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Toda Molecule and Tomimatsu-Sato Solution

$\sim$ Toward the complete proof of Nakamura's conjecture$\sim$

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Abstract

We discuss the Nakamura's conjecture stating that the Tomimatsu-Sato black hole solution with deformation parameter n is composed of the special solutions of the Toda molecule equation at the n-th lattice site. From the previous work, in which the conjecture was partly analytically proved, we goes further toward final full proof by rearranging the rotation parameter. The proof is explicitly performed for the highest and lowest orders. Though the proof for the full orders is still remained unsolved, the prospect to the full proof becomes transparent and workable by our method.

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1 Introduction

There are close and unexpected relations between a variety of integrable systems. The interplay between black hole and the Toda molecule solutions is one of such examples. A. Nakamura found an important relation [1] between the Tomimatsu-Sato (hereafter we refer it as TS) solutions [2] and the Toda molecule solutions [3]. That is, he asserts that a series of TS solutions with deformation parameter n and rotation parameters p, q ($p^2+q^2=1$) are obtained from the special solutions of the Toda molecule equation at the n-th lattice site (Nakamura's conjecture). The Ernst equation is summarized as two sets of equations, {Eqs. (13) and (14)} and {Eqs. (15) and (16)}. Nakamura checked that this conjecture is numerically satisfied for small n. One of the present authors (T.F.) tried to prove analytically this conjecture for general n [4].

In [4], we proved for general n case that the special solutions of Toda molecule satisfy the first set of equations without using the explicit form of the solution. However, the second set was proved to satisfy for the restricted case of $q=0$ by using the explicit form of the solution. The $q=0$ black hole is corresponding to the (extended) Weyl solutions. ($q=0,\ n=1$ correspond to the Schwarzschild solution.) Thus the full proof was remained unsolved for the generic case, $q \neq 0$. For general n, the two solutions of Toda molecule equation, g_n and f_n (see Eq. (4)), are given by the polynomials of p and q of homogeneous degree of n and $n-1$, respectively, that is, $p^i q^{n-i},\ (i=0,\cdots n)$ and $p^i q^{n-i-1},\ (i=0,\cdots n-1)$. If we change these parameters p, q (independent parameter is one) to t defined by (37), the special solutions of Toda molecule equation g_n and f_n are described by the Laurent polynomials whose highest (lowest) degrees are n ($-n$) and $n-1$ ($-n+1$), respectively. We give the explicit form of the Nakamura's conjecture order by order on t. In this paper we prove the second set of equations at the highest and lowest orders of t. This may be the step toward the complete proof. This paper is organized as follows. In the next section we briefly review our previous work [4]. Sec. 3 is the central part of this paper, where the Toda molecule equations and the Nakamura's conjecture are expanded by the Laurent polynomials and its conjecture is proved at the highest and lowest orders of t. Some detailed calculations are developed in the Appendix. In Sec. 4, we discuss on the results and their implications of the present paper.

2 Nakamura's conjecture and our previous results

The two dimensional Toda molecule equation is described by

$$\frac{\partial^2}{\partial S \partial T} log V_n \equiv (log V_n)_{ST} = V_{n-1} - 2V_n + V_{n+1}. \tag{1}$$

Here S and T are light cone coordinates related with the Cartesian coordinates, X and Y, by

$$S = \frac{1}{2}(X+Y), \quad T = \frac{1}{2}(X-Y) \tag{2}$$

and n indicates the n-th lattice site. If we introduce τ function defined by

$$V_n = (log\tau_n)_{ST}, \tag{3}$$

the Toda molecule equation is expressed in terms of the Hirota's bilinear forms [5] as

$$D_S D_T \tau_n \cdot \tau_n - 2\tau_{n+1}\tau_{n-1} = 0, \tag{4}$$

where the Hirota derivative is defined by $D(f \cdot g) = (\partial f)g - f(\partial g)$. The bilinear form is very useful for the integrable system, which is also the case in the present problem. n is positive integer and the boundary condition is chosen as $\tau_0 = 1$ corresponding to the finite and semi-infinite lattices. The general solution of Eq. (4) is expressed in a form of the two-directional Wronskian [7]

$$\tau_n = \det \begin{pmatrix} \psi & L_-\psi & \dots & L_-^{n-1}\psi \\ L_+\psi & L_+L_-\psi & \dots & L_+L_-^{n-1}\psi \\ \vdots & \vdots & \vdots & \vdots \\ L_+^{n-1}\psi & L_+^{n-1}L_-\psi & \dots & L_+^{n-1}L_-^{n-1}\psi \end{pmatrix}. \tag{5}$$

with the boundary condition $\tau_0 = 1$ and initial condition $\tau_1 = \psi$. Here $L_+(L_-) \equiv \partial_S(\partial_T) = \frac{\partial}{\partial X} + \frac{\partial}{\partial Y}(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y})$. For the finite lattice, the initial condition ψ takes a form

$$\psi = \sum_{k=1}^{N+1} H_k(S)G_k(T). \tag{6}$$

So that V_n satisfies the boundary condition

$$V_0 = V_{N+1} = 0, \tag{7}$$

where N stands for the total number of lattice sites. In the present paper, the semi-infinite lattice corresponding to the $N \to \infty$ case is treated. We first show how it appears as a result of a Pfaffian identity for the help of later discussions, though it is a known fact.

We introduce $\mathcal{D}$ as a determinant of $(n+1) \times (n+1)$ matrix τ_{n+1};

$$\mathcal{D} \;\equiv\; \tau_{n+1}. \tag{8}$$

The minor $\mathcal{D}\begin{bmatrix} i \\ j \end{bmatrix}$ is defined by deleting the i-th row and the j-th column from $\mathcal{D}$. Similarly $\mathcal{D}\begin{bmatrix} i,k \\ j,l \end{bmatrix}$ is defined by deleting the i and k-th rows and the j and l-th columns from $\mathcal{D}$ and so on. The Toda molecule equation (4) is now expressed as

$$\mathcal{D}\begin{bmatrix} n \\ n \end{bmatrix}\mathcal{D}\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} - \mathcal{D}\begin{bmatrix} n+1 \\ n \end{bmatrix}\mathcal{D}\begin{bmatrix} n \\ n+1 \end{bmatrix} - \mathcal{D}\mathcal{D}\begin{bmatrix} n,n+1 \\ n,n+1 \end{bmatrix} = 0. \tag{9}$$

It holds since it is nothing but the Jacobi's (Sylvester's) formula for matrix minors, which is one of Pfaffian identities. Thus the Toda molecule equation has been reduced to Pfaffian identity in direct method. It is also the case in the Einstein equation as will be shown in the following.

The Ernst equation for axially symmetric metric of Einstein equation is

$$(\xi\xi^* - 1)\nabla^2\xi - 2\xi^*\nabla\xi\cdot\nabla\xi = 0. \tag{10}$$

It is shown that Eq. (10) is invariant under the global SU(1,1) transformation

$$\xi' = \frac{\alpha\xi + \beta^*}{\beta\xi + \alpha^*}, \quad (|\alpha|^2 - |\beta|^2 \neq 0). \tag{11}$$

Setting

$$\xi_n = \frac{g_n}{f_n}, \tag{12}$$

the Nakamura's conjecture on TS solutions consists of two ingredients [1];

(i) Eq. (10) has a decomposition into two sets [1]

$$D_x(g_n\cdot f_n - g_n^*\cdot f_n^*) = 0, \tag{13}$$
$$D_y(g_n\cdot f_n + g_n^*\cdot f_n^*) = 0, \tag{14}$$

and

$$F(g_n^*\cdot f_n) = 0, \tag{15}$$
$$F(g_n^*\cdot g_n + f_n^*\cdot f_n) = 0. \tag{16}$$

Here the bi-linear operator F is

$$F = (x^2-1)D_x^2 + 2x\partial_x + (y^2-1)D_y^2 + 2y\partial_y + c_n \tag{17}$$

and x and y are usual prolate spheroidal coordinates defined by

$$\partial_X = (x^2-1)\partial_x \text{ and } \partial_Y = (y^2-1)\partial_y. \tag{18}$$

(ii) From the solutions of the Toda molecule equation τ_n, a set of TS solutions with deformation parameter n, $\xi_n = \frac{g_n}{f_n}$, are obtained as

$$g_n \;=\; \tau_n \;=\; \mathcal{D}\begin{bmatrix} n+1 \\ n+1 \end{bmatrix}, \qquad f_n \;=\; \tau_{n-1}\,|_{\psi\to L_+L_-\psi} \;=\; \mathcal{D}\begin{bmatrix} 1, n+1 \\ 1, n+1 \end{bmatrix} \tag{19}$$

by choosing the arbitrary function ψ and the constant c_n in (6) as

$$\psi = px - iqy, \qquad p^2 \;+\; q^2 \;=\; 1, \qquad c_n \;=\; -\;2\;n^2. \tag{20}$$

The first set of Eqs. (13) and (14) is proved in the general case whose detailed proof should be referred to [4]. While, the second set of Eqs. (15) and (16) is shown for a restricted case of $q = 0$ in [4]. The present paper is its extension to generic case $q \neq 0$ and we review for $q = 0$ first.

To prove the second set of decomposition equations Eqs. (15) and (16), the explicit form of ψ in Eq. (20) is required in contrast to the case of the first set. $g_n(f_n)$ is the determinant of matrix whose (i,j) $((i-1, j-1))$ element is

$$\begin{aligned} L_+^{i-1} L_-^{j-1} \psi &= L_+^{i-1} L_-^{j-1} (px - iqy) \\ &= pW_{i+j-1}(x) + (-1)^j iqW_{i+j-1}(y), \end{aligned} \tag{21}$$

where

$$W_{n+1}(z) = (z^2 - 1)\frac{d}{dz} W_n(z) \qquad \text{with} \qquad W_1(z) = z. \tag{22}$$

The polynomial expression for $W_n(z)$ is given by

$$W_n(z) = \sum_{m=0}^{n-2} \left\{ \sum_{l=0}^{m} (-1)^l (m-l+1)^{n-1} \begin{pmatrix} n \\ l \end{pmatrix} \right\} (z+1)^{m+1} (z-1)^{n-m-1}, \quad (n \geq 2), \tag{23}$$

where $\binom{n}{l}$ is the binomial coefficient.

In case of $p = 1$ and $q = 0$, $\psi = x$ and g_n and f_n are real functions depending only on x. Explicit forms of g_n and f_n are

$$g_n = \det \begin{pmatrix} W_1(x) & W_2(x) & \dots & W_n(x) \\ W_2(x) & W_3(x) & \dots & W_{n+1}(x) \\ \vdots & \vdots & \vdots & \vdots \\ W_n(x) & W_{n+1}(x) & \dots & W_{2n-1}(x) \end{pmatrix}, \quad f_n = \det \begin{pmatrix} W_3(x) & \dots & W_{n+1}(x) \\ \vdots & \vdots & \vdots \\ W_{n+1}(x) & \dots & W_{2n-1}(x) \end{pmatrix}. \tag{24}$$

Using Eq. (22) we can evaluate the determinant,

$$\begin{aligned} g_n &= (x^2-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} x & 1 & 0 & 0 & \dots & 0 \\ x^2 - 1 & 2x & 2 & 0 & \dots & 0 \\ 2x(x^2-1) & 6x^2 - 2 & 12x & 12 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \end{pmatrix} \\ &= \frac{A_n}{2} (x^2-1)^{\frac{n(n-1)}{2}} \Big((x+1)^n + (x-1)^n \Big), \end{aligned} \tag{25}$$

where the coefficient A_n is

$$A_n = \left(\prod_{k=0}^{n-1} \Gamma(n-k) \right)^2 = (n-1)^2 (n-2)^4 \cdots 2^{2(n-1)}. \tag{26}$$

Eq. (25) is proved by induction. For $n = 1$ and $n = 2$ cases Eq. (25) holds. Assuming it for $n = l - 1$ and $n = l$ cases we prove it for the case of $n = l + 1$. Applying Jacobi's formula (9) to g_n we obtain, corresponding to the Toda molecule equation in Eq. (5),

$$g_{l-1}\ g_{l+1} = (L_X^2\ g_l)\ g_l - (L_X\ g_l)^2, \tag{27}$$

where $L_X \equiv (x^2-1)\partial_x$. Using the assumed forms for g_l and g_{l-1} we find the expected form for g_{l+1};

$$g_{l+1} = \frac{A_{l+1}}{2}(x^2-1)^{\frac{(l+1)l}{2}}\Big((x+1)^{l+1} + (x-1)^{l+1}\Big). \tag{28}$$

Here we have used an equality obtained from the definition of A_n in Eq. (26);

$$A_{n-1}A_{n+1} = n^2 A_n. \tag{29}$$

Thus Eq. (25) is proved for $n = l+1$. Quite analogously f_n is shown to be

$$f_n = \frac{A_n}{2}(x^2-1)^{\frac{n(n-1)}{2}}\Big((x+1)^n - (x-1)^n\Big). \tag{30}$$

The second set of Eqs. (15) and (16) is

$$\begin{aligned} &F(g_n \cdot f_n) = 0, \\ &F(g_n \cdot g_n + f_n \cdot f_n) = 0, \end{aligned} \tag{31}$$

and F becomes in $q = 0$ and $c_n = -2n^2$ case

$$F a \cdot b = \frac{1}{x^2-1}\Big((L_X^2 a)b + a(L_X^2 b) - 2(L_X a)(L_X b)\Big) - 2n^2 ab. \tag{32}$$

It is straight forward to show g_n and f_n in Eqs. (25) and (30) satisfy Eq. (31).

3 Proof of Nakamura's Conjecture at the highest and lowest orders

If the Nakamura's conjecture is true, the global symmetry (11) of the Ernst's equation must be reflected in the Toda molecule equation. The symmetry (11) with (12) is rewritten as

$$\frac{g'_n}{f'_n} = \frac{\alpha g_n + \beta^* f_n}{\beta g_n + \alpha^* f_n}. \tag{33}$$

This may be divided into the following two transformations:

$$g'_n = \alpha g_n + \beta^* f_n, \tag{34}$$
$$f'_n = \beta g_n + \alpha^* f_n. \tag{35}$$

Indeed it is easily shown that two sets (13)-(16) in the Nakamura's conjecture remain invariant under the above transformations (34) and (35). Furthermore, it turns out that the transformed functions g'_n and f'_n are also solutions of the Toda molecule equations, *i.e.* any linear superposition of the solutions g_n and f_n given by Eq. (19) satisfies the Toda molecule equation. The proof for the linear property is confirmed by several ways. As the first proof, we consider that the transformation $\psi \to \psi + \frac{b}{a}$ in Eq. (5) generates different solution of Toda molecule equation. Under the transformation, the function τ_n becomes $\tau_n + \frac{b}{a}\tau_{n-1}|_{\psi\to L_+L_-\psi}$. Taking the transformation and the rescaling of the tau function into account, one can show that the function $a\tau_n + b\tau_{n-1}|_{\psi\to L_+L_-\psi}$ satisfies the Toda molecule equation, where a and b are any complex constants. Because $\tau_n = g_n$ and $\tau_{n-1}|_{\psi\to L_+L_-\psi} = f_n$ under the choices (20), it is shown that $ag_n + bf_n$ is also a solution of Toda molecule equation. As the second proof, it follows from this linear combination (34) ((35)) that

$$D_S D_T f_n \cdot g_n = f_{n+1}g_{n-1} + f_{n-1}g_{n+1}, \tag{36}$$

which can be reduced to a Pfaffian identity.

Here let us consider expansions of the functions g_n and f_n by the parameters p and q. For general n, g_n and f_n are given by polynomials of p and q of homoheneous degree of n and $n-1$, respectively. As we mentioned, we must prove order by order. However, in terms of p, q, it is rather complicated. So let us introduce one parameter t defined by

$$p = \frac{t + t^{-1}}{2}, \quad q = \frac{t - t^{-1}}{2i}. \tag{37}$$

Then ψ is rewritten as

$$\psi = t\left(\frac{x-y}{2}\right) + t^{-1}\left(\frac{x-y}{2}\right) = tv + t^{-1}u, \quad u = \frac{x+y}{2}, \; v = \frac{x-y}{2}. \tag{38}$$

To make the t dependence of g_n, f_n clear, we write them as

$$g_n = g_n(x,y;t), \quad f_n = f_n(x,y;t). \tag{39}$$

Then, $g_n(x,y;t)$ and $f_n(x,y;t)$ are expressed as the Laurent polynomials of n and $n-1$ orders of t, respectively:

$$g_n(x,y;t) = \sum_{m=-n}^{n} \tilde{g}_n^{(m)}(x,y)t^m, \quad f_n(x,y;t) = \sum_{m=-n+1}^{n-1} \tilde{f}_n^{(m)}(x,y)t^m. \tag{40}$$

They have the following properties

$$g_n^* = g_n(x,y;t^{-1}), \quad f_n^* = f_n(x,y;t^{-1}) \tag{41}$$

$$g_n^* = g_n(x,-y;t), \quad f_n^* = f_n(x,-y;t) \tag{42}$$

$$g_n(x,y;-t) = (-1)^n g_n(x,y;t), \quad f_n(x,y;-t) = (-1)^{n-1} f_n(x,y;t) \tag{43}$$

$$g_n(x,y;it) = (-i)^{n^2} g_n(y,x;t), \quad f_n(y,x;it) = (-i)^{n^2-1} f_n(y,x;t) \tag{44}$$

as are shown from their definitions. It follows from these identities that

$$\tilde{g}_n^{(n-2m-1)}(x,y) = 0, \quad \tilde{f}_n^{(-n+2m+1)}(x,y) = \tilde{f}_n^{(n-2m-1)}(x,-y), \quad (m = 0, \cdots, n-1) \tag{45}$$

$$\tilde{f}_n^{(n-2m)}(x,y) = 0, \quad \tilde{g}_n^{(-n+2m)}(x,y) = \tilde{g}_n^{(n-2m)}(x,-y), \quad (m = 0, \cdots, n) \tag{46}$$

and we obtain the expansions,

$$g_n(x,y;t)=\sum_{m=0}^{n}\tilde{g}_n^{(n-2m)}(x,y)t^{n-2m},\quad f_n(x,y;t)=\sum_{m=0}^{n-1}\tilde{f}_n^{(n-2m-1)}(x,y)t^{n-2m-1} \tag{47}$$

and

$$g_n^*(x,y;t)=\sum_{m=0}^{n}\tilde{g}_n^{(-n+2m)}(x,y)t^{n-2m},\quad f_n^*(x,y;t)=\sum_{m=0}^{n-1}\tilde{f}_n^{(-n+2m+1)}(x,y)t^{n-2m-1} \tag{48}$$

with

$$\tilde{g}_n^{(-m)}(x,y)=\tilde{g}_n^{(m)}(x,-y),\quad \tilde{f}_n^{(-m)}(x,y)=\tilde{f}_n^{(m)}(x,-y). \tag{49}$$

Because the Toda molecule equations for f_n and g_n and Eq. (36) hold for an arbitrary value of the parameter t, we obtain order by order equations of the Laurent expansions of f_n and g_n (see the Appendix). The highest order equations of them, which correspond to the $I=0$ case in the Appendix, are given by

$$D_SD_T\tilde{g}_n^{(n)}(x,y)\cdot\tilde{g}_n^{(n)}(x,y)=2\tilde{g}_{n+1}^{(n+1)}(x,y)\tilde{g}_{n-1}^{(n-1)}(x,y), \tag{50}$$

$$D_SD_T\tilde{f}_n^{(n-1)}(x,y)\cdot\tilde{f}_n^{(n-1)}(x,y)=2\tilde{f}_{n+1}^{(n)}(x,y)\tilde{f}_{n-1}^{(n-2)}(x,y), \tag{51}$$

and

$$D_SD_T\tilde{f}_n^{(n-1)}(x,y)\cdot\tilde{g}_n^{(n)}(x,y)=\tilde{f}_{n+1}^{(n)}(x,y)\tilde{g}_{n-1}^{(n-1)}(x,y)+\tilde{f}_{n-2}^{(n-1)}(x,y)\tilde{g}_{n+1}^{(n+1)}(x,y). \tag{52}$$

In fact, because by definitions the highest order terms $\tilde{g}_n^{(n)}(x,y)$ and $\tilde{f}_n^{(n-1)}(x,y)$ are written by the following determinants

$$\tilde{g}_n^{(n)}(x,y)=\begin{vmatrix} v & L_-v & \cdots & L_-^{n-1}v \\ L_+v & L_+L_-v & \cdots & L_+L_-^{n-1}v \\ \vdots & \vdots & \ddots & \vdots \\ L_+^{n-1}v & L_+^{n-1}L_-v & \cdots & L_+^{n-1}L_-^{n-1}v \end{vmatrix}, \tag{53}$$

$$\tilde{f}_n^{(n-1)}(x,y)=\begin{vmatrix} L_+L_-v & \cdots & L_+L_-^{n-1}v \\ \vdots & \ddots & \vdots \\ L_+^{n-1}L_-v & \cdots & L_+^{n-1}L_-^{n-1}v \end{vmatrix} \tag{54}$$

with $u=(x+y)/2$ and $v=(x-y)/2$, it is shown that all of these equations (50)-(52) reduce to Pfaffian identities.

In general, these functions g_n and f_n should be expressed as finite polynomials of the prolate spheroidal coordinates x,y. To determine the explicit forms for all orders is seemingly a tedious work. However, after some numerical caluculations, we know that the highest term $\tilde{g}_n^{(n)}(x,y)$ and the lowest term $\tilde{g}_n^{(-n)}(x,y)$ have the forms,

$$\tilde{g}_n^{(n)}(x,y)=\begin{vmatrix} v & L_-v & \cdots & L_-^{n-1}v \\ L_+v & L_+L_-v & \cdots & L_+L_-^{n-1}v \\ \vdots & \vdots & \ddots & \vdots \\ L_+^{n-1}v & L_+^{n-1}L_-v & \cdots & L_+^{n-1}L_-^{n-1}v \end{vmatrix}=2^{n(n-1)}A_nu^{\frac{n(n-1)}{2}}v^{\frac{n(n+1)}{2}} \tag{55}$$

and

$$\tilde{g}_n^{(-n)}(x,y) = \begin{vmatrix} u & L_-u & \cdots & L_-^{n-1}u \\ L_+u & L_+L_-u & \cdots & L_+L_-^{n-1}u \\ \vdots & \vdots & \ddots & \vdots \\ L_+^{n-1}u & L_+^{n-1}L_-u & \cdots & L_+^{n-1}L_-^{n-1}u \end{vmatrix} = 2^{n(n-1)}A_n u^{\frac{n(n+1)}{2}} v^{\frac{n(n-1)}{2}}, \tag{56}$$

where A_n is given by Eq. (26). Also it turns out that the highest term $\tilde{f}_n^{(n-1)}(x,y)$ and the lowest term $\tilde{f}_n^{(-n+1)}(x,y)$ are

$$\begin{aligned} \tilde{f}_n^{(n-1)}(x,y) &= \begin{vmatrix} L_+L_-v & \cdots & L_+L_-^{n-1}v \\ \vdots & \ddots & \vdots \\ L_+^{n-1}L_-v & \cdots & L_+^{n-1}L_-^{n-1}v \end{vmatrix} \\ &= \frac{\tilde{g}_n^{(n)}(x,y)}{\sqrt{\pi}} \times \sum_{m=0}^{n-1}\sum_{l=0}^{m}(-1)^{l-m}\frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{2(n-l)+1}{2})}{\Gamma(\frac{2(m-l)+3}{2})\Gamma(l+1)\Gamma(m-l+1)\Gamma(n-m)}u^{2l}v^{-2m-1} \end{aligned} \tag{57}$$

and

$$\begin{aligned} \tilde{f}_n^{(-n+1)}(x,y) &= \begin{vmatrix} L_+L_-u & \cdots & L_+L_-^{n-1}u \\ \vdots & \ddots & \vdots \\ L_+^{n-1}L_-u & \cdots & L_+^{n-1}L_-^{n-1}u \end{vmatrix} \\ &= \frac{\tilde{g}_n^{(-n)}(x,y)}{\sqrt{\pi}} \times \sum_{m=0}^{n-1}\sum_{l=0}^{m}(-1)^{l-m}\frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{2(n-l)+1}{2})}{\Gamma(\frac{2(m-l)+3}{2})\Gamma(l+1)\Gamma(m-l+1)\Gamma(n-m)}u^{-2m-1}v^{2l}. \end{aligned} \tag{58}$$

These polynomial expressions (55)-(58) are proved by applying the mathematical induction and using the identities (49). In the $n=1$ and $n=2$ cases these expressions (55)-(58) hold. Assuming it for $n=l-1$ and $n=l$ cases, we prove them for the case of $n=l+1$. Substituting the assumed polynomial forms $\tilde{g}_{l-1}^{(l-1)}$ and $\tilde{g}_l^{(l)}$ into the Toda molecule equation at the highest order,

$$\tilde{g}_{l+1}^{(l+1)}(x,y) = \frac{1}{2\tilde{g}_{l-1}^{(l-1)}(x,y)}(D_S D_T \tilde{g}_l^{(l)}(x,y)\cdot\tilde{g}_l^{(l)}(x,y)), \tag{59}$$

we obtain the expected polynomial form for $\tilde{g}_{l+1}^{(l+1)}$. By using the identities (49), it is easily shown Eq. (56) is correct. Eq. (52) is used instead of Eq. (51) to prove the parts $\tilde{f}_n^{(n-1)}(x,y)$ and $\tilde{f}_n^{(-n+1)}(x,y)$, with the expressions (55) and (56) shown in the above. Likewise, it is shown the expressions for $\tilde{f}_n^{(n-1)}(x,y)$ and $\tilde{f}_n^{(-n+1)}(x,y)$ are correct by substituting them into Eq. (52) and using the identities (49). Thus these polynomial expressions are correct. The linearity property for g_n and f_n ensures that the expression for $\tilde{f}_n^{(n-1)}(x,y)$ satisfies Eq. (51).

Let us prove the Nakamura's conjecture at the highest order of t. Substituting the Laurent expansions (47) and (48) into it's conjecture (13)-(16), the equations in each order of t corresponding to its conjecture are derived (see the Appendix). The highest orders of t for Eqs. (13)-(15) and Eq. (16) are $2n-1$ and $2n$, respectively. They are described as the following equations at the highest order:

$$D_x(\tilde{g}_n^{(n)}(x,y)\cdot\tilde{f}_n^{(n-1)}(x,y)-\tilde{g}_n^{(-n)}(x,y)\cdot\tilde{f}_n^{(-n+1)}(x,y))=0 \tag{60}$$

$$D_y(\tilde{g}_n^{(n)}(x,y)\cdot\tilde{f}_n^{(n-1)}(x,y)+\tilde{g}_n^{(-n)}(x,y)\cdot\tilde{f}_n^{(-n+1)}(x,y))=0, \tag{61}$$

and

$$F(\tilde{g}_n^{(-n)}(x,y)\cdot\tilde{f}_n^{(n-1)}(x,y))=0 \tag{62}$$

$$F(\tilde{g}_n^{(-n)}(x,y)\cdot\tilde{g}_n^{(n)}(x,y))=0. \tag{63}$$

It is proved by the straightforward calculation that the polynomial expressions (55)-(58) satisfy the Nakamura's conjecture at the highest order (60)-(61) and (62)-(63). Although the validities of Eqs. (60) and (61) are clear due to a Pfaffian identiy for the first set of equations Eqs. (13) and(14), here they are reproduced from this approach.

It is also proved by straightforward calculations that the lowest order equations of the Nakamura's conjecture are right. As an another way, it is easily proved by using the property (49).

4 Discussions

In this paper we have discussed a proof of the Nakamura's conjecture on TS solutions. In the previous work [4], the first set of Eqs. (13) and (14) was proved completely without using the explicit forms of f_n and g_n. Whereas the proof of the second set of Eqs. (15) and (16) needed the explicit forms of f_n and g_n and was given for the restricted case $p=1, q=0$, deformed but non-rotating black hole. It is corresponding to the (extended) Weyl solution. In that case, th key point was that the functions g_n and f_n depends only on x and their polynomial expressions are explicitly required. This work is the extension of that work. That is, we have discussed the deformed and rotating black hole ($q \neq 0$) in this paper, extending the previous results to more generic case of $pq \neq 0$. To prove the deformed and rotating case we have rearranged the original parameters p, q ($p^2+q^2=1$) to t. Thanks to this arrangement, some properties of the functions g_n and f_n become transparent and workable (for instance (41)-(44)). Using the property (44), the explicit forms of functions g_n and f_n are obtained for $p=0, q=1$ ($t=\sqrt{-1}$) case and proved to satisfy the Nakamura's conjecture. For this case it is known that the metric indicates extremal TS solution, not asympotetically flat and describes a region of the TS metric near its ergosphere [8].

For the most generic case, $pq \neq 0$, the functions g_n and f_n are embedded in two dimensions and the situation is drastically changed compared with the $pq=0$ case. The introduction of t enables us to prove the conjecture order by order on t. In fact, by the Laurent expansions of the functions g_n and f_n we obtain the polynomial expressions of x, y (55)-(58) for the highest and lowest orders of g_n and f_n. As the result, it is proved that the highest and lowest orders of the Nakamura's conjecture are valid by the straightforward calculations. Unfortunately the proof for the second set of the conjecture has not been completed at full orders though their explicit forms are described in the Appendix.

Here some comments are in order. It seems that the equations for the next highest order ($I = 1$ in the Appendix) will be an important key to solve the problem. Though it is considerably complex, it is very straightforward. Then Eqs. (B.1)-(B.6) and our polynomial expressions for the highest and lowest order terms of g_n and f_n will be useful, because Eqs. (B.1)-(B.6) were already proved and the highest orders and the following orders for g_n and f_n are coupled in the next highest order equations ($I = 1$ in the Appendix). Futheremore, we observe that the expression for the highest order of f_n is connected with the Gamma functions. It is strongly expected that the polynomial expressions for g_n and f_n are described by two variables hypergeometric functions. As the other approaches, those by the Kramer-Neugebauer and Bäcklund transformations are considered. Although these transformations change properties of special solutions, the approach to the TS solutions as the Kramer-Neugebauer limit [9] will be useful to prove the conjecture.

It is also worth noting that the SU(1,1) symmetry plays the important role for the proof of the polynomial expressions (57) and (58). The Nakamura's conjecture is invariant under the transformation. The physical meanings of its symmetry is that a NUT parameter is generated by the transformation, (for instance the Schwartzschild geometry becomes Taub-NUT geometry by the transformation.). Therefore, if the Nakamura's conjecture is correct, its conjecture is also correct even for a generalization of TS solutions with a NUT parameter. The study of relations between Toda molecule and Ernst equation is very important. It explains the reason why the deformation parameter n is limitted as integers for TS solutions, though generalized TS solutions with non integers deformation parameters were considered by Hori [10] and Cosgrove [11]. On the physical natures of TS geometries, for instance, geometrical difference between odd and even deformation parameters will be also clarified from this point of views [12]. The explorations of these relations are expected to shed new light on the integrable systems and the Ernst equation themselves.

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Appendix

We have seen the functions g_n and f_n are expanded by the one parameter t. In the appendix, we extract order by order equations for the Toda molecule equations and the Nakamura's conjecture by substituing the expressions (47) into them.

A. Order by order equations for Toda molecule equation

The Toda molecule equation for g_n are expressed by

$$D_S D_T \left(\sum_{m=0}^{n} \tilde{g}_n^{(n-2m)}(x,y) t^{n-2m} \right) \cdot \left(\sum_{l=0}^{n} \tilde{g}_n^{(n-2l)}(x,y) t^{n-2l} \right) = 2 \left(\sum_{m=0}^{n+1} \tilde{g}_{n+1}^{(n-2m+1)}(x,y) t^{n-2m+1} \right) \left(\sum_{l=0}^{n-1} \tilde{g}_{n-1}^{(n-2l-1)}(x,y) t^{n-2l-1} \right) \quad \text{(A.1)}$$

in terms of the Laurent expansions by t. The parameter t is an arbitrary constant, so that the above equation must hold at each order of t. For the orders t^{2n-2I}, $(I = 0, \cdots 2n)$, we derive the following coupled equations:

(I) for $0 \le I \le n-1$,

$$\sum_{J=0}^{I} D_S D_T \; \tilde{g}_n^{(n-2J)}(x,y) \cdot \tilde{g}_n^{(n-2I+2J)}(x,y) = 2 \sum_{J=0}^{I} \tilde{g}_{n+1}^{(n-2J+1)}(x,y) \tilde{g}_{n-1}^{(n-2I+2J-1)}(x,y), \qquad \text{(A.2)}$$

(II) for $I = n$,

$$\sum_{J=0}^{n} D_S D_T \; \tilde{g}_n^{(n-2J)}(x,y) \cdot \tilde{g}_n^{(-n+2J)}(x,y) = 2 \sum_{J=1}^{n} \tilde{g}_{n+1}^{(n-2J+1)}(x,y) \tilde{g}_{n-1}^{(-n+2J-1)}(x,y),$$

(III) for $n+1 \le I \le 2n$, $y \to -y$ in (I) and (II). (A.3)

In the similar way, the Toda molecule equation for f_n becomes the following equations at the order $t^{2n-2I-2}$, $(I = 0, \cdots, 2(n-1))$:

(IV) for $0 \le I \le n-2$,

$$\sum_{J=0}^{I} D_S D_T \; \tilde{f}_n^{(n-2J-1)}(x,y) \cdot \tilde{f}_n^{(n-2I+2J-1)}(x,y) = 2 \sum_{J=0}^{I} \tilde{f}_{n+1}^{(n-2J)}(x,y) \tilde{f}_{n-1}^{(n-2I+2J-2)}(x,y), \qquad \text{(A.4)}$$

(V) for $I = n-1$,

$$\sum_{J=0}^{n-1} D_S D_T \ \tilde{f}_n^{(n-2J-1)}(x,y)\cdot \tilde{f}_n^{(-n+2J+1)}(x,y) = 2\sum_{J=1}^{n-1} \tilde{f}_{n+1}^{(n-2J)}(x,y)\tilde{f}_{n-1}^{(-n+2J)}(x,y), \tag{A.5}$$

$$\text{(VI)} \quad \text{for} \quad n \le I \le 2(n-1), \quad y \to -y \quad \text{in (IV)} \quad \text{and} \quad \text{(V)}. \tag{A.6}$$

For $t^{2n-2I-1}$, $(I = 0, \cdots, 2n-1)$, Eq. (36) is reduced to

(VII) for $0 \le I \le n-2$,

$$\sum_{J=0}^{I} D_S D_T \ \tilde{f}_n^{(n-2J-1)}(x,y)\cdot \tilde{g}_n^{(n-2I+2J)}(x,y) = 2\sum_{J=0}^{I}\Big(\tilde{f}_{n+1}^{(n-2J)}(x,y)\tilde{g}_{n-1}^{(n-2I+2J-1)}(x,y) + \tilde{f}_{n-1}^{(n-2J-2)}(x,y)\tilde{g}_{n+1}^{(n-2I+2J+1)}(x,y)\Big), \tag{A.7}$$

(VIII) for $I = n-1$,

$$\sum_{J=0}^{n-1} D_S D_T \ \tilde{f}_n^{(n-2J-1)}(x,y)\cdot \tilde{g}_n^{(-n+2J+2)}(x,y) = 2\sum_{J=0}^{n-1} \tilde{f}_{n+1}^{(n-2J)}(x,y)\tilde{g}_{n-1}^{(-n+2J+1)}(x,y) + \sum_{J=0}^{n-2} \tilde{f}_{n-1}^{(n-2J-2)}(x,y)\tilde{g}_{n+1}^{(-n+2J+3)}(x,y), \tag{A.8}$$

$$\text{(IX)} \quad \text{for} \quad n \le I \le 2n-1, \quad y \to -y \quad \text{in (VII)} \quad \text{and} \quad \text{(VIII)}. \tag{A.9}$$

The highest order equations correspond to the $I = 0$ case.

B. Order by order equations for Nakamura's conjecture

Eqs. (13)-(16) of the Nakamura's conjecture are also rewritten by the Laurent expansions of t. Eq. (13) becomes the following equations for the order $t^{2n-2I-1}, (I = 0, \cdots, 2n-1)$:

(I) for $0 \le I \le n-1$,

$$\sum_{J=0}^{I} D_x \Big(\tilde{g}_n^{(n-2J)}(x,y)\cdot \tilde{f}_n^{(n-2I+2J-1)}(x,y) - \tilde{g}_n^{(-n+2J)}(x,y)\cdot \tilde{f}_n^{(-n+2I-2J+1)}(x,y)\Big) = 0 \tag{B.1}$$

(II) for $I = n$,

$$\sum_{J=1}^{n} D_x \Big(\tilde{g}_n^{(n-2J)}(x,y)\cdot \tilde{f}_n^{(-n+2J-1)}(x,y)$$

$$-\tilde{g}_n^{(-n+2J)}(x,y)\cdot\tilde{f}_n^{(n-2J+1)}(x,y)\Big)=0 \tag{B.2}$$

$$(\text{III})\ \text{for}\quad n+1\le I\le 2n-1,\quad y\to -y\quad \text{in (I)}\quad\text{and}\quad(\text{II}). \tag{B.3}$$

Eq. (14) reduces to the following equations at the order $t^{2n-2I-1}, (I=0,\cdots,2n-1)$:

(IV) for $\quad 0\le I\le n-1,$

$$\sum_{J=0}^{I} D_y\Big(\tilde{g}_n^{(n-2J)}(x,y)\cdot\tilde{f}_n^{(n-2I+2J-1)}(x,y)$$
$$+\tilde{g}_n^{(-n+2J)}(x,y)\cdot\tilde{f}_n^{(-n+2I-2J+1)}(x,y)\Big)=0 \tag{B.4}$$

(V) for $\quad I=n,$

$$\sum_{J=1}^{n} D_y\Big(\tilde{g}_n^{(n-2J)}(x,y)\cdot\tilde{f}_n^{(-n+2J-1)}(x,y)$$
$$+\tilde{g}_n^{(-n+2J)}(x,y)\cdot\tilde{f}_n^{(n-2J+1)}(x,y)\Big)=0 \tag{B.5}$$

$$(\text{VI})\ \text{for}\quad n+1\le I\le 2n-1,\quad y\to -y\quad \text{in (IV)}\quad\text{and}\quad(\text{V}). \tag{B.6}$$

For Eq. (15) at the order $t^{2n-2I-1}, (I=0,\cdots,2n-1)$, it is

(VII) for $\quad 0\le I\le n-1,$

$$\sum_{J=0}^{I} F\ \tilde{g}_n^{(-n+2J)}(x,y)\cdot\tilde{f}_n^{(n-2I+2J-1)}(x,y)=0 \tag{B.7}$$

(VIII) for $\quad I=n,$

$$\sum_{J=1}^{n} F(\tilde{g}_n^{(-n+2J)}(x,y)\cdot\tilde{f}_n^{(-n+2J-1)}(x,y)=0 \tag{B.8}$$

$$(\text{IX})\ \text{for}\quad n+1\le I\le 2n-1,\quad y\to -y\quad \text{in (VII)}\quad\text{and}\quad(\text{VIII}). \tag{B.9}$$

For Eq. (16) at the order $t^{2n-2I}, (I=0,\cdots,2n)$, it is

(X) for $\quad 1\le I\le n,$

$$\sum_{J=0}^{I} F\Big(\tilde{g}_n^{(-n+2J)}(x,y)\cdot\tilde{g}_n^{(n-2I+2J)}(x,y)$$
$$+\tilde{f}_n^{(-n+2J+1)}(x,y)\cdot\tilde{f}_n^{(n-2I+2J+1)}(x,y)\Big)=0 \tag{B.10}$$

(XI) for $\quad I=0,$

$$F(\tilde{g}_n^{(-n)}(x,y)\cdot\tilde{g}_n^{(n)}(x,y))=0 \tag{B.11}$$

$$(\text{XII})\ \text{for}\quad n+1\le I\le 2n,\quad y\to -y\quad \text{in (X)}\quad\text{and}\quad(\text{XI}). \tag{B.12}$$

Eqs. (B.1)-(B.6) are corresponding to the first set (13) and (14) of the Nakamura's conjecture. As already mentioned, the first set (13) and (14) is proved by using a Paffinian identity [4]. Thus Eqs. (B.1)-(B.6) are correct. The highest order equations of the Nakamura's conjecture also correspond to the $I = 0$ case.

References

[1] A. Nakamura, J. Phys. Soc. Japan. **62** (1993) 368.

[2] A.Tomimatsu and H.Sato, Prog.Theor.Phys. **50**, 95 (1973)

[3] M. Toda, Prog.Theor.Phys. Suppl. bf 59, 1 (1976)

[4] T. Fukuyama, K. Kamimura and S. Yu, Journ. Phys. Soc. Jpn. **64**, 3201 (1995), gr-qc/9505013,

[5] R. Hirota, Phys. Rev. Lett. **27** (1971) 1192,
R. Hirota, "Mathematical Theory of Solitons by Direct Method (Iwanami Pub. 1992; in Japanese).

[6] M. Sato, Publ. RIMS Kokyuroku Kyoto Univ. **439** (1981) 30.

[7] R. Hirota, Y.Ohta and J. Satsuma, Prog. Theor. Phys. Supp. **94** (1988) 59.

[8] W. Kinnersley and E. F. Kelley, J. Math. Phys. **15** (1974) 2121.

[9] P. Dale, J. Phys. **A37** (2004) 2299.

[10] S. Hori, Prog.Theor. Phys. **95** (1996) 65.
O. Tanimura and S. Hori, Prog. Theor. Phys. **99** (1998) 783.

[11] C.M. Cosgrove, J. Phys. **A10** (1977) 1481.

[12] A.Nakamura and Y.Ohta, J.Phys.Soc.Jpn. **60** (1991) 1835.