

R -matrices and Hamiltonian Structures for Certain Lax Equations

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Abstract

In this paper R -matrices on a certain class of coupled Lie algebras are obtained. With one of these R -matrices, we construct infinitely many bi-Hamiltonian structures for both the two-component BKP hierarchy and the Toda lattice hierarchy. We also show that, when the above two hierarchies are reduced to their subhierarchies, these bi-Hamiltonian structures are reduced correspondingly.

Key words: R -matrix, Hamiltonian structure, two-component BKP hierarchy, Toda lattice hierarchy

1 Introduction

The existence of Hamiltonian (or Poisson) structures reveals a very important property of nonlinear evolutionary equations, see [9] and references therein. An efficient method for introducing Hamiltonian structures of evolutionary equations written in the Lax form is the so-called classical R -matrix formalism. The classical R -matrix formalism was derived by Semenov-Tyan-Shanskii [22] to construct Poisson brackets on a Lie algebra of an associative algebra on which the Lax equations are defined. This method yields two compatible Poisson brackets, i.e., any linear combination of them is still a Poisson bracket. Consequently one obtains a bi-Hamiltonian structure of the Lax equation by performing a Dirac reduction [19] if needed. Such a formalism was first established for anti-symmetric R -matrices satisfying the modified Yang-Baxter equation [22]. As later developed by Li and Parmentier [17], also by Oevel and Ragnisco [20], the R -matrix formalism becomes available for a wider class of R -matrices. Moreover, this formalism can produce three compatible Poisson brackets on an associative algebra.

In this paper we study R -matrices on a “coupled” Lie algebra $\mathfrak{g} = \mathcal{G}^- \times \mathcal{G}^+$, where \mathcal{G}^\pm are appropriately defined Lie algebras (see Section 3 below). Our goal is to apply the R -matrix formalism to construct Hamiltonian structures of Lax equations defined on the Lie algebra \mathfrak{g} . Here we consider two typical examples of such Lax equations: the two-component BKP hierarchy [6, 18] and the Toda lattice hierarchy [25].

The two-component BKP hierarchy, which is the popular abbreviation of the two-component Kadomtsev-Petviashvili (KP) hierarchy of type B, was proposed by the Kyoto school as a bilinear equation [6]. This hierarchy was recently represented into a Lax form in [18] (cf. [23]) with two types of pseudo-differential operators. The sets \mathcal{D}^\pm

of these two types of operators compose a coupled Lie algebra $\mathcal{D}^- \times \mathcal{D}^+$ of the form \mathfrak{g} . By using an R -matrix on \mathfrak{g} introduced by Carlet [2], a bi-Hamiltonian structure of the two-component BKP hierarchy was derived in [26]. The two-component BKP hierarchy was also shown in [18] to be the universal hierarchy of Drinfeld-Sokolov hierarchies associated to untwisted affine Kac-Moody algebra of type D with the zeroth vertex c_0 of its Dynkin diagram marked [10]. Such Drinfeld-Sokolov hierarchies are bi-Hamiltonian systems. However, it seems that the bi-Hamiltonian structure given in [26] has no corresponding reduction when the two-component BKP hierarchy is reduced to the Drinfeld-Sokolov hierarchies of type D. With the help of an R -matrix on \mathfrak{g} introduced below, we will see that the two-component BKP hierarchy in fact carries a series of bi-Hamiltonian structures. Moreover, each of these bi-Hamiltonian structures can be reduced to the bi-Hamiltonian structure of a corresponding Drinfeld-Sokolov hierarchy of type D. Such a relation is analogous to the reduction from the KP hierarchy to Gelfand-Dickey hierarchies [9].

As another example, the Toda lattice hierarchy [25] also has a Lax representation defined on a coupled Lie algebra of the form \mathfrak{g} . This hierarchy is known to be equipped with three compatible Hamiltonian structures found by Carlet [2], but the reduction property of its Hamiltonian structures has not been considered before. We will show below that the Toda lattice hierarchy possesses infinitely many bi-Hamiltonian structures, and, under suitable constraint such bi-Hamiltonian structures are reduced to those of the extended bigraded Toda hierarchies [3, 5].

Our motivation is also from the study of Frobenius manifold. The concept of Frobenius manifold was introduced by Dubrovin to give a geometrical description of WDVV equations in topological field theory [11]. In the finite-dimensional case, Frobenius manifolds link integrable hierarchies with relevant research branches via the fact that with an arbitrary Frobenius manifold there is an associated bi-Hamiltonian structure of hydrodynamic type [11, 15]. In 2009, by considering the Hamiltonian structures of the Toda lattice hierarchy, Carlet, Dubrovin and Mertens [4] proposed the first example of infinite-dimensional Frobenius manifold. To find more infinite-dimensional Frobenius manifolds as well to consider their relation with Frobenius manifolds of finitely dimensions, it is natural to study the Hamiltonian structures of hierarchies like the Toda lattice hierarchy and clarify their reductions.

This paper is arranged as follows. In next section we state some facts on classical R -matrices, including the R -matrix formalism for deriving Poisson brackets and how R -matrices are acted by the so-called intertwining operators (see below for the definition). In Section 3 we find out all certain linear R -matrices on the Lie algebra $\mathfrak{g} = \mathcal{G}^- \times \mathcal{G}^+$, and classify them according to the action of intertwining involutions. With a selected R -matrix on \mathfrak{g} , we apply the R -matrix formalism to the examples of the two-component BKP hierarchy and the Toda lattice hierarchy in Section 4 and Section 5 respectively. We will construct Hamiltonian structures of these two hierarchies and study their reduction property. The final section is devoted to the summary and outlook.

2 Classical R -matrices

Let us recall briefly the R -matrix formalism developed in [22, 17, 20] and lay out some relevant facts.

2.1 The R -matrix formalism

Let \mathfrak{g} be a complex Lie algebra. A linear transformation $R : \mathfrak{g} \rightarrow \mathfrak{g}$ is called an R -matrix if it defines a Lie bracket as

$$[X, Y]_R = [R(X), Y] + [X, R(Y)], \quad X, Y \in \mathfrak{g}. \quad (2.1)$$

A sufficient condition for a linear transformation R being an R -matrix is that it solves the following modified Yang-Baxter equation

$$[R(X), R(Y)] - R([X, Y]_R) = -[X, Y] \quad (2.2)$$

for any $X, Y \in \mathfrak{g}$.

Remark 2.1 Generally the right hand side of (2.2) is written as $\alpha [X, Y]$ with some constant α . Note that the case $\alpha \neq 0$ is equivalent to $\alpha = -1$ by scaling R , and this is the special case we consider here.

Assume \mathfrak{g} is an associative algebra, whose Lie bracket is defined naturally by the commutator, and there is a function $\langle \cdot \rangle : \mathfrak{g} \rightarrow \mathbb{C}$ that gives a non-degenerate symmetric invariant bilinear form (inner product) $\langle \cdot, \cdot \rangle$ by

$$\langle X, Y \rangle = \langle XY \rangle = \langle YX \rangle, \quad X, Y \in \mathfrak{g}.$$

Via this inner product \mathfrak{g} can be identified with its dual space \mathfrak{g}^* . Let $T\mathfrak{g}$ and $T^*\mathfrak{g}$ denote the tangent and the cotangent bundles of \mathfrak{g} respectively, and their fibers have the form $T_A\mathfrak{g} = \mathfrak{g}$ and $T_A^*\mathfrak{g} = \mathfrak{g}^*$ at any point $A \in \mathfrak{g}$.

Given an R -matrix R on \mathfrak{g} , there define three brackets:

$$\{f, g\}_1(A) = \frac{1}{2} (\langle [A, df], R(dg) \rangle - \langle [A, dg], R(df) \rangle), \quad (2.3)$$

$$\{f, g\}_2(A) = \frac{1}{4} (\langle [A, df], R(A \cdot dg + dg \cdot A) \rangle - \langle [A, dg], R(A \cdot df + df \cdot A) \rangle), \quad (2.4)$$

$$\{f, g\}_3(A) = \frac{1}{2} (\langle [A, df], R(A \cdot dg \cdot A) \rangle - \langle [A, dg], R(A \cdot df \cdot A) \rangle), \quad (2.5)$$

where $f, g \in C^\infty(\mathfrak{g})$ have gradients $df, dg \in T_A^*\mathfrak{g}$ at $A \in \mathfrak{g}$. The brackets (2.3)–(2.5) are called respectively the linear, the quadratic and the cubic brackets according to [17, 20].

Let R^* be the adjoint transformation of R with respect to the above inner product. Then the anti-symmetric part of R is $R_a = \frac{1}{2}(R - R^*)$. The R -matrix formalism on the algebra \mathfrak{g} is as follows.

Theorem 2.2 ([17, 20]) *(1) For any R -matrix R the linear bracket is a Poisson bracket.*

(2) If both R and its anti-symmetric part R_a solve the modified Yang-Baxter equation (2.2), then the quadratic bracket is a Poisson bracket.

(3) If R satisfies the modified Yang-Baxter equation (2.2) then the cubic bracket is a Poisson bracket.

Moreover, these three Poisson brackets are compatible whenever all the above conditions are fulfilled.

On a non-commutative algebra \mathfrak{g} the R -matrix formalism gives no more Poisson brackets of order higher than 3; however, when \mathfrak{g} is a commutative associative algebra, by using R -matrices one can have Poisson brackets with order being any positive integers, see [16].

Theorem 2.3 ([16]) *Let \mathfrak{g} be a Lie algebra of a commutative associative algebra with Lie bracket $[\cdot, \cdot]$ satisfying $[X, YZ] = [X, Y]Z + Y[X, Z]$ for all $X, Y, Z \in \mathfrak{g}$. Assume \mathfrak{g} is equipped with an ad-invariant inner product $\langle \cdot, \cdot \rangle$ that is symmetric with respect to the multiplication, i.e., $\langle XY, Z \rangle = \langle X, YZ \rangle$ for all $X, Y, Z \in \mathfrak{g}$. If $R \in \text{End}(\mathfrak{g})$ is an R -matrix, then on \mathfrak{g} there exist compatible Poisson brackets defined as follows:*

$$\{f, g\}_r(A) = \frac{1}{2} (\langle [A, df], R(A^{r-1}dg) \rangle - \langle [A, dg], R(A^{r-1}df) \rangle) \quad (2.6)$$

with $f, g \in C^\infty(\mathfrak{g})$ and all positive integers r .

2.2 R -matrices and intertwining operators

A linear operator $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ is called intertwining if it satisfies

$$\sigma[X, Y] = [\sigma X, Y] = [X, \sigma Y], \quad X, Y \in \mathfrak{g}. \quad (2.7)$$

Proposition 2.4 ([21]) *If R is an R -matrix and σ is an intertwining operator, then $R \circ \sigma$ is also an R -matrix.*

Note that intertwining operators form a linear family. Hence the R -matrices $R \circ \sigma$, with R fixed and σ being intertwining operators, induce compatible Poisson brackets.

Definition 2.5 A linear operator $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ is called an intertwining involution if it satisfies (2.7) and $\sigma \circ \sigma = \text{id}$.

Proposition 2.6 *Let R be a solution of the modified Yang-Baxter equation (2.2) and σ be an intertwining involution, then $R \circ \sigma$ solves equation (2.2).*

Proof. This proposition follows from a simple calculation:

$$\begin{aligned} & [R \circ \sigma X, R \circ \sigma Y] - R \circ \sigma([R \circ \sigma X, Y] + [X, R \circ \sigma Y]) \\ &= [R \circ \sigma X, R \circ \sigma Y] - R([R \circ \sigma X, \sigma Y] + [\sigma X, R \circ \sigma Y]) \\ &= -[\sigma X, \sigma Y] = -\sigma \circ \sigma[X, Y] = -[X, Y]. \end{aligned}$$

□

3 R -matrices on a coupled Lie algebra

Let \mathcal{G} be a complex linear space with two subspaces \mathcal{G}^- and \mathcal{G}^+ . Assume that on each \mathcal{G}^\pm there is a Lie bracket, and these two brackets coincide on $\mathcal{G}^- \cap \mathcal{G}^+$. Moreover, we assume \mathcal{G}^\pm admit the following decompositions of Lie subalgebras:

$$\mathcal{G}^- = (\mathcal{G}^-)_- \oplus (\mathcal{G}^-)_+, \quad \mathcal{G}^+ = (\mathcal{G}^+)_- \oplus (\mathcal{G}^+)_+ \quad (3.1)$$

such that $(\mathcal{G}^-)_+ \subset (\mathcal{G}^+)_+$ and $(\mathcal{G}^+)_- \subset (\mathcal{G}^-)_-$. We consider the following coupled Lie algebra

$$\mathfrak{g} = \mathcal{G}^- \times \mathcal{G}^+ \quad (3.2)$$

whose Lie bracket is defined diagonally by the brackets on \mathcal{G}^\pm as

$$[(X, \hat{X}), (Y, \hat{Y})] = ([X, Y], [\hat{X}, \hat{Y}]), \quad (X, \hat{X}), (Y, \hat{Y}) \in \mathfrak{g}.$$

Introduce a linear transformation

$$R: \mathfrak{g} \rightarrow \mathfrak{g}, \quad (X, \hat{X}) \mapsto (aX_+ + bX_- + c\hat{X}_-, d\hat{X}_+ + e\hat{X}_- + fX_+) \quad (3.3)$$

with $a, b, c, d, e, f \in \mathbb{C}$. Here we use the subscripts \pm to denote the projections onto the Lie subalgebras $(\mathcal{G}^\pm)_\pm$ or $(\mathcal{G}^\pm)_\mp$ respectively. We substitute (3.3) into the modified Yang-Baxter equation

$$[R(\mathbf{X}), R(\mathbf{Y})] - R([\mathbf{X}, \mathbf{Y}]_R) = -[\mathbf{X}, \mathbf{Y}], \quad \mathbf{X} = (X, \hat{X}), \mathbf{Y} = (Y, \hat{Y}) \in \mathfrak{g}. \quad (3.4)$$

The left hand side is expanded to

l.h.s.

$$\begin{aligned} &= -(a^2[X_+, Y_+] + a^2[X_+, Y_-] + a^2[X_-, Y_+] + b^2[X_-, Y_-] \\ &\quad - (a^2 - b^2)([X_+, Y_-]_- + [X_-, Y_+]_-) + c(-a + b + f)([X_+, \hat{Y}_-]_- + [\hat{X}_-, Y_+]_-) \\ &\quad - c(d + e)[\hat{X}, \hat{Y}]_- + c(e - d - c)[\hat{X}_-, \hat{Y}_-], \\ &\quad d^2[\hat{X}_+, \hat{Y}_+] + d^2[\hat{X}_+, \hat{Y}_-] + d^2[\hat{X}_-, \hat{Y}_+] + e^2[\hat{X}_-, \hat{Y}_-] \\ &\quad - (d^2 - e^2)([\hat{X}_+, \hat{Y}_-]_- + [\hat{X}_-, \hat{Y}_+]_-) + f(d - e + c)([X_+, \hat{Y}_-]_+ + [\hat{X}_-, Y_+]_+) \\ &\quad + f(a + b)[X, Y]_+ + f(a - b - f)[X_+, Y_+]. \end{aligned}$$

By comparing the coefficients in equation (3.4) we have

$$a^2 = b^2 = d^2 = e^2 = 1, \quad (3.5)$$

$$c(-a + b + f) = 0, \quad c(e - d - c) = 0, \quad c(d + e) = 0, \quad (3.6)$$

$$f(a - b - f) = 0, \quad f(d - e + c) = 0, \quad f(a + b) = 0. \quad (3.7)$$

These equations are easily solved, which leads to the the following result.

Proposition 3.1 *The transformation R in (3.3) solves the modified Yang-Baxter equation (3.4) if and only if (a, b, c, d, e, f) is one of the following:*

$$\pm(1, -1, -2, 1, -1, 2), \quad \pm(1, -1, 2, -1, 1, 2), \quad (3.8)$$

$$\pm(1, -1, 0, 1, 1, 2), \quad \pm(1, -1, 0, -1, -1, 2), \quad (3.9)$$

$$\pm(1, 1, -2, 1, -1, 0), \quad \pm(1, 1, 2, -1, 1, 0), \quad (3.10)$$

$$(\pm 1, \pm 1, 0, \pm 1, \pm 1, 0). \quad (3.11)$$

On \mathfrak{g} there exist two intertwining involutions σ_1 and σ_2 defined by

$$\sigma_1(X, \hat{X}) = (-X, \hat{X}), \quad \sigma_2(X, \hat{X}) = (X, -\hat{X}). \quad (3.12)$$

They generate a group $G = \{\text{id}, \sigma_1, \sigma_2, \sigma_1 \circ \sigma_2\}$ of intertwining involutions. Up to the action of G (see Proposition 2.6), the solutions in each line of (3.8)-(3.10) are equivalent, while the solutions in line (3.11) are divided to four equivalence classes.

Among the R -matrices given in Proposition 3.1, we are particularly interested in the first one:

$$R(X, \hat{X}) = (X_+ - X_- - 2\hat{X}_-, \hat{X}_+ - \hat{X}_- + 2X_+). \quad (3.13)$$

Below we will fix R as in (3.13), and use it to construct Hamiltonian structures for integrable hierarchies whose Lax representation is defined on a coupled Lie algebra of the form (3.2).

Example 3.2 The R -matrix corresponding to the solution $(1, -1, 2, -1, 1, 2)$ in (3.8), denoted by \tilde{R} , was first introduced by Carlet [2] on a coupled Lie algebra of difference operators (see Section 5 below). This R -matrix was used in [2, 26] to construct Hamiltonian structures for the Toda lattice hierarchy and the two-component BKP hierarchy respectively. It is easy to see that $\tilde{R} = R \circ \sigma_2$, where R is given in (3.13).

Example 3.3 Every solution in line (3.11) splits into R -matrices on \mathcal{G}^- and \mathcal{G}^+ .

Remark 3.4 When $\mathcal{G}^- = \mathcal{G}^+$, an R -matrix of the form (3.13) was used in [1] to show the Liouville integrability of the Toda lattice defined on semi-simple Lie algebras. In this case, $\mathcal{G}^- \times \mathcal{G}^-$ is called the classical double of the Lie algebra \mathcal{G}^- , on which the ‘‘Adler-Kostant-Symes’’ R -matrices and the corresponding commutative hamiltonian flows were studied recently in [13]. It seems interesting to consider the action of intertwining involutions to such R -matrices.

4 Hamiltonian structures of the two-component BKP hierarchy

In this section we employ the R -matrix (3.13) to construct Hamiltonian structures for the two-component BKP hierarchy, then consider the reduction property of these Hamiltonian structures.

4.1 Notations and Lax representation

We review some notations in [18, 26] first. Let \mathcal{A} be a commutative algebra over \mathbb{C} , and $D : \mathcal{A} \rightarrow \mathcal{A}$ a derivation. Assume \mathcal{A} is equipped with a gradation $\mathcal{A} = \prod_{i \geq 0} \mathcal{A}_i$ such that $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$ and $D(\mathcal{A}_i) \subset \mathcal{A}_{i+1}$. Denote $\mathcal{D} = \{\sum_{i \in \mathbb{Z}} f_i D^i \mid f_i \in \mathcal{A}\}$ and consider its subspaces

$$\mathcal{D}^- = \left\{ \sum_{i < \infty} f_i D^i \mid f_i \in \mathcal{A} \right\}, \quad \mathcal{D}^+ = \left\{ \sum_{i \in \mathbb{Z}} \sum_{j \geq \max\{0, m-i\}} a_{i,j} D^i \mid a_{i,j} \in \mathcal{A}_j, m \in \mathbb{Z} \right\}. \quad (4.1)$$

Then \mathcal{D}^- and \mathcal{D}^+ are called the algebras of pseudo-differential operators of the first type and second type respectively, in which two elements are multiplied as series of the following product of monomials:

$$fD^i \cdot gD^j = \sum_{r \geq 0} \binom{i}{r} f D^r(g) D^{i+j-r}, \quad f, g \in \mathcal{A}.$$

Given a pseudo-differential operator $A = \sum_{i \in \mathbb{Z}} f_i D^i \in \mathcal{D}^\pm$, its positive part, negative part, residue and adjoint operator are respectively

$$A_+ = \sum_{i \geq 0} f_i D^i, \quad A_- = \sum_{i < 0} f_i D^i, \quad (4.2)$$

$$\text{res } A = f_{-1}, \quad A^* = \sum_{i \in \mathbb{Z}} (-D)^i \cdot f_i. \quad (4.3)$$

The projections given in (4.2) induce the following subalgebra decompositions

$$\mathcal{D}^\pm = (\mathcal{D}^\pm)_+ \oplus (\mathcal{D}^\pm)_-. \quad (4.4)$$

Clearly $(\mathcal{D}^-)_+ \subset (\mathcal{D}^+)_+$ and $(\mathcal{D}^+)_- \subset (\mathcal{D}^-)_-$.

Now suppose \mathcal{A} is the algebra of smooth functions on the circle S^1 , and $D = d/dx$ with x being the coordinate of S^1 . Introduce two pseudo-differential operators over \mathcal{A} :

$$P = D + \sum_{i \geq 1} u_i D^{-i}, \quad \hat{P} = D^{-1} \hat{u}_{-1} + \sum_{i \geq 1} \hat{u}_i D^i \quad (4.5)$$

such that $P^* = -DPD^{-1}$ and $\hat{P}^* = -D\hat{P}D^{-1}$. Then the two-component BKP hierarchy can be defined by the following Lax equations [18]:

$$\frac{\partial P}{\partial t_k} = [(P^k)_+, P], \quad \frac{\partial \hat{P}}{\partial t_k} = [(P^k)_+, \hat{P}], \quad (4.6)$$

$$\frac{\partial P}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, P], \quad \frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, \hat{P}] \quad (4.7)$$

with $k \in \mathbb{Z}_+^{\text{odd}}$.

To study Hamiltonian structures of this hierarchy, we need more notations. An element of the quotient space $\mathcal{F} = \mathcal{A}/D(\mathcal{A})$ is called a local functional. The map

$$\langle \rangle : \mathcal{D} \rightarrow \mathcal{F}, \quad A \mapsto \langle A \rangle = \int \text{res } A \, dx \quad (4.8)$$

induces an inner product on each of \mathcal{D}^\pm by

$$\langle A, B \rangle = \langle AB \rangle = \langle BA \rangle. \quad (4.9)$$

With respect to this inner product, the dual space of any subspace $\mathcal{S} \subset \mathcal{D}^\pm$ is denoted by \mathcal{S}^* . For example, one has

$$(\mathcal{D}^\pm)^* = \mathcal{D}^\pm, \quad ((\mathcal{D}^\pm)_\pm)^* = (\mathcal{D}^\pm)_{\mp}. \quad (4.10)$$

The spaces \mathcal{D}^\pm can also be decomposed as

$$\mathcal{D}^\pm = \mathcal{D}_0^\pm \oplus \mathcal{D}_1^\pm, \quad \mathcal{D}_\nu^\pm = \{A \in \mathcal{D}^\pm \mid A^* = (-1)^\nu A\}. \quad (4.11)$$

One sees that the dual subspaces of \mathcal{D}_ν^\pm are $(\mathcal{D}_\nu^\pm)^* = \mathcal{D}_{1-\nu}^\pm$. Every element of \mathcal{D}_ν^\pm can be expressed in the form

$$\sum_{i \in \mathbb{Z}} (a_i D^{2i+\nu} + D^{2i+\nu} a_i), \quad a_i \in \mathcal{A},$$

hence for any $l \in \mathbb{Z}$ we have the following subspace decompositions:

$$\mathcal{D}_\nu^\pm = (\mathcal{D}_\nu^\pm)_{\geq l} \oplus (\mathcal{D}_\nu^\pm)_{< l}, \quad \nu = 0, 1, \quad (4.12)$$

where

$$\begin{aligned} (\mathcal{D}_\nu^\pm)_{\geq l} &= \left\{ \sum_{2i+\nu \geq l} (a_i D^{2i+\nu} + D^{2i+\nu} a_i) \in \mathcal{D}^\pm \mid a_i \in \mathcal{A} \right\}, \\ (\mathcal{D}_\nu^\pm)_{< l} &= \left\{ \sum_{2i+\nu < l} (a_i D^{2i+\nu} + D^{2i+\nu} a_i) \in \mathcal{D}^\pm \mid a_i \in \mathcal{A} \right\}. \end{aligned}$$

4.2 bi-Hamiltonian representations

Let us represent the two-component BKP hierarchy (4.6), (4.7) in a bi-Hamiltonian form. The procedure is almost the same with that in [26], where only one bi-Hamiltonian structure was obtained with the help of the R -matrix \hat{R} (see Example 3.2). Now we use the R -matrix (3.13) instead, and will derive infinitely many bi-Hamiltonian structures.

Take \mathfrak{g} in (3.2) to be the coupled Lie algebra

$$\mathfrak{D} = \mathcal{D}^- \times \mathcal{D}^+, \quad (4.13)$$

where \mathcal{D}^- and \mathcal{D}^+ are the sets of pseudo-differential operators of the first type and the second type over the differential algebra \mathcal{A} . Recall (4.8) and (4.9), on \mathfrak{D} there exists an inner product define as

$$\langle (X, \hat{X}), (Y, \hat{Y}) \rangle = \langle (X, \hat{X})(Y, \hat{Y}) \rangle = \langle X, Y \rangle + \langle \hat{X}, \hat{Y} \rangle, \quad (X, \hat{X}), (Y, \hat{Y}) \in \mathfrak{D}. \quad (4.14)$$

On the algebra \mathfrak{D} we have the R -matrix (3.13), in which the subscripts \pm mean the projections onto $(\mathcal{D}^-)_\pm$ or $(\mathcal{D}^+)_\pm$. It is easy to see that R is anti-symmetric, i.e., $R^* = -R$, with respect to the inner product (4.14). Hence its anti-symmetric part $R_a = R$ clearly satisfies the modified Yang-Baxter equation (2.2).

To state the result we need a bit more preparation. Generally every element in \mathfrak{D} has the following expression

$$\mathbf{A} = \left(\sum_{i \in \mathbb{Z}} w_i D^i, \sum_{i \in \mathbb{Z}} \hat{w}_i D^i \right) \in \mathfrak{D}. \quad (4.15)$$

With the coefficients being a coordinate, \mathfrak{D} can be viewed as an infinite-dimensional manifold. For any local functional $F = \int f dx$ on this manifold, its variational gradient $\delta F/\delta \mathbf{A}$ at \mathbf{A} is defined by $\delta F = \langle \delta F/\delta \mathbf{A}, \delta \mathbf{A} \rangle$. More explicitly,

$$\frac{\delta F}{\delta \mathbf{A}} = \left(\sum_{i \in \mathbb{Z}} D^{-i-1} \frac{\delta F}{\delta w_i}, \sum_{i \in \mathbb{Z}} D^{-i-1} \frac{\delta F}{\delta \hat{w}_i} \right),$$

where $\delta F/\delta w = \sum_{j \geq 0} (-D)^j (\partial f/\partial w^{(j)})$. It shall be indicated that in this section we only consider functionals with their gradients lying in \mathfrak{D} .

By using the second part of Theorem 2.2 we have the following result.

Lemma 4.1 *Let F and H be two arbitrary functionals. On the algebra \mathfrak{D} there is a quadratic Poisson bracket*

$$\{F, H\}(\mathbf{A}) = \left\langle \frac{\delta F}{\delta \mathbf{A}}, \mathcal{P}_{\mathbf{A}} \left(\frac{\delta H}{\delta \mathbf{A}} \right) \right\rangle, \quad \mathbf{A} = (A, \hat{A}) \in \mathfrak{D}, \quad (4.16)$$

where the Poisson tensor $\mathcal{P} : T\mathfrak{D}^* \rightarrow T\mathfrak{D}$ is defined by

$$\begin{aligned} \mathcal{P}_{(A, \hat{A})}(X, \hat{X}) = & (- (AX + \hat{A}\hat{X})_- A + A(XA + \hat{X}\hat{A})_- , \\ & (AX + \hat{A}\hat{X})_+ \hat{A} - \hat{A}(XA + \hat{X}\hat{A})_+). \end{aligned} \quad (4.17)$$

We proceed to reduce the Poisson bracket (4.16) to an appropriate submanifold of \mathfrak{D} where the flows of the two-component BKP hierarchy are defined.

First, according to the decompositions (4.11), one can decompose the space \mathfrak{D} as

$$\mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}_1, \quad \mathfrak{D}_\nu = \mathcal{D}_\nu^- \times \mathcal{D}_\nu^+ \text{ for } \nu = 0, 1. \quad (4.18)$$

The subspaces \mathfrak{D}_0 and \mathfrak{D}_1 are dual to each other, hence at any point $\mathbf{A} \in \mathfrak{D}_\nu$ one has $T_{\mathbf{A}}^* \mathfrak{D}_\nu = (\mathfrak{D}_\nu)^* = \mathfrak{D}_{1-\nu}$. It is easy to check the following lemma.

Lemma 4.2 *The subspaces \mathfrak{D}_0 and \mathfrak{D}_1 are Poisson submanifolds of \mathfrak{D} with respect to the Poisson structure (4.16).*

Second, given an arbitrary positive integer m , we let

$$\mathbf{A} = (A, \hat{A}) = (DP^{2m}, D\hat{P}^2) \quad (4.19)$$

with P and \hat{P} introduced in (4.5). One sees that

$$\begin{aligned} A = DP^{2m} &= D^{2m+1} + \sum_{i \leq m} (v_i D^{2i-1} + f_i D^{2i-2}), \\ \hat{A} = D\hat{P}^2 &= \rho D^{-1} \rho + \sum_{i \geq 1} (\hat{v}_i D^{2i-1} + \hat{f}_i D^{2i-2}), \quad \rho = \hat{u}_{-1}. \end{aligned}$$

Denote $\hat{v}_0 = \rho^2$ and $\mathbf{v} = (v_m, v_{m-1}, \dots, \hat{v}_0, \hat{v}_1, \dots)$. Observe that the coordinate \mathbf{v} is related to $\mathbf{u} = (u_1, u_3, \dots, \hat{u}_{-1}, \hat{u}_1, \hat{u}_3, \dots)$ given in (4.5) via a Miura-type transformation, and that the coefficients f_{-i} and \hat{f}_i are linear functions of derivatives of \mathbf{v} determined by the symmetry $(A^*, \hat{A}^*) = -(A, \hat{A})$.

Operators of the form (4.19) compose a coset $(D^{2m+1}, 0) + \mathcal{U}_m$, where

$$\mathcal{U}_m = (\mathcal{D}_1^-)_{<2m} \times ((\mathcal{D}_1^+)_{\geq 0} \times \mathcal{M}), \quad \mathcal{M} = \{\rho D^{-1}\rho \mid \rho \in \mathcal{A}\}, \quad (4.20)$$

recalling (4.12). Here \mathcal{M} is considered as a 1-dimensional manifold with coordinate ρ and its tangent spaces

$$T_\rho \mathcal{M} = \{\rho D^{-1}f + f D^{-1}\rho \mid f \in \mathcal{A}\}. \quad (4.21)$$

Then the tangent bundle of the coset $(D^{2m+1}, 0) + \mathcal{U}_m$, denoted by $T\mathcal{U}_m$, has fibers

$$T_{\mathbf{A}}\mathcal{U}_m = (\mathcal{D}_1^-)_{<2m} \times ((\mathcal{D}_1^+)_{\geq 0} \oplus T_\rho \mathcal{M}). \quad (4.22)$$

Their dual spaces

$$T_{\mathbf{A}}^*\mathcal{U}_m = (\mathcal{D}_0^-)_{\geq -2m} \times ((\mathcal{D}_0^+)_{<-1} \oplus T_\rho^* \mathcal{M}), \quad T_\rho^* \mathcal{M} = \mathcal{A} \quad (4.23)$$

compose the cotangent bundle $T^*\mathcal{U}_m$ of $(D^{2m+1}, 0) + \mathcal{U}_m$. One sees that a functional F on the coset $(D^{2m+1}, 0) + \mathcal{U}_m$ has variational gradient in $T_{\mathbf{A}}^*\mathcal{U}_m$ as

$$\frac{\delta F}{\delta \mathbf{A}} = \frac{1}{2} \left(\sum_{i \leq m} \left(\frac{\delta F}{\delta v_i} D^{-2i} + D^{-2i} \frac{\delta F}{\delta v_i} \right), \sum_{i \geq 0} \left(\frac{\delta F}{\delta \hat{v}_i} D^{-2i} + D^{-2i} \frac{\delta F}{\delta \hat{v}_i} \right) \right).$$

Lemma 4.3 *The map $\mathcal{P} : T^*\mathcal{U}_m \rightarrow T\mathcal{U}_m$ defined by the formula (4.17) is a Poisson tensor on the coset $(D^{2m+1}, 0) + \mathcal{U}_m$ that consists of operators of the form (4.19).*

Proof. One can show that the Poisson tensor \mathcal{P} on \mathfrak{D}_1 can be properly restricted to the coset $(D^{2m+1}, 0) + \mathcal{U}_m$, see the proof of Lemma 5.1 in [26] for details. Or, perform a Dirac reduction from \mathfrak{D}_1 to the coset $(D^{2m+1}, 0) + \mathcal{U}_m$. That is, decompose

$$\mathfrak{D}_1 = T_{\mathbf{A}}\mathcal{U}_m \oplus \mathcal{V}_{\mathbf{A}}, \quad \mathfrak{D}_1^* = \mathfrak{D}_0 = T_{\mathbf{A}}^*\mathcal{U}_m \oplus \mathcal{V}_{\mathbf{A}}^*,$$

where

$$\begin{aligned} \mathcal{V}_{\mathbf{A}} &= (\mathcal{D}_1^-)_{\geq 2m+1} \times ((\mathcal{D}_1^+)_{<0} / T_\rho \mathcal{M}), \\ \mathcal{V}_{\mathbf{A}}^* &= (\mathcal{D}_0^-)_{<-2m-1} \times (T_\rho^*)^\perp \mathcal{M}, \quad (T_\rho^*)^\perp \mathcal{M} = \{\hat{Y} \in (\mathcal{D}_0^+)_{+} \mid \hat{Y}(\rho) = 0\}, \end{aligned}$$

and then check that the map

$$\mathcal{P}_{\mathbf{A}} = \begin{pmatrix} \mathcal{P}_{\mathbf{A}}^{\mathcal{U}\mathcal{U}} & \mathcal{P}_{\mathbf{A}}^{\mathcal{U}\mathcal{V}} \\ \mathcal{P}_{\mathbf{A}}^{\mathcal{V}\mathcal{U}} & \mathcal{P}_{\mathbf{A}}^{\mathcal{V}\mathcal{V}} \end{pmatrix} : T_{\mathbf{A}}^*\mathcal{U}_m \oplus \mathcal{V}_{\mathbf{A}}^* \rightarrow T_{\mathbf{A}}\mathcal{U}_m \oplus \mathcal{V}_{\mathbf{A}}$$

defined in (4.17) is diagonal. The lemma is proved. \square

Introduce a shift transformation on the coset $(D^{2m+1}, 0) + \mathcal{U}_m$ as

$$\mathcal{S} : (A, \hat{A}) \mapsto (A + sD, \hat{A} + sD), \quad (4.24)$$

where s is a parameter. The push-forward of the Poisson tensor \mathcal{P} in Lemma 4.3 has the form $\mathcal{S}_* \mathcal{P} = \mathcal{P}_2 - s\mathcal{P}_1 + s^2 \mathcal{P}_0$. It is straightforward to show $\mathcal{P}_0 = 0$, hence we derive the following lemma.

Lemma 4.4 *On the coset $(D^{2m+1}, 0) + \mathcal{U}_m$ there exist two compatible Poisson tensors defined as:*

$$\begin{aligned} \mathcal{P}_1(X, \hat{X}) = & \left(- (DX + D\hat{X})_- A - (AX + \hat{A}\hat{X})_- D \right. \\ & + A(XD + \hat{X}D)_- + D(XA + \hat{X}\hat{A})_-, \\ & (DX + D\hat{X})_+ \hat{A} + (AX + \hat{A}\hat{X})_+ D \\ & \left. - \hat{A}(XD + \hat{X}D)_+ - D(XA + \hat{X}\hat{A})_+ \right), \end{aligned} \quad (4.25)$$

$$\begin{aligned} \mathcal{P}_2(X, \hat{X}) = & \left(- (AX + \hat{A}\hat{X})_- A + A(XA + \hat{X}\hat{A})_- \right. \\ & \left. (AX + \hat{A}\hat{X})_+ \hat{A} - \hat{A}(XA + \hat{X}\hat{A})_+ \right) \end{aligned} \quad (4.26)$$

with $(X, \hat{X}) \in T_{\mathbf{A}}^* \mathcal{U}_m$ at any point $\mathbf{A} = (A, \hat{A}) \in (D^{2m+1}, 0) + \mathcal{U}_m$.

Let $\{\cdot, \cdot\}_{1,2}^m$ denote the Poisson brackets on $(D^{2m+1}, 0) + \mathcal{U}_m$ given by the Poisson tensors $\mathcal{P}_{1,2}$ respectively. With the same method as used in the proof of Theorem 5.4 in [26], we arrive at the following result.

Theorem 4.5 *For any positive integer m , the two-component BKP hierarchy (4.6), (4.7) can be expressed in a bi-Hamiltonian recursion form as follows:*

$$\frac{\partial F}{\partial t_k} = \{F, H_{k+2m}\}_1^m = \{F, H_k\}_2^m, \quad \frac{\partial F}{\partial \hat{t}_k} = \{F, \hat{H}_{k+2}\}_1^m = \{F, \hat{H}_k\}_2^m \quad (4.27)$$

with $k \in \mathbb{Z}_+^{\text{odd}}$ and Hamiltonians

$$H_k = \frac{2m}{k} \langle P^k \rangle, \quad \hat{H}_k = \frac{2}{k} \langle \hat{P}^k \rangle. \quad (4.28)$$

Thus by using the R -matrix (3.13) we obtain a series of bi-Hamiltonian structures for the two-component BKP hierarchy. They are different from the bi-Hamiltonian structure given in [26].

Remark 4.6 Observe that the densities of Hamiltonian in (4.28) (cf. $H_k = \frac{2}{k} \langle P^k \rangle$ and $\hat{H}_k = -\frac{2}{k} \langle \hat{P}^k \rangle$ in (5.19) of [26]) satisfy the so-called tau-symmetry condition [15], hence they define a tau function. This tau function satisfies the bilinear equation of the two-component BKP hierarchy in [6], see also [18].

4.3 Reductions of the bi-Hamiltonian structures

We now study the reductions of the bi-Hamiltonian structures in Theorem 4.5.

First, suppose the pseudo-differential operator $\mathbf{A} = (A, \hat{A})$ in (4.19) satisfies

$$A = \hat{A}, \quad \text{i.e.,} \quad P^{2m} = \hat{P}^2. \quad (4.29)$$

It was shown in [18] that this constraint reduces the two-component BKP hierarchy (4.6), (4.7) to the Drinfeld-Sokolov hierarchy associated to the affine Lie algebra $D_{m+1}^{(1)}$ with the zeroth vertex c_0 of its Dynkin diagram marked [10].

Operators $L = A = \hat{A}$ form a coset contained in $\mathcal{D}^- \cap \mathcal{D}^+$; on this coset we let $F_X(L)$ denote the functional that has gradient X with respect to L . Suppose $\delta F_X(L)/\delta \mathbf{A} = (W, \hat{W})$ whenever $F_X(L)$ is viewed as a functional on the coset $(D^{2m+1}, 0) + \mathcal{U}_m$, then we have $X = (W + \hat{W})|_{A=\hat{A}=L}$. Thus the Poisson brackets in Theorem 4.5 under the constraint (4.29) are as follows (the superscripts m omitted):

$$\begin{aligned} \{F_X(L), F_Y(L)\}_1 &= \langle (LX)_- (DY)_+ + (DX)_- (LY)_+ - (XL)_- (YD)_+ - (XD)_- (YL)_+ \rangle, \end{aligned} \quad (4.30)$$

$$\{F_X(L), F_Y(L)\}_2 = \langle (LX)_- (LY)_+ - (XL)_- (YL)_+ \rangle. \quad (4.31)$$

They are just the bi-Hamiltonian structure of the Drinfeld-Sokolov hierarchy of type $(D_{m+1}^{(1)}, c_0)$ given in Proposition 8.3 of [10], with Hamiltonians H_k and \hat{H}_k in (4.28), see also [18]. Thanks to the facts $(DY)_\pm = DY_\pm$ and $\text{res}[L, Y] = 0$, the first bracket (4.30) can be rewritten to

$$\begin{aligned} \{F_X(L), F_Y(L)\}_1 &= \langle X, DY_+L - LY_+D - D(YL)_+ + (LY)_+D \rangle \\ &= \langle X, DY_+L - LY_+D - (DYL)_+ + (LYD)_+ - \text{res}[L, Y] \rangle \\ &= \langle L(X_+DY_+ - Y_+DX_+ + Y_-DX_- - X_-DY_-) \rangle, \end{aligned}$$

which is just the formula (7.0) in [12].

Second, when $\hat{P} = 0$, the hierarchy (4.6), (4.7) becomes the BKP hierarchy [8]. In this case, the formulae (4.48) are reduced to bi-Hamiltonian representations for the BKP hierarchy given by the following Poisson brackets:

$$\{F_X(A), F_Y(A)\}_1^m = \langle X(-DY_-A - (AY)_-D + AY_-D + D(YA)_-) \rangle, \quad (4.32)$$

$$\{F_X(A), F_Y(A)\}_2^m = \langle -(AX)_+(AY)_- + (XA)_+(YA)_- \rangle, \quad (4.33)$$

where $A = DP^{2m}$.

Furthermore, by setting $A_- = (DP^{2m})_- = 0$ to the BKP hierarchy, what one has is the Drinfeld-Sokolov hierarchy of type $(B_m^{(1)}, c_0)$ (see [10]). Distinctively we denote $L = DP^{2m} = (DP^{2m})_+$. Then the bi-Hamiltonian structure of Drinfeld-Sokolov hierarchy of type $(B_m^{(1)}, c_0)$ is reduced from (4.32), (4.33) to

$$\{F_X(L), F_Y(L)\}_1 = \langle L(YDX - XDY) \rangle, \quad (4.34)$$

$$\{F_X(L), F_Y(L)\}_2 = \langle (LX)_-(LY)_+ - (XL)_-(YL)_+ \rangle. \quad (4.35)$$

4.4 More Hamiltonian structures and their reductions

The R -matrix (3.13) produces even more Hamiltonian structures for the two-component BKP hierarchy.

For any positive integer m , we replace (4.19) by $\mathbf{A} = (DP^{2m-1}, D\hat{P})$. All such operators form a coset $(D^{2m}, 0) + \mathcal{V}_m$, where

$$\mathcal{V}_m = (\mathcal{D}_0^-)_{<2m} \times (\mathcal{D}_0^+)_{\geq 0}.$$

With the same method as in §4.2, one can restrict the Poisson bracket (4.16) properly to the coset $(D^{2m}, 0) + \mathcal{V}_m$. Denote the restricted bracket by $\{, \}^m$, then we have the following proposition.

Proposition 4.7 For any positive integer m , the two-component BKP hierarchy (4.6), (4.7) has the following Hamiltonian representation:

$$\frac{\partial F}{\partial t_k} = \{F, H_k\}^m, \quad \frac{\partial F}{\partial \hat{t}_k} = \{F, \hat{H}_k\}^m, \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (4.36)$$

where $H_k = \frac{2m-1}{k} \langle P^k \rangle$ and $\hat{H}_k = \frac{1}{k} \langle \hat{P}^k \rangle$.

When the two-component BKP hierarchy (4.6), (4.7) is constrained by

$$L = DP^{2m-1} = D\hat{P}, \quad (4.37)$$

it becomes the Drinfeld-Sokolov hierarchy of type $(A_{2m-1}^{(2)}, c_0)$ (see [10]), which possesses a Hamiltonian structure reduced from (4.36) as:

$$\{F_X(L), F_Y(L)\} = \langle (LX)_-(LY)_+ - (XL)_-(YL)_+ \rangle. \quad (4.38)$$

Example 4.8 In the particular case of $m = 1$ so that $L = D + D^{-1}\rho$, we have the reduced hierarchy

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (4.39)$$

The bracket (4.38) can be rewritten as

$$\{\rho(x), \rho(y)\} = \frac{1}{2}\rho(x)\delta'(x-y) + \rho'(x)\delta(x-y) - \frac{1}{2}\delta'''(x-y).$$

Considering together the Hamiltonians $H_k = \frac{1}{k} \langle L^k \rangle$, one sees that (4.39) is just the Korteweg-de Vries hierarchy, see e.g. [9].

4.5 Dispersionless case

Let us consider Hamiltonian structures of the dispersionless two-component BKP hierarchy.

Consider two algebras $\mathcal{H}^- = \mathcal{A}((z^{-1}))$ and $\mathcal{H}^+ = \mathcal{A}((z))$ of Laurent series in $z \in S^1$. On each \mathcal{H}^\pm there exist a Lie bracket

$$[a, b] = \frac{\partial a}{\partial z} \frac{\partial b}{\partial x} - \frac{\partial b}{\partial z} \frac{\partial a}{\partial x}, \quad (4.40)$$

and an ad-invariant inner product

$$\langle a, b \rangle = \langle a b \rangle, \quad \langle a \rangle = \frac{1}{2\pi\sqrt{-1}} \oint_{S^1} \oint_{S^1} a(z) dz dx. \quad (4.41)$$

Let

$$p(z) = z + \sum_{i \leq 0} u_i z^{2i-1} \in \mathcal{H}^-, \quad \hat{p}(z) = \sum_{i \geq 0} \hat{u}_i z^{2i-1} \in \mathcal{H}^+, \quad (4.42)$$

then the dispersionless two-component BKP hierarchy is defined as

$$\frac{\partial \alpha(z)}{\partial t_k} = [(p(z)^k)_+, \alpha(z)], \quad \frac{\partial \alpha(z)}{\partial \hat{t}_k} = [-\langle \hat{p}(z)^k \rangle_-, \alpha(z)], \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (4.43)$$

where $\alpha(z) \in \{p(z), \hat{p}(z)\}$. Here we use the subscripts \pm to stand for the projections of a series in \mathcal{H}^\pm to its nonnegative and negative part respectively. Recall that the hierarchy (4.43) was first written down by Takasaki [24] as the hierarchy underlying the D-type topological Landau-Ginzburg models.

Introduce the coupled Lie algebra $\mathfrak{H} = \mathcal{H}^- \times \mathcal{H}^+$, which is equipped with an inner product

$$\langle (a, \hat{a}), (b, \hat{b}) \rangle = \langle a, b \rangle + \langle \hat{a}, \hat{b} \rangle, \quad (a, \hat{a}), (b, \hat{b}) \in \mathfrak{H}.$$

Define on \mathfrak{H} an R -matrix R as given in (3.13). Now we apply Theorem 2.3 with $r = 2$ to construct bi-Hamiltonian structures for the dispersionless two-component BKP hierarchy.

Given any positive integers m and n , let

$$\mathbf{a}(z) = (a(z), \hat{a}(z)) = (z p(z)^{2m}, z \hat{p}(z)^{2n}). \quad (4.44)$$

All such series form a coset $(z^{2m+1}, 0) + U_{m,n}$, where

$$U_{m,n} = \left\{ \left(\sum_{i \leq m} v_i z^{2i-1}, \sum_{i \geq 1-n} \hat{v}_i z^{2i-1} \right) \in \mathcal{H}^- \times \mathcal{H}^+ \right\}. \quad (4.45)$$

In the same way as for the dispersive case, on the coset $(z^{2m+1}, 0) + U_{m,n}$ there exist two compatible Poisson brackets $\{\cdot, \cdot\}_\nu^{m,n}$ ($\nu = 1, 2$) given by the following tensors:

$$\begin{aligned} & \mathcal{P}_1(X(z), \hat{X}(z)) \\ &= (-z([a(z), X(z)]_- + [\hat{a}(z), \hat{X}(z)]_-) - a(z)(\partial_x X(z) + \partial_x \hat{X}(z))_- \\ & \quad + [a(z), z(X(z) + \hat{X}(z))]_- + \partial_x(X(z)a(z) + \hat{X}(z)\hat{a}(z))_-, \\ & \quad z([a(z), X(z)]_+ + [\hat{a}(z), \hat{X}(z)]_+) + \hat{a}(z)(\partial_x X(z) + \partial_x \hat{X}(z))_+ \\ & \quad - [\hat{a}(z), z(X(z) + \hat{X}(z))]_+ - \partial_x(X(z)a(z) + \hat{X}(z)\hat{a}(z))_+), \end{aligned} \quad (4.46)$$

$$\begin{aligned} & \mathcal{P}_2(X(z), \hat{X}(z)) \\ &= (-a(z)([a(z), X(z)]_- + [\hat{a}(z), \hat{X}(z)]_-) + [a(z), (X(z)a(z) + \hat{X}(z)\hat{a}(z))]_-, \\ & \quad \hat{a}(z)([a(z), X(z)]_+ + [\hat{a}(z), \hat{X}(z)]_+) - [\hat{a}(z), (X(z)a(z) + \hat{X}(z)\hat{a}(z))]_+). \end{aligned} \quad (4.47)$$

where $(X(z), \hat{X}(z)) \in T_{\mathbf{a}}^* U_{m,n}$ (equal to the dual space of $U_{m,n}$). Thus we have

Proposition 4.9 *For any positive integers m and n , the dispersionless two-component BKP hierarchy (4.43) can be expressed as*

$$\frac{\partial F}{\partial t_k} = \{F, H_{k+2m}\}_1^{m,n} = \{F, H_k\}_2^{m,n}, \quad \frac{\partial F}{\partial \hat{t}_k} = \{F, \hat{H}_{k+2n}\}_1^{m,n} = \{F, \hat{H}_k\}_2^{m,n} \quad (4.48)$$

with $k \in \mathbb{Z}_+^{\text{odd}}$ and

$$H_k = \frac{2m}{k} \langle p(z)^k \rangle, \quad \hat{H}_k = \frac{2n}{k} \langle \hat{p}(z)^k \rangle.$$

If the dispersionless two-component BKP hierarchy (4.43) is constrained by

$$p(z)^{2m} = \hat{p}(z)^{2n} = l(z)$$

with $l(z) = z^{2m} + \sum_{i=1-n}^m v_i z^{2i-2}$, then we have the following integrable hierarchy

$$\frac{\partial l(z)}{\partial t_k} = [(p(z)^k)_+, l(z)], \quad \frac{\partial l(z)}{\partial \hat{t}_k} = [-(\hat{p}(z)^k)_-, l(z)], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (4.49)$$

This dispersionless hierarchy possesses a bi-Hamiltonian structure reduced from (4.46), (4.47) as:

$$\{F_X(l), F_Y(l)\}_1 = \langle X(z), [l(z), Y(z)]_+ - [l(z), Y(z)_+] \rangle, \quad (4.50)$$

$$\{F_X(l), F_Y(l)\}_2 = \langle X(z), l(z)[l(z), Y(z)]_+ - [l(z), (l(z)Y(z))_+] \rangle. \quad (4.51)$$

Remark 4.10 The quantization of the dispersionless hierarchy (4.49) is the two-component BKP hierarchy (4.6), (4.7) constrained by $P^{2m} = \hat{P}^{2n}$. This is called the $(2m, 2n)$ -reduction, which corresponds to the reduction of Lie algebras from $\mathfrak{go}(2\infty)$ to $D_{m+n}^{(1)}$ in the notation of [7]. In more details, due to the closedness of the 1-form

$$\omega = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} (\text{res } P^k dt_k + \text{res } \hat{P}^k d\hat{t}_k),$$

we introduce a tau function $\tau = \tau(\mathbf{t}, \hat{\mathbf{t}})$ by

$$\omega = d(2 \partial_x \log \tau) \quad \text{with} \quad x = t_1. \quad (4.52)$$

With the same dressing method as in [18], one can show that the $(2m, 2n)$ -reduction of the two-component BKP hierarchy is equivalent to the following bilinear equation of tau function:

$$\begin{aligned} & \text{res}_z z^{2mj-1} X(\mathbf{t}; z) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\hat{\mathbf{t}}'; -z) \tau(\mathbf{t}', \hat{\mathbf{t}}') \\ & = \text{res}_z z^{2nj-1} X(\hat{\mathbf{t}}; z) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\hat{\mathbf{t}}'; -z) \tau(\mathbf{t}', \hat{\mathbf{t}}'), \quad j \geq 0. \end{aligned} \quad (4.53)$$

Here X is a vertex operator given as

$$X(\mathbf{t}; z) = \exp \left(\sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k \right) \exp \left(- \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{2}{kz^k} \frac{\partial}{\partial t_k} \right),$$

and formally $\text{res}_z \sum f_i z^i = f_{-1}$. Note that the bilinear equation (4.53) with $j = 0$ is the original form of the two-component BKP hierarchy [6], and the case $j = 1$ was written down in [7] (see equation (2.25) there).

When $n = 1$, the bilinear equation (4.53) coincides with the Drinfeld-Sokolov hierarchy of type $(D_{m+1}^{(1)}, c_0)$ (see [18]), whose bi-Hamiltonian structure (4.30), (4.31) can be reduced from that of the two-component BKP hierarchy. When $n > 1$, however, up to now we only obtain the bi-Hamiltonian structures (4.50), (4.51) for the dispersionless Lax equations. The difficulty to consider the dispersive case lies in the description of the manifold consisting operators of the form $(\hat{P}^{2n})_-$ with \hat{P} given in (4.5), which obstructs a Dirac reduction needed.

5 Hamiltonian structures of the Toda lattice hierarchy

In this section we apply the R -matrix (3.13) to the case of the Toda lattice hierarchy.

5.1 Lax representation

Let \mathcal{A} be the set of discrete functions whose support is a finite subset of \mathbb{Z} , and Λ be a shift operator on \mathcal{A} such that $\Lambda(f(n)) = f(n+1)$. Denote $\mathcal{E} = \{\sum_{i \in \mathbb{Z}} f_i \Lambda^i \mid f_i \in \mathcal{A}\}$, and for $A = \sum_{i \in \mathbb{Z}} f_i \Lambda^i \in \mathcal{E}$ one has

$$A_{\geq k} = A_{>k-1} = \sum_{i \geq k} f_i \Lambda^i, \quad A_{<k} = A_{\leq k+1} = \sum_{i < k} f_i \Lambda^i, \quad (5.1)$$

$$\text{Res } A = f_0, \quad \langle A \rangle = \sum_{n \in \mathbb{Z}} \text{Res } A(n) = \sum_{n \in \mathbb{Z}} f_0(n). \quad (5.2)$$

The space \mathcal{E} contains the following two subspaces:

$$\mathcal{E}^- = \left\{ \sum_{i < \infty} f_i \Lambda^i \mid f_i \in \mathcal{A} \right\}, \quad \mathcal{E}^+ = \left\{ \sum_{i > -\infty} f_i \Lambda^i \mid f_i \in \mathcal{A} \right\}. \quad (5.3)$$

Define a product by $f(m)\Lambda^i \cdot g(n)\Lambda^j = f(m)g(n+i)\Lambda^{i+j}$, then \mathcal{E}^\pm become associative algebras. Hence they are Lie algebras with Lie bracket given by the commutator. It is easy to see that the formula

$$\langle A, B \rangle = \langle AB \rangle = \langle BA \rangle, \quad A, B \in \mathcal{E}^\pm \quad (5.4)$$

defines a non-invariant inner product on each of \mathcal{E}^\pm .

One has the decompositions of Lie subalgebras:

$$\mathcal{E}^\pm = (\mathcal{E}^\pm)_{\geq 0} \oplus (\mathcal{E}^\pm)_{<0}$$

Introduce

$$L = \Lambda + \sum_{i \leq 0} u_i \Lambda^i \in \mathcal{E}^-, \quad \hat{L} = \sum_{i \geq -1} \hat{u}_i \Lambda^i \in \mathcal{E}^+, \quad (5.5)$$

then the Toda lattice hierarchy [25] is defined as

$$\frac{\partial L}{\partial t_k} = [(L^k)_{\geq 0}, L], \quad \frac{\partial \hat{L}}{\partial t_k} = [(L^k)_{\geq 0}, \hat{L}], \quad (5.6)$$

$$\frac{\partial L}{\partial \hat{t}_k} = [-(\hat{L}^k)_{<0}, L], \quad \frac{\partial \hat{L}}{\partial \hat{t}_k} = [-(\hat{L}^k)_{<0}, \hat{L}], \quad (5.7)$$

where k runs over all positive integers.

5.2 Hamiltonian structures

Consider the coupled Lie algebra

$$\mathfrak{E} = \mathcal{E}^- \times \mathcal{E}^+, \quad (5.8)$$

whose Lie bracket is defined diagonally. Similar to the case for \mathfrak{D} in the previous section, one defines on \mathfrak{E} an inner product according to (5.4), and hence the gradient of functionals in $C^\infty(\mathfrak{E})$. Here only functionals with gradient lying in \mathfrak{E} will be considered.

The R -matrix (3.13) on the Lie algebra \mathfrak{E} reads

$$R(X, \hat{X}) = (X_{\geq 0} - X_{< 0} - 2\hat{X}_{< 0}, \hat{X}_{\geq 0} - \hat{X}_{< 0} + 2X_{\geq 0}). \quad (5.9)$$

One can check that the adjoint transformation of R satisfies

$$R^*(X, \hat{X}) = -R(X, \hat{X}) + 2R_0(X, \hat{X}),$$

where $R_0(X, \hat{X}) = (\text{Res}(X + \hat{X}), \text{Res}(X + \hat{X}))$. Hence the anti-symmetric part of R is

$$R_a(X, \hat{X}) = \frac{1}{2}(R(X, \hat{X}) - R^*(X, \hat{X})) = R(X, \hat{X}) - R_0(X, \hat{X}). \quad (5.10)$$

Claim The transformation R_a satisfies the modified Yang-Baxter equation (3.4).

Proof. Since R solves the modified Yang-Baxter equation, for any $\mathbf{X} = (X, \hat{X}), \mathbf{Y} = (Y, \hat{Y}) \in \mathfrak{E}$ we have

$$\begin{aligned} & [R_a(\mathbf{X}), R_a(\mathbf{Y})] - R_a([R_a(\mathbf{X}), \mathbf{Y}] + [\mathbf{X}, R_a(\mathbf{Y})]) + [\mathbf{X}, \mathbf{Y}] \\ &= -[R_0(\mathbf{X}), R(\mathbf{Y})] - [R(\mathbf{X}), R_0(\mathbf{Y})] + R([R_0(\mathbf{X}), \mathbf{Y}] + [\mathbf{X}, R_0(\mathbf{Y})]) \\ & \quad + R_0([R(\mathbf{X}), \mathbf{Y}] + [\mathbf{X}, R(\mathbf{Y})]) - R_0([R_0(\mathbf{X}), \mathbf{Y}] + [\mathbf{X}, R_0(\mathbf{Y})]). \end{aligned} \quad (5.11)$$

On the right hand side, the first three terms cancel due to $[R_0(\mathbf{X}), R(\mathbf{Y})] = R([R_0(\mathbf{X}), \mathbf{Y}])$; the fourth term is equal to (f, f) with

$$\begin{aligned} f &= \text{Res}([X_{\geq 0} - X_{< 0} - 2\hat{X}_{< 0}, Y] + [X, Y_{\geq 0} - Y_{< 0} - 2\hat{Y}_{< 0}]) \\ & \quad + \text{Res}([\hat{X}_{\geq 0} - \hat{X}_{< 0} + 2X_{\geq 0}, \hat{Y}] + [\hat{X}, \hat{Y}_{\geq 0} - \hat{Y}_{< 0} + 2Y_{\geq 0}]) \\ &= 2 \text{Res}([X_{\geq 0} - \hat{X}_{< 0}, Y] + [X, -Y_{< 0} - \hat{Y}_{< 0}]) \\ & \quad + [\hat{X}_{\geq 0} + X_{\geq 0}, \hat{Y}] + [\hat{X}, -\hat{Y}_{< 0} + Y_{\geq 0}] \\ &= 2 \text{Res}([X, Y_{\leq 0}] - [\hat{X}, Y_{> 0}] + [X, -Y_{< 0} - \hat{Y}_{< 0}]) \\ & \quad + [\hat{X} + X, \hat{Y}_{\leq 0}] + [\hat{X}, -\hat{Y}_{< 0} + Y_{\geq 0}] \\ &= \text{Res}([X + \hat{X}, \text{Res}(Y + \hat{Y})]) = 0; \end{aligned}$$

clearly the last term vanishes. Therefore the claim is proved. \square

According to Theorem 2.2 we have the following result.

Lemma 5.1 *Let F and H be two arbitrary functionals. On \mathfrak{E} there exist three compatible Poisson brackets:*

$$\{F, H\}_\nu(\mathbf{A}) = \left\langle \frac{\delta F}{\delta \mathbf{A}}, \mathcal{P}_\nu \left(\frac{\delta H}{\delta \mathbf{A}} \right) \right\rangle, \quad \nu = 1, 2, 3, \quad (5.12)$$

where $\mathbf{A} \in \mathfrak{E}$ and the Poisson tensors $\mathcal{P}_\nu : T\mathfrak{D}^* \rightarrow T\mathfrak{D}$ are defined by

$$\mathcal{P}_1(X, \hat{X}) = ([-X_{< 0} - \hat{X}_{< 0}, A] + [X, A]_{\leq 0} + [\hat{X}, \hat{A}]_{\leq 0},$$

$$[X_{\geq 0} + \hat{X}_{\geq 0}, \hat{A}] - [X, A]_{>0} - [\hat{X}, \hat{A}]_{>0}, \quad (5.13)$$

$$\begin{aligned} \mathcal{P}_2(X, \hat{X}) = & \frac{1}{2}([- (AX + XA)_{<0} - (\hat{A}\hat{X} + \hat{X}\hat{A})_{<0}, A] \\ & + A([X, A]_{\leq 0} + [\hat{X}, \hat{A}]_{\leq 0}) + ([X, A]_{\leq 0} + [\hat{X}, \hat{A}]_{\leq 0})A, \\ & [(AX + XA)_{\geq 0} + (\hat{A}\hat{X} + \hat{X}\hat{A})_{\geq 0}, \hat{A}] \\ & - \hat{A}([X, A]_{>0} + [\hat{X}, \hat{A}]_{>0}) - ([X, A]_{>0} + [\hat{X}, \hat{A}]_{>0})\hat{A}). \end{aligned} \quad (5.14)$$

$$\begin{aligned} \mathcal{P}_3(X, \hat{X}) = & ([- (AXA + \hat{A}\hat{X}\hat{A})_{<0}, A] + A([X, A]_{\leq 0} + [\hat{X}, \hat{A}]_{\leq 0})A, \\ & [(AXA + \hat{A}\hat{X}\hat{A})_{\geq 0}, \hat{A}] - \hat{A}([X, A]_{>0} + [\hat{X}, \hat{A}]_{>0})\hat{A}). \end{aligned} \quad (5.15)$$

As before these Poisson structures need to be reduced to appropriate subsets of \mathfrak{E} . Given two arbitrary positive integers N and M , with L and \hat{L} in (5.5) we let

$$\mathbf{A} = (A, \hat{A}) = (L^N, \hat{L}^M). \quad (5.16)$$

All such operators form a coset $(\Lambda^N, 0) + \mathcal{U}_{N,M}$ of \mathfrak{E} , where

$$\mathcal{U}_{N,M} = (\mathcal{E}^-)_{<N} \times (\mathcal{E}^+)_{\geq -M}.$$

On this coset, the tangent bundle $T\mathcal{U}_{N,M}$ and the cogtangent bundle $T^*\mathcal{U}_{N,M}$ have their fibers respectively

$$T_{\mathbf{A}}\mathcal{U}_{N,M} = \mathcal{U}_{N,M}, \quad T_{\mathbf{A}}^*\mathcal{U}_{N,M} = \mathcal{U}_{N,M}^* = (\mathcal{E}^-)_{>-N} \times (\mathcal{E}^+)_{\leq M}.$$

As in Lemma 4.3, we perform a Dirac reduction for Poisson structures \mathcal{P}_ν in Lemma 5.1 from \mathfrak{E} to the coset $(\Lambda^N, 0) + \mathcal{U}_{N,M}$. The procedure is similar with that in [2], so we only sketch the main steps. First, we have the decompositions of subspaces

$$\mathcal{E} = \mathcal{U}_{N,M} \oplus \mathcal{V}_{N,M} = \mathcal{U}_{N,M}^* \oplus \mathcal{V}_{N,M}^*.$$

Where $\mathcal{V}_{N,M} = (\mathcal{E}^-)_{\geq N} \times (\mathcal{E}^+)_{<-M}$ and $\mathcal{V}_{N,M}^* = (\mathcal{E}^-)_{\leq -N} \times (\mathcal{E}^+)_{>M}$. Then, the Poisson tensors

$$\mathcal{P}_\nu = \begin{pmatrix} \mathcal{P}_\nu^{\mathcal{U}\mathcal{U}} & \mathcal{P}_\nu^{\mathcal{U}\mathcal{V}} \\ \mathcal{P}_\nu^{\mathcal{V}\mathcal{U}} & \mathcal{P}_\nu^{\mathcal{V}\mathcal{V}} \end{pmatrix} : \mathcal{U}_{N,M}^* \oplus \mathcal{V}_{N,M}^* \rightarrow \mathcal{U}_{N,M} \oplus \mathcal{V}_{N,M}$$

are reduced onto the coset $(\Lambda^N, 0) + \mathcal{U}_{N,M}$ as

$$\mathcal{P}_\nu^{\text{red}} = \mathcal{P}_\nu^{\mathcal{U}\mathcal{U}} - \mathcal{P}_\nu^{\mathcal{U}\mathcal{V}} \circ (\mathcal{P}_\nu^{\mathcal{V}\mathcal{V}})^{-1} \circ \mathcal{P}_\nu^{\mathcal{V}\mathcal{U}}.$$

After a long but straightforward calculation, we conclude that, the first tensor \mathcal{P}_1 can be restricted to the coset directly, for \mathcal{P}_2 one needs a correction term, but the reduction of \mathcal{P}_3 is not clear except for $N = M = 1$.

Lemma 5.2 *On the coset $(\Lambda^N, 0) + \mathcal{U}_{N,M}$ there are two compatible Poisson structures*

$$\mathcal{P}_\nu^{\text{red}} : T^*\mathcal{U}_{N,M} \rightarrow T\mathcal{U}_{N,M}, \quad \nu = 1, 2 \quad (5.17)$$

defined as

$$\mathcal{P}_1^{\text{red}}(X, \hat{X}) = \mathcal{P}_1(X, \hat{X}), \quad (5.18)$$

$$\mathcal{P}_2^{\text{red}}(X, \hat{X}) = \mathcal{P}_2(X, \hat{X}) - ([f, A], [f, \hat{A}]). \quad (5.19)$$

where $(X, \hat{X}) \in T_{\mathbf{A}}^* \mathcal{U}_{N,M}$ and

$$f = \frac{1}{2}(1 + \Lambda^N)(1 - \Lambda^N)^{-1}(\text{Res}([X, A] + [\hat{X}, \hat{A}]))$$

with $(1 - \Lambda^N)^{-1} = 1 + \Lambda^N + \Lambda^{2N} + \dots$.

Let $\{\cdot, \cdot\}_{\nu}^{N,M}$ be the Poisson brackets on the coset $(\Lambda^N, 0) + \mathcal{U}_{N,M}$ given by the tensors $\mathcal{P}_{\nu}^{\text{red}}$ in the above lemma. The following result can be verified directly.

Theorem 5.3 *For any positive integers N and M , the Toda lattice hierarchy (5.6), (5.7) has the following bi-Hamiltonian representation:*

$$\frac{\partial F}{\partial t_k} = \{F, H_{k+N}\}_1^{N,M} = \{F, H_k\}_2^{N,M}, \quad \frac{\partial F}{\partial \hat{t}_k} = \{F, \hat{H}_{k+M}\}_1^{N,M} = \{F, \hat{H}_k\}_2^{N,M} \quad (5.20)$$

with $k > 0$ and arbitrary functional F and

$$H_k = \frac{N}{k} \langle L^k \rangle, \quad \hat{H}_k = \frac{M}{k} \langle \hat{L}^k \rangle. \quad (5.21)$$

In the particular case $N = M = 1$, on the coset $(\Lambda, 0) + \mathcal{U}_{1,1}$ there exists another Poisson structure $\mathcal{P}_3^{\text{red}}$ that is compatible with $\mathcal{P}_1^{\text{red}}$ and $\mathcal{P}_2^{\text{red}}$. More precisely,

$$\mathcal{P}_3^{\text{red}}(X, \hat{X}) = \mathcal{P}_3(X, \hat{X}) - ([Z, A], [Z, \hat{A}]), \quad (X, \hat{X}) \in T_{\mathbf{A}}^* \mathcal{U}_{N,M} \quad (5.22)$$

where $Z = (A(g \Lambda^{-1} + h \Lambda^{-2})A)_{\geq 0}$ with functions g and h determined by

$$\begin{aligned} (1 - \Lambda)(g) &= \text{Res}([X, A] + [\hat{X}, \hat{A}]), \\ (1 - \Lambda)(h) - g(1 - \Lambda^{-1})(\text{Res } A) &= \text{Res}([X, A]\Lambda + [\hat{X}, \hat{A}]\Lambda). \end{aligned}$$

In this case, the derivatives $\partial/\partial t_k$ and $\partial/\partial \hat{t}_k$ with $k \geq 2$ of the Toda lattice hierarchy can also be represented into Hamiltonian flows of $\mathcal{P}_3^{\text{red}}$, cf. [2].

5.3 Reduction to Hamiltonian structures of the extended bigraded Toda hierarchies

We constrain the Toda lattice hierarchy as follows:

$$L^N = \hat{L}^M = \mathcal{L}, \quad (5.23)$$

where \mathcal{L} has the form

$$\mathcal{L} = \Lambda^N + v_{N-1} \Lambda^{N-1} + v_{N-2} \Lambda^{N-2} + \dots + v_{-M} \Lambda^{-M}. \quad (5.24)$$

With the same method as in §4.3, under the constraint (5.23) the Poisson brackets in Theorem 5.3 are reduced to:

$$\{F_X(\mathcal{L}), F_Y(\mathcal{L})\}_1 = \langle X, [Y_{\geq 0}, \mathcal{L}]_{\leq 0} - [Y_{< 0}, \mathcal{L}]_{> 0} \rangle, \quad (5.25)$$

$$\begin{aligned} \{F_X(\mathcal{L}), F_Y(\mathcal{L})\}_2 &= \left\langle X, [-(\mathcal{L}Y + Y\mathcal{L})_{<0}, \mathcal{L}] + \frac{1}{2}\mathcal{L}[Y, \mathcal{L}]_{\leq 0} + \frac{1}{2}[Y, \mathcal{L}]_{\leq 0}\mathcal{L} \right\rangle \\ &\quad - \left\langle X, \frac{1}{2}[(1 + \Lambda^N)(1 - \Lambda^N)^{-1}(\text{Res}[Y, \mathcal{L}])] \right\rangle. \end{aligned} \quad (5.26)$$

These formulae has the same expression with the bi-Hamiltonian structure of the extended bigraded Toda hierarchy [5, 3].

We recall briefly the construction of the extended bigraded Toda hierarchy. First, certain continuation procedure needs to be performed. That is, replace \mathcal{A} by the algebra of analytic functions, such that $f(n)$ is replaced by $f(n\epsilon)$ with some small constant ϵ , and the shift operator Λ becomes $e^{\epsilon D}$ with $D = d/dx$. Second, to obtain a complete integrable hierarchy from \mathcal{L} in (5.24), one needs not only the factorizations $\mathcal{L} = L^N = \hat{L}^M$ with L and \hat{L} as in (5.5), but also a logarithm operator $\text{Log } \mathcal{L}$ defined in an appropriate way, see [5, 3] for details. Thus up to a scalar transformation of the time variables, the extended bigraded Toda hierarchy is composed of the Hamiltonian flows given by the Poisson brackets (5.25), (5.26) together with Hamiltonians (5.21) and

$$H_k^L = \frac{2}{(k-1)!} \left\langle \mathcal{L}^{k-1} \left(\text{Log } \mathcal{L} - \frac{1}{2} \left(\frac{1}{M} + \frac{1}{N} \right) c_{k-1} \right) \right\rangle, \quad k \geq 1, \quad (5.27)$$

where $c_0 = 0$ and $c_k = 1 + \dots + \frac{1}{k}$.

One can also consider Hamiltonian structures of the dispersionless Toda hierarchy. However, in contrast to the case of the two-component BKP hierarchy, nothing is obtained beyond the dispersionless limit of the above result. Mention that, the dispersionless limit of $\mathcal{P}_2^{\text{red}}$ for $M = N = 1$ was written down in Proposition 3.3 of [4], for the purpose of constructing an infinite dimensional Frobenius manifold there.

6 Summary and outlook

On the Lie algebra \mathfrak{g} in (3.2), we write down all R -matrices of the form (3.3) that satisfy the modified Yang-Baxter equation (2.2). These R -matrices are connected by the action of the group G of intertwining involutions generated by σ_1 and σ_2 in (3.12). Among the R -matrices we have found, the one (3.13) is particularly efficient in deriving Hamiltonian structures for Lax equations defined on \mathfrak{g} . Two typical examples, the two-component BKP hierarchy and the Toda lattice hierarchy, are considered. It is interesting to apply the R -matrix (3.13) to other hierarchies, such as multicomponent Toda lattice hierarchies [25] and their generalizations. We leave it to a subsequent publication.

For each of the two-component BKP hierarchy and the Toda lattice hierarchy, we obtain infinitely many bi-Hamiltonian structures, which correspond to Poisson brackets on different submanifolds of Lie algebras of the form (3.2). Moreover, they can be reduced to bi-Hamiltonian structures (4.30), (4.31) for Drinfeld-Sokolov hierarchies of type D and (5.25), (5.26) for the extended bigraded Toda hierarchies respectively. This is an advantage of R in (3.13) comparing with the other R -matrices given in Proposition 3.1. For instance, a nonidentity intertwining involutions σ may not preserve the constraints (4.29) or (5.23), hence Hamiltonian structures derived from the R -matrix $R \circ \sigma$ would not admit such constraints.

As suggested by [4], we hope that there also exist infinite-dimensional Frobenius manifolds underlying the bi-Hamiltonian structures (4.48) and (5.20) of the two-component BKP hierarchy and the Toda lattice hierarchy. Furthermore, since the bi-Hamiltonian structures (4.30), (4.31) and (5.25), (5.26) reduced from them are associated to finite-dimensional Frobenius manifolds defined on the orbit space of corresponding (extended) affine Weyl groups [3, 11, 14, 12], the reduction property of the bi-Hamiltonian structures (4.48) and (5.20) probably gives insights into the connection between Frobenius manifolds of finite and infinite dimensions. We will consider it in subsequent publications.

Acknowledgment. The author is grateful to Boris Dubrovin, Si-Qi Liu and Youjin Zhang for helpful discussions and comments.

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