Hierarchical Gompertzian growth maps with application in astrophysics

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The Gompertz model describes the growth in time of the size of significant quantities associated to a large number of systems, taking into account nonlinearity features by a linear equation satisfied by a nonlinear function of the size. Following this scheme, we introduce a class of hierarchical maps which describe discrete sequences of intermediate characteristic scales. We find the general solutions of the maps, which account for a rich set of possible phenomena. Eventually, we provide an important application, by showing that a map belonging to the class so introduced generates all the observed astrophysical length and mass scales.

The laws of growth of many systems, and the deep origin of their characteristic scales of, e. g., length, mass, energy, or numerosity, are intensively investigated in many branches of science, such as biomedicine, economy, and population dynamics. The Gompertz model was originally introduced in 1825 by B. Gompertz [1] as a model of human mortality: Gompertz found empirically the fitting distribution of human age for a given community. From the first half of the twentieth century the Gompertz model, and the associated Gompertz equation, have become a frequently used tool to account for mechanisms of growth [2]. The form of the Gompertz equation is:

$$z^{-1}\frac{dz}{dt} = \beta - \alpha \ln\left(\frac{z}{\tilde{z}}\right), \qquad (1)$$

where z describes the "size" (not necessarily a spatial size) of some quantity characterizing the system, β and α denote two positive constants with the dimensions of the inverse of time, and $\tilde{z}$ is a constant with the same dimensions of z. Eq. (1) can be recast as:

$$\frac{d(\ln s)}{dt} = -\alpha \ln s, \qquad (2)$$

where $s(t) \doteq z(t)/z_\infty$, and $z_\infty = \tilde{z}\exp{(\beta/\alpha)}$. The Gompertz equation is then characterized by four parameters (all dependent on the specific system): α, β, z_∞, and the initial condition ("scale") $z(0) = z_0$. Its solution is:

$$z(t) = z_\infty \exp{[(\ln\gamma)\cdot e^{-\alpha t}]}, \qquad (3)$$

where $\gamma \doteq z_0/z_\infty$. It is immediately verified that this solution always approaches monotonically in time z_∞. Depending on the conditions $z_0 < z_\infty \equiv \tilde{z}\exp\beta/\alpha$ $(\gamma < 1)$, or $z_0 > z_\infty \equiv \tilde{z}\exp\beta/\alpha$ $(\gamma > 1)$, the system monotonically grows or monotonically decreases, respectively, from the dimension z_0 to the dimension z_∞, approaching the asymptotic value z_∞ with the characteristic time α^{-1}. In TABLE I we list the main symbols which have been exploited, or which will be exploited in the following, with their meanings.

Symbol	Meaning
α, β	Parameters of the Gompertz eq.
$z(t)$	Size at time t
z_0	Initial size
z_∞	Asymptotic (or reference) size
γ	Ratio z_0/z_∞
$s(t)$	Relative size $z(t)/z_\infty$
s_n	n-th relative size z_n/z_∞ in the discrete map
y_n	$\ln s_n$
$\tilde{\alpha}$	Parameter of the discrete map
R_n	n-th astrophysical length scale
R	Max. astroph. length scale (Radius of the obs. univ.)
λ	Min. length scale R_0 (radius of a nucleon)
M_n	n-th astrophysical Mass scale
M	Max. astroph. mass scale (Mass of the obs. univ.)
m	Min. mass scale M_0 (mass of a nucleon)
c	Velocity of the light
G	Gravitational constant

TABLE I: Meaning of the symbols

We observe that the Gompertz model includes in a very interesting and peculiar way nonlinearity, which is a macroscopic mirror of a nonlinear microscopic background ruling the growth of natural systems. In fact, the equation looks linear in a suitable function $y(s)$ of the original (relative) size s, while the nonlinearity is introduced by the fact that the function y is itself a nonlinear function of s; in particular, $y(s) = \ln s$, or, equivalently, $s(y) = \exp y$. On the other hand, while the Gompertz equation describes a continuous growth of a size in time, we know that many systems are placed on some discrete sequence of sizes, ranging from a minimum to a maximum scale. Therefore, following the Gompertz model, we try to describe these sequence of scales by a sequence $\{s_n\}$ of relative sizes resulting from the solution of a *linear* map fulfilled by a *nonlinear* function, $y(s_n) \equiv y_n$, of s_n:

$$y_{n+1} = \delta_n y_n, \qquad (4)$$

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where δ_n are proportionality coefficients. The solution of the map is obviously:

$$y_n = y_0 \prod_{k=0}^{n} \delta_k. \qquad (5)$$

Here, s_n will be defined by $s_n \doteq z_n/z_\infty$, where z_n is the n-th size, and z_∞ is an asymptotic, or reference, size. Moreover, for further generalization, we at first allow the dependence of the proportionality constant on n (analogously, we could promote α to a suitable function of the time in the Gompertz equation). The choice of the nonlinear function $y(s_n) \equiv y_n$ is obviously a crucial question: this choice, in principle, will depend on the specific system, and it should be made following physical criteria and phenomenological observations. But the Gompertz model suggests that the logarithm can play a privileged and somewhat universal role. This suggestion is strengthened by the connection among Gompertz model, lognormality and stability, as we will briefly discuss in the conclusions [3]. We then select $y_n = \ln s_n$ (or $s_n = \exp y_n$). Thus, if γ is defined as in the Gompertz equation, $y_0 = \ln \gamma$, and the solution (5) gives:

$$s_{n+1} = s_n{}^{\delta_n}, \qquad (6)$$

i. e.

$$s_n = (\gamma)^{\prod_{k=0}^{n} \delta_k}. \qquad (7)$$

Furthermore, resorting again to the Gompertz model, we make the simplifying choice $\delta_n \equiv \delta \equiv 1 - \tilde{\alpha}$, obtaining

$$y_{n+1} = (1 - \tilde{\alpha})y_n,, \qquad (8)$$

$$y_n = (1 - \tilde{\alpha})^n \ln \gamma,, \qquad (9)$$

$$s_{n+1} = s_n^{(1-\tilde{\alpha})}, \qquad (10)$$

and

$$s_n = \gamma^{(1-\tilde{\alpha})^n}. \qquad (11)$$

In fact, Eq. (8) looks as the form of the discretized Gompertz equation. It is, however, worth to be remarked that the physical meaning of the discrete map is different from that of the continuous equation; in particular, we are not necessarily authorized to interpret Eq. (8) as an equation in (discretized) time.

We now comment on the behavior of the solutions (11). We see that a monotone sequence (increasing if $\gamma < 1$, decreasing if $\gamma > 1$) is assured only if $\tilde{\alpha} < 1$ (while monotonicity is always assured by the Gompertz equation). If, instead, $\tilde{\alpha} > 1$, the map (8), (with solution(9)) changes its sign at each step, and, correspondingly, the values of z_n *oscillate*, but with two further possible behaviors. If $\tilde{\alpha}$ is not an integer number, the size of the system approaches the asymptotic value z_∞ by oscillating in alternating way above and below it, with more and more damped oscillations. If $\tilde{\alpha}$ is an integer number, the size of the system oscillates indefinitely above and below the reference value z_∞, assuming in alternating way the pair of values $\gamma^{(\tilde{\alpha}-1)}$ and $\gamma^{(1-\tilde{\alpha})}$, and does not converge to any asymptotic limit. Therefore, we see that the discrete maps so introduced describe a variety of situations.

In order to investigate the usefulness of the maps (10), (11), we move towards an application, by premising some considerations. We observe that, if the growth is monotone ($\tilde{\alpha} < 1$), usually one knows (phenomenologically) the initial and the final sizes (z_0, z_∞) of a system, and that, if $\gamma < 1$ (aggregation process) the initial value z_0 is noting but the size of the more elementary constituent of the system, and z_∞ is the maximum size, while if $\gamma > 1$ (fragmentation process), z_∞ is the size of the more elementary constituent, and z_0 is instead the maximum size. The (phenomenological) knowledge of the extremal sizes, together with a good fit for the value of $\tilde{\alpha}$, is sufficient to determine all the intermediate scales. Obviously, a more ambitious goal could be to obtain the hierarchical sequence of characteristic scales z_n by fixing phenomenologically only the initial size z_0, and obtaining the values of $\tilde{\alpha}$ and of z_∞ by independent theoretical hypotheses. This, as we will show, can be done when the physical background underlying the system is well established.

Now, we show that Eq. (11) can be applied in the framework of astrophysics: we assume that the components on different scales of the observed universe organize themselves according to our scheme, and proceed to verify this hypothesis. We choose for z_n the sequence of the length scales ("radii") of the astrophysical aggregations, and replace the symbols z_n with R_n and z_∞ with R, and all the others in a consistent manner. Moreover, we fix $\tilde{\alpha} = 1/2$. Being $\tilde{\alpha} < 1$, the map is monotone. We assume also that R_0 is the size of the elementary constituent (i. e. the minimum size). Thus, the radii R_n are monotonically increasing, and their asymptotic limit R is the maximum radius (i. e. the observed radius of the universe). We remark that in our computation we consider only the order of magnitude of the radii, expressed (in centimeters) as powers of 10. It is natural to choose as elementary constituent a nucleon, because the system is ruled by gravitation, which in turn is determined by the mass distribution; and nucleons contain all the significant (observed) mass. Our minimum size is then the radius of a nucleon as measured by α-particles scattering: $R_0 \equiv \lambda = 10^{-13} cm$. It is also known that R, the observed radius of the universe, is $R = 10^{26} cm$ [4]. Then $\gamma = 10^{-39}$. The map (10), with $\tilde{\alpha} = 1/2$, becomes a *square-root map*, while if we use $s_n = R_n/R$ we obtain for the radii a *geometric-mean map*:

$$R_{n+1} = (R_n\, R)^{\frac{1}{2}}. \qquad (12)$$

Now, exploiting Eq. (11) with $\gamma = 10^{-39}$ and $\tilde{\alpha} = 1/2$, or, equivalently, Eq. (12), we obtain the rapidly convergent sequence in the first column of TABLE II.

These length scales represent, in order of magnitude, just the main six observed astrophysical length scales [4] (the presence of two aggregates both for radius R_1 and for radius R_2 is discussed later). Therefore, we have obtained a first important result: we can fit the intermediate astrophysical length scales by choosing a specific value of $\tilde{\alpha}$, and by exploiting the

RADIUS	MASS	AGGREGATE
$10^6 cm$	$10^{15} g$	planetesimal
$10^6 cm$	$10^{34} g$	neutron star
$10^{16} cm$	$10^{34} g$	solar system
$10^{16} cm$	$10^{44} g$	s.m. black hole
$10^{21} cm$	$10^{44} g$	typical galaxy
$10^{23} cm$	$10^{49} g$	cluster of galaxies
$10^{25} cm$	$10^{52} g$	supercluster of galaxies
$10^{26} cm$	$10^{54} g$	observed universe

TABLE II: List of astrophysical length scales and of the corresponding mass scales, with the associated astrophysical aggregates

experimentally determined values of the two extremal length scales.

However, as remarked previously, a scientific theory with some *predictive* power should be able to determine, on the basis of few other assumptions, both the (apparently arbitrary) choice $\tilde{\alpha} = 1/2$, and the value of the maximum size R, by the mere phenomenological knowledge of the minimum scale. We now show that this is actually possible by resorting to a suitable assumption naturally suggested by the physics of the gravitational systems: *We assume that the dimension of the aggregate with radius R_1 (and mass $\mathbf{M}$) is given by the Schwartzschild radius of a three-dimensional close packing of elementary constituents (nucleons: radius λ, mass m), i. e. by a "collaps" restricted by the constraint that the escape velocity is the maximum one: the velocity c of the light.* This hypothesis leads to the following two conditions:

$$\frac{\mathbf{M}}{R_1^3} = \frac{m}{\lambda^3}; \quad G\frac{\mathbf{M}}{R_1} = c^2, \quad (13)$$

where G is the gravitational constant. The second equation is the Schwartzschild condition for the radius, while the first one requires that the mass density per unit volume of the first aggregate coincide with that of a nucleon (three-dimensional close packing). By eliminating the mass M, we obtain:

$$R_1 = (\lambda R)^{\frac{1}{2}}, \quad (14)$$

with $R = (\lambda c)^2/Gm$. Inserting the numerical values of λ, c, g, m provides $R = 10^{26} cm$, i. e. just the observed radius of the universe. Thus, we see that our simple hypothesis leads both to an independent determination of the maximum size R, *and* to the value of $\tilde{\alpha}$ which coincides with our original choice (Note in fact that Eq. (14) is Eq. (12) with $n = 0$). Here we comment also that, by defining (in order of magnitude) the age T of the observed universe by $R = cT$, from the expression soon established for R we get: $T = \lambda^2 c/Gm \equiv 10^{16} s$, i. e. just the right order of magnitude.

Finally, we aim to determine, besides the length scales, also the masses of the astrophysical aggregates. To this purpose, we note that, if some other quantity of the system, say M_n, is connected to z_n by the "allometric" relation: $M_n = bz_n^{\delta}$, then the quantity $q_n \doteq M_n/M_\infty$ satisfies the same map as

s_n, and has the same form of solution with the replacement $\gamma \to \gamma' = \gamma^\delta$:

$$q_n = (\gamma')^{(1-\tilde{\alpha})^n} \equiv \gamma^{\delta(1-\tilde{\alpha})^n}. \quad (15)$$

Moreover, putting together the allometric relations at a generic n and at $n = 0$, we obtain also the proportionality constant b as: $b = M_0/z_0^\delta$.

In the case of the universe, we consider the sequence of the masses M_n, and the sequence of their radii R_n, and find their "allometric" relation by introducing another assumption (of "minimum fluctuation"), probably less intuitive with respect to the first one, but reasonable if one considers that the very complex character of the gravitational systems, and the huge numbers of elementary constituents induce instabilities and fluctuations: *An aggregate can bind a test particle up to a distance where the gravitational force generated by the aggregate on a test particle reduces to a value comparable with the background fluctuating force, whose magnitude is determined by the elementary constituent*. The claim gives, for each n:

$$G\frac{M_n}{R_n^2} = G\frac{m}{\lambda^2}, \quad (16)$$

i. e.:

$$M_n = bR_n^2, \quad (17)$$

where $b = m/\lambda^2 = 100g/cm^2$. Then, $\delta = 2$, and, being $\tilde{\alpha} = 1/2$, we obtain the sequence:

$$M_n = \gamma^{2(2^{-n}-1)}m, \quad (18)$$

where, by denoting M (instead of M_∞) the asymptotic value providing the total mass of the observed universe, we have exploited: $m/M = \gamma^2$. Then, once the sequence for the radii is established, the allometric relation, and the knowlege of the minimum mass m, automatically provide the sequence of the intermediate mass scales until the maximum one. From $\gamma = 10^{-39}$ and $m = 10^{-24}$ g, we obtain (in order of magnitude) the sequence of the second column in TABLE II, which represents just the sequence of the observed astrophysical masses. In the last column of TABLE II we describe the corresponding structures. Note also that the total mass of the observed universe corresponds to that of $\gamma^{-2} \equiv 10^{78}$ nucleons, in perfect agreement with the central value obtained from nucleosynthesis calculations [4].

We conclude this letter with the following observations and remarks:

Being the macroscopic behavior of complex systems the result of the collective behavior of a huge number of elementary constituents, it must be a mean effect deriving by a suitable microscopic (stochastic) model. In fact, in ref. [3] it has been recently proved that the Gompertz equation is a *macroscopic consequence* of the action of a microscopic, lognormally distributed diffusion process performed by the elementary constituents of the system, because this equation holds for the *median* of the process, which then provides the evolution in time of the macroscopic characteristic size.

We remark that hypotheses and results on the astrophysical structures here discussed, and the hierarchical maps for the sequence of astrophysical radii and masses are contained in papers previously published by the authors in collaboration with other researchers [5]. However, we as well remark that in this letter we frame these results in a model based on discrete maps deduced by the Gompertz model of growth. The background stochastic origin of the Gompertz model [3], together with the derivation from physical hypothesis of the values of $\tilde{\alpha}$ and of the asymptotic length R, lead us to interpret our scheme as a first important step towards the discovery of the underlying process performed by the elementary constituents, and responsible for the observed sequence of astrophysical scales. Note, in fact, that it clearly emerges from our model the role of a geometrical progression, whose relevance in the framework of natural phenomena and in a variety of experimental environments was highlighted already at the end of nineteenth century by F. Galton [6] and by D. McAlister [7], which showed that the geometrical mean (median) describes the behavior of a large set of natural phenomena better than the arithmetic one.

The discrete maps here introduced can provide a general model that can be potentially exploited to search for the enlightenment of a large number of growth phenomena in many fields of research. Namely searching for sequences of significant scales associated to different systems, which possibly develop these intermediate scales step by step in time, or which organize themselves on all these scales simultaneously. It is also of great interest to investigate the existence of natural quantities whose evolution is characterized by oscillations, as described by the maps with $\tilde{\alpha} > 1$.

We remark that the apparent discrepancy between the first relation in Eq. (13) and the relation (16) is solved by the fact that to the same radius R_n can be associated the mass M_n *and also* the mass M_{n+1}, where the mass M_n satisfies the relation (16) (two-dimensional close packing), while the mass M_{n+1} satisfies the relation (13) (three-dimensional close packing, "critical" aggregate) [5]. For example, as is well known, a neutron star (three dimensional close packing with mass $M_2 = 10^{34}g$) has, in order of magnitude, the same radius ($R_1 = 10^6 m$) of a planetesimal (two dimensional close packing with mass $M_1 = 10^{15}g$). Similar considerations can be extended by comparing a typical galaxy with a supermassive black-hole at the center of the galaxy bulge, whose estimated mass (until billions of solar masses, i. e. $10^{42}g$) is in fact comparable with that of the whole galaxy [8]. This explains the two structures associated to the first two radii in Table I. Summing up, our model allows two *possible* sequences, given by: $\{R_n, M_n\}$ (two-dimensional close-packed aggregates) and $\{R_n, M_{n+1}\}$ (three-dimensional close-packed, critical aggregates), and this scheme is confirmed at least on scale R_1, and on scale R_2.

A question which is worth to be deepened is the meaning of the proportionality parameters in the linearized maps. But this question is connected to the more general, and already raised, question of the possible stochastic background underlying the maps, and will be addressed in forthcoming papers.

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