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The asymptotic expansion of Tracy-Widom GUE law and symplectic invariants

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Abstract

We establish the relation between two objects: an integrable system related to Painlevé II equation, and the symplectic invariants of a certain plane curve Σ_{TW}. This curve describes the average eigenvalue density of a random hermitian matrix spectrum near a hard edge (a bound for its maximal eigenvalue). This shows that the $s \to -\infty$ asymptotic expansion of Tracy-Widow law $\mathrm{F}_{\mathrm{GUE}}(s)$, governing the distribution of the maximal eigenvalue in hermitian random matrices, is given by symplectic invariants.

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Introduction

The Tracy-Widom law $\mathrm{F}_{\mathrm{GUE}}(s)$ governs the distribution of the maximal eigenvalue in large hermitian matrices, sampled randomly in the GUE ensemble [32]. In fact, this law is universal, it is also valid (under some hypothesis) for non gaussian ensembles in the unitary symmetry class [9], for large Wigner matrices [30] and for a certain class of large random complex covariance matrices [31]. Just like the Gumbel, Fréchet and Weibull distribution are the possible universality class for the maximum of a large number of independent variables [16, 20], $\mathrm{F}_{\mathrm{GUE}}$ is a possible universality class for the maximum of strongly correlated variables. Thus, it is a very important law in physics and mathematics, and it has been observed in numerous phenomena and experiments [24, 3, 29]. Tracy and Widom have written in 2002 a concise review [33] and provide article references for the situations where $\mathrm{F}_{\mathrm{GUE}}(s)$ appears.

The GUE ensemble [25] is defined as the space of $N \times N$ hermitian matrices $\mathcal{H}_N$ endowed with probability law:

$$\mathrm{d}\mu_N(M) = \frac{1}{\widetilde{Z}_N}\,\mathrm{d}M\, e^{-N\,\frac{Tr\,M^2}{2t}}$$

where $\mathrm{d}M$ is the canonical Lebesgue measure on the real vector space $\mathcal{H}_N$. The decomposition $M = U\,\mathrm{diag}(\lambda_1, \ldots, \lambda_N)\,U^\dagger$ induces the probability law on the eigenvalues:

$$\mathrm{d}\mu_N(\lambda) = \frac{1}{Z_N(\infty)} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \cdot \prod_{i=1}^{N} \mathrm{d}\lambda_i\, e^{-N\,\frac{\lambda_i^2}{2t}} \tag{0-1}$$

The expected maximum eigenvalue is $\lim_{N\to\infty} \mathbb{E}(\lambda_{\max}) = 2\sqrt{t}$, and the scale of fluctuations of the maximum around this value is $N^{-2/3}$. Here, we set $t = 1$. The following limit exists and defines the GUE Tracy-Widom law $\mathrm{F}_{\mathrm{GUE}}$:

$$\mathrm{F}_{\mathrm{GUE}}(s) = \lim_{N\to\infty} \mathrm{Prob}[\lambda_{\max} \leq 2 + s\,N^{-2/3}] \tag{0-2}$$

In their original paper [32], Tracy and Widom were the first to characterize this law:

Theorem 0.1 *Let $q(s)$ be the Hastings-McLeod solution of Painlevé II, uniquely defined by:*

$$q'' = 2q^3 + sq, \qquad q(s) \underset{s\to-\infty}{\sim} \sqrt{-s/2}.$$

If we define $H(s) = \int_s^\infty q^2(\sigma)\,\mathrm{d}\sigma$,

$$\mathrm{F}_{\mathrm{GUE}}(s) = \exp\left(-\int_s^\infty H(\sigma)\,\mathrm{d}\sigma\right).$$

$H(s)$ can be identified with a Hamiltonian for PII [27], and $\mathrm{F}_{\mathrm{GUE}}(s)$ with a τ-function associated to this family of Hamiltonians ([18] in the sense of Okamoto, [5] in the sense of Jimbo-Miwa-Ueno). From Theorem 0.1, $\mathrm{F}_{\mathrm{GUE}}(s)$ can be shown to have the following asymptotic expansion when $s \to -\infty$:

$$\mathrm{F}_{\mathrm{GUE}}(s) = C\,\exp\left(-\frac{|s|^3}{12} - \frac{\ln|s|}{8} + \sum_{g\geq 2}(-s/2)^{3(1-g)}\,A_g\right). \tag{0-3}$$

In some sense, $A_0 = -\frac{2}{3}$, $A_1 = -\frac{1}{8}$, and the higher order coefficients A_g can be computed recursively thanks to Painlevé II equation. This method does not give the constant $C = 2^{1/24}\,e^{\zeta'(-1)}$, which was obtained later by Deift et al. by Riemann-Hilbert asymptotic analysis [10]. This article aims to provide an alternative description of A_g:

Proposition 0.1 *A_g are the symplectic invariants $F^g(\Sigma_{\mathrm{PII}})$ associated to the spectral curve of equation:*

$$(\Sigma_{\mathrm{PII}}) : \qquad y^2 = \frac{1}{4}\,x^2(x^2+4).$$

We will explain briefly the notion of spectral curve Σ and symplectic invariants $F^g(\Sigma)$ (Section 1.1). Those algebro-geometric objects were introduced axiomatically in [12]. One of their important property is their invariance under transformations of the spectral curve $(x,y) \to (x_o, y_o)$ such that $|\mathrm{d}x_o \wedge \mathrm{d}y_o| = |\mathrm{d}x \wedge \mathrm{d}y|$. So, we have equivalently:

Proposition 0.2 *A_g are the symplectic invariants $F^g(\Sigma_{\mathrm{TW}})$ associated to the spectral curve of equation:*

$$(\Sigma_{\mathrm{TW}}) : \qquad y^2 = x + \frac{1}{x} - 2$$

In the chronology, we first obtained Proposition 0.2 heuristically by studying large deviations in an ensemble of random hermitian matrices conditioned by $\lambda_{\max} \leq a$ [6]. Then, we realized that the curve Σ_{TW} is equivalent up to symplectic transformation to Σ_{PII}, and the latter appears in relation with Painlevé II [17]. Now, we give a direct proof of Proposition 0.1 relying on earlier results in integrable systems.

Meanwhile, we felt the need to prove the claim that Tau functions for integrable systems are related to symplectic invariants. As far as Tracy-Widom law is concerned, we only need this claim for 2×2 Lax pairs, and it was proved in a longer version of this article [7]. We are currently working on the generalization to $d \times d$ Lax pairs.

Outline

We begin with an introduction to "symplectic invariants" and the formalism of the "topological recursion" (Section 1.1). We explain its relation to integrable systems in the case of 2×2 systems (Section 1.2-1.3, with Thm. 1.1 and 1.2 as main tools). Next, we show how this can be applied to an 2×2 integrable system whose compatibility condition is the Painlevé II equation, and which admits $\mathrm{F}_{\mathrm{GUE}}(s)$ as Tau function (Section 2). This will prove Proposition 0.1.

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1 Topological recursion and 2×2 integrable systems

1.1 Axiomatic of the topological recursion

Let us recall briefly what the topological recursion consists in (for a extensive review, see [14]). It is an algorithm associating some numbers $F^g(\Sigma)$ and differential forms $\omega_n^g(\Sigma)(z_1,\ldots,z_n)$ to a regular spectral curve Σ. For our purposes, we call *spectral curve* the data of:

- a plane curve $(\mathcal{C},x,y)$, in other words a Riemann surface $\mathcal{C}$, with y and $\mathrm{d}x$ meromorphic functions $\mathcal{C}\to\mathbb{C}P^1$ (In most of our examples, $\mathcal{C}=\mathbb{C}P^1$ is the Riemann sphere, and thus x and y are complex rational functions).

- a maximal open domain $U\subseteq\mathcal{C}$ on which x is a coordinate patch, called *physical sheet*.

- a Bergman kernel $B(z_1,z_2)$, i.e. a differential form in $z_1\in\mathcal{C}$ and in $z_2\in\mathcal{C}$, such that, in a local coordinate ξ:

$$B(z_1,z_2)\underset{z_1\to z_2}{=}\frac{\mathrm{d}\xi(z_1)\mathrm{d}\xi(z_2)}{(\xi(z_1)-\xi(z_2))^2}+O(1)$$

The zeroes of $\mathrm{d}x$ lying on ∂U are called *branchpoints* (name them a_i), and a spectral curve is said to be *regular* when these zeroes are simple and $\mathrm{d}y(a_i)\neq 0$. In other words, when $\sqrt{x-x(a_i)}$ is a good coordinate around $a_i\in\mathcal{C}$ and $y-y(a_i)\propto\sqrt{x-x(a_i)}$. Before giving the full definitions of $\omega_n^g(\Sigma)$ and F^g, we need to introduce:

- The local conjugation. For z in a neighborhood of a_i, there exists a unique $\overline{z}\neq z$ such that $x(z)=x(\overline{z})$.

- The recursion kernel. We define, for $z_0\in\mathcal{C}$, and z in a neighborhood of a_i:

$$R(z_0,z)=\frac{-\frac{1}{2}\int_{z'=\overline{z}}^{z}B(z',z_0)}{\big(y(z)-y(\overline{z})\big)\mathrm{d}x(z)}\tag{1-1}$$

  R is a differential form in z_0, and the inverse of a differential form in z.

- For z in a neighborhood of a_i, we introduce ϕ an (arbitrary) primitive of $y\mathrm{d}x$.

Then, we construct:

$$\omega_1^0(z)=-y(z)\mathrm{d}x(z),\qquad \omega_2^0(z_1,z_2)(\Sigma)=B(z_1,z_2)\tag{1-2}$$

and by recursion on $|\chi| = 2g+n-2 > 0$:

$$\omega_n^g(z_1,\underbrace{z_2,\dots,z_n}_{z_J}) \;=\; \sum_i \mathop{\mathrm{Res}}_{\xi\to a_i} R(z_1,\xi)\left[\omega_{n+1}^{g-1}(\xi,\overline{\xi},z_J) + \sum_{K\subseteq J,\,0\leq h\leq g}^{\prime} \omega_{|K|+1}^{h}(\xi,z_K)\,\omega_{n-|K|}^{g-h}(\overline{\xi},z_{J\setminus K})\right] \tag{1-3}$$

$\sum'$ means that we exclude the terms where ω_n^g itself appears. Eventually, for $g\geq 2$:

$$F^g = \frac{1}{2-2g}\sum_i \mathop{\mathrm{Res}}_{z\to a_i} \phi(z)\,\omega_1^g(z) \tag{1-4}$$

We refer to [12] for the construction of F^1 and F^0, which is more involved. Let us state the main properties of $F^g(\Sigma)$ and $\omega_n^g(\Sigma)$:

- For $2-2g-n<0$, $\omega_n^g(\Sigma)\in T^*(\mathcal{C})\otimes\cdots\otimes T^*(\mathcal{C})$, i.e. $\omega_n^g(\Sigma)(z_1,\dots,z_n)$ is a meromorphic differential form in each $z_i\in\mathcal{C}$, symmetric in all z_i's, and it has poles only at the branchpoints. There is no residue at these poles, and their maximal order is $2(3g+n-2)$.

- $F^g(\Sigma)$ is invariant under any transformation $(x,y)\to(x_o,y_o)$ such that $|\mathrm{d}x\wedge\mathrm{d}y| = |\mathrm{d}x_o\wedge\mathrm{d}y_o|$. For this reason, the $F^g(\Sigma)$ are called *symplectic invariants*[3].

- $\omega_n^g(\Sigma)$ have nice scaling properties when the spectral curve approaches a singular spectral curve. We will be more precise when needed.

- $\omega_n^g(\Sigma)$ have nice properties under variation of the spectral curve.

1.2 Relation to 2×2 integrable systems

Let $\mathbf{L}(x)$ be a 2×2 matrix whose entries are rational in x. We consider a solution $\Psi(x)$ to the differential system:

$$\frac{1}{N}\,\partial_x\Psi(x) = \mathbf{L}(x)\Psi(x),\qquad \Psi = \begin{pmatrix} \psi & \phi \\ \overline{\psi} & \overline{\phi}\end{pmatrix} \tag{1-5}$$

N is a parameter of $\mathbf{L}$. The $1/N$ in front of the derivative is here for convenience, and can always be absorbed in a redefinition of x and $\mathbf{L}$. It is not restrictive to assume $\mathrm{Tr}\,\mathbf{L}=0$. So, $\partial_x\big(\det\Psi\big)=0$, and we can choose the normalization:

$$\det\Psi = 1$$

[3]However, $\omega_n^g(\Sigma)(z_1,\dots,z_n)$ is shifted by an exact differential form in each z_i under symplectic transformations

To this system is associated the Christoffel-Darboux kernel:

$$\mathcal{K}(x_1,x_2)=\frac{\psi(x_1)\overline{\phi}(x_2)-\overline{\psi}(x_1)\phi(x_2)}{x_1-x_2} \tag{1-6}$$

In [4] were also introduced correlators $\overline{\mathcal{W}}_n(x_1,\dots,x_n)$ and connected correlators $\mathcal{W}_n(x_1,\dots,x_n)$. The connected correlators are defined by:

$$\begin{aligned}
\mathcal{W}_1(x) &= \lim_{x'\to x}\left(\mathcal{K}(x,x')-\frac{1}{x-x'}\right)\\
\mathcal{W}_2(x_1,x_2) &= -\mathcal{K}(x_1,x_2)\mathcal{K}(x_2,x_1)-\frac{1}{(x_1-x_2)^2}\\
\mathcal{W}_n(x_1,\dots,x_n) &= (-1)^{n+1}\sum_{\sigma\,\text{cycles of}\,\mathfrak{S}_n}\prod_{i=1}^{n}\mathcal{K}(x_i,x_{\sigma(i)})
\end{aligned} \tag{1-7}$$

and the correlators by:

$$\overline{\mathcal{W}}_n(x_1,\dots,x_n)=\text{"det"}\,\mathcal{K}(x_i,x_j) \tag{1-8}$$

where "det" means that each occurence of $\mathcal{K}(x_i,x_i)$ in the determinant should be replaced by $\mathcal{W}_1(x_i)$, and each occurence of $\mathcal{K}(x_i,x_j)\mathcal{K}(x_j,x_i)$ by $-\mathcal{W}_2(x_i,x_j)$. For example:

$$\mathcal{W}_1=\psi\,\partial_x\overline{\phi}-\overline{\psi}\,\partial_x\phi=-\big(\partial_x\psi\,\overline{\phi}-\partial_x\overline{\psi}\,\phi\big)$$

It can be checked that the correlators are symmetric in the x_i's, and that they do not have poles at coinciding points $x_i=x_j$, $i\neq j$. The *spectral curve of a first order differential system* is defined by the plane curve Σ_N of equation:

$$\det(y-\mathbf{L}(x))=0 \tag{1-9}$$

(since this is an algebraic equation, this defines a Riemann surface $\mathcal{C}$ and two meromorphic functions on it, related by the algebraic equation). As physical sheet, we take the open maximal domain $U\subseteq\mathcal{C}$ containing one preimage of the poles of $\mathbf{L}(x)$ (here $x=\infty$), and as Bergman kernel:

$$B(z_1,z_2)=-\mathrm{d}x(z_1)\mathrm{d}x(z_2)\,\mathcal{K}(x(z_1),x(z_2))\,\mathcal{K}(x(z_2),x(z_1)) \tag{1-10}$$

These objects can be studied with the following theorem.

Theorem 1.1 *Assume that:*

(i) $\mathbf{L}$ *has a limit when* $N\to\infty$.

(ii) The spectral curve Σ_N of the system Eqn. 1-5 has a large N limit Σ_∞ which is regular, and has genus[4] *0.*

$(iii)'$ $\mathcal{W}_1(x)$ admits an asymptotic expansion when $N \to \infty$ of the form : $\mathcal{W}_1 = \sum_{g\geq 0} N^{1-2g}\,\mathcal{W}_1^g$, and for $g \geq 1$, each coefficient $\mathcal{W}_1^g(x)$ for $g \neq 0$ may have singularities only at branchpoints of Σ_∞.

Then, $\mathcal{W}_n(x_1,\dots,x_n)$ admits an expansion of the form:

$$\mathcal{W}_n = \sum_{g\geq 0} N^{2-2g-n}\,\mathcal{W}_n^g \tag{1-11}$$

The expansion coefficients of the correlators have only singularities at the branchpoints of Σ_∞, and are computed by the topological recursion applied to Σ_∞:

$$\begin{aligned}\mathcal{W}_n^g(x(z_1),\dots,x(z_n))\mathrm{d}x(z_1)\cdots\mathrm{d}x(z_n) &= \omega_n^g(\Sigma_\infty)(z_1,\dots,z_n)\\ &\quad -\delta_{g,0}\,\delta_{n,2}\,\frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\end{aligned} \tag{1-12}$$

The result was first proved in [4] with a stronger hypothesis (iii), which was weakened to hypothesis $(iii)'$ in [7]. Let us explain its origin in brief. Following [1], we can always embed $\frac{1}{N}\,\partial_x\Psi = \mathbf{L}\Psi$ in a hierarchy of compatible differential systems, with a full family of times $\vec{t} = (t_j)_{j\in\mathbb{N}}$:

$$\frac{1}{N}\,\partial_x\Psi(x,\vec{t}) = \mathbf{L}(x,\vec{t})\Psi(x,\vec{t}), \qquad \frac{1}{N}\,\partial_{t_j}\Psi(x,\vec{t}) = \mathbf{M}_j(x,\vec{t})\Psi(x,\vec{t}) \tag{1-13}$$

Then, one can show that the expansion of Eqn. 1-11 and the analytical structure of its coefficients for all $n \geq 1$ is implied by (iii). The correlators $\mathcal{W}_n$ were defined such that the coefficients of the expansion $\mathcal{W}_n^g$ satisfy loop equations. Actually, the solution of loop equations with prescribed analytical structure is unique. This analytical structure is encoded in the large N limit spectral curve Σ_∞. And, the topological recursion was designed precisely to produce the unique solution to the loop equations with analytical structure given by Σ_∞.

1.3 Tau function and symplectic invariants

It was claimed in [4] and justified in [7] that the Tau function of the integrable system Eqn. 1-13 is given by a resummation of the symplectic invariants of Σ_∞.

Theorem 1.2 *We make the following assumptions:*

[4] A similar result holds when Σ_∞ is not of genus 0 under an extra hypothesis on $\mathcal{W}_1^g$. This will not be needed here.

(*i-iii'*) *The hypothesis of Thm. 1.1.*

(*iv*) *Eqn. 1-13 has a solution* $\Psi(x,\vec{t})$ *which behaves like:*

$$\Psi(x,\vec{t}) \sim \big(\mathbf{1}+o(1)\big)e^{N\mathbf{T}_\alpha(x,\vec{t})} \tag{1-14}$$

in a given sector when $x \to x_\alpha$, *a pole of* $\mathbf{L}(x,\vec{t})$.

(*v*) $\mathbf{T}_\alpha(x,\vec{t})$ *is a* 2×2 *diagonal matrix.*

Then, the Jimbo-Miwa-Ueno Tau function, $\tau(\vec{t})$, *associated to that solution admits has the same large* N *asymptotic as:*

$$\mathcal{F}(\vec{t}) = C \ \exp\left(\sum_{g\geq 0} N^{2-2g}\, F^g(\Sigma_\infty(\vec{t}))\right) \tag{1-15}$$

up to a constant C *which does not depend on* $\vec{t}$.

2 Application to Painlevé II

2.1 Integrable system associated to Painlevé II

The Painlevé II equation $q''(s) = 2q^3(s) + sq(s)$ appears [19] as the compatibility condition of the following system for $\Psi(x,s)$:

$$\left\{\begin{array}{rcl} \partial_x\Psi &=& \mathbf{L}\,\Psi \\ \partial_s\Psi &=& \mathbf{M}\,\Psi \\ \Psi &=& \widetilde{\Psi}\,\exp\left[i\left(\frac{4}{3}x^3+sx\right)\sigma_3\right] \quad \text{when } x\to+\infty \end{array}\right. \tag{2-1}$$

where $\widetilde{\Psi} = \mathbf{1}+O(1/x)$ when $x\to+\infty$. The Lax pair $(\mathbf{L},\mathbf{M})$ is given by:

$$\mathbf{L}(x,s) = \begin{pmatrix} -4ix^2 - i(s+2q^2(s)) & 4xq(s)+2ip(s) \\ 4xq(s)-2ip(s) & 4ix^2+i(s+2q^2(s)) \end{pmatrix} \tag{2-2}$$

$$\mathbf{M}(x,s) = \begin{pmatrix} -ix & q(s) \\ q(s) & ix \end{pmatrix} \tag{2-3}$$

The necessary condition of existence of Ψ is $\partial_s\mathbf{L} - \partial_x\mathbf{M} + [\mathbf{L},\mathbf{M}] = 0$. This implies that $q(s)$ is solution of Painlevé II, and $p(s) = q'(s)$. The asymptotic behavior of Ψ (Eqn. 2-1) determined the asymptotic behavior of $q(s)$, namely:

$$q(s) \underset{s\to+\infty}{\sim} \mathrm{Ai}(s) \sim \frac{\exp(-\frac{2}{3}s^{3/2})}{2\sqrt{\pi}s^{1/4}} \tag{2-4}$$

Hastings and McLeod [21] have shown that requiring this asymptotic determines a unique solution of Painlevé II. Moreover, this solution is also the unique one with left tail asymptotic:

$$q(s) \underset{s\to-\infty}{\sim} \sqrt{-s/2} \tag{2-5}$$

The existence of the solution $\Psi(x,s)$ was shown in [19]. This system is an example of isomonodromy problem, itself closely related to a Riemann-Hilbert problem. Many authors have contributed to the study of these systems, and we refer to [17] for a review of the theory. In this section, our goal is to do some simple checks to put ourselves in the framework on Section 1.

Let us introduce a redundant parameter N. We define:

$$\left\{\begin{array}{l} x = N^{1/3}\,X \\ s = N^{2/3}\,S \end{array}\right. , \qquad q(s) = N^{1/3}Q(S) \tag{2-6}$$

Then, Eqn. 2-1 is equivalent to:

$$\left\{\begin{array}{rcl} \frac{1}{N}\,\partial_X\Psi & = & \mathbf{L}\,\Psi \\ \frac{1}{N}\,\partial_S\Psi & = & \mathbf{M}\,\Psi \\ \Psi & = & \widetilde{\Psi}\,\exp\left[iN\left(\frac{4}{3}X^3+SX\right)\sigma_3\right] \quad \text{when } X\to+\infty \end{array}\right. \tag{2-7}$$

with the Lax pair:

$$\begin{aligned} \mathbf{L}(X,S) &= \begin{pmatrix} -4iX^2 - i(S+2Q^2(S)) & 4XQ(S) + \frac{2iQ'(S)}{N} \\ 4XQ(S) - \frac{2iQ'(S)}{N} & 4iX^2 + i(S+2Q^2(S)) \end{pmatrix} \\ \mathbf{M}(X,S) &= \begin{pmatrix} -iX & Q(S) \\ Q(S) & iX \end{pmatrix} \end{aligned} \tag{2-8}$$

and the compatibility equation:

$$\frac{1}{N^2}Q''(S) = 2Q(S)^3 + SQ(S) \tag{2-9}$$

Again, $Q(S)$ is given by the Hastings-McLeod solution of Painlevé II, which is such that (see Eqn. 2-5) $\lim_{N\to\infty} Q(S)^2 = -\frac{S}{2}$ for any $S<0$. The reason for introducing N is that the $s\to-\infty$ asymptotics for the system Eqn. 2-1 corresponds to the $N\to\infty$ asymptotics for the system of Eqn. 2-7. We have now the notations of Thm. 1.1.

2.2 Spectral curve

After Eqn. 1-9, the finite N spectral curve for this system is:

$$Y^2 = -16X^4 - 8X^2S - S^2 - 4Q^4(S) - 4SQ^2(S) + \frac{4\big(Q'(S)\big)^2}{N^2} \tag{2-10}$$

In the large N limit, since $Q^2(S)\sim -S/2$ (we assume $S<0$), we obtain:

$$(\Sigma_\infty)\,: \qquad Y^2 = -16X^2\left(X^2+\frac{S}{2}\right) \tag{2-11}$$

It can be brought in a canonical form by rescaling:

$$(\Sigma_{\mathrm{PII}})\,: \qquad y^2 = \frac{1}{4}\,x^2(x^2+4) \tag{2-12}$$

where

$$x = i\sqrt{\frac{8}{-S}}\,X, \qquad y = \frac{i}{S}\,Y \tag{2-13}$$

For this transformation, $Y\,\mathrm{d}X = (-S/2)^{3/2}\,y\mathrm{d}x$. Going from the topological recursion of Σ_∞ to that of Σ_{PII} is only a matter of rescaling:

$$F^g(\Sigma_\infty) = (-S/2)^{3(1-g)}\,F^g(\Sigma_{\mathrm{PII}}) \tag{2-14}$$

$$\omega_n^{(g)}(\Sigma_\infty) = (-S/2)^{3(1-g-n/2)}\,\omega_n^g(\Sigma_{\mathrm{PII}}) \tag{2-15}$$

Thus:

$$\begin{aligned}\mathcal{F}(\Sigma_\infty) &= \sum_{g\geq 0} N^{2-2g}\,F^g(\Sigma_\infty) \\ &= \sum_{g\geq 0}\left(N^{2/3}\,(-S/2)\right)^{3(1-g)}\,F^g(\Sigma_{\mathrm{PII}}) \\ &= \sum_{g\geq 0}(-s/2)^{3(1-g)}\,F^g(\Sigma_{\mathrm{PII}})\end{aligned} \tag{2-16}$$

2.3 Proof of Proposition 0.1

In our case, $\mathbf{L}(X,S)$ has only one pole, at $X = \infty$, which is already present in $\mathbf{L}^{(0)}(X,S)$. We define $Y(x)$ by Eqn. 2-11 with the choice of the branch of the square root imposed by the $X \to \infty$ asymptotics in Eqn. 2-7:

$$Y(X) = -4iX\sqrt{X^2 + \frac{S}{2}}, \qquad Y(x) \underset{X\to+\infty}{\sim} -4iX^2 \tag{2-17}$$

If the hypothesis of Theorem 1.2 were satisfied, it would prove the identity (in the sense of $s \to -\infty$ asymptotic) between

$$\ln \mathrm{F}_{\mathrm{GUE}}(s) = \ln\tau(s) \qquad \text{and} \qquad \mathcal{F}(\Sigma_\infty) = \sum_{g\geq 0}(-s/2)^{3(1-g)}\,F^g(\Sigma_{\mathrm{PII}})$$

which is the content of Proposition 0.1. For the moment, the state of our checklist is:

- Hypothesis (i), (iv) and (v) are automatic with Eqn. 2-7.
- Σ_∞ is a regular spectral curve of genus 0, i.e fulfils hypothesis (ii). This is readily seen with the rational parametrization:

$$X(z) = \gamma\left(z + \frac{1}{z}\right), \qquad Y(z) = \frac{S}{2}\left(z^2 - \frac{1}{z^2}\right)$$

- $\mathcal{W}_1$ has an expansion in odd powers of N (see Lemma 2.2 below), and $\mathcal{W}_1^g$ has only singularities at branchpoints of Σ_∞ (see Lemma 2.1 below). Hence $(iii)'$.

We now turn to the technical lemmas.

Lemma 2.1 *$\psi(X,S)$ (resp. $\overline{\psi}(X,S)$, $\phi(X,S)$, $\overline{\phi}(X,S)$) admits a $1/N$ expansion:*

$$\psi(X,S)=\left(\widetilde{\psi}^{(0)}(X,S)+\sum_{l\geq 1}N^{-l}\,\widetilde{\psi}^{(l)}(X,S)\right)\exp\left[iN\left(\frac{4}{3}X^3+SX\right)\right]$$

and for all $l\geq 1$, $\widetilde{\psi}^{(l)}(X,S)$ has only poles at $X=\pm\sqrt{\frac{-S}{2}}$ (the branchpoints of Σ_∞). In particular, it does not have singularities at $X=0$, the other zero of Y.

proof:
We only sketch the proof. We already know that $Q^{(0)}(S)=\sqrt{\frac{-S}{2}}$, and the Hastings-McLeod solution of Eqn. 2-9 has a $1/N^2$ expansion:

$$Q(S)=Q^{(0)}(S)+\sum_{l\geq 1}N^{-2l}\,Q^{(l)}(S)$$

Hence, $\mathbf{L}$ and $\mathbf{M}$ have a $1/N$ expansion. The large N limit spectral curve Σ_∞ associated to the system $\frac{1}{N}\partial_X\Psi=\mathbf{L}\Psi$ is:

$$Y(X,S)=\pm 4iX\sqrt{X^2+S/2}$$

and ψ to leading order is given by:

$$\begin{aligned}\psi^{(0)}(X,S) &= \mathrm{cte}^{(0)}(S)\sqrt{\frac{\mathbf{L}^{(0)}_{12}(X,S)}{2Y(X)}}\exp\left[-N\int^X Y(\xi)\mathrm{d}\xi\right]\\ &= \mathrm{cte}^{(0)}(S)\left(\frac{-S/2}{X^2+S/2}\right)^{1/4}\exp\left[\mp iN\left(\frac{4}{3}X^3+SX\right)\right]\end{aligned}$$

To agree with the asymptotic required in Eqn. 2-7, we must choose the minus sign. On the other hand, the Iarge N limit spectral curve $\underline{\Sigma}_\infty$ associated to the system $\frac{1}{N}\partial_S\Psi=\mathbf{M}\Psi$ is:

$$\underline{Y}(X,S)=\pm\sqrt{-\det\mathbf{M}^{(0)}}=\pm i\sqrt{X^2+S/2}$$

Hence, ψ is also given to leading order by:

$$\psi^{(0)}(X,S)=\underline{\mathrm{cte}}^{(0)}(X)\sqrt{\frac{\mathbf{M}^{(0)}_{12}(X,S)}{2\underline{Y}(X,S)}}\exp\left[-N\int^S\underline{Y}(X,\sigma)\mathrm{d}\sigma\right]$$

One can determine recursively the solution of each differential system (with respect to X, or with respect to S), and one finds an expansion of the form of Eqn. 2-18. By plugging this expansion alternatively in the two systems, it is possible to show that $\psi^{(l)}$ can have singularities only at common zeroes of $Y(X,S)$ and $\underline{Y}(X,S)$. In other words, the possible singularity at $X = 0$ obtained with the system with respect to X, is shown to disappear if we use the system with respect to S. At all orders in $1/N$, only the singularities at the simple branchpoints of Σ_∞ remains, i.e. at $X = \pm\sqrt{-S/2}$. The discussion for ϕ, $\overline{\phi}$ and $\overline{\psi}$ is similar. $\square$

Lemma 2.2 *$\mathcal{W}_1(X)$ admits an expansion in odd powers of $1/N$:*

$$\mathcal{W}_1(X) = \sum_{g\geq 0} N^{1-2g}\, \mathcal{W}_1^g(X)$$

and the singularities of $\mathcal{W}_1^g(X)$ for $g \neq 0$ are found only at branchpoints $X = \sqrt{-S/2}$.

proof:

From the asymptotics of ψ, we see that $\mathcal{W}_1$ has at least an expansion in powers of $1/N$, and the position of the singularities away from $X = \infty$ at all orders is a consequence of Lemma 2.1. One can also check that a singularity at $X = \infty$ can only appear in the leading order $\mathcal{W}_1^0$. Let us stress the dependence in N by writing $\mathbf{L}_N$, and Ψ_N for the solution of Eqn. 2-7. We observe that ${}^t\mathbf{L}_{-N} = \mathbf{L}_N$, which implies that ${}^t\Psi_{-N}^{-1}$ obey the same differential system as Ψ. Moreover, ${}^t\Psi_{-N}^{-1}$ has the same asymptotic behavior near $x \to \infty$ as Ψ_N, and is also of determinant 1. So, ${}^t\Psi_{-N}^{-1} = \Psi_N$, and at the level of the integrable kernel:

$$\begin{aligned}
\mathcal{K}_N(x_1, x_2) &= \frac{\psi_N(x_1)\overline{\phi}_N(x_2) - \overline{\psi}_N(x_1)\phi_N(x_2)}{x_1 - x_2} \\
&= \frac{\overline{\phi}_{-N}(x_1)\psi_{-N}(x_2) - \phi_{-N}(x_1)\overline{\psi}_{-N}(x_2)}{x_1 - x_2} \\
&= \mathcal{K}_{-N}(x_2, x_1)
\end{aligned}$$

But, we can see on definition (Eqn. 1-7) that the correlators take a $(-1)^n$ sign if we revert the orientation of all the cycles. Thus:

$$\mathcal{W}_n(x_1, \ldots, x_n)_{-N} = (-1)^n\, \mathcal{W}_n(x_1, \ldots, x_n)_N$$

In particular, $\mathcal{W}_1$ must be odd in N $\square$

References

[1] O. Babelon, D. Bernard, M. Talon, *Introduction to classical integrable systems*, Cambridge Monographs on Mathematical Physics (2003)

[2] J. Baik, R. Buckingham, J. DiFranco, *Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function*, math.FA/0704.3636 (2007)

[3] J. Baik, A. Borodin, P. Deift, T. Suidan, *A model for the bus system in Cuernevaca (Mexico)*, J. Phys. A, **39** pp 8965-8975 (2006), math.PR/0510414

[4] M. Bergère, B. Eynard, *Determinantal formulae and loop equations*, math-ph/0901.3273 (2009)

[5] A. Borodin, P. Deift, *Fredholm determinants, Jimbo-Miwa-Ueno tau-functions, and representation theory*, math-ph/0111007 (2001)

[6] G. Borot, B. Eynard, S.N. Majumdar, C. Nadal, *Large deviations of the maximal eigenvalue of random matrices*, math-ph/1009.1945 (2010)

[7] G. Borot, B. Eynard, *Tracy-Widom GUE law and symplectic invariants*, nlin.SI/1011.1418 (2010)

[8] M. Cafasso, O. Marchal, *Double scaling limits of random matrices and minimal* $(2m, 1)$ *models: the merging of two cuts in a degenerate case*, math-ph/1002.3347 (2010)

[9] P. Deift, D. Gioev, *Universality at the edge of the spectrum for unitary, orthogonal and symplectic ensembles of random matrices*, Comm. Pure Appl. Math. **60** no. 6 pp 867-910 (2007), math-ph/0507023

[10] P. Deift, A. Its, I. Krasovsky, *Asymptotics of the Airy-kernel determinant*, math.FA/0609451 (2006)

[11] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, JHEP 024A:0904, hep-th/0407261

[12] B. Eynard, N. Orantin, *Invariants of algebraic curves and topological recursion*, math-ph/0702045 (2007)

[13] , B. Eynard, *Recursion between Mumford volumes of moduli spaces*, math.AG/0706.4403 (2007)

[14] B. Eynard, N. Orantin, *Algebraic methods in random matrices and enumerative geometry*, math-ph/0811.3531 (2008)

[15] B. Eynard, *A matrix model for plane partitions*, J. Stat. Mech. 0910:P10011 (2009), math-ph/0905.0535

[16] R.A. Fisher, L.H.C. Tippett, *Limiting forms of the frequency distribution of the largest or smallest member of a sample*, Proceedings of the Cambridge Philosophical Society, vol 24, pp 180-190 (1928)

[17] A. Fokas, A. Its, A. Kapaev, V. Novokshenov, *Painlevé transcendents: the Riemann-Hilbert approach*, Mathematical Surveys and Monographs, Vol. 128 (2006)

[18] P.J. Forrester, N.S. Witte, *Application of the τ-Function Theory of Painlev Equations to Random Matrices: PIV, PII and the GUE*, math-ph/0103025 (2001)

[19] H. Flaschka, A.C. Newell, *Monodromy- and spectrum-preserving deformations, I*, Comm. Math. Phys. Volume 76, Number 1 pp 65-116 (1980), available here

[20] B.V. Gnedenko, *Sur la distribution limite du terme maximum d'une série aléatoire*, Annals of Mathematics, **44**, pp 423 453 (1943)

[21] S.P. Hastings, J.B. McLeod, *A boundary value problem associated with the second Painlevé transcendent and the Korteweg-deVries equation*, Archive for Rational Mechanics and Analysis, Volume 73, Issue 1, pp 31-51, available here

[22] M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients : I. General theory and τ-function*, Physica D, Volume 2, Issue 2, pp 306-352 (1981)

[23] M. Kontsevich, *Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function*, Commun. Math. Phys. 147, pp 1-23 (1992)

[24] M. Krbálek, P. Čeba, *The statistical properties of the city transport in Cuernavaca (Mexico) and random matrix ensembles*, J. Phys. A: Math. Gen. **33** L229 (2000), nlin.CD/0001015

[25] M.L. Mehta, *Random matrices*, 3rd edition, Pure and Applied Mathematics, Vol. 142, 3rd edn (Amsterdam: Elsevier/Academic)

[26] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and Geometry (M. Artin, J. Tate, editors), vol. 2, Birkhäuser, Boston, pp 271-328 (1983)

[27] K. Okamoto, *On the τ-function of the Painlevé equations*, Physica D, Nonlinear Phenomena, Volume 2, Issue 3, pp 525-535 (1981)

[28] D. Korotkin, H. Samtleben, *Generalization of Okamoto's equation to arbitrary 2×2 Schlesinger system*, Adv. Math. Phys. 461860 (2009) nlin.SI/0906.1962

[29] K. Takeuchi, M. Sano, *Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals*, PRL **104**, 230601 (2010), cond-mat.stat-mech/1001.5121

[30] A. Soshnikov, *Universality at the edge of the spectrum in Wigner random matrices*, Commun. Math. Phys. **207** pp 697-733 (1999), math-ph/9907013

[31] A. Soshnikov, *A note on universality of the distribution of the largest eigenvalues in certain sample covariance matrices*, J. Stat. Phys **108** pp 1033-1056 (2002), math.PR/0104113

[32] C. Tracy, H. Widom, *Level spacing distributions and the Airy kernel*, Commun.Math.Phys. **159** pp 151-174 (1994), hep-th/9211141 (1992)

[33] C. Tracy, H. Widom, *Distribution functions for largest eigenvalues and their applications*, Proceedings of the ICM, Beijing 2002, vol. 1 pp 587-596, math-ph/0210034