Reflection K-matrices related to Temperley-Lieb R-matrices

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Abstract

The general solutions of the reflection equation associated with Temperley-Lieb R-matrices are constructed. Their parametrization is defined and the Hamiltonians of corresponding integrable spin systems are given.

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1 Introduction

The Temperley-Lieb algebra $TL_N(q)$ [1, 2], hereafter denoted TL-algebra, plays a central role in the construction and derivation of quantum integrable models of great interest in statistical mechanics and solid state physics (see e.g. [2, 3]). In particular it is well known that special representations of the TL algebra give rise to constant solutions R of the Yang-Baxter equation. From such R-matrices one then constructs integrable quantum spin chains [4] on the space of state $\mathcal{H} = \overset{N}{\underset{1}{\otimes}}\mathbb{C}^n$ for any integer n. These spin chains are very similar to the spin 1/2 XXZ-model.

In order to formulate the generalization of this construction to TL-related *open* spin chains one is lead to consider scalar (i.e. non-operatorial) solutions to the related reflection equation [5]. The complete resolution, and classification of solutions to such equations, are therefore key issues in the definition of new quantum integrable models with a symmetry algebra related to TL algebra. Relevance of such quantum systems is manifold and the associated algebraic structures present several interesting features: indeed their corresponding quantum algebra $\mathcal{U}_q(n)$ is different from $U_q(sl(2))$ for $n \geq 2$, while the integrals of motion are elements of $TL_N(q)$ [6].

To construct these spin chains a (reducible) representation of $TL_N(q)$ on $\mathcal{H}$ the tensor product of local state spaces $\mathbb{C}^n$ will be used. Let us be more specific: The TL R-matrix considered throughout this paper is parametrized by an invertible $n \times n$ matrix b while the parameter q of the corresponding "XXZ-type" TL spin model is given by:

$$tr({}^t b\, b^{-1}) = -(q + \frac{1}{q}). \tag{1.1}$$

It was pointed out [4] that the $n \times n$ matrices K solving the reflection equation for this TL R-matrix actually satisfy a quadratic equation:

$$q\, K^2 + \, c_1\, K + (q + \frac{1}{q})^{-1}(c_1^2 + q\, c_2)\, I = 0 \tag{1.2}$$

with appropriate central elements c_1 and c_2 depending on K and b.

The aim of the paper is to describe a complete parametrization of these K-matrices. In addition, once the constant R- and K-matrices are known, the Yang-Baxterization procedure then yields the spectral parameter dependent matrices, which are cornerstones of the quantum inverse scattering method [7, 8, 9, 5].

We shall first recall more precisely the context of TL R-matrices construction from braid groups and Hecke algebras, prepare the notation and formulate a derivation of the quadratic equation (1.2). The complete classification, and full parametrization of constant solutions shall be obtained (Section 3). Finally the Yang-Baxterized form for the K-matrices will be given. The integrable spin systems with boundary interactions will be constructed and some comments on their spectral properties will be given (Section 4).

2 Hecke and Temperley-Lieb Algebras

Both Hecke algebra $H_N(q)$ and TL algebra $TL_N(q)$ are quotients of the group algebra of the braid group $\mathcal{B}_N$ generated by $(N-1)$ generators $\check{R}_j$, $j = 1, 2, \ldots, N-1$, their inverses $\check{R}_j^{-1}$ and the relations (see [10]):

$$\check{R}_j \check{R}_k \check{R}_j = \check{R}_k \check{R}_j \check{R}_k, \text{ for } |j-k| = 1 \quad \text{and} \quad \check{R}_j \check{R}_k = \check{R}_k \check{R}_j, \text{ for } |j-k| > 1. \tag{2.1}$$

The Hecke algebra $H_N(q)$ is obtained by adding to these relations the following constraints obeyed by each generator $\check{R}_j$ (q-deformation of the symmetric group):

$$\left(\check{R}_j - q\right)\left(\check{R}_j + 1/q\right) = 0. \tag{2.2}$$

Equation (2.2) is equivalent to write $\check{R}_j$ in term of some idempotent X_j, namely:

$$\check{R}_j = q\mathbb{I} + X_j \tag{2.3}$$

with

$$X_j^2 = -\left(q + \frac{1}{q}\right) X_j. \tag{2.4}$$

$\mathbb{I}$ denotes the identity in the Hecke algebra. The braid group relations (2.1) read in terms of the idempotents X_j and X_k such that $|j-k| = 1$:

$$X_j X_k X_j - X_j = X_k X_j X_k - X_k. \tag{2.5}$$

Finally the TL algebra $TL_N(q)$ is obtained as the quotient algebra of the Hecke algebra $H_N(q)$ by the set of equations requiring that each side of (2.5) be zero. To sum up, $TL_N(q)$ is defined by the generators X_j, $j = 1, 2, \ldots, N-1$ and their relations:

$$\begin{aligned} &X_j^2 = -\nu(q) X_j, \\ &X_j X_k X_j = X_j, \quad |j-k| = 1, \\ &X_j X_k = X_k X_j, \quad |j-k| > 1 \end{aligned} \tag{2.6}$$

with $\nu(q) = q + 1/q$.

The dimension of the Hecke algebra, $N!$, is the same as the dimension of the symmetric group, whereas the dimension of $TL_N(q)$ is equal to the Catalan number $C_N = (2N)!/N!(N+1)!$. Implementation of the TL constraint thus considerably reduces the dimension of the algebra.

In connection with integrable spin systems we will be interested in representations of $TL_N(q)$ on the tensor product space $\mathcal{H} = \overset{N}{\underset{1}{\otimes}} \mathbb{C}^n$. We will consider in the following a particular representation (reducible) defined by a single complex invertible $n \times n$ matrix b which can also be seen as a vector of $\mathbb{C}^n \otimes \mathbb{C}^n$ (with n^2 entries: $\{b_{cd}\}$) [4]. We use the notation $\bar{b} := b^{-1}$ and view this matrix also as a vector of $\mathbb{C}^n \otimes \mathbb{C}^n$ with entries $\{\bar{b}_{cd}\}$.

The matrix realization on $\mathcal{H}$ of the idempotent generator X_j now reads in terms of b:

$$X_j = \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{j-1} \otimes \left(\sum_{\substack{c,\, d,\, c',\, d \\ \in \{1 \ldots n\}}} b_{cd} \bar{b}_{c'd'} E_{cc'} \otimes E_{dd'} \right) \otimes \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{N-j-1} \tag{2.7}$$

where we have now denoted by $\mathbb{I}$ the identity matrix in $End(\mathbb{C}^n)$ and we have used the canonical basis of $n \times n$ matrices, $E_{cc'}$ denoting the $n \times n$ matrix with entries $(E_{cc'})_{xx'} = \delta_{cx}\, \delta_{c'x'}$.

One sees here that $\left(\sum_{c,d,c',d'=1,\ldots,n} b_{cd} \bar{b}_{c'd'} E_{cc'} \otimes E_{dd'} \right)$ is proportional to a rank-1 projector on $\mathbb{C}^n \otimes \mathbb{C}^n$. Direct computation shows that the set of relations (2.6) are satisfied and they fix the value of the parameter q up to a duality $q \to 1/q$:

$$-\nu(q) = \mathrm{tr}\, {}^t b \bar{b} = -\left(q + \frac{1}{q}\right). \tag{2.8}$$

The $\check{R}_j$ generators are now also represented in terms of endomorphisms on $\mathcal{H}$. From the particular form (2.7) these endomorphisms can be consistently denoted as $\check{R}_{jj+1}$. Conditions (2.1) are in particular represented as the braided Yang-Baxter equation:

$$\check{R}_{12}\ \check{R}_{23}\ \check{R}_{12} = \check{R}_{23}\ \check{R}_{12}\ \check{R}_{23}. \tag{2.9}$$

The R-matrix is then defined from this representation of the braid group generators by $R_{jj+1} = \mathcal{P}_{jj+1}\check{R}_{jj+1}$, with $\mathcal{P}(v \otimes v') = v' \otimes v$ for any couple of vectors of $\mathbb{C}^n$. The indexation $jj+1$ of $\mathcal{P}$ is self-explanatory. The notation R_{jj+1} is then straightforwardly extended to define general endomorphisms R_{ij} of $\mathcal{H}$ labeled by any non-adjacent pair of "site indices" (i, j), using the time-honored notation [7] for such elements of $End(\mathcal{H})$ with indices labelling the spaces.

Equation (2.9) then immediately becomes the Yang-Baxter equation for R:

$$R_{12}\ R_{13}\ R_{23} = R_{23}\ R_{13}\ R_{12}. \tag{2.10}$$

Let us finally formulate the Yang-Baxterization procedure of these R-matrices. In fact the Yang-Baxterization procedure is already valid at the stage of abstract Hecke algebra generators: Indeed if one defines the spectral parameter-dependant R-matrix as

$$\check{R}_j(u) = u\check{R}_j - \frac{1}{u}\check{R}_j^{-1} = (u - \frac{1}{u})\check{R}_j + \frac{\omega(q)}{u}\mathbb{I}; \quad \omega(q) = q - \frac{1}{q} \tag{2.11}$$

one sees that it obeys the cubic equation in braid group form with multiplicative spectral parameter u (additive spectral parameter is of course obtained as $u \equiv e^{\lambda}$:

$$\check{R}_j(u)\check{R}_k(uw)\check{R}_j(w) = \check{R}_k(w)\check{R}_j(uw)\check{R}_k(u), \text{ for } |j-k| = 1. \tag{2.12}$$

Now once the generators $\check{R}$ of the Hecke algebra $H_N(q)$ itself have been represented as R-matrices acting on some tensor product of two finite-dimensional vector spaces, this procedure will immediately give rise to solutions of the non-constant braided Yang-Baxter equation with multiplicative spectral parameters:

$$\check{R}_{12}(u)\check{R}_{23}(uw)\check{R}_{12}(w) = \check{R}_{23}(w)\check{R}_{12}(uw)\check{R}_{23}(u). \tag{2.13}$$

3 Classification of the solutions of the constant reflection equation

We present here a complete classification of the solutions to the constant reflection equation (boundary Yang-Baxter equation) associated to the TL R-matrix R (viewed as an endomorphism of $\mathbb{C}^n \otimes \mathbb{C}^n$) for any value of n. The Yang-Baxterization procedure defined in [11, 12, 13] will then allow to obtain general spectral-parameter dependant reflection matrices $K(u)$ (section 4.1) which will then be of direct use for the construction of TL spin chains (section 4.2).

Let us first recall how one derives the quadratic equation (1.2) satisfied by matrix solutions K of the constant reflection equation:

$$R_{12}\ K_1\ R_{21}\ K_2 = K_2\ R_{12}\ K_1\ R_{21} \tag{3.1}$$

whenever R is a constant solution of the Yang-Baxter equation, associated with the TL representation obtained in the previous section.

Since the R-matrix is obtained from the braid group generators by $R = \mathcal{P}\check{R}$, (3.1) can be rewritten:

$$\check{R}_{12}\ K_1\ \check{R}_{12}\ K_1 = K_1\ \check{R}_{12}\ K_1\ \check{R}_{12}. \tag{3.2}$$

Using $\check{R} = q\mathbb{I} + X$, it yields:

$$q\ X_{12}\ K_1^2 +\ X_{12}\ K_1\ X_{12}\ K_1 = q\ K_1^2\ X_{12} + K_1\ X_{12}\ K_1\ X_{12}. \tag{3.3}$$

This equation reads in term of the b matrix:

$$q\ b \otimes ({}^tK^2\bar{b})\ +\ tr({}^tb^tK\bar{b})\ b \otimes ({}^tK\bar{b}) = q\ ({}^tK^2b) \otimes \bar{b}\ +\ tr({}^tb^tK\bar{b})\ ({}^tKb) \otimes \bar{b}. \tag{3.4}$$

Since matrices b and $\bar{b}$ are invertible, this is equivalent (after taking the transposition) to:

$$\mathbb{I} \otimes (q\ K^2 +\ tr({}^tb^tK\bar{b})\ K) = (q\ K^2 +\ tr({}^tb^tK\bar{b})\ K) \otimes \mathbb{I}. \tag{3.5}$$

This establishes that $q\ K^2 +\ tr({}^t\bar{b}Kb)\ K$ is proportional to identity.

The value of the coefficient of the identity term is immediately obtained as a consistency condition for the value of the linear form (q-trace) $tr({}^tb\bar{b} - -)$ applied to both sides of the equality. One finally gets the normalized quadratic polynomial annihilating K as:

$$K^2 + \frac{1}{q}\ tr({}^t\bar{b}Kb)\ K = k_2\mathbb{I} \tag{3.6}$$

with

$$k_2 = -\frac{1}{qtr({}^t\bar{b}b)}(tr({}^t\bar{b}Kb)^2 + qtr({}^t\bar{b}K^2b)). \tag{3.7}$$

It follows that the complete resolution of the reflection equation for these constant TL R-matrices will be realized in two steps:

1. Parametrize all matrices K with a minimal polynomial of degree 2 (or less).
2. Fix the value of the coefficient of the linear term to its expression in (3.6).

The coefficient of the constant term, as we have seen, is the result of a self-consistent evaluation of a trace and therefore does not represent a supplementary independant constraint on K. These two steps are thus necessary AND sufficient to obtain all solutions of the reflection equation.

Step 1 is separated into three obvious subcases:

1a: Minimal polynomial of degree 1. The matrix K is then proportional to the Identity and automatically solves the reflection equation without further conditions.

1b: Minimal polynomial of degree 2 with two distinct roots. The matrix K is then diagonalizable with the same two zeroes as eigenvalues.

1c: Minimal polynomial of degree 2 with a double root. The matrix K is then only trigonalizable (i.e. is written with Jordanian cells) with a single eigenvalue and an order-2 nilpotency on the corresponding eigenspace.

We now consider in detail cases 1b and 1c.

3.1 Diagonalizable K-matrices

Any diagonalizable $n \times n$ matrix with two distinct eigenvalues denoted λ and μ can be parametrized as follows:

$$K = \lambda\mathbb{I} + (\mu - \lambda)P \tag{3.8}$$

where P is the projector parallel to the eigenspace V_λ with eigenvalue λ, onto the eigenspace V_μ with eigenvalue μ. One can always choose μ such that the dimension of V_μ, hereafter denoted m, is lower than (or at most equal to) the dimension of V_λ, hence $m \leq [\frac{n}{2}]$.

The projector P is then constructed from two sets of data encapsulating all the information on V_μ and V_λ albeit with redundancies:

a: a set of m independent vectors building a basis of V_μ, defining in this way an $n \times m$ rectangular matrix B of maximal rank m. The redundancy in this parametrization correspond to the arbitrariness in the choice of the basis in V_μ , described by the transformation $B \rightarrow Bg$ for any g in $Gl(m)$.

b: a set of m independent vectors building a basis of $\bar{V_\lambda}$ defined as the m-dimensional vector space of solutions to the rank $n-m$ homogeneous linear system:

$$ {}^t vC = 0 \tag{3.9} $$

where v is the unknown n dimensional vector and C is an $n \times (n-m)$ rectangular matrix defined from any basis of vectors for V_λ in the same way as B is defined for V_μ. C is of course defined up to rhs multiplication by g' in $Gl(n-m)$ which does not affect v. This second set of m n-dimensional vectors allows then to build a second $n \times m$ rectangular matrix A of maximal rank m. A is also defined up to a rhs multiplication by any h in $Gl(m)$. From (3.9) one sees that ${}^t AC = 0$.

In addition one must impose that the intersection of V_μ and V_λ is empty, which is equivalent to asking that no vector of V_μ be a solution of (3.9), or finally to requiring that the square $m \times m$ matrix ${}^t AB$ be invertible.

P is then built as:

$$ P = B({}^t AB)^{-1} A^t \tag{3.10} $$

as is immediately checked by operating P on B (vectors of V_μ), yielding again B, and C (vectors of V_λ), yielding 0.

We recall that this parametrization is defined up to separate rhs multiplication of A and B by any matrix of $Gl(m)$. This redundancy shall be presently used to simplify (3.10).

Hence, any diagonalizable K-matrix with 2 eigenvalues can be written as:

$$ K = \lambda \mathbb{I} + (\mu - \lambda) B({}^t AB)^{-1} A^t. \tag{3.11} $$

The number of relevant parameters is thus 2 (eigenvalues) $+2nm$ (matrices A and B) $-2m^2$ (2 changes of basis in $Gl(m)$). The redundancy of the parametrization under $A \rightarrow Ag$ and $B \rightarrow Bh$ is manifest in (3.11).

We now realize Step 2 by imposing that the value of the coefficient of the linear term in the minimal polynomial, which is identified with the sum of the two zeroes $\lambda + \mu$, be identified to its expression in (3.6), i.e.:

$$ \lambda + \mu = -\frac{1}{q} tr(b^t \bar{b} K) \tag{3.12} $$

that is:

$$ \lambda(1 + \frac{1}{q} tr(b^t \bar{b}) - \frac{1}{q} tr(b^t \bar{b} B({}^t AB)^{-1} A^t)) + \mu(1 + \frac{1}{q} tr(b^t \bar{b} B({}^t AB)^{-1} A^t) = 0. \tag{3.13} $$

This fixes univocally the ratio $\frac{\lambda}{\mu}$ unless both coefficients in (3.13) vanish. This in turn implies that $tr(b^t\bar{b}) = -2q$ hence from the TL trace condition (2.8) $q = 1$. In this case $\check{R}$ is triangular and indeed no condition may relate the eigenvalues (see e.g. [14]).

We can now use the arbitrariness in the choice of A and B to impose that the $m \times m$ matrix $({}^tAB)$ be set to $\mathbb{I}$. This leaves a set of m^2 non-relevant parameters. K is then given as:

$$K = \lambda\{\mathbb{I} + \{\frac{-q + 1/q}{q + tr(b^t\bar{b}B^tA)}\}B^tA\}; \quad ({}^tAB) = \mathbb{I}. \tag{3.14}$$

The overall number of relevant parameters in K is now $2(n-m)m+1$ (except in the triangular case where one extra parameter occurs as just seen). Using a spin notation which is useful in the context of spin chain construction using TL algebra, one equivalently rewrites $n = 2s+1$ leading to $2(2s+1-m)m+1$ parameters. The irrelevant parameters are the components of the diagonal global $Gl(m)$ gauge transformation $A \to A(g^{-1})^t \;\; ; \;\; B \to Bg$.

This enables us to identify the solutions recently proposed in [15] (at least the infinite-spectral parameter limit thereof, which solve the constant reflection equation). They are precisely the diagonalizable solutions corresponding to the choice $m = [\frac{2s+1}{2}]$. For instance when $s = 3/2$ ($n = 4$) one obtains a 9 parameter solution which can be shown to have two eigenvalues of multiplicity 2. Explicit formulation of the eigenvectors is also available but the particular form of the parametrization used in [15] yields cumbersome formulae which we shall not give here.

3.2 Non-diagonalizable K-matrices

In this case K is automatically of the form $\lambda\mathbb{I} + N$ where N is a nilpotent $n \times n$ matrix: $N^2 = 0$. A similar parametrization for N as in the diagonalizable case exists. Set the dimension of the kernel of N to be $n-m$ with of course $m \leq n/2$ since $ImN \subset KerN$. This time the m-dimensional image of the cokernel of N yields a rectangular matrix B up to rhs multiplication by g in $Gl(m)$. The $n-m$-dimensional kernel of N, as in (3.9) can again be characterized by another $n \times m$ rectangular matrix A. However this time one must impose a complete inclusion condition of the image vectors defining B in the kernel, in other words ${}^tAB = 0$. N is then immediately obtained as $N = B^tA$. Because of the condition ${}^tAB = 0$ the scale of N is not fixed; this scale fixing is here obtained by the implementation of Step 2 to impose:

$$\lambda(q - q^{-1}) = tr(b^t\bar{b}B^tA). \tag{3.15}$$

The irrelevant parameters are again the components of the diagonal global $Gl(m)$ gauge transformation $A \to A(g^{-1})^t \;\; ; \;\; B \to Bg$. The number of relevant parameters is thus 1 (eigenvalue) $+2nm$ (matrices A and B) $-m^2$ (changes of basis in $Gl(m)$) $-m^2$ (inclusion relation ${}^tAB = 0$) -1 (trace relation) $= 2nm - 2m^2$.

3.3 Complete parametrization

Both situations can now be summarized into a single representation:

Proposition

Any solution to the Temperley-Lieb constant reflection equation (3.1) takes the form:

$$K = \lambda \mathbb{I} + B^t A \tag{3.16}$$

where A and B are rectangular $n \times m$ matrices of rank m, $m \leq [\frac{n}{2}$ defined up to a diagonal $Gl(m)$ gauge transformation g:

$$A \to A(g^{-1})^t \;\; ; \;\; B \to Bg \tag{3.17}$$

and submitted to the condition:

$${}^t AB = (\mu - \lambda)\mathbb{I} \tag{3.18}$$

and

$$-\frac{1}{q}\lambda + q\mu = -\frac{1}{q} tr(b^t \bar{b} B^t A). \tag{3.19}$$

If $\mu = \lambda$ one recovers the non-diagonalizable case
If $\mu \neq \lambda$ one recovers the diagonalizable case.

4 Spectral parameter dependent K matrices

4.1 Yang-Baxterization

The Yang-Baxterization of the Hecke R-matrices was formulated in Section 2, eqn. (2.11). In order to define the corresponding Yang-Baxterization for the associated K-matrices obeying the reflection equation one is lead to define the extension of the Hecke algebra to the affine Hecke algebra $\hat{H}_N(q)$. It has one more generator K with relations:

$$\check{R}_1 K \check{R}_1 K = K \check{R}_1 K \check{R}_1 \qquad K\check{R}_j = \check{R}_j K, j > 1. \tag{4.1}$$

As in the Hecke case an extra polynomial constraint imposed on K as $p_n(K) = 0$, will define a quotient of $\hat{H}_N(q)$ known as "cyclotomic Hecke algebra" [16]. There exists then consistent realizations of the Yang-Baxterized K-matrix by Laurent polynomials $K(u)$ in u, u^{-1}, depending on the coefficients of p_n and K^m, $m = 0, 1, ...n-1$ [12, 13]. They are solutions to the algebraic reflection equation [5]:

$$\check{R}_1(u/w)K(u)\check{R}_1(uw)K(w) = K(w)\check{R}_1(uw)K(u)\check{R}_1(u/w) \tag{4.2}$$

where $\check{R}_1(u/w)$ is the Yang-Baxterized $\check{R}$ matrix (2.11). Once suitable matrix representations of both R and K are considered, respectively in $End(\mathbb{C}^n \otimes \mathbb{C}^n)$ and $End(\mathbb{C}^n)$ this becomes the well-known Sklyanin reflection equation

$$\check{R}_{12}(u/w)K_1(u)\check{R}_{12}(uw)K_1(w) = K_1(w)\check{R}_{12}(uw)K_1(u)\check{R}_{12}(u/w). \tag{4.3}$$

We are here considering specifically the case of a represented TL-type R-matrix built from a rank-1 projector. We have seen in Section 2 that the constant solution of the reflection equation then satisfies a quadratic constraint. It follows from general arguments

[11, 12, 13] that the corresponding spectral parameter-dependent $K(u)$ is given by the expression:

$$K(u) = u^2 K - \frac{1}{u^2} K^{-1} + c\mathbb{I} \tag{4.4}$$

with an arbitrary central element c. Due to the relation (4.4), after a suitable normalization of K, one gets the regularity property of $K(u)$: $K(u)|(u=1) = \mathbb{I}$. This property is important to construct an integrable spin chain Hamiltonian with nearest-neighbour interaction and a boundary interaction on the left and right boundary sites described by matrices $K^-(u)$ and $K^+(u)$ respectively [5].

4.2 Spin chains

The construction of the spin chain Hamiltonian proceeds now from general principles. Taking the R-matrix $R_{0j}(u) = \mathcal{P}_{0j}\check{R}_{0j}$ as an L-operator at each site j with auxiliary space labeled by 0 index, one constructs the monodromy matrix [7, 8]:

$$T(u) = L_{0N}(u) L_{0N-1}(u) \cdots L_{01}(u) \tag{4.5}$$

and the two-row monodromy matrix [5]:

$$\mathcal{T}(u) = T(u) K_0^-(u) T^{-1}(1/u) \tag{4.6}$$

where $K_0^-(u)$ is a solution of the reflexion equation. The generating functional of integrals of motions (including the Hamiltonian) is:

$$\tau(u) = tr K_0^+(u) T(u) K_0^-(u) T^{-1}(1/u). \tag{4.7}$$

where in addition $K_0^+(u)$ is a solution of the suitably defined dual reflexion equation. All solutions thereof can be obtained straightforwardly from the set of solutions K^- due to crossing-unitarity of R-matrix. In terms of the R-matrix $\tau(u)$ reads:

$$\tau(u) = tr K_0^+(u) R_{0N}(u) R_{0N-1}(u) \cdots R_{01}(u) K_0^-(u) R_{10}(u) R_{20}(u) \cdots R_{N0}(u). \tag{4.8}$$

With an appropriate normalization one fixes $\tau(1) = tr K_0^+(1)$ (remember that in the multiplicative spectral parameter representation the critical value for regularity properties is 1). The spin chain hamiltonian becomes then proportional to the local expression:

$$H = \Sigma_{k=1}^{N-1} \frac{d}{du} \check{R}_{kk+1}(u=1) + \frac{1}{2}\frac{d}{du} K_1^-(u=1) + (tr K_0^+(1))^{-1} tr K_0^+(1) \frac{d}{du} \check{R}_{N0}(u=1) \tag{4.9}$$

where the contribution of the boundary conditions is explicit.

If one chooses general $n \times n$ K-matrices K^+ and K^- with the full arbitrariness parametrized in Section 3 this contribution makes it difficult to characterize quantitatively properties (such as spectrum and eigenvectors) of the spin system. At this time we lack a proper framework to deal with this general case and this is the key issue which must be adressed in the future.

To illustrate what could be done, were the suitable algebraic tools available, let us finally concentrate on the simplest particular case which indeed can be treated very extensively using general algebraic arguments.

Restricting oneself to the free ends case:

$$K_1^-(u) = \mathbb{I} \qquad K_1^+(u) = M = b^t \bar{b} \tag{4.10}$$

where M is the matrix entering into the crossing-unitarity relation for R, the spin chain Hamiltonian and the higher conserved quantities then lose altogether their boundary contributions and become elements of the TL algebra, for instance:

$$H = \Sigma_{k=1}^{N-1} \frac{d}{du} \check{R}_{kk+1}(u=1) \in TL_N(q) \tag{4.11}$$

This Hamiltonian is now symmetric w.r.t. the quantum algebra $\mathcal{U}_q(n)$ and can be restricted to the irreducible representation subspaces of $TL_N(q)$ in a decomposition of the phase space:

$$\mathcal{H} = \overset{N}{\underset{1}{\otimes}} \mathbb{C}^n = \overset{[N/2]}{\underset{k=0}{\oplus}} W_k \otimes \mathbb{C}^{\nu(k)} \tag{4.12}$$

where W_k denotes the irrep of $TL_N(q)$ corresponding to the two-row Young diagramme with partition $\{(\lambda_1, \lambda_2)|\lambda_1 + \lambda_2 = N, \lambda_2 = k\}$ and $\nu(k)$ is the multiplicity of this irrep in the decomposition. Hence the spectrum of H consists here of multiplets of subspaces $\{E_k^{(j)}\}, j = 1, 2, \cdots dimW_k$. associated with the irreps W_k, each with multiplicity $\nu(k)$.

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