Comment on paper by L. M. Malyshkin and S. Boldyrev, "Magnetic dynamo action at low magnetic Prandtl numbers", PRL 105, 215002 (2010)

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Is the scaling, $\lambda \propto Rm^{1/2}$, for the growth rate of small-scale dynamo instability at low magnetic Prandtl numbers and large magnetic Reynolds numbers, Rm, valid in the vicinity of the threshold? Our analysis and even numerical solution [1] of the dynamo equations for a Gaussian white-noise velocity field (the Kazantsev-Kraichnan model) imply that the answer is negative. Contrary to the claim in [1], there are two different asymptotics for the dynamo growth rate: in the vicinity of the threshold and far from the threshold.

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Let us discuss the asymptotic behaviour of the growth rate of magnetic fluctuations with a zero mean field for small magnetic Prandtl numbers in a homogeneous, isotropic, non-helical, incompressible and Gaussian white-noise velocity field (the Kazantsev-Kraichnan model). The equation for the longitudinal correlation function, $W(r) = \langle b_r(\mathbf{x})\, b_r(\mathbf{y})\rangle$ of the magnetic field reads:

$$18\, r^2\, W'' + 96\, r\, W' + \left(104 - 27\,\lambda\, r^{2/3}\right) W = 0\,, \qquad (1)$$

(see [2]), where b_r is the component of magnetic field $\mathbf{b}$ in the direction $\mathbf{r} = \mathbf{x} - \mathbf{y}$, $W' = dW(r)/dr$, λ is the growth rate of small-scale dynamo instability, and velocity fluctuations have Kolmogorov scaling from viscous scale to integral scale. Equation (1) is written in dimensionless variables: length and velocity are measured in units of ℓ_0 and u_0, where u_0 is the characteristic turbulent velocity in the integral scale ℓ_0. The solution of Eq. (1) is $W(r) = C\, r^{-13/6} K_\alpha \left(\sqrt{27\,\lambda/2}\; r^{1/3}\right)$ (see [2]), where $K_\alpha(y)$ is the real part of the modified Bessel function (Macdonald function) with $\alpha = (i/2)\sqrt{39}$. This solution is chosen to be finite at large r, with positively defined spectrum, and it has the following asymptotics: $W(r) = A_1\, r^{-13/6}\, \cos(\ln r + \varphi_0)$ at scales $\lambda^{1/2}\, r^{1/3} \ll 1$ (see [3]), and $W(r) = A_2\, r^{-7/3} \exp\left(-\sqrt{27\,\lambda/2}\; r^{1/3}\right)$ at scales $\lambda^{1/2}\, r^{1/3} \gg 1$ (see [4]).

For $\ell \geq \ell_\eta$ the scaling for the growth rate of small-scale dynamo instability which is far from the threshold, is $\lambda \sim u_\eta/\ell_\eta \sim (u_0/\ell_0)\, Rm^{1/2}$ (see [5]), where $\ell_\eta = \ell_0/Rm^{3/4}$ is the resistive scale, $u_\eta = (\varepsilon\, \ell_\eta)^{1/3}$ is the characteristic turbulent velocity at the resistive scale, $u_0 = (\varepsilon\, \ell_0)^{1/3}$, $\varepsilon = u_0^3/\ell_0$ is the dissipation rate of turbulent kinetic energy, $Rm = u_0\, \ell_0/\eta \gg 1$ is the magnetic Reynolds number and η is the magnetic diffusion due to electrical conductivity of the fluid. For the scaling $\lambda \propto Rm^{1/2}$, the condition $\lambda^{1/2}\, r^{1/3} \gg 1$ implies $r \gg Rm^{-3/4}$.

However, the scaling, $\lambda \propto Rm^{1/2}$, is not valid in the vicinity of the threshold of the dynamo instability. Indeed, in the vicinity of the threshold when $\lambda \to 0$, there is only one range of the solution of Eq. (1), $\lambda^{1/2}\, r^{1/3} \ll 1$, which determines the growth rate of the small-scale dynamo instability, $\lambda = \beta\, \ln(Rm/Rm_{\rm cr})$ (see [3]), where $\beta = 4/3$ is the exponent of the turbulent diffusivity scaling, $D(\ell) \propto \ell^\beta$. In Fig. 1 we plot the growth rate of small-scale dynamo instability versus $\ln(Rm/Rm_{\rm cr})$ in the vicinity of the threshold, which demonstrates perfect agreement between the scaling $\lambda = \beta\, \ln(Rm/Rm_{\rm cr})$ (solid line) and the numerical solution [1] of the dynamo equations for the Kazantsev-Kraichnan model (squares).

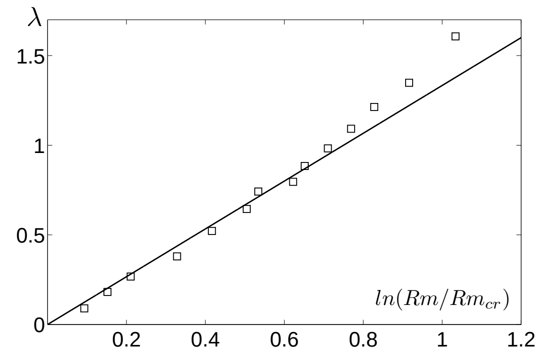

FIG. 1: The growth rate of small-scale dynamo instability versus $\ln(Rm/Rm_{\rm cr})$ in the vicinity of the instability threshold: solid line corresponds to the scaling $\lambda = \beta\, \ln(Rm/Rm_{\rm cr})$ and squares are the results of the numerical solution of the dynamo equations for the Kazantsev-Kraichnan model of velocity field with zero kinetic helicity taken from Fig. 1 in [1].

[1] L. M. Malyshkin and S. Boldyrev, Phys. Rev. Lett. **105**, 215002 (2010).
[2] H. Arponen and P. Horvai, J. Statist. Phys. **129**, 205 (2007).
[3] I. Rogachevskii and N. Kleeorin, Phys. Rev. E **56**, 417 (1997).
[4] S. Boldyrev and F. Cattaneo, Phys. Rev. Lett. **92**, 144501 (2004).
[5] K. Moffatt, J. Fluid Mech. **11**, 625 (1961).