_1^2|\Psi|}{|\Psi|}\Psi \tag{20}$$

We called this equation the Resonant Nonlinear Schrodinger equation (RNLS). It appears in the study of low-dimensional gravity model on a line, the Jackiw-Teitelboim model [1], and in description of cold plasma [3]. For under-critical case $s < 1$ it reduces to the standard NLS equation (16), and is integrable model. For the over-critical case $s > 1$ it reduces to the couple of cubic reaction-diffusion equations (18), (19), which is also integrable as SL(2,R) NLS from the AKNS hierarchy. In the last case new resonance phenomena for envelope solitons take place [1]. In the next section we will discuss another reduction of Chern-Simons theory with dynamical field B, and show that in this case resonant versions of the JNLS and DNLS equations appear.

3 Dynamical BF theory

To do the gauge field component B in Section 2 to be dynamical, following [11] we introduce the corresponding kinetic term so that

$$\begin{gathered} L = \kappa B(\partial_0 A_1 - \partial_1 A_0) + \theta\,\partial_0 B\partial_1 B + \tfrac{i}{2}(\bar\psi(\partial_0 + ieA_0)\psi \\ -\psi(\partial_0 - ieA_0)\bar\psi) - (\partial_1 - ieA_1)\bar\psi(\partial_1 + ieA_1)\psi + s\partial_1|\psi|\partial_1|\psi| + V(\bar\psi\psi), \end{gathered} \tag{21}$$

Then equations of motion are

$$i(\partial_0 + ieA_0)\psi + (\partial_1 + ieA_1)^2\psi + V'\psi = s\frac{\partial_1^2|\psi|}{|\psi|}\psi, \tag{22}$$

$$\partial_0 A_1 - \partial_1 A_0 = \frac{2\theta}{\kappa}\partial_0\partial_1 B \tag{23}$$

$$\partial_1 B = \frac{e}{\kappa}\rho \tag{24}$$

$$\partial_0 B = -\frac{e}{\kappa}J \tag{25}$$

where the particle and the momentum density are

$$\rho = \bar\psi\psi, \;\; J = \frac{1}{i}[\bar\psi(\partial_1 + ieA_1)\psi - \psi(\partial_1 - ieA_1)\bar\psi] = j + 2eA_1\rho \tag{26}$$

and $j = -i[\bar\psi\partial_1\psi - \psi\partial_1\bar\psi]$. Equation (22) implies the conservation law

$$\partial_0\rho + \partial_1 J = 0 \tag{27}$$

This conservation law is the compatibility condition for the system (24),(25). and allows us to write

$$\partial_0\partial_1 B = \frac{e}{\kappa}[\,\alpha\partial_0\rho - (1-\alpha)\partial_1 J\,] \tag{28}$$

where α is an arbitrary real constant. Substituting (28) to (23) and combining terms under the same derivatives we have

$$\partial_0\left(A_1 - \frac{2\theta e}{\kappa^2}\alpha\rho\right) - \partial_1\left(A_0 - \frac{2\theta e}{\kappa^2}(1-\alpha)J\right) = 0 \tag{29}$$

The system (22)-(25) is invariant under the local $U(1)$ gauge transformations

$$\psi \to \psi' = e^{-ie\phi(x,t)}\psi, \quad A_\mu \to A'_\mu = A_\mu + \partial_\mu\phi \tag{30}$$

Then solving (29) we have

$$A_1 = \frac{2\theta e}{\kappa^2}\alpha\rho + \partial_1\phi, \quad A_0 = \frac{2\theta e}{\kappa^2}(1-\alpha)J + \partial_0\phi \tag{31}$$

and for the gauge invariant field $\Psi = e^{ie\phi}\psi$ it gives

$$i\left(\partial_0 + i\frac{2\theta e^2}{\kappa^2}(1-\alpha)J\right)\Psi + \left(\partial_1 + i\frac{2\theta e^2}{\kappa^2}\alpha\rho\right)^2\Psi + V'\Psi = s\frac{\partial_1^2|\psi|}{|\Psi|}\Psi, \tag{32}$$

where $J = j + \frac{4\theta e^2}{\kappa^2}\alpha\rho^2$, $j = -i[\bar{\Psi}\Psi_x - \Psi\bar{\Psi}_x]$, $\rho = \bar{\Psi}\Psi$. Finally we have

$$\begin{aligned} &i\Psi_t + \Psi_{xx} + i\tfrac{2\theta e^2}{\kappa^2}\left[(2\alpha+1)|\Psi|^2\Psi_x + (2\alpha-1)\Psi^2\bar{\Psi}_x\right] \\ &+4\tfrac{\theta^2 e^4}{\kappa^4}\alpha(\alpha-2)|\Psi|^4\Psi + V'\Psi = s\tfrac{|\Psi|_{xx}}{|\Psi|}\Psi \end{aligned} \tag{33}$$

where partial differentiation notations are evident. The remaining gauge transformation for this equation is just the global $U(1)$ transformation: $\Psi \to e^{i\lambda}\Psi$, $\lambda = const.$

3.1 Reductions of general RDNLS

The behavior of equation (33) depends on value of parameter s. If we replace $\Psi = e^{R-iS}$ then we have couple of equations

$$R_t - (S_{xx} + 2R_x S_x) + \frac{2\theta e^2}{\kappa^2}4\alpha R_x e^{2R} = 0,$$

$$S_t - S_x^2 + (1-s)(R_{xx} + R_x^2) + \frac{2\theta e^2}{\kappa^2}2S_x e^{2R} + \frac{4\theta^2 e^4}{\kappa^4}\alpha(\alpha-2)e^{4R} + V' = 0,$$

determining the Madelung fluid representation

$$\rho_t + (\rho v + \frac{2\theta e^2}{\kappa^2}2\alpha\rho^2)_x = 0, \tag{34}$$

$$v_t + vv_x = 2\left[(1-s)\frac{\sqrt{\rho}_{xx}}{\sqrt{\rho}} - \frac{2\theta e^2}{\kappa^2}\rho v + \left(\frac{2\theta e^2}{\kappa^2}\right)^2\alpha(\alpha-2)\rho^2 + V'\right]_x \tag{35}$$

For $s<1$ for redefined variables $t\sqrt{1-s}\equiv\tilde{t}$, $S/\sqrt{1-s}\equiv\tilde{S}$, $\tilde{\Psi}\equiv e^{R-i\tilde{S}}$, we have

$$\begin{gathered} i\tilde{\Psi}_{\tilde{t}}+\tilde{\Psi}_{xx}+i\tfrac{2\theta e^2}{\kappa^2\sqrt{1-s}}\left[(2\alpha+1)|\tilde{\Psi}|^2\tilde{\Psi}_x+(2\alpha-1)\tilde{\Psi}^2\bar{\tilde{\Psi}}_x\right] \\ +4\tfrac{\theta^2 e^4}{\kappa^4(1-s)}\alpha(\alpha-2)|\tilde{\Psi}|^4\tilde{\Psi}+\tfrac{V'}{1-s}\tilde{\Psi}=0 \end{gathered} \tag{36}$$

Similar to the Chern-Simons 2+1 dimensional case [4] we have effective result of the quantum potential in the rescaling of the statistical parametr $\kappa^2\to\kappa^2\sqrt{1-s}$, but in contrast no quantization of this parameter now appears. Transformation between wave functions has nonlinear form

$$\Psi(x,t)=|\tilde{\Psi}|\left(\frac{\tilde{\Psi}}{|\tilde{\Psi}|}\right)^{\sqrt{1-s}}(x,t\sqrt{1-s}) \tag{37}$$

For $s>1$ it is impossible to reduce the system to the Schrodinger type form. However for redefined parameters $t\sqrt{s-1}\equiv\tilde{t}$, $S/\sqrt{s-1}\equiv\tilde{S}$ and two real functions $E^+=e^{R+\tilde{S}}$, $E^-=e^{R-\tilde{S}}$ we get

$$\begin{gathered} \mp E^{\pm}_{\tilde{t}}+E^{\pm}_{xx}\mp\tfrac{2\theta e^2}{\kappa^2\sqrt{s-1}}\left[(2\alpha+1)E^+E^-E^{\pm}_x+(2\alpha-1)E^{\pm 2}E^{\mp}_x\right] \\ -4\tfrac{\theta^2e^4}{\kappa^4(s-1)}\alpha(\alpha-2)(E^+E^-)^2E^{\pm}-\tfrac{V'}{s-1}\tilde{E}^{\pm}=0 \end{gathered} \tag{38}$$

3.2 Gauge transformation

We notice that in the gauge potential representation (31) , the gauge function $\phi=\phi^{(\alpha)}$ depends on α:

$$A_1=\frac{2\theta e}{\kappa^2}\alpha\rho+\partial_1\phi^{(\alpha)},\ \ A_0=\frac{2\theta e}{\kappa^2}(1-\alpha)J+\partial_0\phi^{(\alpha)} \tag{39}$$

Comparison with the case $\alpha=0$

$$A_1=\partial_1\phi^{(0)},\ \ A_0=\frac{2\theta e}{\kappa^2}J+\partial_0\phi^{(0)} \tag{40}$$

gives relations

$$\partial_1(\phi^{(0)}-\phi^{(\alpha)})=\frac{2\theta e}{\kappa^2}\alpha\,\rho,\ \ \partial_0(\phi^{(0)}-\phi^{(\alpha)})=-\frac{2\theta e}{\kappa^2}\alpha\,J \tag{41}$$

Compatibility of this system is ensured by the continuity equation (27). Then corresponding gauge transformed wave functions $\Psi^{(\alpha)}$ and $\Psi^{(0)}$

$$\psi=e^{-ie\phi^{(\alpha)}}\Psi^{(\alpha)}=e^{-ie\phi^{(0)}}\Psi^{(0)} \tag{42}$$

are related by

$$\Psi^{(\alpha)} = e^{-ie(\phi^{(0)}-\phi^{(\alpha)})}\Psi^{(0)} \tag{43}$$

Integrating (41) and substituting to (43) we have gauge transformation between equations (33) and the same equation with $\alpha = 0$:

$$\Psi^{(\alpha)} = \exp\left(-i\frac{2\theta e^2}{\kappa^2}\alpha\int^x \rho dx'\right)\Psi^{(0)} \tag{44}$$

From this relation we can connect two samples of equation (33) with different constants α and β

$$\Psi^{(\alpha)} = \exp\left(-i\frac{2\theta e^2}{\kappa^2}(\alpha-\beta)\int^x \rho dx'\right)\Psi^{(\beta)} \tag{45}$$

Indeed one can check easily from

$$\bar{\Psi}^{(\alpha)}\Psi^{(\alpha)} = \bar{\Psi}^{(\beta)}\Psi^{(\beta)},\ \ \bar{\Psi}^{(\alpha)}(\partial_1 + i\nu\alpha\rho)\Psi^{(\alpha)} = \bar{\Psi}^{(\beta)}(\partial_1 + i\nu\alpha\rho)\Psi^{(\beta)} \tag{46}$$

that $\rho^{(\alpha)} = \rho^{(\beta)}$, $J^{(\alpha)} = J^{(\beta)}$, and

$$(\partial_1 + i\nu\alpha\rho)\Psi^{(\alpha)} = e^{-i\nu(\alpha-\beta)\int^x \rho dx'}(\partial_1 + i\nu\beta\rho)\Psi^{(\beta)} \tag{47}$$

$$(\partial_0 + i\nu(1-\alpha)J)\Psi^{(\alpha)} = e^{-i\nu(\alpha-\beta)\int^x \rho dx'}(\partial_0 + i\nu(1-\beta)J)\Psi^{(\beta)} \tag{48}$$

where $\nu \equiv \frac{2\theta e^2}{\kappa^2}$

The gauge transformation (45) for the Madelung representation implies

$$S^{(\alpha)} - S^{(\beta)} = \nu(\alpha-\beta)\int^x \rho dx' + 2\pi n,\ \ R^{(\alpha)} = R^{(\beta)} \tag{49}$$

For $s < 1$ it gives $U(1)$ gauge transformation for (36) in the form

$$\tilde{\Psi}^{(\alpha)} = \tilde{\Psi}^{(\beta)} e^{-i\frac{\nu}{\sqrt{1-s}}(\alpha-\beta)\int^x \rho dx'} e^{-i\frac{2\pi n}{\sqrt{1-s}}} \tag{50}$$

The last multiplier can be absorbed by the global phase transformation on Ψ.

For $s > 1$ the above $U(1)$ gauge transformation give rise to the local $SO(1,1)$ scale transformation (the Weyl transformation) for equation (38)

$$E^{\pm(\alpha)} = E^{\pm(\beta)} e^{\pm\frac{\nu}{\sqrt{s-1}}(\alpha-\beta)\int^x \rho dx'} e^{\pm\frac{2\pi n}{\sqrt{s-1}}} \tag{51}$$

4 Integrable DRD Systems

It was shown above that the one dimensional problem of anyons in quantum potential with a specific form of the three-body interaction, can be reduced to the general resonant DNLS equation.

4.1 General Resonant DNLS

This equation

$$
\begin{aligned}
i\Psi_{\tilde{t}} + \Psi_{xx} + i\tilde{\nu}\left[(2\alpha+1)|\Psi|^2\Psi_x + (2\alpha-1)\Psi^2\bar{\Psi}_x\right] \\
+4\tilde{\nu}^2(\alpha-\frac{1}{2})(\alpha-\frac{3}{2})|\Psi|^4\Psi = s\frac{|\Psi|_{xx}}{|\Psi|}\Psi
\end{aligned}
\tag{52}
$$

is integrable for any values of parameter α.

4.2 The Resonant Case

For special case $s > 1$, by the Madelung transformation $\Psi = e^{R-iS}$ and introduction of two new real functions $E^+ = e^{R+S}$, $E^- = e^{R-S}$ we get the general DRD system

$$
\begin{gathered}
\mp E_t^{\pm} + E_{xx}^{\pm} \\
\mp \tfrac{2\theta e^2}{\kappa^2\sqrt{s-1}}\left[(2\alpha+1)E^+E^-E_x^{\pm} + (2\alpha-1){E^{\pm}}^2E_x^{\mp}\right] \\
-4\tfrac{\theta^2 e^4}{\kappa^4(s-1)}(\alpha-\tfrac{1}{2})(\alpha-\tfrac{3}{2})(E^+E^-)^2E^{\pm} = 0,
\end{gathered}
\tag{53}
$$

where θ is the statistical parameter.

This system has particular reductions

1. DRD-I, $(\alpha = 3/2)$

$$ -E_t^+ + E_{xx}^+ - 2\nu(E^+E^-E^+)_x = 0, \tag{54} $$

$$ +E_t^- + E_{xx}^- + 2\nu(E^+E^-E^-)_x = 0 \tag{55} $$

2. DRD-II, $(\alpha = 1/2)$

$$ -E_t^+ + E_{xx}^+ - 2\nu E^+E^-E_x^+ = 0, \tag{56} $$

$$ +E_t^- + E_{xx}^- + 2\nu E^+E^-E_x^- = 0 \tag{57} $$

3. DRD-III, $(\alpha = -1/2)$

$$ -E_t^+ + E_{xx}^+ + 2\nu {E^+}^2E_x^- - 2\nu^2(E^+E^-)^2E^+ = 0, \tag{58} $$

$$ +E_t^- + E_{xx}^- - 2\nu {E^-}^2E_x^+ - 2\nu^2(E^+E^-)^2E^- = 0 \tag{59} $$

4. JRD, $(\alpha = 0)$

$$
\begin{aligned}
\mp E_t^{\pm} + E_{xx}^{\pm} - \nu\left[E_x^+E^- - E^+E_x^-\right]E^{\pm} \\
-\frac{3\nu^2}{4}(E^+E^-)^2E^{\pm} = 0
\end{aligned}
\tag{60}
$$

5 Resonant Hydrodynamic Systems

To find hydrodynamic form of the above equations we introduce velocity variables according to the Cole-Hopf transformation

$$v^+ = (\ln E^+)_x, \;\; v^- = (\ln E^-)_x \tag{61}$$

and density

$$\rho = E^+ E^-. \tag{62}$$

Then by identity

$$\rho_x = \rho v^+ + \rho v^-, \tag{63}$$

we can rewrite the DRD system in a closed form for only one of the couples of hydrodynamic variables (ρ, v^+) or (ρ, v^-).

5.1 Hydrodynamic Form for DRD-I

For DRD-I case it gives the new hydrodynamic system

$$\begin{aligned} v_t^+ &= [v_x^+ + (v^+)^2 - 2\nu(\rho_x + \rho v^+)]_x, \\ \rho_t + \rho_{xx} &= [2\rho v^+ - 3\nu\rho^2]_x. \end{aligned} \tag{64}$$

5.2 Hydrodynamic Form for DRD-II

For DRD-II case first we get the coupled heat equation with transport

$$\begin{aligned} -E_t^+ + E_{xx}^+ - 2\nu\rho E_x^+ &= 0, \\ \rho_t + \rho_{xx} &= (2\rho(\ln E^+)_x - \nu\rho^2)_x. \end{aligned} \tag{65}$$

Then the hydrodynamic form for this system is

$$\begin{aligned} v_t^+ &= [v_x^+ + (v^+)^2 - 2\nu\rho v^+]_x, \\ \rho_t + \rho_{xx} &= [2\rho v^+ - \nu\rho^2]_x. \end{aligned} \tag{66}$$

5.3 Hydrodynamic Form for DRD-III

For DRD-III case it gives the new hydrodynamic system

$$\begin{aligned} v_t^+ &= [v_x^+ + (v^+)^2 + 2\nu(\rho - \rho v^+) - 2\nu^2\rho^2]_x, \\ \rho_t + \rho_{xx} &= [2\rho v^+ + \nu\rho^2]_x. \end{aligned} \tag{67}$$

5.4 Hydrodynamic Form for JRD

For JRD case it gives the new hydrodynamic system

$$\begin{aligned} v_t^+ &= [v_x^+ + (v^+)^2 - \nu(2\rho v^+ - \rho_x) - \frac{3}{4}\nu^2\rho^2]_x, \\ \rho_t + \rho_{xx} &= [2\rho v^+]_x. \end{aligned} \tag{68}$$

In all above cases for v^- we have the system with replaced $t \to -t$, $\nu \to -\nu$.

5.5 Generic Case

For the generic case of arbitrary α firstly we have the system

$$\begin{aligned} - E_t^+ + E_{xx}^+ - \nu[2\rho E_x^+ + (2\alpha - 1)\rho_x E^+] \\ -\nu^2(\alpha - \frac{1}{2})(\alpha - \frac{3}{2})\rho^2 E^+ = 0, \\ \rho_t = [2\rho(\ln E^+)_x - \rho_x - 2\nu\alpha\rho^2]_x. \end{aligned} \tag{69}$$

It gives the new hydrodynamic system

$$\begin{aligned} v_t^+ = [v_x^+ + (v^+)^2 - \nu(2\rho v^+ + (2\alpha - 1)\rho_x) \\ -\nu^2(\alpha - \frac{1}{2})(\alpha - \frac{3}{2})\rho^2]_x, \\ \rho_t + \rho_{xx} = [2\rho v^+ - 2\nu\alpha\rho^2]_x. \end{aligned} \tag{70}$$

6 RNLS and Broer-Kaup system

The RNLS for $s > 1$ can be transformed to the reaction-diffusion system

$$R_t^+ = R_{xx}^+ + 2\nu R^+ R^- R^+, \tag{71}$$

$$-R_t^- = R_{xx}^- + 2\nu R^+ R^- R^-. \tag{72}$$

By substitution $v^+ = (\ln E^+)_x,\ \ \rho = E^+ E^-$, it can be transformed to the the hydrodynamic form as the Broer-Kaup system, [9], [10],

$$\begin{aligned} v_t^+ = (v_x^+ + (v^+)^2)_x + 2\nu\rho_x, \\ \rho_t + \rho_{xx} = (2\rho v^+)_x. \end{aligned} \tag{73}$$

If $v^- = (\ln E^-)_x,\ \ \rho = E^+ E^-$, then we have

$$\begin{aligned} - v_t^- = (v_x^- + (v^-)^2)_x + 2\nu\rho_x, \\ -\rho_t + \rho_{xx} = (2\rho v^-)_x. \end{aligned} \tag{74}$$

7 Relation with Broer-Kaup System

Given $E^+(x,t)$, $E^-(x,t)$ satisfying general DRD system (53), then real functions

$$\begin{aligned} R^+ &= E^+ e^{-(\alpha+\frac{1}{2})\nu \int^x E^+ E^-}, \\ R^- &= \left[E_x^- + (\alpha - \frac{1}{2})\nu E^+ E^- E^+\right] e^{(\alpha+\frac{1}{2})\nu \int^x E^+ E^-} \end{aligned} \tag{75}$$

or

$$
\begin{aligned}
R^+ &= \left[-E_x^+ + (\alpha - \frac{1}{2})\nu E^+ E^- E^+\right] e^{-(\alpha+\frac{1}{2})\nu \int^x E^+ E^-}, \\
R^- &= E^- e^{(\alpha+\frac{1}{2})\nu \int^x E^+ E^-} \qquad (76)
\end{aligned}
$$

satisfy the reaction-diffusion (RD) system

$$
\begin{aligned}
R_t^+ &= R_{xx}^+ + 2\nu R^+ R^- R^+, \qquad (77) \\
-R_t^- &= R_{xx}^- + 2\nu R^+ R^- R^-. \qquad (78)
\end{aligned}
$$

From this fact we can get next result.

If v_E^+ and ρ_E satisfy (70) then v_R^+ and ρ_R determined by

$$
\begin{aligned}
v_R^+ &= v_E^+ - (\alpha + \frac{1}{2})\nu \rho_E, \qquad (79) \\
\rho_R &= (\rho_E)_x - \rho_E v_E^+ + (\alpha - \frac{1}{2})\nu \rho_E^2, \qquad (80)
\end{aligned}
$$

is solution of the Broer-Kaup system (73). For v_E^- and ρ_E satisfying the analog of system (70),

$$
\begin{aligned}
v_R^- &= v_E^- + (\alpha + \frac{1}{2})\nu \rho_E + [\ln(v_E^- + (\alpha - \frac{1}{2}))\nu \rho_E]_x, \qquad (81) \\
\rho_R &= \rho_E v_E^- + (\alpha - \frac{1}{2})\nu \rho_E^2 \qquad (82)
\end{aligned}
$$

is solution of (74).

Similar way we can get result.

If v_E^+ and ρ_E satisfy (70) then v_R^+ and ρ_R determined by

$$
\begin{aligned}
v_R^+ &= v_E^+ - (\alpha + \frac{1}{2})\nu \rho_E + [\ln(-v_E^+ + (\alpha - \frac{1}{2}))\nu \rho_E]_x, \qquad (83) \\
\rho_R &= -\rho_E v_E^+ + (\alpha - \frac{1}{2})\nu \rho_E^2 \qquad (84)
\end{aligned}
$$

is solution of the Broer-Kaup system (73). For v_E^- and ρ_E satisfying the analog of system (70),

$$
\begin{aligned}
v_R^- &= v_E^- + (\alpha + \frac{1}{2})\nu \rho_E, \qquad (85) \\
\rho_R &= -(\rho_E)_x + \rho_E v_E^- + (\alpha - \frac{1}{2})\nu \rho_E^2 \qquad (86)
\end{aligned}
$$

is solution of (74).

8 Bäcklund Transformation

When $\rho \equiv 0$, both systems (70) and (73) reduce to the Burgers equation. Then the above Miura type transformations reduce to the auto-Bäcklund transformations

$$v_R^+ = v_E^+ + (\ln v_E^+)_x, \;\; v_R^- = v_E^- + (\ln v_E^-)_x \tag{87}$$

for the Burgers and anti-Burgers equations correspondingly.

9 Classical Bousinesque Systems

If in (73) we change variables

$$p^+ = v_x^+ + 2\nu\rho, \tag{88}$$

then we get the classical Bousinesque system

$$v_t^+ = ((v^+)^2 + p^+)_x, \tag{89}$$

$$p_t^+ = (v_{xx}^+ + 2p^+v^+)_x. \tag{90}$$

Similar way in (74) by variable change

$$p^- = v_x^- + 2\nu\rho \tag{91}$$

we get

$$-v_t^- = ((v^-)^2 + p^-)_x, \tag{92}$$

$$-p_t^- = (v_{xx}^- + 2p^-v^-)_x. \tag{93}$$

10 Bilinear Form and Solitons

By substitution $E^\pm = g^\pm/f^\pm$ to (69) we have bilinear representation

$$(\mp D_{\tilde{t}} + D_x^2)(g^\pm \cdot f^\pm) = 0, \tag{94}$$

$$D_x^2(f^+ \cdot f^-) + \frac{1}{2}D_x(g^+ \cdot g^-) = 0, \tag{95}$$

$$D_x(f^+ \cdot f^-) + \alpha g^+ g^- = 0, \tag{96}$$

where $\alpha = \frac{1}{2}$ (DRD-II case), or $\alpha = -\frac{1}{2}$ (DRD-III case). We note that only in these two cases the Hirota substitution has simple bilinear form. Then for solution of the hydrodynamics systems (66) and (67) we have

$$v^+ = (\ln E^+)_x = \frac{g_x^+}{g^+} - \frac{f_x^+}{f^+}, \tag{97}$$

$$\rho = E^+E^- = (\ln \frac{f^+}{f^-})_x. \tag{98}$$

Bilinearization for arbitrary α can be derived by the gauge transformation, so that

$$E^+ = \frac{g^+}{(f^+)^{\frac{1}{2}+\alpha}(f^-)^{\frac{1}{2}-\alpha}}, E^- = \frac{g^-}{(f^+)^{\frac{1}{2}-\alpha}(f^-)^{\frac{1}{2}+\alpha}} \tag{99}$$

It implies next substitution for equation (70)

$$v^+ = (\ln E^+)_x = \frac{g_x^+}{g^+} - (\frac{1}{2}+\alpha)\frac{f_x^+}{f^+} - (\frac{1}{2}-\alpha)\frac{f_x^-}{f^-}, \tag{100}$$

$$\rho = E^+E^- = (\ln\frac{f^+}{f^-})_x. \tag{101}$$

10.1 One Soliton Solution

For one soliton solution we have

$$g^\pm = e^{\eta_1^\pm},\ f^\pm = 1 + e^{\phi_{11}^\pm} e^{\eta_1^+ + \eta_1^-}, \tag{102}$$

where $e^{\phi_{11}^\pm} = \mp\frac{k_1^\pm}{(k_1^+ + k_1^-)^2}$, $\eta_1^\pm = k_1^\pm x \pm (k_1^\pm)^2 t + \eta_1^{\pm(0)}$. For regularity of this solution we choose conditions $k_1^- > 0$ and $k_1^+ < 0$, then $-\tilde{v} < k < \tilde{v}$, where $k = k_1^+ + k_1^-$, $\tilde{v} = k_1^- - k_1^+$, $-kx_0^\pm = \eta_1^{+(0)} + \eta_1^{-(0)} + \phi_{11}^\pm$. Then velocity is positive $\tilde{v} > 0$, so that our dissipaton is chiral. For the density we have soliton solution

$$\rho = E^+E^- = \frac{k^2}{\sqrt{\tilde{v}^2 - k^2}\cosh k(x - \tilde{v}t - x_0) + \tilde{v}} \tag{103}$$

where $2x_0 = x_0^+ + x_0^-$, and for velocity field

$$v^+ = \frac{k_1^+ - k_1^- e^{\phi_{11}^+} e^{\eta_1^+ + \eta_1^-}}{1 + e^{\phi_{11}^+} e^{\eta_1^+ + \eta_1^-}}, \tag{104}$$

the kink solution

$$v^+ = -\frac{\tilde{v}}{2} - \frac{k}{2}\tanh\frac{k}{2}(x - \tilde{v}t - x_0). \tag{105}$$

10.2 Integrals of Motion

The particle number, momentum and energy integrals are given respectively

$$N = \int_{-\infty}^{\infty} \rho dx = -\frac{1}{\nu}\ln\frac{f^+}{f^-}|_{-\infty}^{\infty} \tag{106}$$

$$P = -\int_{-\infty}^{\infty} \rho v^+ dx = \frac{1}{2\nu}\ln(f^+f^-)_x|_{-\infty}^{\infty} \tag{107}$$

$$E = -\int_{-\infty}^{\infty} [\rho(v^+)^2 - \rho_x v^+ - \nu\rho^2 v^+]dx. \tag{108}$$

Then substituting for one soliton solution we find

$$N = \frac{1}{\nu} \ln \frac{\tilde{v} + |k|}{\tilde{v} - |k|}, \;\; P = \frac{|k|}{\nu}, \;\; E = \frac{\tilde{v}|k|}{2\nu}. \tag{109}$$

The mass of soliton $M = |k|/(\nu\tilde{v})$ in terms of particle number becomes $M = \frac{1}{\nu} \tanh \frac{N\nu}{2}$, and for the momentum and the energy we have non-relativistic free particle form $P = M\tilde{v}, \;\; E = \frac{M\tilde{v}^2}{2}$.

For the process of fusion or fission of two solitons then the next conditions should be valid

$$N = N_1 + N_2, \;\; P = P_1 + P_2, \;\; E = E_1 + E_2 \tag{110}$$

Using (109) after some algebraic manipulations we get the resonance condition

$$|\tilde{v}_1 - \tilde{v}_2| = |k_1| + |k_2| \tag{111}$$

where $\tilde{v}_a = k_a^- - k_a^+$, $k_a = k_a^- + k_a^+$, $a = 1, 2$.

10.3 Two Soliton Solution

For two soliton solution we have

$$g^{\pm} = e^{\eta_1^{\pm}} + e^{\eta_2^{\pm}} + \alpha_1^{\pm} e^{\eta_2^+ + \eta_2^- + \eta_1^{\pm}} + \alpha_2^{\pm} e^{\eta_1^+ + \eta_1^- + \eta_2^{\pm}}, \tag{112}$$

$$f^{\pm} = 1 + \sum_{i,j=1}^{2} e^{\phi_{ij}^{\pm}} e^{\eta_i^+ + \eta_j^-} + \beta^{\pm} e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \tag{113}$$

where $\eta_i^{\pm} = k_i^{\pm} x \pm (k_i^{\pm})^2 t + \eta_{i0}^{\pm}$, $k_{ij}^{nm} \equiv (k_i^n + k_j^m)$ and

$$\alpha_1^{\pm} = \pm \frac{1}{2} \frac{k_2^{\mp}(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-})^2 (k_{12}^{\pm\mp})^2}, \;\; \alpha_2^{\pm} = \pm \frac{1}{2} \frac{k_1^{\mp}(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-})^2 (k_{21}^{\pm\mp})^2}, \tag{114}$$

$$\beta^{\pm} = \frac{(k_1^+ - k_2^+)^2 (k_1^- - k_2^-)^2}{4(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-})^2} k_1^{\pm} k_2^{\pm}, \tag{115}$$

$$e^{\phi_{ii}^{\pm}} = \mp \frac{k_i^{\pm}}{2(k_{ii}^{+-})^2}, \;\; e^{\phi_{ij}^{+}} = \frac{-k_i^{+}}{2(k_{ij}^{+-})^2}, \;\; e^{\phi_{ij}^{-}} = \frac{k_j^{-}}{2(k_{ij}^{+-})^2}. \tag{116}$$

By regularity we have $k_i^+ \leq 0$, $k_i^- \geq 0$ in the Case 1, and $k_i^+ \geq 0$, $k_i^- \leq 0$ in the Case 2. Then solving the resonance condition (111) we find that for every solution of this algebraic equation, the coefficient β vanishes or becomes infinite. In both cases two soliton solution reduces to the one soliton solution. Hence the solution describes a collision of two solitons propagating in the same direction and at some value of parameters creating the resonance states (see Fig.1 and Fig.2).

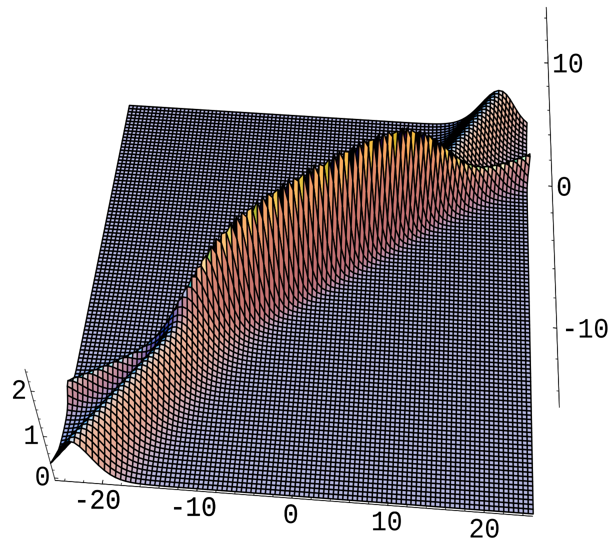

Figure 1: 3D plot of typical soliton resonant state with one soliton resonance

11 Conclusions

The problem of chiral solitons in quantum potential, as a reduction of 2+1 dimensional Chern-Simons theory, was formulated in terms of family of integrable derivative NLS equations by the Madelung fluid representation. By using new, non-Madelung fluid representation we constructed integrable family of hydrodynamical systems of the Kaup-Broer type. By bilinear method we found resonance character of corresponding chiral soliton mutual interaction.

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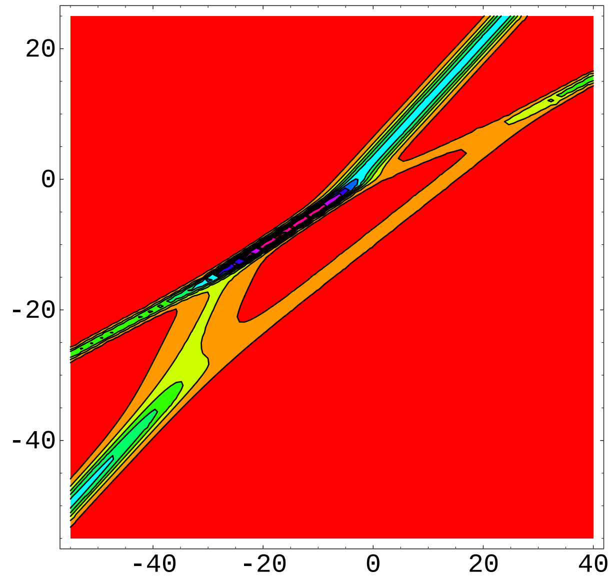

Figure 2: Contour plot of four soliton resonances

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