Ultradiscrete Plücker Relation Specialized for Soliton Solutions

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Abstract

We propose an ultradiscrete analogue of Plücker relation specialized for soliton solutions. It is expressed by an ultradiscrete permanent which is obtained by ultradiscretizing the permanent, that is, the signature-free determinant. Using this relation, we also show soliton solutions to the ultradiscrete KP equation and the ultradiscrete two-dimensional Toda lattice equation respectively.

1 Introduction

Soliton equations have been researched for several decades. There are many equations expressed by different levels of discreteness. Now we have continuous, semi-discrete, discrete and ultradiscrete soliton equations. The continuous soliton equation is expressed by a partial differential equation and the semi-discrete soliton equation by a system of ordinary or partial differential equations. The Kadomtsev-Petviashvili (KP) equation and the two-dimensional Toda lattice equation are continuous and semi-discrete respectively and they are fundamental for the soliton theory[1, 2]. These equations are transformed into bilinear forms, and their solutions are expressed by Wronski determinants.

In general, soliton solutions in the determinant form obey Plücker relations and the relations are transformed into the soliton equations replacing the operations on the determinants by the differential or difference operators[3, 4]. This structure enables us to view the hierarchy and the common structure of soliton equations. In fact, many soliton equations including the Korteweg-de Vries (KdV) equation, the Toda lattice equation and the sine-Gordon equation are obtained from the KP equation or the two-dimensional Toda lattice equation by the reduction of variables.

Discrete soliton equation is an equation of which independent variables are all discrete. The discrete soliton equation is also expressed by the bilinear form and its determinant solution satisfies the Plücker relation. In this case, the solution is expressed by the Casorati determinant.

Ultradiscrete soliton equation is an equation of which all dependent and independent variables can take integer values. It is derived from a discrete soliton equation by the ultradiscretization[5], which is a limiting procedure of dependent variable using a key formula,

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{a/\varepsilon} + e^{b/\varepsilon}) = \max(a, b). \tag{1}$$

Ultradiscrete soliton equation has also soliton solutions[6, 7]. Some interesting properties on the equation are discovered recently. For instance, Nakamura discovered a soliton solution with a periodic phase for the ultradiscrete hungry Lotka-Volterra equation[8]. Nakata proposed the vertex operator for the ultradiscrete KdV (uKdV) equation or the non-autonomous ultradiscrete KP (uKP) equation and showed their solutions[9, 10].

Moreover, the authors and Hirota proposed the ultradiscrete analogue of determinant solutions though the determinant cannot be ultradiscretized directly[11, 12, 13]. Instead of the determinant,

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they used an ultradiscrete permanent (UP) defined by

$$\max[a_{ij}]_{1\le i,j\le N} \equiv \max_{\pi} \sum_{1\le i\le N} a_{i\pi_i}, \tag{2}$$

where a_{ij} is an arbitrary $N\times N$ matrix and $\pi = \{\pi_1, \pi_2, \dots, \pi_N\}$ is an arbitrary permutation of 1, 2, …, N. The soliton solutions in the UP form for the uKdV equation and the ultradiscrete Toda equation are shown in [11, 12]. There exist Bäcklund transformations of ultradiscrete soliton equations[13].

The (i, j) element of these UP soliton solutions is generally expressed by $|y_i + jr_i|$, where y_i and r_i are arbitrary parameters, and $|x|$ denotes an absolute value of x. For example, the soliton solution to the uKdV equation is given by

$$f_i^n = \max \begin{bmatrix} |s_1(n,i)+2p_1| & |s_1(n,i)+4p_1| & \dots & |s_1(n,i)+2Np_1| \\ \dots & \dots & \dots & \dots \\ |s_N(n,i)+2p_N| & |s_N(n,i)+4p_N| & \dots & |s_N(n,i)+2Np_N| \end{bmatrix}, \tag{3}$$

where

$$s_j(n,i) = p_j n - q_j i + c_j \qquad q_j = \frac{1}{2}(|p_j+1| - |p_j-1|). \tag{4}$$

Though the expression of an ultradiscrete solution is analogous to that of discrete solution, we have not established the ultradiscretized Plücker relation. Therefore, we have used the individual method to find the solution for every ultradiscrete soliton equation.

This is due to the differences of basic operations between the determinant and the UP. We show an example of such differences as follows. The determinant satisfy

$$\begin{vmatrix} a_{11} & a_{11}+a_{12} \\ a_{21} & a_{21}+a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{5}$$

for any a_{ij} $(1 \le i, j \le 2)$. When we consider the UP corresponding to the left-hand side of (5), we have

$$\max \begin{bmatrix} a_{11} & \max(a_{11}, a_{12}) \\ a_{21} & \max(a_{21}, a_{22}) \end{bmatrix}. \tag{6}$$

Then, using a property of UP

$$\begin{aligned} &\max[\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{j-1} \ \max(\boldsymbol{b}_j, \boldsymbol{b}'_j) \ \boldsymbol{b}_{j+1} \ \dots \ \boldsymbol{b}_N] \\ =&\max\big(\max[\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{j-1} \ \boldsymbol{b}_j \ \boldsymbol{b}_{j+1} \ \dots \ \boldsymbol{b}_N], \ \max[\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{j-1} \ \boldsymbol{b}'_j \ \boldsymbol{b}_{j+1} \ \dots \ \boldsymbol{b}_N]\big), \end{aligned} \tag{7}$$

where $\boldsymbol{b}_j$ and $\boldsymbol{b}'_j$ $(1 \le j \le N)$ are arbitrary N-dimensional vectors and $\max(\boldsymbol{b_j}, \boldsymbol{b'_j})$ denotes

$$\max(\boldsymbol{b_j}, \boldsymbol{b'_j}) \equiv \begin{pmatrix} \max(b_1, b'_1) \\ \max(b_2, b'_2) \\ \dots \\ \max(b_N, b'_N) \end{pmatrix}, \tag{8}$$

we can expand (6),

$$\begin{aligned} \max \begin{bmatrix} a_{11} & \max(a_{11}, a_{12}) \\ a_{21} & \max(a_{21}, a_{22}) \end{bmatrix} &= \max\left(\max \begin{bmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{bmatrix}, \ \max \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \\ &= \max(a_{11}+a_{21}, a_{11}+a_{22}, a_{12}+a_{21}). \end{aligned} \tag{9}$$

In contrast to the determinant case, the first argument in the right-hand side cannot be neglected. Hence (6) is not always equal to

$$\max\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{10}$$

and it means UP does not have the relation such as (5).

The above kind of differences cause many troubles when we verify the solutions. For example, one of the simplest Plücker relations is

$$\begin{aligned} &|\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_1| \times |\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_2\ \boldsymbol{b}_3| \\ -&|\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_2| \times |\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_1\ \boldsymbol{b}_3| \\ +&|\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_3| \times |\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_1\ \boldsymbol{b}_2| = 0, \end{aligned} \tag{11}$$

for any N-dimensional column vectors $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$. However, the similar identity does not exist for the UP case. Instead, Hirota showed UP's satisfy the following identity[3][14]:

$$\begin{aligned} &\max\big(\max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_1] + \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_2\ \boldsymbol{b}_3], \\ &\qquad \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_2] + \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_1\ \boldsymbol{b}_3]\big) \\ &= \max\big(\max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_1] + \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_2\ \boldsymbol{b}_3], \\ &\qquad \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_3] + \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_1\ \boldsymbol{b}_2]\big) \\ &= \max\big(\max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_2] + \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_1\ \boldsymbol{b}_3], \\ &\qquad \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_3] + \max[\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_1\ \boldsymbol{b}_2]\big). \end{aligned} \tag{12}$$

This identity is not useful for the verification on ultradiscrete solutions since the anti-symmetry does not hold as shown in (5) for determinants.

In this article, we consider a general UP expression specialized for ultradiscrete soliton solutions. The (i, j) element of the specialized UP is defined by $|y_i + jr_i|$ where y_i and r_i are arbitrary constants. Imposing this condition, we give a relation which corresponds to (11) in Section 2. We call this relation the conditional ultradiscrete Plücker relation. In Section 3 and 4, we present UP soliton solutions to the uKP equation and the ultradiscrete two-dimensional (u2D) Toda lattice equation respectively, and show that these solutions are verified by means of the conditional uPlücker relation. Finally, we give the concluding remarks in Section 5.

2 Conditional ultradiscrete Plücker relation

We give the following theorem in this section.

Theorem 2.1 *Let $\boldsymbol{x}_j$ be an N-dimensional vector defined by*

$$\boldsymbol{x}_j = \begin{pmatrix} |y_1 + jr_1| \\ |y_2 + jr_2| \\ \dots \\ |y_N + jr_N| \end{pmatrix} \qquad (y_i,\ r_i\ :\ \text{arbitrary constants}). \tag{13}$$

Then

$$\begin{aligned} &\max[\boldsymbol{x}_1\ \dots\ \widehat{\boldsymbol{x}_{k_2}}\ \dots\ \boldsymbol{x}_{N+1}] + \max[\boldsymbol{x}_1\ \dots\ \widehat{\boldsymbol{x}_{k_1}}\ \dots\ \widehat{\boldsymbol{x}_{k_3}}\ \dots\ \boldsymbol{x}_{N+2}] \\ = \max\big(&\max[\boldsymbol{x}_1\ \dots\ \widehat{\boldsymbol{x}_{k_3}}\ \dots\ \boldsymbol{x}_{N+1}] + \max[\boldsymbol{x}_1\ \dots\ \widehat{\boldsymbol{x}_{k_1}}\ \dots\ \widehat{\boldsymbol{x}_{k_2}}\ \dots\ \boldsymbol{x}_{N+2}], \\ &\max[\boldsymbol{x}_1\ \dots\ \widehat{\boldsymbol{x}_{k_1}}\ \dots\ \boldsymbol{x}_{N+1}] + \max[\boldsymbol{x}_1\ \dots\ \widehat{\boldsymbol{x}_{k_2}}\ \dots\ \widehat{\boldsymbol{x}_{k_3}}\ \dots\ \boldsymbol{x}_{N+2}]\big) \end{aligned} \tag{14}$$

[3]Hirota gives an identity of ultradiscrete analogue of Pfaffian in [14], and it reduces to (12) with proper conditions. We give another proof in terms of UP in Appendix A.

holds. Here $1 \le k_1 < k_2 < k_3 \le N+1$ and the symbol $\widehat{\boldsymbol{x}_{k_j}}$ means that $\boldsymbol{x}_{k_j}$ is omitted.

Let us call (14) 'conditional ultradiscrete Plücker(uPlücker) relation'. We note (14) can be rewritten as

$$\begin{aligned}&\max[M\ \boldsymbol{x}_{k_1}\ \boldsymbol{x}_{k_3}] + \max[M\ \boldsymbol{x}_{k_2}\ \boldsymbol{x}_{N+2}]\\ =&\max\big(\max[M\ \boldsymbol{x}_{k_1}\ \boldsymbol{x}_{k_2}] + \max[M\ \boldsymbol{x}_{k_3}\ \boldsymbol{x}_{N+2}],\ \max[M\ \boldsymbol{x}_{k_2}\ \boldsymbol{x}_{k_3}] + \max[M\ \boldsymbol{x}_{k_1}\ \boldsymbol{x}_{N+2}]\big)\end{aligned} \tag{15}$$

with an $N \times (N-2)$ matrix M defined by

$$M \equiv [\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_1}}\ \ldots\ \widehat{\boldsymbol{x}_{k_2}}\ \ldots\ \widehat{\boldsymbol{x}_{k_3}}\ \ldots\ \boldsymbol{x}_{N+1}]. \tag{16}$$

In order to prove Theorem 2.1, we give several lemmas.

Lemma 2.1 *If an inequality*

$$\begin{aligned}&\max[\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_2}}\ \ldots\ \boldsymbol{x}_{N+1}] + \max[\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_1}}\ \ldots\ \widehat{\boldsymbol{x}_{k_3}}\ \ldots\ \boldsymbol{x}_{N+2}]\\ \ge&\max[\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_3}}\ \ldots\ \boldsymbol{x}_{N+1}] + \max[\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_1}}\ \ldots\ \widehat{\boldsymbol{x}_{k_2}}\ \ldots\ \boldsymbol{x}_{N+2}]\end{aligned} \tag{17}$$

holds, then (14) holds.

Lemma 2.2 *The relation (14) can be rewritten as*

$$\begin{aligned}&\max[\boldsymbol{x}_2\ \ldots\ \widehat{\boldsymbol{x}_{k_2}}\ \ldots\ \boldsymbol{x}_{N+2}] + \max[\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_1}}\ \ldots\ \widehat{\boldsymbol{x}_{k_3}}\ \ldots\ \boldsymbol{x}_{N+2}]\\ =\max\big(&\max[\boldsymbol{x}_2\ \ldots\ \widehat{\boldsymbol{x}_{k_3}}\ \ldots\ \boldsymbol{x}_{N+2}] + \max[\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_1}}\ \ldots\ \widehat{\boldsymbol{x}_{k_2}}\ \ldots\ \boldsymbol{x}_{N+2}],\\ &\max[\boldsymbol{x}_2\ \ldots\ \widehat{\boldsymbol{x}_{k_1}}\ \ldots\ \boldsymbol{x}_{N+2}] + \max[\boldsymbol{x}_1\ \ldots\ \widehat{\boldsymbol{x}_{k_2}}\ \ldots\ \widehat{\boldsymbol{x}_{k_3}}\ \ldots\ \boldsymbol{x}_{N+2}]\big),\end{aligned} \tag{18}$$

where $1 < k_1 < k_2 < k_3 \le N+2$.

Lemma 2.3 *If*

$$0 \le |r_1| \le |r_2| \le \cdots \le |r_{N-1}| \le r_N, \tag{19}$$

then the Nth-order UP can be reduced to the $(N-1)$th-order UP as

$$\begin{aligned}\max[\boldsymbol{x}_{j_1}\ \boldsymbol{x}_{j_2}\ \ldots\ \boldsymbol{x}_{j_N}] = \max\big(&y_N + j_N r_N + \max[\tilde{\boldsymbol{x}}_{j_1}\ \tilde{\boldsymbol{x}}_{j_2}\ \ldots\ \tilde{\boldsymbol{x}}_{j_{N-1}}],\\ &-y_N - j_1 r_N + \max[\tilde{\boldsymbol{x}}_{j_2}\ \tilde{\boldsymbol{x}}_{j_3}\ \ldots\ \tilde{\boldsymbol{x}}_{j_N}]\big),\end{aligned} \tag{20}$$

where $j_1 < j_2 < \cdots < j_N$ and $\tilde{\boldsymbol{x}}_j$ denotes an $(N-1)$-dimensional vector

$$\tilde{\boldsymbol{x}}_j = \begin{pmatrix} |y_1 + jr_1| \\ |y_2 + jr_2| \\ \cdots \\ |y_{N-1} + jr_{N-1}| \end{pmatrix}. \tag{21}$$

Lemma 2.1 is derived from (12). Lemma 2.2 is obtained since each $\boldsymbol{x}_j$ of (14) can be rewritten as $\boldsymbol{x}_{-j+N+3}$ with suitable transformations. About Lemma 2.3, the UP is expressed by

$$\begin{aligned}\max[\boldsymbol{x}_{j_1}\ \boldsymbol{x}_{j_2}\ \ldots\ \boldsymbol{x}_{j_N}] &= \max_{\rho_i=\pm1,\pi_i} \sum_{1\le i\le N} \rho_i(y_i + \pi_i r_i)\\ &= \max_{\rho_i=\pm1}\Big(\sum_{1\le i\le N} \rho_i y_i + \max_{\pi_i} \sum_{1\le i\le N} \rho_i \pi_i r_i\Big),\end{aligned} \tag{22}$$

where $(\pi_1, \pi_2, \ldots, \pi_N)$ denotes an arbitrary permutation of 1, 2, …, N. The maximum of (22) is given by $\pi_N = j_N$ in the case of $\rho_N = 1$, and $\pi_N = j_1$ in the case of $\rho_N = -1$[11]. Thus we obtain Lemma 2.3.

For Lemma 2.1, Theorem 2.1 is proved if we show (17). Then let us prove (17) with a mathematical induction. Hereafter, we adopt a simple notation j for $\boldsymbol{x}_j$. For $N = 2$, one can prove

$$\max[1\ 3] + \max[2\ 4] \geq \max[1\ 2] + \max[3\ 4]. \tag{23}$$

Then let us show the inequality

$$\begin{aligned} &\max[1\ \ldots\ \widehat{k_2}\ \ldots\ N+2] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3}\ \ldots\ N+3] \\ \geq &\max[1\ \ldots\ \widehat{k_3}\ \ldots\ N+2] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_2}\ \ldots\ N+3] \end{aligned} \tag{24}$$

for $1 \leq k_1 < k_2 < k_3 \leq N+2$ under the assumptions (17) and

$$0 \leq |r_1| \leq |r_2| \leq \cdots \leq |r_N|. \tag{25}$$

We note (25) can be assumed without loss of generality.

In the case of $1 < k_1 < k_2 < k_3 < N+2$ and $r_{N+1} > |r_N|$, the UP's of the left-hand side in (24) are rewritten as

$$\begin{aligned} \max[1\ \ldots\ \widehat{k_2}\ \ldots\ N+2] = \max\big(&y_{N+1} + (N+2)r_{N+1} + \max[1\ \ldots\ \widehat{k_2}\ \ldots\ N+1], \\ &- y_{N+1} - r_{N+1} + \max[2\ \ldots\ \widehat{k_2}\ \ldots\ N+2]\big), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+3] = \max\big(&y_{N+1} + (N+3)r_{N+1} + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+2], \\ &- y_{N+1} - r_{N+1} + \max[2\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+3]\big), \end{aligned} \tag{27}$$

respectively by Lemma 2.3. Therefore, a sum of (26) and (27) is expressed by

$$\begin{aligned} &\max[1\ \ldots\ \widehat{k_2}\ \ldots\ N+2] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+3] \\ = \max\big(&2y_{N+1} + (2N+5)r_{N+1} + \max[1\ \ldots\ \widehat{k_2}\ \ldots\ N+1] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+2], \\ &- 2y_{N+1} - 2r_{N+1} + \max[2\ \ldots\ \widehat{k_2}\ \ldots\ N+2] + \max[2\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+3], \\ &(N+2)r_{N+1} + \max[2\ \ldots\ \widehat{k_2}\ \ldots\ N+2] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+2], \\ &(N+1)r_{N+1} + \max[1\ \ldots\ \widehat{k_2}\ \ldots\ N+1] + \max[2\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+3]\big). \end{aligned} \tag{28}$$

Similarly, the right-hand side in (24) is expressed by

$$\begin{aligned} \max\big(&2y_{N+1} + (2N+5)r_{N+1} + \max[1\ \ldots\ \widehat{k_3}\ \ldots\ N+1] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_2}\ \ldots\ N+2], \\ &- 2y_{N+1} - 2r_{N+1} + \max[2\ \ldots\ \widehat{k_3}\ \ldots\ N+2] + \max[2\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_2}\ \ldots\ N+3], \\ &(N+2)r_{N+1} + \max[2\ \ldots\ \widehat{k_3}\ \ldots\ N+2] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_2}\ \ldots\ N+2], \\ &(N+1)r_{N+1} + \max[1\ \ldots\ \widehat{k_3}\ \ldots\ N+1] + \max[2\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_2}\ \ldots\ N+3]\big). \end{aligned} \tag{29}$$

The first and second arguments of (28) in the right-hand side are greater than those of (29) from the assumption. The third argument of (28) in the right-hand side is also greater than that of (29) from Lemma 2.2. Moreover, the following lemma holds.

Lemma 2.4 *Inequalities*

$$\begin{aligned} r_{N+1} &+ \max[2\ \ldots\ \widehat{k_2}\ \ldots\ N+2] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+2] \\ &\geq \max[1\ \ldots\ \widehat{k_2}\ \ldots\ N+1] + \max[2\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_3} \ldots\ N+3] \end{aligned} \tag{30}$$

and

$$\begin{aligned} r_{N+1} &+ \max[2\ \ldots\ \widehat{k_3}\ \ldots\ N+2] + \max[1\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_2}\ \ldots\ N+2] \\ &\geq \max[1\ \ldots\ \widehat{k_3}\ \ldots\ N+1] + \max[2\ \ldots\ \widehat{k_1}\ \ldots\ \widehat{k_2}\ \ldots\ N+3] \end{aligned} \tag{31}$$

hold for $1 < k_1 < k_2 < k_3 < N+2$.

Lemma 2.4 is proved by a mathematical induction shown in Appendix B. Thus, the fourth argument is smaller than the third one in (28) and (29) respectively. Therefore, (24) holds in the case of $1 < k_1 < k_2 < k_3 < N+2$ and $r_{N+1} > |r_N|$. The similar procedure enable us to prove in the other cases. Hence, we obtain the conditional uPlücker relation.

3 The ultradiscrete KP equation and its UP solution

Let us consider the following tau function defined by UP.

$$\tau(l, m, n) = \max[\phi_i(l, m, n, s+j-1)]_{1 \leq i,j \leq N}, \tag{32}$$

where s is an auxiliary variable, and $\phi_i(l, m, n, s)$ is defined by

$$\phi_i(l, m, n, s) = \max(\eta_i(l, m, n, s), \eta'_i(l, m, n, s)) \tag{33}$$

with

$$\begin{aligned} \eta_i(l, m, n, s) &= p_i s + \max(0, p_i - a_1)l + \max(0, p_i - a_2)m + \max(0, p_i - a_3)n + c_i, \\ \eta'_i(l, m, n, s) &= -p_i s + \max(0, -p_i - a_1)l + \max(0, -p_i - a_2)m + \max(0, -p_i - a_3)n + c'_i. \end{aligned} \tag{34}$$

Here a_1, a_2 and a_3 are parameters satisfying $a_1 > a_2 > a_3$, and p_i, c_i and c'_i are arbitrary parameters. One can obtain the following relations:

$$\phi_i(l+1, m, n, s) = \max(\phi_i(l, m, n, s),\ \phi_i(l, m, n, s+1) - a_1), \tag{35}$$

$$\phi_i(l, m+1, n, s) = \max(\phi_i(l, m, n, s),\ \phi_i(l, m, n, s+1) - a_2), \tag{36}$$

$$\phi_i(l, m, n+1, s) = \max(\phi_i(l, m, n, s),\ \phi_i(l, m, n, s+1) - a_3) \tag{37}$$

and

$$\begin{aligned} \phi_{i_1}(l, m, n, s+j) + \phi_{i_2}(l, m, n, s+j) \leq \max(&\phi_{i_1}(l, m, n, s+j-1) + \phi_{i_2}(l, m, n, s+j+1), \\ &\phi_{i_2}(l, m, n, s+j-1) + \phi_{i_1}(l, m, n, s+j+1)) \end{aligned} \tag{38}$$

for $1 \leq i, i_1, i_2 \leq N$. We first rewrite the tau function with (35), (36), (37) and (38) in Subsection 3.1. Second we give the relation shown by the conditional uPlücker relation in Subsection 3.2. Finally, in Subsection 3.3, we give the UP solution for the uKP equation.

3.1 Rewriting the tau function

Using (35), $\tau(l+1, m, n)$ is expanded as

$$\begin{aligned} \tau(l+1, m, n) &= \max[\phi_i(l+1, m, n, s+j-1)]_{1 \leq i,j \leq N} \\ &= \max[\max(\phi_i(l, m, n, s+j-1),\ \phi_i(l, m, n, s+j) - a_1)]_{1 \leq i,j \leq N} \end{aligned} \tag{39}$$

In particular, using the simple notations,

$$\begin{pmatrix} \phi_1(l,m,n,s+j) \\ \phi_2(l,m,n,s+j) \\ \dots \\ \phi_N(l,m,n,s+j) \end{pmatrix} \equiv \begin{pmatrix} \phi_1(j) \\ \phi_2(j) \\ \dots \\ \phi_N(j) \end{pmatrix} \equiv \boldsymbol{\phi}(j), \tag{40}$$

(39) is expressed by

$$\tau(l+1,m,n) = \max[\max(\boldsymbol{\phi}(j-1),\ \boldsymbol{\phi}(j) - a_1 \cdot \mathbf{1})]_{1\le j\le N}, \tag{41}$$

where $\mathbf{1}$ and $\max(\boldsymbol{\phi}(j-1), \boldsymbol{\phi}(j))$ denote

$$\mathbf{1} \equiv \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} \tag{42}$$

and (8) respectively. Furthermore, by applying a property of UP (7) to each column in (41), $\tau(l+1,m,n)$ is expanded as the maximum of the following 2^N UP's,

$$\begin{aligned} &\max[\boldsymbol{\phi}(0) \quad \boldsymbol{\phi}(1) \quad \boldsymbol{\phi}(2) \quad \dots \quad \boldsymbol{\phi}(N-1)], \\ &\max[\boldsymbol{\phi}(1) - a_1\cdot\mathbf{1} \quad \boldsymbol{\phi}(1) \quad \boldsymbol{\phi}(2) \quad \dots \quad \boldsymbol{\phi}(N-1)], \\ &\max[\boldsymbol{\phi}(0) \quad \boldsymbol{\phi}(2) - a_1\cdot\mathbf{1} \quad \boldsymbol{\phi}(2) \quad \dots \quad \boldsymbol{\phi}(N-1)], \\ &\dots \\ &\max[\boldsymbol{\phi}(1) - a_1\cdot\mathbf{1} \quad \boldsymbol{\phi}(2) - a_1\cdot\mathbf{1} \quad \boldsymbol{\phi}(3) - a_1\cdot\mathbf{1} \quad \dots \quad \boldsymbol{\phi}(N) - a_1\cdot\mathbf{1}]. \end{aligned} \tag{43}$$

Let us call a set of the above UP's S. Moreover, using another property of UP,

$$\max[\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{j-1} \quad \boldsymbol{b}_j + c\cdot\mathbf{1} \quad \boldsymbol{b}_{j+1} \ \dots \ \boldsymbol{b}_N] = \max[\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{j-1} \ \ \boldsymbol{b}_j \ \ \boldsymbol{b}_{j+1} \ \dots \ \boldsymbol{b}_N] + c, \tag{44}$$

where $\boldsymbol{b}_j$ $(1 \le j \le N)$ is an arbitrary N-dimensional vector and c arbitrary constant, we can divide S into $N+1$ sets as

$$S = \{S_0, S_1 - a_1, S_2 - 2a_1, \dots, S_N - Na_1\}. \tag{45}$$

For example, S_0 is expressed by

$$S_0 = \{\max[0\ 1\ 2\ \dots\ N-1]\} \tag{46}$$

where j denotes $\boldsymbol{\phi}(j)$, and S_1 is

$$S_1 = \{\max[1\ 1\ 2\ \dots\ N-1],\ \max[0\ 2\ 2\ 3\ \dots\ N-1],\ \dots,\ \max[0\ 1\ 2\ \dots\ N-2\ N]\}. \tag{47}$$

About these sets of UP's, we give the following lemma.

Lemma 3.1 *An inequality*

$$\max[M\ j\ j] \le \max[M\ j-1\ j+1] \tag{48}$$

holds for any j, where M denotes an arbitrary $N \times (N-2)$ matrix.

Lemma 3.1 is proved since each UP is expanded as

$$\begin{aligned}
\max[M\ j\ j] &= \max_{\substack{1\le i_1,i_2\le N\\ i_1\ne i_2}} \Big(\max[M\ j\ j]_{\substack{i_1,i_2\\ N-1,N}} + \phi_{i_1}(j)+\phi_{i_2}(j)\Big),\\
\max[M\ j-1\ j+1] &= \max_{\substack{1\le i_1,i_2\le N\\ i_1\ne i_2}} \Big(\max[M\ j-1\ j+1]_{\substack{i_1,i_2\\ N-1,N}} + \phi_{i_1}(j-1)+\phi_{i_2}(j+1)\Big),
\end{aligned} \tag{49}$$

where $\max A_{\substack{i_1,i_2\\ N-1,N}}$ denotes the $(N-2)$th-order UP obtained by eliminating the i_1-th and i_2-th rows and the $(N-1)$-th and N-th columns from $N\times N$ matrix A. Inequality (48) is derived from

$$\max[M\ j\ j]_{\substack{i_1,i_2\\ N-1,N}} = \max[M\ j-1\ j+1]_{\substack{i_1,i_2\\ N-1,N}} \tag{50}$$

and (38).

Therefore, $\max S_1$ is determined as $\max[0\ 1\ 2\ \ldots\ N-2\ N]$ since

$$\begin{aligned}
&\max[1\ 1\ 2\ \ldots\ N-1]\\
\le&\max[0\ 2\ 2\ 3\ \ldots\ N-1]\\
\le&\cdots\\
\le&\max[0\ 1\ 2\ \ldots\ N-1\ N-1]\\
\le&\max[0\ 1\ 2\ \ldots\ N-2\ N].
\end{aligned} \tag{51}$$

holds. Similarly, other $\max S_{k_1}$ $(0\le k_1\le N)$ are determined, and $\tau(l+1,m,n)$ is reduced to the maximum of $(N+1)$ UP's and we obtain the following lemma.

Lemma 3.2 *Tau function $\tau(l+1,m,n)$ is reduced to*

$$\tau(l+1,m,n) = \max_{0\le k_1\le N}(\tau_c(N-k_1,N+1)-k_1a_1), \tag{52}$$

where $\tau_c(\alpha,\beta)$ $(\alpha<\beta)$ is the UP defined by

$$\tau_c(\alpha,\beta) = \max[0\ \ldots\ \widehat{\alpha}\ \ldots\ \widehat{\beta}\ \ldots\ N+1]. \tag{53}$$

Furthermore, using (36) and (37) respectively, $\tau(l,m+1,n+1)$ is also reduced to the maximum of $(N+1)^2$ UP's as follows.

Lemma 3.3 *Tau function $\tau(l,m+1,n+1)$ is reduced to*

$$\tau(l,m+1,n+1) = \max_{0\le k_2,k_3\le N}(\Psi(k_2,k_3)-k_2a_2-k_3a_3), \tag{54}$$

where $\Psi(k_2,k_3)$ is defined by

$$\Psi(k_2,k_3) = \begin{cases}
\max_{0\le i\le N-k_3}(\tau_c(N-k_3-i,N-k_2+1+i)) & (k_3\ge k_2, N-k_2)\\
\max_{0\le i\le k_2}(\tau_c(N-k_2-k_3+i,N+1-i)) & (N-k_2\ge k_3\ge k_2)\\
\max_{0\le i\le N-k_2}(\tau_c(N-k_2-i,N-k_3+1+i)) & (k_2\ge k_3\ge N-k_2)\\
\max_{0\le i\le k_3}(\tau_c(N-k_2-k_3+i,N+1-i)) & (k_2, N-k_2\ge k_3)
\end{cases}. \tag{55}$$

for $0\le k_2,k_3\le N$. Especially, (55) gives

$$\begin{aligned}
&\Psi(k_2-1,k_3) = \max(\Psi(k_2,k_3-1),\ \tau_c(N-k_3+1,N-k_2+1)), && (k_2>k_3)\\
&\Psi(k_2-1,k_3) = \Psi(k_2,k_3-1), && (k_2=k_3)\\
&\max(\Psi(k_2-1,k_3),\tau_c(N-k_2+1,N-k_3+1)) = \Psi(k_2,k_3-1), && (k_2<k_3)
\end{aligned} \tag{56}$$

for $1\le k_2,k_3\le N$.

The proof of Lemma 3.3 is shown in Appendix C. We can obtain the similar expressions for $\tau(l,m+1,n)$, $\tau(l,m,n+1)$, $\tau(l+1,m,n+1)$ and $\tau(l+1,m+1,n)$.

3.2 Identity for τ_c

About the function τ_c, the following identity holds.

$$\tau_c(k_2,N+1)+\tau_c(k_1,k_3)=\max(\tau_c(k_1,N+1)+\tau_c(k_2,k_3),\ \tau_c(k_3,N+1)+\tau_c(k_1,k_2)), \tag{57}$$

where $0<k_1<k_2<k_3<N+1$. It is proved as below. Equation (57) is rewritten by

$$\begin{aligned}
&\max[\boldsymbol{\phi}(0)\ \dots\ \widehat{\boldsymbol{\phi}(k_2)}\ \dots\ \boldsymbol{\phi}(N)]+\max[\boldsymbol{\phi}(0)\ \dots\ \widehat{\boldsymbol{\phi}(k_1)}\ \dots\ \widehat{\boldsymbol{\phi}(k_3)}\ \dots\ \boldsymbol{\phi}(N+1)]\\
=\max\big(&\max[\boldsymbol{\phi}(0)\ \dots\ \widehat{\boldsymbol{\phi}(k_3)}\ \dots\ \boldsymbol{\phi}(N)]+\max[\boldsymbol{\phi}(0)\ \dots\ \widehat{\boldsymbol{\phi}(k_1)}\ \dots\ \widehat{\boldsymbol{\phi}(k_2)}\ \dots\ \boldsymbol{\phi}(N+1)],\\
&\max[\boldsymbol{\phi}(0)\ \dots\ \widehat{\boldsymbol{\phi}(k_1)}\ \dots\ \boldsymbol{\phi}(N)]+\max[\boldsymbol{\phi}(0)\ \dots\ \widehat{\boldsymbol{\phi}(k_2)}\ \dots\ \widehat{\boldsymbol{\phi}(k_3)}\ \dots\ \boldsymbol{\phi}(N+1)]\big).
\end{aligned} \tag{58}$$

Especially, let us recall the definition of $\boldsymbol{\phi}(j)$,

$$\boldsymbol{\phi}(j)=\begin{pmatrix}\max(\eta_1+jp_1,\eta_1'-jp_1)\\ \max(\eta_2+jp_2,\eta_2'-jp_2)\\ \dots\\ \max(\eta_N+jp_N,\eta_N'-jp_N)\end{pmatrix}, \tag{59}$$

where η_i and η_i' denote $\eta_i(l,m,n,s)$ and $\eta_i'(l,m,n,s)$ for short. By adding $\sum_{1\le i\le N}(-\eta_i-\eta_i')/2$ to both sides in (58), it is reduced to the conditional uPlücker relation, hence, proved.

3.3 Equations for the tau functions

Substituting the expression of tau functions into

$$\begin{aligned}
\max(&\tau(l+1,m,n)+\tau(l,m+1,n+1)-a_1-a_2,\\
&\tau(l,m+1,n)+\tau(l+1,m,n+1)-a_2-a_3,\\
&\tau(l,m,n+1)+\tau(l+1,m+1,n)-a_1-a_3)
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
\max(&\tau(l+1,m,n)+\tau(l,m+1,n+1)-a_1-a_3,\\
&\tau(l,m+1,n)+\tau(l+1,m,n+1)-a_1-a_2,\\
&\tau(l,m,n+1)+\tau(l+1,m+1,n)-a_2-a_3)
\end{aligned} \tag{61}$$

respectively, we obtain

$$\begin{aligned}
\max_{0\le k_1,k_2,k_3\le N}\big(&\tau_c(N-k_1,N+1)+\Psi(k_2,k_3)-(k_1+1)a_1-(k_2+1)a_2-k_3a_3,\\
&\tau_c(N-k_2,N+1)+\Psi(k_1,k_3)-k_1a_1-(k_2+1)a_2-(k_3+1)a_3,\\
&\tau_c(N-k_3,N+1)+\Psi(k_1,k_2)-(k_1+1)a_1-k_2a_2-(k_3+1)a_3\big)
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
\max_{0\le k_1,k_2,k_3\le N}\big(&\tau_c(N-k_1,N+1)+\Psi(k_2,k_3)-(k_1+1)a_1-k_2a_2-(k_3+1)a_3,\\
&\tau_c(N-k_2,N+1)+\Psi(k_1,k_3)-(k_1+1)a_1-(k_2+1)a_2-k_3a_3,\\
&\tau_c(N-k_3,N+1)+\Psi(k_1,k_2)-k_1a_1-(k_2+1)a_2-(k_3+1)a_3\big).
\end{aligned} \tag{63}$$

Let us show that (62) is equal to (63). For this purpose, we compare the arguments which have the same $-k_1a_1 - k_2a_2 - k_3a_3$ in both.

In the case of $k_1 = 0$, the argument in (62) is expressed by

$$\tau_c(N - k_2, N + 1) + \Psi(0, k_3) - (k_2 + 1)a_2 - (k_3 + 1)a_3. \tag{64}$$

On the other hand, that in (63) is expressed by

$$\tau_c(N - k_3, N + 1) + \Psi(0, k_2) - (k_2 + 1)a_2 - (k_3 + 1)a_3. \tag{65}$$

They are equivalent for (55). Similarly, if $k_2 = 0$ or $k_3 = 0$, then the arguments are equivalent.

Next, we consider in the case of $k_1 = N + 1$. When k_2 or k_3 is also $N + 1$, both are obviously equivalent. When $1 \le k_2, k_3 \le N$, each argument is expressed by

$$\begin{aligned}\max\bigl(&\tau_c(0, N + 1) + \Psi(k_2 - 1, k_3) - (N + 1)a_1 - k_2a_2 - k_3a_3,\\ &\tau_c(N - k_3 + 1, N + 1) + \Psi(N, k_2) - (N + 1)a_1 - k_2a_2 - k_3a_3\bigr),\end{aligned} \tag{66}$$

$$\begin{aligned}\max\bigl(&\tau_c(0, N + 1) + \Psi(k_2, k_3 - 1) - (N + 1)a_1 - k_2a_2 - k_3a_3,\\ &\tau_c(N - k_2 + 1, N + 1) + \Psi(N, k_3) - (N + 1)a_1 - k_2a_2 - k_3a_3\bigr)\end{aligned} \tag{67}$$

respectively. It is trivial that they coincide when $k_2 = k_3$. When $k_2 > k_3$, (66) and (67) reduce to

$$\begin{aligned}\max\bigl(&\tau_c(0, N + 1) + \max(\Psi(k_2, k_3 - 1), \tau_c(N - k_3 + 1, N - k_2 + 1)),\\ &\tau_c(N - k_3 + 1, N + 1) + \tau_c(0, N - k_2 + 1)\bigr) - (N + 1)a_1 - k_2a_2 - k_3a_3,\end{aligned} \tag{68}$$

$$\begin{aligned}\max\bigl(&\tau_c(0, N + 1) + \Psi(k_2, k_3 - 1),\\ &\tau_c(N - k_2 + 1, N + 1) + \tau_c(0, N - k_3 + 1)\bigr) - (N + 1)a_1 - k_2a_2 - k_3a_3\end{aligned} \tag{69}$$

for (55) and (56). They also coincide since

$$\begin{aligned}&\max\bigl(\tau_c(0, N + 1) + \tau_c(N - k_3 + 1, N - k_2 + 1), \tau_c(N - k_3 + 1, N + 1) + \tau_c(0, N - k_2 + 1)\bigr)\\ =&\tau_c(N - k_2 + 1, N + 1) + \tau_c(0, N - k_3 + 1),\end{aligned} \tag{70}$$

holds for $1 \le k_3 < k_2 \le N$ because of (57). It is also shown in the case of $k_2 < k_3$.

Finally, we consider in the case of $1 \le k_1, k_2, k_3 \le N$. The arguments in (62) and (63) are expressed by

$$\begin{aligned}\max(&\tau_c(N - k_1 + 1, N + 1) + \Psi(k_2 - 1, k_3),\\ &\tau_c(N - k_2 + 1, N + 1) + \Psi(k_1, k_3 - 1),\\ &\tau_c(N - k_3 + 1, N + 1) + \Psi(k_1 - 1, k_2)),\end{aligned} \tag{71}$$

$$\begin{aligned}\max(&\tau_c(N - k_1 + 1, N + 1) + \Psi(k_2, k_3 - 1),\\ &\tau_c(N - k_2 + 1, N + 1) + \Psi(k_1 - 1, k_3),\\ &\tau_c(N - k_3 + 1, N + 1) + \Psi(k_1, k_2 - 1)).\end{aligned} \tag{72}$$

It is clear that both correspond if $k_i = k_j$ ($i, j = 1, 2, 3$ and $i \ne j$). Then, we assume $k_1 > k_2 > k_3$ and have

$$\begin{aligned}\max(&\tau_c(N - k_1 + 1, N + 1) + \max(\Psi(k_2, k_3 - 1), \tau_c(N - k_2 + 1, N - k_3 + 1)),\\ &\tau_c(N - k_2 + 1, N + 1) + \Psi(k_1, k_3 - 1),\\ &\tau_c(N - k_3 + 1, N + 1) + \max(\Psi(k_1, k_2 - 1), \tau_c(N - k_1 + 1, N - k_2 + 1))),\end{aligned} \tag{73}$$

$$
\begin{aligned}
&\max(\tau_c(N-k_1+1,N+1)+\Psi(k_2,k_3-1),\\
&\qquad \tau_c(N-k_2+1,N+1)+\max(\Psi(k_1,k_3-1),\tau_c(N-k_1+1,N-k_3+1)),\\
&\qquad \tau_c(N-k_3+1,N+1)+\Psi(k_1,k_2-1)).
\end{aligned} \tag{74}
$$

They coincide since

$$
\begin{aligned}
&\max(\tau_c(N-k_1+1,N+1)+\tau_c(N-k_2+1,N-k_3+1),\\
&\qquad \tau_c(N-k_3+1,N+1)+\tau_c(N-k_1+1,N-k_2+1))\\
&\quad =\tau_c(N-k_2+1,N+1)+\tau_c(N-k_1+1,N-k_3+1).
\end{aligned} \tag{75}
$$

holds by (57).

Therefore, we obtain the following lemma.

Lemma 3.4 *The UP (32) defined by (33) and (34) satisfies the equation,*

$$
\begin{aligned}
&\max(\tau(l+1,m,n)+\tau(l,m+1,n+1)-a_1-a_2,\\
&\qquad \tau(l,m+1,n)+\tau(l+1,m,n+1)-a_2-a_3,\\
&\qquad \tau(l,m,n+1)+\tau(l+1,m+1,n)-a_1-a_3)\\
&=\max(\tau(l+1,m,n)+\tau(l,m+1,n+1)-a_1-a_3,\\
&\qquad \tau(l,m+1,n)+\tau(l+1,m,n+1)-a_1-a_2,\\
&\qquad \tau(l,m,n+1)+\tau(l+1,m+1,n)-a_2-a_3).
\end{aligned} \tag{76}
$$

In particular, it can be reduced to the uKP equation[15, 10],

$$
\begin{aligned}
&\tau(l,m+1,n)+\tau(l+1,m,n+1)-a_2\\
&=\max(\tau(l+1,m,n)+\tau(l,m+1,n+1)-a_1,\tau(l,m,n+1)+\tau(l+1,m+1,n)-a_2)
\end{aligned} \tag{77}
$$

since

$$
\begin{aligned}
&\tau(l+1,m,n)+\tau(l,m+1,n+1)-a_1-a_2<\tau(l+1,m,n)+\tau(l,m+1,n+1)-a_1-a_3,\\
&\tau(l,m+1,n)+\tau(l+1,m,n+1)-a_2-a_3>\tau(l,m+1,n)+\tau(l+1,m,n+1)-a_1-a_2,\\
&\tau(l,m,n+1)+\tau(l+1,m+1,n)-a_1-a_3<\tau(l,m,n+1)+\tau(l+1,m+1,n)-a_2-a_3
\end{aligned} \tag{78}
$$

hold for $a_1>a_2>a_3$. We obtain therefore Theorem 3.1.

Theorem 3.1 *The UP (32) defined by (33) and (34) satisfies the uKP equation (77).*

4 The ultradiscrete 2D Toda lattice equation and its UP solution

In this section, we give the UP soliton solution to the u2D Toda lattice equation[2],

$$
\begin{aligned}
\tau(l,m-1,n)+\tau(l+1,m,n)=&\max(\tau(l,m,n)+\tau(l+1,m-1,n),\\
&\tau(l,m-1,n+1)+\tau(l+1,m,n-1)-\delta-\varepsilon),
\end{aligned} \tag{79}
$$

where $\delta,\varepsilon>0$. The procedure is similar to the previous section. We only show the points of the proof.

Considering the tau function defined by UP

$$
\tau(l,m,n)=\max[\phi_i(l,m,n+j-1)]_{1\le i,j\le N}, \tag{80}
$$

where $\phi_i(l,m,n+j-1)$ is defined by

$$\phi_i(l,m,n)=\max(\eta_i(l,m,n),\eta'_i(l,m,n)) \tag{81}$$

with

$$\begin{aligned}\eta_i(l,m,n)&=\max(0,r_i-\delta)l-\max(0,\ -r_i-\varepsilon)m+r_in+c_i,\\ \eta'_i(l,m,n)&=\max(0,\ -r_i-\delta)l-\max(0,\ r_i-\varepsilon)m-r_in+c'_i.\end{aligned} \tag{82}$$

Here, r_i, c_i and c'_j are arbitrary parameters. In particular, $\phi_i(l,m,n)$ satisfies

$$\begin{aligned}\phi_i(l+1,m,n)&=\max(\phi_i(l,m,n),\ \phi_i(l,m,n+1)-\delta),\\ \phi_i(l,m-1,n)&=\max(\phi_i(l,m,n),\ \phi_i(l,m,n-1)-\varepsilon).\end{aligned} \tag{83}$$

Moreover, using the notation $\phi_i(l,m,n+j)\equiv\phi_i(j)$, we have

$$\phi_{i_1}(j)+\phi_{i_2}(j)\le\max(\phi_{i_1}(j-1)+\phi_{i_2}(j+1),\ \phi_{i_2}(j-1)+\phi_{i_1}(j+1)), \tag{84}$$

where $1\le i_1,i_2\le N$. The above relation gives the reduced expression of tau functions.

Lemma 4.1 *Tau functions are reduced to*

$$\tau(l+1,m,n)=\max_{0\le k_1\le N}(\tau_c(-1,N-k_1)-k_1\delta), \tag{85}$$

$$\tau(l,m-1,n)=\max_{0\le k_2\le N}(\tau_c(k_2-1,N)-k_2\varepsilon), \tag{86}$$

$$\tau(l+1,m,n-1)=\max_{0\le k_1\le N}(\tau_c(N-k_1-1,N)-k_1\delta), \tag{87}$$

$$\tau(l,m-1,n+1)=\max_{0\le k_2\le N}(\tau_c(-1,k_2)-k_2\varepsilon), \tag{88}$$

$$\tau(l,m,n)=\tau_c(-1,N), \tag{89}$$

and

$$\tau(l+1,m-1,n)=\max_{0\le k_1,k_2\le N}(\Psi(k_1,k_2)-k_1\delta-k_2\varepsilon), \tag{90}$$

where $\tau_c(\alpha,\beta)$ $(\alpha<\beta)$ is defined by

$$\tau_c(\alpha,\beta)=\max[-1\ \ldots\ \widehat{\alpha}\ \ldots\ \widehat{\beta}\ \ldots\ N]. \tag{91}$$

We use j for $(\phi_i(j))_{1\le i\le N}$ and define $\Psi(k_1,k_2)$ as follows:

$$\Psi(k_1,k_2)=\begin{cases}\max\limits_{0\le i\le k_2}(\tau_c(k_2-i-1,N-k_1+i)) & (k_1\ge k_2 \text{ and } N-k_1\ge k_2)\\ \max\limits_{0\le i\le k_1}(\tau_c(k_2-i-1,N-k_1+i)) & (N-k_1\ge k_2\ge k_1)\\ \max\limits_{0\le i\le N-k_1}(\tau_c(i-1,N-k_1+k_2-i)) & (k_1\ge k_2\ge N-k_1)\\ \max\limits_{0\le i\le N-k_2}(\tau_c(N-k_1-i-1,k_2+i)) & (k_2\ge N-k_1 \text{ and } k_2\ge k_1).\end{cases} \tag{92}$$

for $1\le k_1,k_2\le N$. In the case of $1\le k_1,k_2\le N$,

$$\begin{aligned}\Psi(k_1,k_2)&=\max(\Psi(k_1-1,k_2-1),\ \tau_c(k_2-1,N-k_1)) && (k_2-1<N-k_1)\\ \Psi(k_1-1,k_2-1)&=\max(\Psi(k_1,k_2),\ \tau_c(N-k_1,k_2-1)) && (k_2-1>N-k_1)\\ \Psi(k_1,k_2)&=\Psi(k_1-1,k_2-1) && (k_2-1=N-k_1)\end{aligned} \tag{93}$$

hold.

Moreover, we can obtain the following equation by the conditional uPlücker relation.

$$\tau_c(k_1, N+1) + \tau_c(0, k_2) = \max(\tau_c(k_2, N+1) + \tau_c(0, k_1), \tau_c(0, N+1) + \tau_c(k_1, k_2)), \tag{94}$$

where $1 \leq k_1 < k_2 < N+1$. Then, comparing the arguments which have the same $-k_1\delta - k_2\varepsilon$ in

$$\max(\tau(l, m-1, n) + \tau(l+1, m, n),\ \tau(l, m, n) + \tau(l+1, m-1, n) - \delta - \varepsilon), \tag{95}$$

and

$$\max(\tau(l, m, n) + \tau(l+1, m-1, n),\ \tau(l, m-1, n+1) + \tau(l+1, m, n-1) - \delta - \varepsilon) \tag{96}$$

with Lemma 4.1 and (94), we get Lemma 4.2.

Lemma 4.2 *The UP (80) defined by (81) and (82) satisfies the equation,*

$$\begin{aligned} &\max(\tau(l, m-1, n) + \tau(l+1, m, n),\ \tau(l, m, n) + \tau(l+1, m-1, n) - \delta - \varepsilon) \\ =&\max(\tau(l, m, n) + \tau(l+1, m-1, n),\ \tau(l, m-1, n+1) + \tau(l+1, m, n-1) - \delta - \varepsilon). \end{aligned} \tag{97}$$

Since (97) can be reduced to the u2D Toda lattice equation (79), we obtain the following theorem.

Theorem 4.1 *The UP (80) defined by (81) and (82) satisfies the u2D Toda lattice equation (79).*

5 Concluding Remarks

In this article, we consider the specialized UP, and give the conditional uPlücker relation. Moreover, we show it solves both the uKP and the u2D Toda lattice equation. Since the determinant solution on continuous or discrete soliton equation are derived from Plücker relation, the conditional uPlücker relation can be regarded as the ultradiscrete analogue of Plücker relation. However, Plücker relations used for continuous or discrete soliton equations are quite general formulae on determinants, but strong conditions are necessary for the entry of UP in the uPlücker relation. In fact, we note there exist a difference between determinant and UP solutions as below. The UP solution for the uKP equation (32) is defined by (33) and (34) and they derive (35), (36) and (37). On the other hand, the discrete KP equation,

$$\begin{aligned} &a_1(a_2 - a_3)\tau(l+1, m, n)\tau(l, m+1, n+1) \\ &+a_2(a_3 - a_1)\tau(l, m+1, n)\tau(l+1, m, n+1) \\ &+a_3(a_1 - a_2)\tau(l, m, n+1)\tau(l+1, m+1, n) = 0, \end{aligned} \tag{98}$$

has the determinant solution

$$\tau(l, m, n) = |\varphi_i(l, m, n, s+j-1)|_{1 \leq i,j \leq N} \tag{99}$$

with

$$\begin{aligned} \varphi_i(l+1, m, n, s) =& \varphi_i(l, m, n, s) + a_1\varphi_i(l, m, n, s+1), \\ \varphi_i(l, m+1, n, s) =& \varphi_i(l, m, n, s) + a_2\varphi_i(l, m, n, s+1), \\ \varphi_i(l, m, n+1, s) =& \varphi_i(l, m, n, s) + a_3\varphi_i(l, m, n, s+1). \end{aligned} \tag{100}$$

Equation (100) corresponds to (35), (36) and (37). Then it is expected that the UP solution with only (35), (36) and (37) also satisfies the uKP equation. However, it does not. In fact, for $N = 2$, when we set the function $\phi_i(l,m,n,s)$ as

$$\begin{array}{ll} \phi_1(l,m,n,s)=10 & \phi_2(l,m,n,s)=30 \\ \phi_1(l,m,n,s+1)=50 & \phi_2(l,m,n,s+1)=0 \\ \phi_1(l,m,n,s+2)=0 & \phi_2(l,m,n,s+2)=40 \\ \phi_1(l,m,n,s+3)=100 & \phi_2(l,m,n,s+3)=0 \end{array} \tag{101}$$

and $(a_1,a_2,a_3)=(30,2,1)$, they satisfy (35), (36), (37) and also (38). Nevertheless, the UP solution provided with the above functions does not satisfy the uKP equation. Thus, it means the form $|y_i+jr_i|$ is necessary for the UP solution. It is one of the future problems to clarify the difference between these structures.

A Identity of UP's

We prove an identity of UP's (12). In this appendix, we use the simple notations of the $N\times N$ matrices

$$\begin{aligned} A_j &= [\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-1}\ \boldsymbol{b}_j] \quad (1\le j\le 3), \\ A_{jj'} &= [\boldsymbol{a}_1 \dots \boldsymbol{a}_{N-2}\ \boldsymbol{b}_j\ \boldsymbol{b}_{j'}] \quad (1\le j<j'\le 3), \end{aligned} \tag{102}$$

and the $(N-1)\times(N-1)$ matrix obtained by eliminating the k_1-th row and the l_1-th column from A_j as $A_{j\,{k \atop l}}$. In the same way, the $(N-n)\times(N-n)$ matrix obtained by eliminating the $k_1,k_2,\dots,$ and k_n-th rows and the $l_1,l_2,\dots,$ and l_n-th columns from A_j is denoted by $A_{j\,{k_1,k_2,\dots,k_n \atop l_1,l_2,\dots,l_n}}$. These notations give

$$A_{1\,{k_1,\dots,k_{n-1},k_n \atop l_1,\dots,l_{n-1},N}} = A_{2\,{k_1,\dots,k_{n-1},k_n \atop l_1,\dots,l_{n-1},N}} = A_{3\,{k_1,\dots,k_{n-1},k_n \atop l_1,\dots,l_{n-1},N}} \tag{103}$$

for $1\le l_1<l_2<\dots<l_{n-1}\le N-1$, and

$$A_{23\,{k_1,\dots,k_{n-1},k_n \atop l_1,\dots,l_{n-1},N-1}} = A_{13\,{k_1,\dots,k_{n-1},k_n \atop l_1,\dots,l_{n-1},N-1}}, \tag{104}$$

$$A_{23\,{k_1,\dots,k_{n-1},k_n \atop l_1,\dots,l_{n-1},N}} = A_{12\,{k_1,\dots,k_{n-1},k_n \atop l_1,\dots,l_{n-1},N-1}}, \tag{105}$$

for $1\le l_1<l_2<\dots<l_{n-1}\le N-2$.

We can expand $\max A_1$ as

$$\max A_1 = \max_{1\le k_1\le N}\Big(\max A_{1\,{k_1 \atop N}} + b_{k_1 1}\Big). \tag{106}$$

Here $b_{k_1 1}$ stands for the k_1-th element of $\boldsymbol{b}_1$. This expansion corresponds the cofactor expansion. Similarly, we can derive $\max A_{23}$ by expanding with respect to the k_1-th row

$$\max A_{23} = \max\big(\max_{1\le l_1\le N-2}(\max A_{23\,{k_1 \atop l_1}} + a_{k_1 l_1}),\ \max A_{23\,{k_1 \atop N-1}} + b_{k_1 2},\ \max A_{23\,{k_1 \atop N}} + b_{k_1 3}\big). \tag{107}$$

for $1\le k_1\le N$. The symbols $a_{k_1 l_1}$, $b_{k_1 2}$, $b_{k_1 3}$ mean the k_1-th element of $\boldsymbol{a}_{l_1}$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ respectively. Thus, we have

$$\begin{aligned} \max A_1 + \max A_{23} = \max_{1\le k_1\le N}\big(&\max A_{1\,{k_1 \atop N}} + b_{k_1 1} + \max_{1\le l_1\le N-2}(\max A_{23\,{k_1 \atop l_1}} + a_{k_1 l_1}), \\ &\max A_{1\,{k_1 \atop N}} + b_{k_1 1} + \max A_{23\,{k_1 \atop N-1}} + b_{k_1 2}, \\ &\max A_{1\,{k_1 \atop N}} + b_{k_1 1} + \max A_{23\,{k_1 \atop N}} + b_{k_1 3}\big). \end{aligned} \tag{108}$$

On the other hand,

$$\max A_2 + \max A_{13} = \max_{1\le k_1\le N}\Big(\max A_{2\,{k_1 \atop N}} + b_{k_1 2} + \max_{1\le l_1\le N-2}(\max A_{13\,{k_1 \atop l_1}} + a_{k_1 l_1}), \\ \max A_{2\,{k_1 \atop N}} + b_{k_1 2} + \max A_{13\,{k_1 \atop N-1}} + b_{k_1 1}, \\ \max A_{2\,{k_1 \atop N}} + b_{k_1 2} + \max A_{13\,{k_1 \atop N}} + b_{k_1 3}\Big). \tag{109}$$

Then using (103), the second argument of (108) is rewritten as

$$\max A_{2\,{k_1 \atop N}} + b_{k_1 1} + \max A_{13\,{k_1 \atop N-1}} + b_{k_1 2}. \tag{110}$$

Hence, the second argument of (108) is equal to that of (109), in other words,

$$\max A_{1\,{k_1 \atop N}} + b_{k_1 1} + \max A_{23\,{k_1 \atop N-1}} + b_{k_1 2} \le \max A_2 + \max A_{13}. \tag{111}$$

Similarly, it follows that the third argument of (108) is smaller than or equal to $\max A_3 + \max A_{12}$. Next, let us consider the first argument of (108),

$$\max A_{1\,{k_1 \atop N}} + \max_{1\le l_1\le N-2}(\max A_{23\,{k_1 \atop l_1}} + a_{k_1 l_1}) + b_{k_1 1}. \tag{112}$$

We can derive the first term by expanding with respect to the $l_1(\neq N)$-th column

$$\max A_{1\,{k_1 \atop N}} = \max_{\substack{1\le k_2\le N,\\ k_2\neq k_1}}\big(\max A_{1\,{k_2,k_1 \atop l_1,N}} + a_{k_2 l_1}\big), \tag{113}$$

and the second term with respect to the k_2-th row

$$\max_{1\le l_1\le N-2}\big(\max A_{23\,{k_1 \atop l_1}} + a_{k_1 l_1}\big) = \max_{1\le l_1\le N-2}\Big(\big(\max_{\substack{1\le l_2\le N-2\\ l_2\neq l_1}}(\max A_{23\,{k_2,k_1 \atop l_2,l_1}} + a_{k_2 l_2}), \\ \max A_{23\,{k_2,k_1 \atop N-1,l_1}} + b_{k_2 2},\ \max A_{23\,{k_2,k_1 \atop N,l_1}} + b_{k_2 3}\big) + a_{k_1 l_1}\Big). \tag{114}$$

Recursively, any argument of $\max A_1 + \max A_{23}$ is expressed by either

$$\max A_{1\,{k_n,\dots,k_2,k_1 \atop l_{n-1},\dots,l_1,N}} + \sum_{1\le i\le n-1} a_{k_{i+1} l_i} + b_{k_1 1} + \max A_{23\,{k_n,k_{n-1},\dots,k_1 \atop N-1,l_{n-1},\dots,l_1}} + \sum_{1\le i\le n-1} a_{k_i l_i} + b_{k_n 2} \tag{115}$$

or

$$\max A_{1\,{k_n,\dots,k_2,k_1 \atop l_{n-1},\dots,l_1,N}} + \sum_{1\le i\le n-1} a_{k_{i+1} l_i} + b_{k_1 1} + \max A_{23\,{k_n,k_{n-1},\dots,k_1 \atop N,l_{n-1},\dots,l_1}} + \sum_{1\le i\le n-1} a_{k_i l_i} + b_{k_n 3}. \tag{116}$$

Using (104) and (105), (115) is expressed by

$$\max A_{2\,{k_{n-1},\dots,k_1,k_n \atop l_{n-1},\dots,l_1,N}} + \sum_{1\le i\le n-1} a_{k_i l_i} + b_{k_n 2} + \max A_{13\,{k_1,k_n,\dots,k_2, \atop N-1,l_{n-1},\dots,l_1}} + \sum_{1\le i\le n-1} a_{k_{i+1} l_i} + b_{k_1 1}, \tag{117}$$

and it is small than or equal to $\max A_2 + \max A_{13}$. We can prove (116) is smaller than or equal to $\max A_3 + \max A_{12}$ similarly. Therefore, we obtain

$$\max A_1 + \max A_{23} \le \max\big(\max A_2 + \max A_{13}, \max A_3 + \max A_{12}\big) \tag{118}$$

since any argument of $\max A_1 + \max A_{23}$ is smaller than or equal to either $\max A_2 + \max A_{13}$ or $\max A_3 + \max A_{12}$. Moreover,

$$\max A_2 + \max A_{13} \le \max\big(\max A_1 + \max A_{23}, \max A_3 + \max A_{12}\big), \\ \max A_3 + \max A_{12} \le \max\big(\max A_1 + \max A_{23}, \max A_2 + \max A_{13}\big) \tag{119}$$

also hold from the symmetry, and we get (12).

B Proofs of inequalities (30) and (31)

We prove only (30) in this appendix since (31) is proved by the similar way. We note that the idea of the proof is given in [9]. Let us define H_1^N by

$$\begin{aligned} H_1^N \equiv & \max[1\ \dots\ \widehat{k_2}\ \dots\ N] + \max[2\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+2] \\ & - \max[2\ \dots\ \widehat{k_2}\ \dots\ N+1] - \max[1\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+1], \end{aligned} \tag{120}$$

where $1 < k_1 < k_2 < k_3 < N+1$ and N is a natural number satisfying $N \geq 4$. We use a mathematical induction to prove $H_1^N \leq r_N$. For $N = 4$, we can calculate

$$\max[1\ 2\ 4] + \max[3\ 5\ 6] - \max[2\ 4\ 5] - \max[1\ 3\ 5] \leq r_4. \tag{121}$$

Let us suppose $H_1^N \leq r_N$ and prove $H_1^{N+1} \leq r_{N+1}$. Using Lemma 2.3, we have

$$\begin{aligned} \max[1\ \dots\ \widehat{k_2}\ \dots\ N+1] = \max\big(& y_N + (N+1)r_N + \max[1\ \dots\ \widehat{k_2}\ \dots\ N], \\ & - y_N - r_N + \max[2\ \dots\ \widehat{k_2}\ \dots\ N+1]\big), \end{aligned} \tag{122}$$

$$\begin{aligned} \max[2\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+3] = \max\big(& y_N + (N+3)r_N + \max[2\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+2], \\ & - y_N - 2r_N + \max[3\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+3]\big), \end{aligned} \tag{123}$$

$$\begin{aligned} \max[2\ \dots\ \widehat{k_2}\ \dots\ N+2] = \max\big(& y_N + (N+2)r_N + \max[2\ \dots\ \widehat{k_2}\ \dots\ N+1], \\ & - y_N - 2r_N + \max[3\ \dots\ \widehat{k_2}\ \dots\ N+2]\big), \end{aligned} \tag{124}$$

$$\begin{aligned} \max[1\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+2] = \max\big(& y_N + (N+2)r_N + \max[1\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+1], \\ & - y_N - r_N + \max[2\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+2]\big). \end{aligned} \tag{125}$$

In the case of $k_1 = 2$, we define

$$\max[3\ \widehat{2}\ \dots\ \widehat{k_3}\dots\ N+3] \equiv -r_N + \max[4\ \dots\ \widehat{k_3}\dots\ N+3]. \tag{126}$$

Here, we have inequalities

$$\begin{aligned} & \max[1\ \dots\ \widehat{k_2}\ \dots\ N+1] - \max[1\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+2] \\ \leq & \max\big(-r_N + \max[1\ \dots\ \widehat{k_2}\ \dots\ N] - \max[1\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+1], \\ & \max[2\ \dots\ \widehat{k_2}\ \dots\ N+1] - \max[2\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+2]\big) \end{aligned} \tag{127}$$

and

$$\begin{aligned} & \max[2\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+3] - \max[2\ \dots\ \widehat{k_2}\ \dots\ N+2] \\ \leq & \max\big(r_N + \max[2\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+2] - \max[2\ \dots\ \widehat{k_2}\ \dots\ N+1], \\ & \max[3\ \dots\ \widehat{k_1}\ \dots\ \widehat{k_3}\dots\ N+3] - \max[3\ \dots\ \widehat{k_2}\ \dots\ N+2]\big) \end{aligned} \tag{128}$$

from a formula $\max(x, y) - \max(z, w) \leq \max(x - z, y - w)$ for any real numbers x, y, z and w. Then, a sum of the above inequalities gives

$$H_1^{N+1} \leq \max(H_1^N,\ r_N,\ -r_N + H_1^N + H_2^N, H_2^N) \leq r_N \tag{129}$$

for the assumption. Therefore, we obtain $H_1^{N+1} \leq r_{N+1}$.

C Proofs of Lemma 3.3

In this appendix, we prove Lemma 3.3. The relation (36) derives

$$\begin{aligned}\tau(l,m+1,n+1) &= \max[\phi_i(l,m+1,n+1,s+j-1)]_{1\le i,j\le N}\\ &= \max_{0\le k_2\le N}(\tau_c(N-k_2,N+1|n+1)-k_2a_2),\end{aligned} \tag{130}$$

where $\tau_c(N-k_2,N+1|n+1)$ is the same as $\tau_c(N-k_2,N+1)$ except that the label n in $\tau_c(N-k_2,N+1)$ replaced by $n+1$. Furthermore, applying (37) to each column in $\tau_c(N-k_2,N+1|n+1)$, we have

$$\tau_c(N-k_2,N+1|n+1) = \max[\max(\boldsymbol{\phi}(j-1),\boldsymbol{\phi}(j)-a_3)]_{\substack{1\le j\le N+1\\ j\ne N-k_2+1}}. \tag{131}$$

Let us consider the maximum of the UP's which have $-k_3a_3$ in (131). In the case of $k_3\ge k_2, N-k_2$, for example, it is expressed by

$$\begin{aligned}\max\Big(&\max[0\ 1\ \ldots\ N-k_3-1\ \underbrace{N-k_3+1\ \ldots\ N-k_2}_{k_3-k_2}\ \underbrace{N-k_2+2\ \ldots\ N\ N+1}_{k_2}],\\ &\max[0\ 1\ \ldots\ N-k_3-2\ \underbrace{N-k_3\ \ldots\ N-k_2}_{k_3-k_2+1}\ N-k_2+1\ \underbrace{N-k_2+3\ \ldots\ N\ N+1}_{k_2-1}],\\ &\ldots,\\ &\max[\underbrace{1\ 2\ \ldots\ N-k_2}_{N-k_2}\ N-k_2+1\ \ldots\ 2N-k_2-k_3\ \underbrace{2N-k_2-k_3+2\ \ldots\ N\ N+1}_{k_3-(N-k_2)}]\Big)\end{aligned} \tag{132}$$

due to (44) and (48). Then, the above is expressed by

$$\max_{0\le i\le N-k_3}(\tau_c(N-k_3-i,N-k_2+1+i)) \tag{133}$$

and it is equal to $\Psi(k_2,k_3)$ in the case of $k_3\ge k_2, N-k_2$. We can derive (55) in the other conditions by similar procedure.

The relations (55) derive (56). For example, in the case of $k_2<k_3$,

$$\begin{aligned}\Psi(k_2-1,k_3) &= \begin{cases}\max\limits_{0\le i\le N-k_3}(\tau_c(N-k_3-i,N-k_2+2+i)) & (k_3\ge N-k_2+1)\\ \max\limits_{0\le i\le k_2-1}(\tau_c(N-k_2+1-k_3+i,N+1-i)) & (N-k_2+1\ge k_3)\end{cases}\\ &= \begin{cases}\max\limits_{1\le i\le N-k_3+1}(\tau_c(N-k_3-i+1,N-k_2+1+i)) & (k_3\ge N-k_2+1)\\ \max\limits_{0\le i\le k_2-1}(\tau_c(N-k_2+1-k_3+i,N+1-i)) & (N-k_2+1\ge k_3)\end{cases}.\end{aligned} \tag{134}$$

On the other hand,

$$\Psi(k_2,k_3-1) = \begin{cases}\max\limits_{0\le i\le N-k_3+1}(\tau_c(N-k_3+1-i,N-k_2+1+i)) & (k_3-1\ge N-k_2)\\ \max\limits_{0\le i\le k_2}(\tau_c(N-k_2-k_3+1+i,N+1-i)) & (N-k_2\ge k_3-1)\end{cases}. \tag{135}$$

The other relations also hold for the symmetry. Therefore, we have completed the proofs. In addition, (92), (93) are also given by the similar procedure.

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