JUCYS–MURPHY ELEMENTS FOR PARTITION ALGEBRAS

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ABSTRACT. An inductive formula is given for a family of elements which are shown to play a role in the partition algebras which is analogous to that played by classical Jucys–Murphy elements in the group algebra of the symmetric group. Using Schur–Weyl duality it is shown that the aforementioned inductive definition is equivalent to the combinatorial definition given by Halverson and Ram for Jucys–Murphy elements of partition algebras. As a consequence of the inductive formula for Jucys–Murphy elements, a new presentation for partition algebras in terms of certain involutions is also derived.

CONTENTS

1. INTRODUCTION

Partition algebras arose in the work of Martin [Ma] and Jones [Jo] as a generalisation of the Temperley–Lieb algebra and the Potts model in statistical mechanics. From the point of view of Schur–Weyl duality, the partition algebras occur as centralisers for the action of the symmetric group on tensor space when the the symmetric group acts diagonally as the subgroup of permutation matrices in the complex general linear group [Jo]. Given their role as centraliser algebras arising from Schur–Weyl duality, several authors have worked to extend classical results from the representation theory of the symmetric group to the partition algebras ([Ma],[Jo],[Xi],[HR],[GG]). Using Schur–Weyl duality to prove conjectures by Owens [Ow], Halverson and Ram [HR] provided a combinatorial definition of Jucys–Murphy elements for the partition algebras which play an exactly analogous role to the classical Jucys–Murphy elements in the symmetric group. The purpose of this note is to revisit the construction of [HR], by providing a recursion for the Jucys–Murphy elements for partition algebras and establishing directly the basic properties of the Jucys–Murphy elements using only the presentation for the partition algebras given in [HR]. We also derive a new presentation for the partition algebras in terms of certain involutions that arise naturally from our recursive definition of Jucys–Murphy elements. The contents of this note are presented as follows:

(1) In §2 we recall the presentation of the partition algebras given by [HR].
(2) In §3 we state a recursion defining a family of operators

$$(1.1) \qquad (L_i, L_{i+\frac{1}{2}} : i = 0, 1, \dots)$$

in terms of the presentation for the partition algebras given by [HR]. By direct calculation with generators and relations, we establish that the operators (1.1) form a commuting family with properties analogous to the Jucys–Murphy elements that arise in the representation theory of the symmetric group. Simultaneously, we establish some basic commutativity results for certain operators denoted

$$(1.2) \qquad (\sigma_i, \sigma_{i+\frac{1}{2}} : i = 1, 2, \dots)$$

which arose in the recursive definition of the Jucys–Murphy elements (1.1).

(3) In §4 we show that the elements of (1.2) are involutions which are related to the Coxeter generators for the symmetric group in a precise way. Using the relation between the involutions (1.2) and the Coxeter generators for the symmetric group, and the properties established in §3, we derive a new presentation for the partition algebras.

(4) In §5 we describe precisely the action of the Jucys–Murphy elements (1.1) and the involutions (1.2) on tensor space. The action of the elements (1.1) on tensor space is then used show that the recursive definition of Jucys–Murphy elements given in §2 is equivalent to the definition of Jucys–Murphy elements in [HR].

Given that the partition algebras are cellular [Xi] and have branching rules for induction and restriction that are described by a multiplicity–free Bratteli diagram, it is natural to expect that the partition algebras also have an integral Murphy–type basis [MS] with respect to which the Jucys–Murphy elements act triangularly. Using a framework for cellularity of algebras related to the Jones basic construction, Goodman and Graber [GG], [GG1] have demonstrated the existence of Murphy bases for the partition algebras, indexed by paths in the Bratteli diagram and compatible with restriction and induction. However, the approach of Goodman and Graber describes the action of Jucys–Murphy elements on the cell modules in terms of the action of certain central elements whose eigenvalues are not explicitly derived. In a separate note in preparation [En] we will provide an explicit construction of Murphy bases for the partition algebras, compute the action of the Jucys–Murphy elements on the cell modules, and use Nazarov's method [Na] of computing generating functions for central elements to write down a seminormal form for the partition algebras.

2. The Partition Algebras

The material of this section follows the exposition given by Halverson and Ram in [HR]. For $k = 1, 2, \dots$, let

$$A_k = \{\text{set partitions of } \{1, 2, \dots, k, 1', 2', \dots, k'\}\},$$

and

$$A_{k-\frac{1}{2}} = \{d \in A_k \mid k \text{ and } k' \text{ are in the same block of } d\}.$$

For convenience, any element $\rho \in A_k$ may be represented as a graph with k vertices in the top row, labelled from left to right, by $1, 2, \dots, k$ and k vertices in the bottom row, labelled, from left to right by $1', 2', \dots, k'$, with vertex i joined to vertex j if i and j belong to the same block of ρ. The representation of a partition by a diagram is not unique; for example the partition

$$\rho = \{\{1, 1', 3, 4', 5', 6\}, \{2, 2', 3', 4, 5, 6'\}\}$$

can be represented by the diagrams:
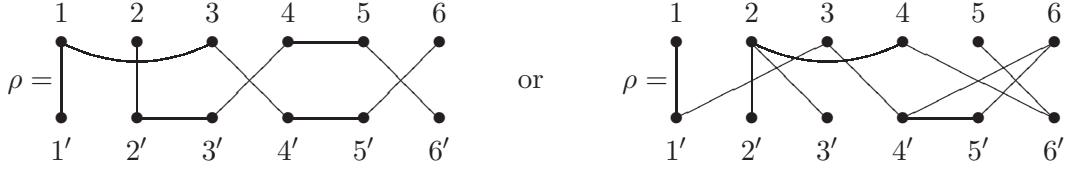
If $\rho_1, \rho_2 \in A_k$, then the composition $\rho_1 \circ \rho_2$ is the partition obtained by placing ρ_1 above ρ_2 and identifying each vertex in the bottom row of ρ_1 with the corresponding vertex in the top row of ρ_2 and deleting any components of the resulting diagram which contains only elements from the middle row. The composition product makes A_k into an associative monoid with identity

$$1 = \Big| \quad \Big| \quad \cdots \quad \Big| .$$

Let z be an indeterminate and $R = \mathbb{Z}[z]$. The partition algebra $\mathcal{A}_k(z)$ is the R–module freely generated by A_k, equipped with the product

$$\rho_1 \rho_2 = z^\ell \rho_1 \circ \rho_2, \qquad\qquad \text{for } \rho_1, \rho_2 \in A_k,$$

where ℓ is the number of blocks removed from the middle row in constructing the composition $\rho_1 \circ \rho_2$. Let $\mathcal{A}_{k-\frac{1}{2}}(z)$ denote the subalgebra of $\mathcal{A}_k(z)$ generated by $A_{k-\frac{1}{2}}$. A presentation for $\mathcal{A}_k(z)$ has been given by Halverson and Ram.

Theorem 2.1 (Theorem 1.11 of [HR]). *If $k = 1, 2, \ldots$, then the partition algebra $\mathcal{A}_k(z)$ is the unital associative R–algebra presented by the generators*

$$p_1, p_2, \ldots, p_k, p_{1+\frac{1}{2}}, p_{2+\frac{1}{2}}, \ldots, p_{k-\frac{1}{2}}, s_1, s_2, \ldots, s_{k-1},$$

and the relations

(1) *(Coxeter relations)*
    (i) $s_i^2 = 1$, *for $i = 1, \ldots, k-1$.*
    (ii) $s_i s_j = s_j s_i$, *if $j \neq i+1$.*
    (iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, *for $i = 1, \ldots, k-2$.*
(2) *(Idempotent relations)*
    (i) $p_i^2 = z p_i$, *for $i = 1, \ldots, k$.*
    (ii) $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}$, *for $i = 1, \ldots, k-1$.*
    (iii) $s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}}$, *for $i = 1, \ldots, k-1$.*
    (iv) $s_i p_i p_{i+1} = p_i p_{i+1} s_i = p_i p_{i+1}$, *for $i = 1, \ldots, k-1$.*
(3) *(Commutation relations)*
    (i) $p_i p_j = p_j p_i$, *for $i = 1, \ldots k$ and $j = 1, \ldots, k$.*
    (ii) $p_{i+\frac{1}{2}} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_{i+\frac{1}{2}}$, *for $i = 1, \ldots k-1$ and $j = 1, \ldots, k-1$.*
    (iii) $p_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_i$, *for $j \neq i, i+1$.*
    (iv) $s_i p_j = p_j s_i$, *for $j \neq i, i+1$.*
    (v) $s_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} s_i$, *for $j \neq i-1, i+1$.*
    (vi) $s_i p_i s_i = p_{i+1}$, *for $i = 1, \ldots, k-1$.*
    (vii) $s_i p_{i-\frac{1}{2}} s_i = s_{i-1} p_{i+\frac{1}{2}} s_{i-1}$, *for $i = 2, \ldots, k-1$.*
(4) *(Contraction relations)*
    (i) $p_{i+\frac{1}{2}} p_j p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, *for $j = i, i+1$.*
    (ii) $p_i p_{j-\frac{1}{2}} p_i = p_i$, *for $j = i, i+1$.*

The above relations also imply that:

$$p_{i+\frac{1}{2}}s_{i\pm1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_{i\pm\frac{1}{2}},$$
$$p_i s_i p_i = p_i p_{i+1} = p_{i+1}s_i p_{i+1},$$
$$p_i p_{i+\frac{1}{2}} p_{i+1} = p_i s_i,$$
$$p_{i+1} p_{i+\frac{1}{2}} p_i = p_{i+1}s_i.$$

In Theorem 2.1, Halverson and Ram have made the following identifications:
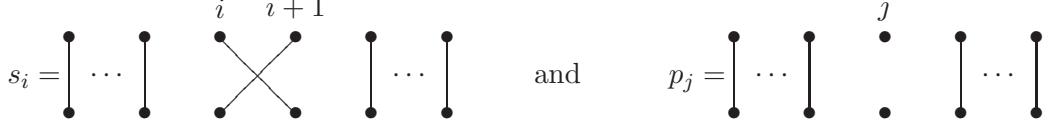
and
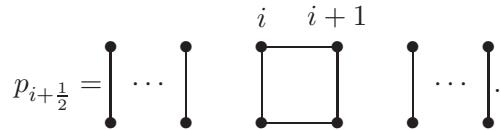
We also let $\mathfrak{S}_k$ denote the symmetric group on k letters which is generated by $s_1,\ldots,s_{k-1}$. If $u \in \mathfrak{S}_k \subset \mathcal{A}_k(z)$, and $\rho \in \mathcal{A}_k(z)$, we will sometimes write $\rho^u = u\rho u^{-1}$. It will be useful to note that R-linear map $* : \mathcal{A}_k(z) \to \mathcal{A}_k(z)$ defined on generators by

$$u^* = u^{-1} \qquad\qquad (\text{for } u \in \mathfrak{S}_k)$$

and

$$p_i^* = p_i \qquad (\text{for } i = 1,\ldots,k) \qquad \text{and} \qquad p_{j+\frac{1}{2}}^* = p_{j+\frac{1}{2}} \qquad (\text{for } j = 1,\ldots,k-1),$$

is an algebra anti-involution of $\mathcal{A}_k(z)$. Restricting the map $*$ from $\mathcal{A}_k(z)$ to $\mathcal{A}_{k-\frac{1}{2}}(z)$, gives an algebra anti involution of $\mathcal{A}_{k-\frac{1}{2}}(z)$ which we also denote by $*$.

3. JUCYS–MURPHY ELEMENTS

The purpose of this section is to give a recursion for Jucys–Murphy elements in $\mathcal{A}_k(z)$ and $\mathcal{A}_{k+\frac{1}{2}}(z)$. It will be shown in §4 that this recursion is equivalent to the combinatorial definition of Jucys–Murphy elements given [HR]. Let $(\sigma_i : i = 1, 2, \ldots)$ and $(L_i : i = 0, 1, \ldots)$ be given by

$$L_0 = 0, \qquad L_1 = p_1, \qquad \sigma_1 = 1, \qquad \text{and,} \qquad \sigma_2 = s_1,$$

and, for $i = 1, 2, \ldots$,

$$(3.1) \qquad L_{i+1} = -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i s_i + \sigma_{i+1},$$

where, for $i = 2, 3, \ldots$,

$$(3.2) \quad \sigma_{i+1} = s_{i-1}s_i\sigma_i s_i s_{i-1} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}$$
$$- s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i.$$

Define $(\sigma_{i+\frac{1}{2}} : i = 1, 2, \ldots)$ and $(L_{i+\frac{1}{2}} : i = 0, 1, \ldots)$ by

$$L_{\frac{1}{2}} = 0, \qquad\qquad \sigma_{\frac{1}{2}} = 1, \qquad\qquad \text{and,} \qquad\qquad \sigma_{1+\frac{1}{2}} = 1,$$

and, for $i = 1, 2, \ldots$,

$$(3.3) \qquad L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}},$$

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where, for $i = 2, 3, \ldots$,

$$(3.4) \quad \sigma_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_{i-\frac{1}{2}}s_is_{i-1} + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_i + s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}$$
$$- p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}} - s_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i.$$

Rewriting the last summand in (3.4) as

$$s_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i = s_{i-1}p_{i+\frac{1}{2}}s_{i-1}s_ip_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_is_{i-1}p_{i+\frac{1}{2}}s_{i-1}$$
$$= s_{i-1}p_{i+\frac{1}{2}}s_{i-1}s_ip_{i+1}p_{i+\frac{1}{2}}s_{i-1}s_iL_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}$$
$$= s_{i-1}p_{i+\frac{1}{2}}s_{i-1}p_{i+1}s_{i-1}s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}$$
$$= s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1},$$

the expression (3.4) becomes

$$(3.5) \quad \sigma_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_{i-\frac{1}{2}}s_is_{i-1} + p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1} + s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}$$
$$- p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}} - s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}.$$

Using induction, it follows that if $i = 0, 1, \ldots$, then $\sigma_{i+\frac{1}{2}} \in \mathcal{A}_{i+\frac{1}{2}}(z)$, and $L_{i+\frac{1}{2}} \in \mathcal{A}_{i+\frac{1}{2}}(z)$. Observe that if $i = 0, 1, \ldots$, then $(L_i)^* = L_i$ and $(\sigma_{i+1})^* = \sigma_{i+1}$. The fact that $(L_{i+\frac{1}{2}})^* = L_{i+\frac{1}{2}}$ and $(\sigma_{i+\frac{1}{2}})^* = \sigma_{i+\frac{1}{2}}$ will be shown in Proposition 3.2.

Example 3.1. In terms of the diagram presentation for $\mathcal{A}_3(z)$, we have expressions for L_2 and σ_3 as:

and

Using the above expressions for L_2 and σ_3, we can write L_3 as

The relations given in the next proposition are fundamental for subsequent calculations.

Proposition 3.1. *For $i = 1, 2, \ldots$, the following statements hold:*

(1) $\sigma_{i+1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$,

(2) $s_{i+1}\sigma_{i+1}p_{i+\frac{3}{2}} = p_{i+\frac{1}{2}}s_{i+1}\sigma_{i+1}$,

(3) $\sigma_{i+1}p_ip_{i+\frac{1}{2}} = s_iL_ip_{i+\frac{1}{2}}$,

(4) $\sigma_{i+1}p_{i+1}p_{i+\frac{1}{2}} = L_ip_{i+\frac{1}{2}}$,

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(5) $p_{i+\frac{1}{2}}L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$.

Proof. (1) We consider each of the terms on the right hand side of the definition

$$\sigma_{i+1}p_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_i s_i s_{i-1}p_{i+\frac{1}{2}} + s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$
$$- s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}},$$

beginning with the argument by induction:

$$s_{i-1}s_i\sigma_i s_i s_{i-1}p_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_i p_{i-\frac{1}{2}}s_i s_{i-1} = s_{i-1}s_i p_{i-\frac{1}{2}}s_i s_{i-1} = p_{i+\frac{1}{2}}.$$

From the fact that $p_{i+\frac{1}{2}}$ commutes with L_{i-1}, $s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$ and $p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$, while

$$s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} = s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$

and, from the relation $p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}p_i s_{i-1}s_i p_{i-\frac{1}{2}}s_i$, we obtain

$$p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}}p_i s_{i-1}s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}s_{i-1}p_{i-1}s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}p_{i-1}s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.$$

Substituting the terms obtained above into the definition of $\sigma_{i+1}p_{i+\frac{1}{2}}$, we observe that all terms vanish except for $p_{i+\frac{1}{2}}$, which completes the proof of (1).

(2) The definition (3.2) gives

$$s_{i+1}\sigma_{i+1}p_{i+\frac{3}{2}} = s_{i+1}s_{i-1}s_i\sigma_i s_i s_{i-1}p_{i+\frac{3}{2}} + s_{i+1}s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_{i+\frac{3}{2}}$$
$$+ s_{i+1}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_{i+\frac{3}{2}} - s_{i+1}s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i+\frac{3}{2}}$$
$$- s_{i+1}p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+\frac{3}{2}}.$$

Now consider each of the terms on the right hand side of the above equality. Firstly,

$$s_{i+1}s_{i-1}s_i\sigma_i s_i s_{i-1}p_{i+\frac{3}{2}} = s_{i+1}s_{i-1}s_i s_{i+1}\sigma_i s_{i+1}s_i s_{i-1}p_{i+\frac{3}{2}}$$
$$= s_{i-1}s_{i+1}s_i s_{i+1}\sigma_i s_{i+1}s_i p_{i+\frac{3}{2}}s_{i-1}$$
$$= s_{i-1}s_i s_{i+1}s_i\sigma_i p_{i+\frac{1}{2}}s_{i+1}s_i s_{i-1}$$
$$= s_{i-1}s_i s_{i+1}p_{i-\frac{1}{2}}s_i\sigma_i s_{i+1}s_i s_{i-1} \qquad \text{(by induction)}$$
$$= s_{i-1}s_i p_{i-\frac{1}{2}}s_{i+1}s_i\sigma_i s_{i+1}s_i s_{i-1}$$
$$= p_{i+\frac{1}{2}}s_{i-1}s_i s_{i+1}s_i\sigma_i s_{i+1}s_i s_{i-1}$$
$$= p_{i+\frac{1}{2}}s_{i-1}s_{i+1}s_i s_{i+1}\sigma_i s_{i+1}s_i s_{i-1}$$
$$= p_{i+\frac{1}{2}}s_{i+1}s_{i-1}s_i\sigma_i s_i s_{i-1}.$$

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Next, use the fact that $s_i p_{i+\frac{3}{2}} s_i$ and $p_{i-\frac{1}{2}}$ commute to observe that

$$
\begin{aligned}
s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} &= s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} (s_i p_{i+\frac{3}{2}} s_i) s_i \\
&= s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i (s_i p_{i+\frac{3}{2}} s_i) p_{i-\frac{1}{2}} s_i \\
&= s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i \\
&= s_{i+1} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
&= p_{i+\frac{1}{2}} s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i,
\end{aligned}
$$

and that

$$
\begin{aligned}
s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} &= s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} \\
&= s_{i+1} p_{i-\frac{1}{2}} L_{i-1} (s_i p_{i+\frac{3}{2}} s_i) s_i p_{i-\frac{1}{2}} \\
&= s_{i+1} p_{i-\frac{1}{2}} (s_i p_{i+\frac{3}{2}} s_i) L_{i-1} s_i p_{i-\frac{1}{2}} \\
&= s_{i+1} (s_i p_{i+\frac{3}{2}} s_i) p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
&= p_{i+\frac{1}{2}} s_{i+1} s_i s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
&= p_{i+\frac{1}{2}} s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}.
\end{aligned}
$$

Since $p_{i+\frac{3}{2}}$ commutes with $\mathcal{A}_{i+\frac{1}{2}}(z)$, we see that

$$
\begin{aligned}
s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} &= s_{i+1} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&= p_{i+\frac{1}{2}} s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}.
\end{aligned}
$$

For the last term, we use the fact that $(s_i p_{i+\frac{3}{2}} s_i)$ and $p_{i-\frac{1}{2}}$ commute to see that

$$
\begin{aligned}
s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} &= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} (s_i p_{i+\frac{3}{2}} s_i) s_i \\
&= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} (s_i p_{i+\frac{3}{2}} s_i) p_{i-\frac{1}{2}} s_i \\
&= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} (s_i p_{i+\frac{3}{2}} s_i) L_{i-1} p_{i-\frac{1}{2}} s_i \\
&= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} (s_{i-1} s_i p_{i+\frac{3}{2}} s_i s_{i-1}) L_{i-1} p_{i-\frac{1}{2}} s_i \\
&= s_{i+1} p_{i-\frac{1}{2}} (s_{i-1} s_i p_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} s_i s_{i-1}) L_{i-1} p_{i-\frac{1}{2}} s_i \\
&= s_{i+1} p_{i-\frac{1}{2}} (s_{i-1} s_i p_{i+\frac{3}{2}} s_i s_{i-1}) p_i p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} s_i \\
&= s_{i+1} p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} s_i s_{i-1} p_i p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} s_i \\
&= p_{i-\frac{1}{2}} s_{i+1} s_i p_{i+\frac{3}{2}} s_i s_{i-1} p_i p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} s_i \\
&= p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i+1} s_i^2 s_{i-1} p_i p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} s_i \\
&= p_{i+\frac{1}{2}} s_{i+1} p_{i-\frac{1}{2}} s_{i-1} p_i p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} s_i.
\end{aligned}
$$

Putting the above together, we have

$$
\begin{aligned}
s_{i+1} \sigma_{i+1} p_{i+\frac{3}{2}} &= p_{i+\frac{1}{2}} s_{i+1} s_{i-1} s_i \sigma_i s_i s_{i-1} + p_{i+\frac{1}{2}} s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
&\quad + p_{i+\frac{1}{2}} s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} - p_{i+\frac{1}{2}} s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&\quad\quad\quad\quad - p_{i+\frac{1}{2}} s_{i+1} p_{i-\frac{1}{2}} s_{i-1} p_i p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} s_i,
\end{aligned}
$$

which completes the proof of (2).

(3) We consider the terms on the right hand side of the equality

$$\sigma_{i+1}p_ip_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_is_is_{i-1}p_ip_{i+\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_ip_{i+\frac{1}{2}}$$
$$+ p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}} - s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}$$
$$- p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_ip_{i+\frac{1}{2}},$$

beginning with

$$s_{i-1}s_i\sigma_is_is_{i-1}p_ip_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_ip_{i-1}s_is_{i-1}p_{i+\frac{1}{2}}$$
$$= s_{i-1}s_i\sigma_ip_{i-1}p_{i+\frac{1}{2}}s_is_{i-1}$$
$$= s_{i-1}s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_is_{i-1}$$
$$= s_is_{i-1}s_iL_{i-1}s_is_{i-1}p_{i+\frac{1}{2}}$$
$$= s_is_{i-1}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}.$$

For the second term, we have

$$s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_ip_{i+\frac{1}{2}} = s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_{i+1}s_ip_{i+\frac{1}{2}}$$
$$= s_ip_{i-\frac{1}{2}}L_{i-1}p_is_ip_{i-\frac{1}{2}}s_ip_{i+\frac{1}{2}}$$
$$= s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}},$$

and for the third,

$$p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i-1}p_is_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.$$

Using the relation $p_ip_{i-\frac{1}{2}}p_i = p_i$, we see that the fourth term satisfies

$$s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}} = s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}},$$

while, for the fifth term,

$$p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_ip_{i+\frac{1}{2}} = p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+1}s_ip_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}p_{i+1}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}s_ip_{i+1}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$= s_is_{i-1}p_{i+\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$= s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.$$

Combining the above terms, we obtain

$$\sigma_{i+1}p_ip_{i+\frac{1}{2}} = s_is_{i-1}L_{i-1}s_{i-1}p_{i+\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$
$$- s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}} - s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}},$$

which completes the proof (3).

(4) We consider each of the terms on the right hand side of the expression

$$\sigma_{i+1}p_{i+1}p_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_is_is_{i-1}p_{i+1}p_{i+\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i+1}p_{i+\frac{1}{2}}$$
$$+ p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} - s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$- p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i+1}p_{i+\frac{1}{2}}.$$

Firstly,

$$
\begin{aligned}
s_{i-1}s_i\sigma_i s_i s_{i-1}p_{i+1}p_{i+\frac{1}{2}} &= s_{i-1}s_i\sigma_i p_i s_i s_{i-1}p_{i+\frac{1}{2}} \\
&= s_{i-1}s_i\sigma_i p_i p_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_{i-1}s_i L_{i-1}p_{i-\frac{1}{2}}s_i s_{i-1} \qquad \text{(by induction)} \\
&= s_{i-1}s_i L_{i-1}s_i s_{i-1}p_{i+\frac{1}{2}} \\
&= s_{i-1}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}.
\end{aligned}
$$

For the second term,

$$
\begin{aligned}
s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}} &= s_i p_{i-\frac{1}{2}}s_i\sigma_i p_i p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}} \qquad &\text{(by induction)} \\
&= s_i s_i\sigma_i p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}} \qquad &\text{(by item (2))} \\
&= \sigma_i p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_i s_i p_{i+\frac{1}{2}} \\
&= \sigma_i p_{i+\frac{1}{2}}p_i s_i p_{i+\frac{1}{2}} \\
&= \sigma_i p_{i+\frac{1}{2}}.
\end{aligned}
$$

For the third term,

$$
\begin{aligned}
p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} &= p_{i-\frac{1}{2}}L_{i-1}p_i s_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.
\end{aligned}
$$

For the fourth term, we use the relation $s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} = s_i p_{i-\frac{1}{2}}p_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$ in

$$
\begin{aligned}
s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} &= s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}p_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= s_i p_{i-\frac{1}{2}}s_i\sigma_i p_i p_{i-\frac{1}{2}}p_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \qquad &\text{(by induction)} \\
&= s_i p_{i-\frac{1}{2}}s_i\sigma_i p_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= s_i s_i\sigma_i p_{i+\frac{1}{2}}p_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \qquad &\text{(by item (2))} \\
&= \sigma_i p_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \qquad &\text{(by item (3))}.
\end{aligned}
$$

For the final term, we use the relation $p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}s_i p_{i-1}p_{i-\frac{1}{2}}s_i$ in

$$
\begin{aligned}
p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}} &= p_{i-\frac{1}{2}}s_i p_{i-1}p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}} \\
&= p_{i-\frac{1}{2}}s_i p_{i-1}p_{i-\frac{1}{2}}s_i\sigma_i p_i p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}} \qquad &\text{(by induction)} \\
&= p_{i-\frac{1}{2}}s_i p_{i-1}p_{i-\frac{1}{2}}s_i\sigma_i p_i p_{i-\frac{1}{2}}p_i s_i p_{i+\frac{1}{2}} \\
&= p_{i-\frac{1}{2}}s_i p_{i-1}p_{i-\frac{1}{2}}s_i\sigma_i p_i p_{i+\frac{1}{2}} \\
&= p_{i-\frac{1}{2}}s_i p_{i-1}s_i\sigma_i p_{i+\frac{1}{2}}p_i p_{i+\frac{1}{2}} \qquad &\text{(by item (2))} \\
&= p_{i-\frac{1}{2}}s_i p_{i-1}s_i\sigma_i p_{i+\frac{1}{2}} \\
&= p_{i-\frac{1}{2}}p_{i-1}\sigma_i p_{i+\frac{1}{2}} \\
&= p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}} \qquad &\text{(by item (3))}.
\end{aligned}
$$

Putting the above together,

$$\sigma_{i+1}p_{i+1}p_{i+\frac{1}{2}} = s_{i-1}L_{i-1}s_{i-1}p_{i+\frac{1}{2}} + \sigma_i p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$
$$- s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}},$$

which proves (4).

(5) Parts (1) and (3) give

$$p_{i+\frac{1}{2}}L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}s_i L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+1}p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}},$$

as required. $\square$

Proposition 3.2. *If $i = 1, 2, \ldots$, then*

(1) $(\sigma_{i+\frac{1}{2}})^* = \sigma_{i+\frac{1}{2}}$,

(2) $(L_{i+\frac{1}{2}})^* = L_{i+\frac{1}{2}}$.

Proof. (1) We show that the summand $p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$ in (3.4) is fixed under the $*$ anti-involution on $\mathcal{A}_{i+\frac{1}{2}}(z)$, using Proposition 3.1, as follows:

$$p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}p_i \sigma_i s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}p_i \sigma_i s_i p_{i-\frac{1}{2}}s_i s_{i-1}p_i p_{i-\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_i s_{i-1}p_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_i p_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}.$$

(2) Given that $(\sigma_{i+\frac{1}{2}})^* = \sigma_{i+\frac{1}{2}}$, it suffices to show that the summand $p_{i+\frac{1}{2}}L_i p_i p_{i+\frac{1}{2}}$ in (3.3) is fixed under the $*$ anti-involution on $\mathcal{A}_{i+\frac{1}{2}}(z)$. Using Proposition 3.1,

$$p_{i+\frac{1}{2}}L_i p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1}p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_{i+1}s_i L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}s_i p_i L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_i L_i p_{i+\frac{1}{2}},$$

gives the required result. $\square$

Proposition 3.3. *If $i = 1, 2, \ldots$, then $\sigma_{i+\frac{1}{2}}s_i = s_i \sigma_{i+\frac{1}{2}} = \sigma_{i+1}$.*

Proof. After noting that σ_2 commutes with s_1 and that $\sigma_{1+\frac{1}{2}} = \sigma_2 s_1$, the statement follows from induction and the fact that $p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$ which was demonstrated in the proof of Proposition 3.2. $\square$

The following observation is made for later reference.

Lemma 3.4. *If $i = 1, 2, \ldots$, then*

$$s_i s_{i+1}\sigma_{i+1}s_{i+1}s_i p_{i+\frac{1}{2}} = s_{i+1}p_{i+\frac{1}{2}}L_i s_i p_{i+\frac{3}{2}}p_{i+1}p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}s_{i+1}p_{i+\frac{1}{2}}.$$

Proof. On the one hand,

$$s_i s_{i+1}\sigma_{i+1}s_{i+1}s_i p_{i+\frac{1}{2}} = s_i s_{i+1}\sigma_{i+1}s_{i+1}p_{i+\frac{1}{2}} = s_i s_{i+1}p_{i+\frac{3}{2}}\sigma_{i+1}s_{i+1}$$
$$= s_i p_{i+\frac{3}{2}}\sigma_{i+1}s_{i+1} = s_i \sigma_{i+1}s_{i+1}p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}s_{i+1}p_{i+\frac{1}{2}},$$

and on the other,

$$s_{i+1}p_{i+\frac{1}{2}}L_i s_i p_{i+\frac{3}{2}}p_{i+1}p_{i+\frac{1}{2}} = s_{i+1}p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1}s_i p_{i+\frac{3}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$= s_{i+1}p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+\frac{1}{2}}p_{i+\frac{3}{2}}p_{i+1}p_{i+\frac{1}{2}} = s_{i+1}p_{i+\frac{1}{2}}p_{i+1}p_{i+\frac{3}{2}}\sigma_{i+\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$= s_i p_{i+\frac{3}{2}}s_i s_{i+1}p_{i+1}p_{i+\frac{3}{2}}\sigma_{i+\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} = s_i p_{i+\frac{3}{2}}s_i p_{i+2}p_{i+\frac{3}{2}}\sigma_{i+\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$= s_i p_{i+\frac{3}{2}}s_i \sigma_{i+\frac{1}{2}}p_{i+2}p_{i+\frac{3}{2}}p_{i+1}p_{i+\frac{1}{2}} = s_i p_{i+\frac{3}{2}}\sigma_{i+1}p_{i+2}p_{i+\frac{3}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$= s_i p_{i+\frac{3}{2}}\sigma_{i+1}s_{i+1}p_{i+1}p_{i+\frac{1}{2}} = s_i \sigma_{i+1}s_{i+1}p_{i+\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}$$
$$= s_i \sigma_{i+1}s_{i+1}p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}s_{i+1}p_{i+\frac{1}{2}},$$

as required. $\square$

We are now in a position to state the first of the main commutativity results of this note.

Theorem 3.5. *The elements σ_{i+1} and $\sigma_{i+\frac{1}{2}}$ satisfy the following commutativity relations:*

(1) $\sigma_{i+1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}\sigma_{i+1} = p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}$ *for $i = 2,3,\ldots$.*
(2) $\sigma_{i+1}p_{i-1} = p_{i-1}\sigma_{i+1} = s_{i-1}\sigma_ip_{i+1}s_is_{i-1}$ *for $i = 2,3,\ldots$.*
(3) $\sigma_{i+1}p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}}\sigma_{i+1}$ *for $i = 3,4,\ldots$.*
(4) $\sigma_{i+1}s_{i-2} = s_{i-2}\sigma_{i+1}$ *for $i = 3,4,\ldots$.*
(5) $\sigma_{i+1}p_{i-2} = p_{i-2}\sigma_{i+1}$ *for $i = 3,4,\ldots$.*
(6) $\sigma_{i+\frac{1}{2}}p_{i-1} = p_{i-1}\sigma_{i+\frac{1}{2}}$ *for $i = 2,3,\ldots$.*
(7) $\sigma_{i+\frac{1}{2}}p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}}\sigma_{i+\frac{1}{2}}$ *for $i = 3,4,\ldots$.*
(8) $\sigma_{i+\frac{1}{2}}s_{i-2} = s_{i-2}\sigma_{i+\frac{1}{2}}$ *for $i = 3,4,\ldots$.*
(9) $\sigma_{i+\frac{1}{2}}p_{i-2} = p_{i-2}\sigma_{i+\frac{1}{2}}$ *for $i = 3,4,\ldots$.*

Proof. (1) Using Lemma 3.4,

$$
\begin{aligned}
\sigma_{i+1}p_{i-\frac{1}{2}} &= -s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}} - p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} \\
&\quad + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + s_{i-1}s_i\sigma_is_is_{i-1}p_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} \\
&= -p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} \\
&= -p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= -p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}p_{i+\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= -p_{i-\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= -p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&= p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}s_iL_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}\sigma_{i+1},
\end{aligned}
$$

as required.

(2) We first show that

(i) $s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i-1}$,
(ii) $p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i-1} = p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_{i-1}$,
(iii) $s_{i-1}s_i\sigma_is_is_{i-1}p_{i-1} = s_{i-1}s_i\sigma_ip_{i+1}s_is_{i-1} = p_{i-1}s_{i-1}s_i\sigma_is_is_{i-1}$.

The left hand side of (i) gives

$$
s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}p_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i-1},
$$

which is the right hand side of (i). Using the relation $p_{i+\frac{1}{2}}s_{i-1}s_i = s_{i-1}s_ip_{i-\frac{1}{2}}$, the left hand side of (ii) gives

$$
\begin{aligned}
p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i-1} &= p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_{i-1}s_i \\
&= p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_is_{i-1}s_i \\
&= p_{i-\frac{1}{2}}p_i\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}s_ip_{i-1} \\
&= p_{i-\frac{1}{2}}p_i\sigma_{i-\frac{1}{2}}s_{i-1}s_ip_{i-\frac{1}{2}}p_{i-1} \\
&= p_{i-\frac{1}{2}}p_i\sigma_is_ip_{i-\frac{1}{2}}p_{i-1} \\
&= p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}p_{i-1},
\end{aligned}
$$

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as required. The item (iii) follows from the relation $s_{i-1}s_i p_{i+1} = p_{i-1}s_{i-1}s_i$. Next, using the definition (3.2),

$$\sigma_{i+1}p_{i-1} = s_{i-1}s_i\sigma_i s_i s_{i-1}p_{i-1} + s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_{i-1} + p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}$$
$$- s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-1} - p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_{i-1}$$
$$= s_{i-1}s_i\sigma_i s_i s_{i-1}p_{i-1},$$

the right hand side of which is fixed under is fixed under the $*$ anti-involution on $\mathcal{A}_{i+1}(z)$.
(3) We first show that

(i) $s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}$,

(ii) $s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}$,

(iii) $s_i p_{i-\frac{1}{2}}\sigma_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = s_{i-1}s_i p_{i-\frac{3}{2}}L_{i-2}s_{i-1}p_{i-\frac{3}{2}}s_i s_{i-1}p_{i-\frac{3}{2}}$,

(iv) $p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}\sigma_{i-1}p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}}$,

(v) $p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}s_{i-2}L_{i-2}s_{i-2}p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}}$,

(vi) $p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}} = s_{i-1}s_i s_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-1}p_{i-\frac{3}{2}}s_{i-1}s_i s_{i-1}p_{i-\frac{3}{2}}$,

(vii) $p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}$,

(viii) $p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}}\sigma_{i-1}s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}$,

(ix) $s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_i p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_i p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}}$,

(x) $s_i p_{i-\frac{1}{2}}\sigma_{i-1}s_i p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_i p_{i-\frac{1}{2}}s_i p_{i-\frac{3}{2}}$.

For the left hand side of (i),

$$s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$$
$$= s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-\frac{1}{2}}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}},$$

and for the right hand side of (i),

$$s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_{i-1}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$$
$$= s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{1}{2}}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}}s_{i-2}p_{i-\frac{1}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$$
$$= s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}.$$

For the left hand side of (ii),

$$s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$$
$$= s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}}s_{i-2}p_{i-\frac{1}{2}}L_{i-2}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$$
$$= s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}},$$

and for the right hand side of (ii),

$$s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_{i-1}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$$
$$= s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}p_i p_{i-\frac{1}{2}}$$
$$= s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = s_i p_{i-\frac{3}{2}}L_{i-2}p_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}}$$
$$= s_i p_{i-\frac{3}{2}}L_{i-2}p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}p_{i+\frac{1}{2}} = p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.$$

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For the left hand side of (iii),

$$s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = s_i \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}}$$
$$= s_i \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = \sigma_{i-1} s_i s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}}$$
$$= \sigma_{i-1} p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} s_i p_{i-1} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}}$$
$$= s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} = s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}},$$

and for the right hand side of (iii),

$$s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} = s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}}$$
$$= s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} = s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}}.$$

For the left hand side of (iv),

$$p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i$$
$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}},$$

and, for the right hand side of (iv),

$$p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i$$
$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}}.$$

For the left hand side of (v),

$$p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i,$$

and, for the right hand side of (v),

$$p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i$$
$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i.$$

For the left hand side of (vi),

$$p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i$$
$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_i s_i p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i$$
$$= p_{i-\frac{1}{2}} s_{i-1} s_i p_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} p_{i-1} s_i L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i$$
$$= p_{i-\frac{1}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} p_i s_{i-1} s_i L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i = p_{i-\frac{1}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} p_i s_{i-1} s_i p_{i-\frac{1}{2}} L_{i-2} p_{i-\frac{3}{2}} s_i$$
$$= s_i s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_i L_{i-2} p_{i-\frac{3}{2}} s_i = s_i s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} s_i L_{i-2} p_{i-\frac{3}{2}},$$

and for the right hand side of (vi),

$$s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} = s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} s_i p_{i-\frac{3}{2}}$$
$$= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} s_i = s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} s_i$$
$$= s_{i-1} s_i s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_i = s_i s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}}.$$

For the left hand side of (vii)

$$p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}}$$

and for the right hand side of (vii)

$$p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}}.$$

For the left hand side of (viii),

$$p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}$$
$$= p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}$$
$$= p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}},$$

and for the right hand side of (viii),

$$p_{i-\frac{1}{2}}\sigma_{i-1}s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}}\sigma_{i-1}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}} = p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.$$

For the left hand side of (ix),

$$s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_ip_{i-\frac{1}{2}}s_ip_{i-\frac{3}{2}} = s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_i$$
$$= s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_i$$

and for the right hand side of (ix),

$$s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_ip_{i-\frac{3}{2}} = s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_i.$$

For the left hand side of (x),

$$s_ip_{i-\frac{1}{2}}\sigma_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i-\frac{3}{2}} = s_ip_{i-\frac{1}{2}}\sigma_{i-1}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_i = p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}},$$

and for the right hand side of (x),

$$s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_ip_{i-\frac{1}{2}}s_ip_{i-\frac{3}{2}} = s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_i$$
$$= s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_i = p_{i-\frac{3}{2}}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.$$

Now, after substituting the expression

$$L_{i-1} = -p_{i-\frac{3}{2}}L_{i-2}s_{i-2} - s_{i-2}L_{i-2}p_{i-\frac{3}{2}} + p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}} - s_{i-1}L_{i-2}s_{i-1} + \sigma_{i-1}$$

into the definition (3.2), items (i)–(x) above imply that

$$\sigma_{i+1}p_{i-\frac{3}{2}} = s_{i-1}s_is_{i-2}s_{i-1}\sigma_{i-1}s_{i-1}s_{i-2}s_is_{i-1}p_{i-\frac{3}{2}} + p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}$$
$$+ s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}s_i - s_{i-1}s_is_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{3}{2}}s_is_{i-1}p_{i-\frac{3}{2}}$$
$$- s_{i-1}s_ip_{i-\frac{3}{2}}p_{i-1}p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}s_is_{i-1}p_{i-\frac{3}{2}}.$$

Since the right hand side of the above expression is fixed under the $*$ anti-involution on $\mathcal{A}_{i+1}(z)$, it follows that $\sigma_{i+1}p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}}\sigma_{i+1}$.

(4) We show that after substituting the expression

$$L_{i-1} = -p_{i-\frac{3}{2}}L_{i-2}s_{i-2} - s_{i-2}L_{i-2}p_{i-\frac{3}{2}} + p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}} + s_{i-2}L_{i-2}s_{i-2} + \sigma_{i-1}$$

into the definition (3.2), conjugation by s_{i-2} permutes the summands of σ_{i+1} as follows:

(i) $(s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}})^{s_{i-2}} = s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}$,

(ii) $(s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}})^{s_{i-2}} = p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{1}{2}}s_i$,

(iii) $(s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}})^{s_{i-2}} = s_ip_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_ip_{i-\frac{1}{2}}s_i$,

(iv) $(s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}})^{s_{i-2}} = s_{i-1}s_is_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{3}{2}}s_is_{i-1}$,

(v) $(s_ip_{i-\frac{1}{2}}\sigma_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}})^{s_{i-2}} = p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}$,

(vi) $(p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}})^{s_{i-2}} = p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_ip_{i-\frac{1}{2}}$,

(vii) $(s_ip_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_ip_{i-\frac{1}{2}}s_i)^{s_{i-2}} = s_{i-1}s_is_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-1}p_{i-\frac{3}{2}}s_{i-1}s_is_{i-1}$,

(viii) $(s_ip_{i-\frac{1}{2}}\sigma_{i-1}s_ip_{i-\frac{1}{2}}s_i)^{s_{i-2}} = p_{i-\frac{1}{2}}\sigma_{i-1}s_ip_{i-\frac{1}{2}}$,

(ix) $(s_{i-1}s_ip_{i-\frac{3}{2}}L_{i-2}s_{i-1}p_{i-\frac{3}{2}}s_is_{i-1})^{s_{i-2}} = p_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_ip_{i-\frac{1}{2}}$,

(x) $(s_{i-1}s_is_{i-2}s_{i-1}\sigma_{i-1}s_{i-1}s_{i-2}s_is_{i-1})^{s_{i-2}} = s_{i-1}s_is_{i-2}s_{i-1}\sigma_{i-1}s_{i-1}s_{i-2}s_is_{i-1}$.

The item (i) follows from the relation $s_{i-2}p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}}s_{i-2} = p_{i-\frac{3}{2}}$.

For the left hand side of (ii),

$$
\begin{aligned}
(s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_{i-2}} &= s_{i-2} s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= s_{i-2} s_i p_{i-\frac{1}{2}} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= s_{i-2} s_i s_{i-2} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= s_i s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{1}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{1}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{1}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{1}{2}} s_i s_{i-1} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{1}{2}} s_i \sigma_{i-1} p_{i-\frac{1}{2}} s_{i-2},
\end{aligned}
$$

and, for the right hand side of (ii),

$$
\begin{aligned}
p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-2} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\
&= p_{i-\frac{1}{2}} p_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-2} \sigma_{i-1} p_{i-\frac{1}{2}} s_i = p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} p_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\
&= p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} p_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i = p_{i-\frac{1}{2}} p_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\
&= p_{i-\frac{1}{2}} p_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i = p_{i-\frac{1}{2}} p_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} s_i \\
&= p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} s_i = p_{i-\frac{1}{2}} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} s_i \\
&= p_{i-\frac{1}{2}} s_{i-2} s_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} \sigma_{i-1} = p_{i-\frac{1}{2}} s_{i-2} s_i p_{i-\frac{1}{2}} \sigma_{i-1} = p_{i-\frac{1}{2}} s_i \sigma_{i-1} p_{i-\frac{1}{2}} s_{i-2},
\end{aligned}
$$

where the last equality follows from $s_{i-2} p_{i-\frac{1}{2}} \sigma_{i-1} = \sigma_{i-1} p_{i-\frac{1}{2}} s_{i-2}$.

For the left hand side of (iii),

$$
\begin{aligned}
(s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_{i-2}} &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2} \\
&= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} s_{i-2} \\
&= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
&= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_i p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
&= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} p_i p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
&= s_i p_{i-\frac{3}{2}} L_{i-2} p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
&= s_i p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-2} \\
&= p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-2} \\
&= p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{3}{2}} p_{i-1} \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
&= p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2},
\end{aligned}
$$

and for the right hand side of (iii),

$$s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i = s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i$$

$$= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} p_{i-\frac{1}{2}} s_i$$

$$= p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} s_i = p_{i-\frac{3}{2}} p_{i-1} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} s_i$$

$$= p_{i-\frac{3}{2}} p_{i-1} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} s_i = p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} s_i$$

$$= p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_i s_{i-2} s_i = p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2}.$$

From the left hand side of (iv), we obtain

$$(s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_{i-2}} = (s_i p_{i-\frac{1}{2}} s_{i-2} s_{i-1} L_{i-2} s_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_{i-2}}$$

$$= (s_i s_{i-2} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_{i-2}}$$

$$= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2}$$

$$= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} s_{i-2} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2}$$

$$= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i s_{i-2} p_{i-\frac{1}{2}} s_{i-2}$$

$$= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1}$$

$$= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i s_{i-1} p_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1}$$

$$= s_i s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{3}{2}} s_{i-1}$$

$$= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{3}{2}} s_{i-1}$$

$$= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1}$$

which is identical to the right hand side of (iv).
From the left hand side of (v), we obtain

$$(s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_{i-2}} = s_{i-2} s_i \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2}$$

$$= s_i \sigma_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} \sigma_{i-\frac{3}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}}$$

$$= p_{i-\frac{1}{2}} \sigma_{i-\frac{3}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-2} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}}$$

$$= p_{i-\frac{1}{2}} \sigma_{i-1} p_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}},$$

which is identical to the right hand side of (v).
The item (vi) follows immediately from the relation $s_{i-2} p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}} s_{i-2} = p_{i-\frac{3}{2}}$.
From the left hand side of (vii), we obtain

$$(s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i)^{s_{i-2}} = s_{i-2} s_i s_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i s_{i-2}$$

$$= s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i s_{i-2} = s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2}$$

$$= s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i = s_i s_{i-1} p_{i-\frac{3}{2}} s_i L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} s_i$$

$$= s_i s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} s_i = s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1},$$

which is identical to the right hand side of (vii).
From the left hand side of (viii),

$$(s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_i)^{s_{i-2}} = s_{i-2} s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2}$$

$$= s_{i-2} s_i \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} = \sigma_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} s_{i-2}$$

$$= \sigma_{i-\frac{3}{2}} s_i s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} = \sigma_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i s_{i-1} s_{i-1} s_{i-2} p_{i-\frac{1}{2}}$$

$$= \sigma_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i s_{i-2} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} \sigma_{i-\frac{3}{2}} s_{i-2} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}},$$

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which is identical to the right hand side of (viii).

From the left hand side of (ix),

$$(s_{i-1}s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1})^{s_{i-2}} = s_{i-2} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} s_{i-2}$$

$$= p_{i-\frac{1}{2}} s_{i-2} s_{i-1} L_{i-2} s_i s_{i-1} s_i s_{i-1} s_{i-2} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-1} s_i s_{i-1} s_i s_{i-1} s_{i-2} p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_i s_{i-2} p_{i-\frac{1}{2}},$$

which is identical to the right hand side of (ix).

The equality (x) follows from the fact that $s_{i-2} s_{i-1} s_i s_{i-2} s_{i-1} = s_{i-1} s_i s_{i-2} s_{i-1} s_i$.

(5) By (2) and (4), σ_{i+1} commutes with $\langle s_{i-2}, p_{i-1} \rangle$, and so with $p_{i-2} = s_{i-2} p_{i-1} s_{i-2}$.

(6) By Proposition 3.3, and (2),

$$\sigma_{i+\frac{1}{2}} p_{i-1} = s_i \sigma_{i+1} p_{i-1} = p_{i-1} s_i \sigma_{i+1} = p_{i-1} \sigma_{i+\frac{1}{2}}.$$

(7)–(9) Can be proved using the same argument as part (6). $\qquad\square$

Theorem 3.6. *The elements L_{i+1} and $L_{i+\frac{1}{2}}$ satisfy the following commutation relations:*

(1) $L_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} L_{i+1} = p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}}$, *for $i = 1, 2, \ldots$.*

(2) $L_{i+1} p_i = p_i L_{i+1}$, *for $i = 1, 2, \ldots$.*

(3) $L_{i+1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} L_{i+1}$, *for $i = 2, 3, \ldots$.*

(4) $L_{i+1} s_{i-1} = s_{i-1} L_{i+1}$, *for $i = 2, 3, \ldots$.*

(5) $L_{i+1} p_{i-1} = p_{i-1} L_{i+1}$, *for $i = 2, 3 \ldots$.*

(6) $L_{i+\frac{1}{2}} p_i = p_i L_{i+\frac{1}{2}}$, *for $i = 1, 2, \ldots$.*

(7) $L_{i+\frac{1}{2}} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} L_{i+\frac{1}{2}}$, *for $i = 2, 3, \ldots$.*

(8) $L_{i+\frac{1}{2}} s_{i-1} = s_{i-1} L_{i+\frac{1}{2}}$, *for $i = 2, 3, \ldots$.*

(9) $L_{i+\frac{1}{2}} p_{i-1} = p_{i-1} L_{i+\frac{1}{2}}$, *for $i = 2, 3 \ldots$.*

Proof. (1) Using Proposition 3.1,

$$L_{i+1} p_{i+\frac{1}{2}} = -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i s_i p_{i+\frac{1}{2}} + \sigma_{i+1} p_{i+\frac{1}{2}}$$

$$= -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}$$

$$= -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}$$

$$= p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}},$$

as required.

(2) From Proposition 3.1, we obtain $\sigma_{i+1} p_i = \sigma_{i+1} p_i p_{i+\frac{1}{2}} p_i = s_i L_i p_{i+\frac{1}{2}} p_i$. Thus

$$L_{i+1} p_i = -s_i L_i p_{i+\frac{1}{2}} p_i - p_{i+\frac{1}{2}} L_i s_i p_i + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} p_i + s_i L_i s_i p_i + \sigma_{i+1} p_i$$

$$= (\sigma_{i+1} p_i - s_i L_i p_{i+\frac{1}{2}} p_i) + (p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} p_i - p_{i+\frac{1}{2}} L_i s_i p_i) + s_i L_i p_{i+1} s_i$$

$$= (\sigma_{i+1} p_i - \sigma_{i+1} p_i) + (p_{i+\frac{1}{2}} L_i s_i p_i - p_{i+\frac{1}{2}} L_i s_i p_i) + s_i L_i p_{i+1} s_i,$$

and the statement now follows from the fact that the right hand side of the above expression is fixed under the $*$ anti-involution on $\mathcal{A}_{i+1}(z)$.

(3) We first show that

(i) $p_{i+\frac{1}{2}} L_i s_i p_{i-\frac{1}{2}} = p_{i+\frac{1}{2}} \sigma_i s_i$,

(ii) $s_i L_i p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$,

(iii) $p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$,

(iv) $\sigma_{i+1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} \sigma_{i+1}$.

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(i) The definition (3.1) gives

$$
\begin{aligned}
p_{i+\frac{1}{2}}L_is_ip_{i-\frac{1}{2}} &= -p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_ip_{i-\frac{1}{2}} \\
&\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}s_ip_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}\sigma_is_ip_{i-\frac{1}{2}} \\
&= -p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i+\frac{1}{2}}s_{i-1}s_i \\
&\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i+\frac{1}{2}}s_{i-1}s_i + p_{i+\frac{1}{2}}\sigma_is_ip_{i-\frac{1}{2}} \\
&= -p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i \\
&\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i + p_{i+\frac{1}{2}}\sigma_is_ip_{i-\frac{1}{2}} \\
&= p_{i+\frac{1}{2}}\sigma_is_i,
\end{aligned}
$$

where the last equality follows from Proposition 3.1.

(ii) The definition (3.1) gives

$$
\begin{aligned}
s_iL_ip_{i+\frac{1}{2}}p_{i-\frac{1}{2}} &= -s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
&\quad + s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} + s_is_{i-1}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} + s_i\sigma_ip_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
&= -s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - s_ip_{i-\frac{1}{2}}L_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
&\quad + s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} + s_is_{i-1}L_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
&= s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}}
\end{aligned}
$$

since $p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}$.

(iii) The definition (3.1) gives

$$
\begin{aligned}
p_{i+\frac{1}{2}}L_ip_{i+1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} &= -p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
&\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
&\quad + p_{i+\frac{1}{2}}\sigma_ip_{i+1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
&= -p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} \\
&\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} \\
&\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} \\
&= -p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} \\
&\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} \\
&= p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}.
\end{aligned}
$$

(iv) Was demonstrated in Theorem 3.5.
Now, using

$$
\begin{aligned}
s_iL_{i-1}s_{i-1}p_{i-\frac{1}{2}} &= -s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_ip_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} \\
&\quad + s_is_{i-1}L_{i-1}s_{i-1}s_ip_{i-\frac{1}{2}} + s_i\sigma_is_ip_{i-\frac{1}{2}} \\
&= -s_is_{i-1}L_{i-1}p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i + s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \\
&\quad + s_is_{i-1}L_{i-1}s_{i-1}s_ip_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}\sigma_is_i,
\end{aligned}
$$

we obtain

$$L_{i+1}p_{i-\frac{1}{2}} = -s_iL_ip_{i+\frac{1}{2}}p_{i-\frac{1}{2}} - p_{i+\frac{1}{2}}L_is_ip_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}L_ip_{i+1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} + s_iL_is_ip_{i-\frac{1}{2}} + \sigma_{i+1}p_{i-\frac{1}{2}}$$

$$= -s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}\sigma_is_i + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}$$

$$+ s_iL_is_ip_{i-\frac{1}{2}} + \sigma_{i+1}p_{i-\frac{1}{2}}$$

$$= p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} + s_is_{i-1}L_{i-1}s_{i-1}s_ip_{i-\frac{1}{2}} + \sigma_{i+1}p_{i-\frac{1}{2}}.$$

Since the right hand side of the last expression is fixed under the $*$ anti-involution on $\mathcal{A}_{i+1}(z)$, the proof of (3) is complete.

(4) We show that after substituting the expression

$$L_i = -p_{i-\frac{1}{2}}L_{i-1}s_{i-1} - s_{i-1}L_{i-1}p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}} + s_{i-1}L_{i-1}s_{i-1} + \sigma_i$$

into the definition (3.1), conjugation by s_{i-1} permutes the summands of L_{i+1} as follows:

(i) $s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_is_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i,$

(ii) $s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_is_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}},$

(iii) $s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}s_is_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+1}p_{i+\frac{1}{2}},$

(iv) $s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}s_is_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i,$

(v) $s_{i-1}p_{i+\frac{1}{2}}\sigma_is_is_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}},$

(vi) $s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}},$

(vii) $s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}s_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}s_i,$

(viii) $s_{i-1}p_{i+\frac{1}{2}}\sigma_ip_{i+1}p_{i+\frac{1}{2}}s_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_i,$

(ix) $s_{i-1}s_is_{i-1}L_{i-1}s_{i-1}s_is_{i-1} = s_is_{i-1}L_{i-1}s_{i-1}s_i,$

(x) $s_{i-1}p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}.$

For item (i),

$$s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_is_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_is_{i-1}s_i = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i.$$

For item (ii),

$$s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_is_{i-1} = s_ip_{i-\frac{1}{2}}s_iL_{i-1}p_{i-\frac{1}{2}}s_is_{i-1}$$

$$= s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_is_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}.$$

For item (iii),

$$s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}s_is_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_is_is_{i-1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}.$$

For item (iv),

$$s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}s_is_{i-1} = s_ip_{i-\frac{1}{2}}s_iL_{i-1}s_{i-1}s_is_{i-1}$$

$$= s_ip_{i-\frac{1}{2}}L_{i-1}s_is_{i-1}s_is_{i-1} = s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i.$$

For the left hand side of item (v),

$$s_{i-1}p_{i+\frac{1}{2}}\sigma_is_is_{i-1} = s_{i-1}\sigma_is_ip_{i-\frac{1}{2}}s_{i-1} = \sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}},$$

and for the right hand side of item (v),

$$s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}} = s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_is_{i-1}p_ip_{i-\frac{1}{2}} = s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i-1}p_{i-\frac{1}{2}}$$

$$= s_ip_{i-\frac{1}{2}}s_iL_{i-1}p_{i-\frac{1}{2}}p_{i-1}s_ip_{i-\frac{1}{2}} = s_ip_{i-\frac{1}{2}}s_i\sigma_ip_ip_{i-\frac{1}{2}}p_{i-1}s_ip_{i-\frac{1}{2}} = s_ip_{i-\frac{1}{2}}s_i\sigma_is_{i-1}p_{i-1}s_ip_{i-\frac{1}{2}}$$

$$= s_is_i\sigma_ip_{i+\frac{1}{2}}s_{i-1}s_ip_{i-1}p_{i-\frac{1}{2}} = \sigma_is_{i-1}s_ip_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{1}{2}} = \sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}.$$

The item (vi) follows from the relation $s_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}$. For item (vii),

$$s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}s_{i-1} = s_ip_{i-\frac{1}{2}}s_iL_{i-1}p_{i+1}s_ip_{i-\frac{1}{2}}s_i = s_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}s_i.$$

For item (viii),

$$s_{i-1}p_{i+\frac{1}{2}}\sigma_ip_{i+1}p_{i+\frac{1}{2}}s_{i-1} = s_{i-1}\sigma_is_ip_{i-\frac{1}{2}}s_ip_{i+1}p_{i+\frac{1}{2}}s_{i-1} = \sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}$$

$$= \sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}p_is_{i-1}s_ip_{i-\frac{1}{2}}s_i = \sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}p_{i-1}s_ip_{i-\frac{1}{2}}s_i = \sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}s_ip_{i-1}p_{i-\frac{1}{2}}s_i$$

$$= \sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}s_ip_{i-1}p_{i-\frac{1}{2}}s_i = \sigma_{i-\frac{1}{2}}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}p_{i-1}p_{i-\frac{1}{2}}s_i = s_{i-1}p_{i+\frac{1}{2}}s_{i-1}\sigma_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{1}{2}}s_i$$

$$= s_ip_{i-\frac{1}{2}}s_i\sigma_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{1}{2}}s_i = s_ip_{i-\frac{1}{2}}s_iL_{i-1}p_{i-\frac{1}{2}}s_i.$$

(ix) Follows from the Coxeter relations, and (x) from the relation $s_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}$.

(5) By parts (2) and (4), L_{i+1} commutes with $\langle p_i, s_{i-1}\rangle$, and so with $p_{i-1} = s_{i-1}p_is_{i-1}$.

(6) From item (3) of Proposition 3.1, $\sigma_{i+\frac{1}{2}}p_i = s_i\sigma_{i+1}p_ip_{i+\frac{1}{2}}p_i = L_ip_{i+\frac{1}{2}}p_i$, and

$$L_{i+\frac{1}{2}}p_i = -p_{i+\frac{1}{2}}L_ip_i - L_ip_{i+\frac{1}{2}}p_i + p_{i+\frac{1}{2}}L_ip_ip_{i+\frac{1}{2}}p_i + s_iL_{i-\frac{1}{2}}s_ip_i + \sigma_{i+\frac{1}{2}}p_i$$

$$= -p_{i+\frac{1}{2}}L_ip_i - L_ip_{i+\frac{1}{2}}p_i + p_{i+\frac{1}{2}}L_ip_i + s_iL_{i-\frac{1}{2}}p_{i+1}s_i + L_ip_{i+\frac{1}{2}}p_i$$

$$= s_iL_{i-\frac{1}{2}}p_{i+1}s_i.$$

Since $s_iL_{i-\frac{1}{2}}p_{i+1}s_i = s_ip_{i+1}L_{i-\frac{1}{2}}s_i$, this completes the proof of (6).

(7) We show that

$$(3.6)\quad L_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = -p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}$$

$$+ s_is_{i-1}L_{i-\frac{3}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i + s_is_{i-1}\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i.$$

From the definition (3.3),

$$(3.7)\qquad L_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = (-L_ip_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}L_i + p_{i+\frac{1}{2}}L_ip_ip_{i+\frac{1}{2}} + s_iL_{i-\frac{1}{2}}s_i + \sigma_{i+\frac{1}{2}})p_{i-\frac{1}{2}}.$$

Using part (1) the first three summands in the right hand side of (3.7) are transformed as:

$$L_ip_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \qquad \text{and} \qquad p_{i+\frac{1}{2}}L_ip_{i-\frac{1}{2}} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}},$$

and

$$p_{i+\frac{1}{2}}L_ip_ip_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = p_{i+\frac{1}{2}}p_iL_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}L_{i-1}p_ip_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$

$$= p_{i+\frac{1}{2}}p_iL_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} = L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}.$$

Next,

$$s_iL_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} = -s_iL_{i-1}p_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}} - s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}p_{i-1}p_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}$$

$$+ s_is_{i-1}L_{i-\frac{3}{2}}s_{i-1}s_ip_{i-\frac{1}{2}} + s_i\sigma_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}$$

$$= -s_iL_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + s_ip_{i-\frac{1}{2}}L_{i-1}p_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$

$$+ s_is_{i-1}L_{i-\frac{3}{2}}s_{i-1}s_ip_{i-\frac{1}{2}} + s_is_{i-1}\sigma_{i-\frac{1}{2}}s_{i-1}s_ip_{i-\frac{1}{2}}$$

$$= -L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}p_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$$

$$+ s_is_{i-1}L_{i-\frac{3}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i + s_is_{i-1}\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i.$$

Substituting each of the above into (3.7), and using part (1) of Theorem 3.5 gives (3.6). The right hand side of (3.6) being fixed under the $*$ anti-involution on $\mathcal{A}_{i+\frac{1}{2}}(z)$, the proof of (7) is complete.

(8) We show that, after substituting the expression

$$L_{i-\frac{1}{2}} = -p_{i-\frac{1}{2}}L_{i-1} - L_{i-1}p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}}L_{i-1}p_{i-1}p_{i-\frac{1}{2}} + s_{i-1}L_{i-\frac{1}{2}}s_{i-1} + \sigma_{i-\frac{1}{2}}$$

into the definition (3.3), conjugation by s_{i-1} permutes the summands of $L_{i+\frac{1}{2}}$ as follows:

(i) $s_{i-1}(p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1})s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}$,

(ii) $s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_{i-1} = s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}$,

(iii) $s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}$,

(iv) $s_{i-1}(p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1})s_{i-1} = s_i p_{i-\frac{1}{2}}L_{i-1}s_i$,

(v) $s_{i-1}p_{i+\frac{1}{2}}\sigma_i s_{i-1} = \sigma_i p_{i+\frac{1}{2}}$,

(vi) $s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_i p_{i+\frac{1}{2}}s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_i p_{i+\frac{1}{2}}$,

(vii) $s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1} = s_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i$,

(viii) $s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}p_i p_{i+\frac{1}{2}}s_{i-1} = s_i p_{i-\frac{1}{2}}L_{i-1}p_{i-1}p_{i-\frac{1}{2}}s_i$,

(ix) $s_{i-1}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}$.

The left hand side of (i) is

$$s_{i-1}(p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1})s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}$$

and the right hand side of (i) is

$$p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}.$$

The left hand side of (ii) is

$$s_{i-1}p_{2+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_{i-1} = s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}},$$

which is the same as the right hand side of (ii). The left hand side of (iii) is

$$s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}},$$

which is the same as the right hand side of (iii). The left hand side of (iv) is

$$s_{i-1}(p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1})s_{i-1} = s_i p_{i-\frac{1}{2}}s_i L_{i-1} = s_i p_{i-\frac{1}{2}}L_{i-1}s_i,$$

which is the same as the right hand side of (iv). The left hand side of (v) is

$$s_{i-1}p_{i+\frac{1}{2}}\sigma_i s_{i-1} = s_{i-1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}} = s_{i-1}\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}} = \sigma_i p_{i+\frac{1}{2}},$$

which is the same as the right hand side of (v). The left hand side of (vi) is

$$s_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_i p_{i+\frac{1}{2}}s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_i s_{i-1}s_i p_{i-\frac{1}{2}}s_i$$
$$= p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i-1}s_i p_{i-\frac{1}{2}}s_i$$
$$= p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i-1}p_{i-\frac{1}{2}},$$

which is the same as the right hand side of (vi). The left hand side of (vii) is

$$s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1} = s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}$$
$$= s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1} = s_i p_{i-\frac{1}{2}}p_i \sigma_i s_{i-1}p_{i+\frac{1}{2}}s_{i-1}s_i p_i p_{i+\frac{1}{2}}s_{i-1}$$
$$= s_i p_{i-\frac{1}{2}}p_i \sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i p_i s_{i-1}s_i p_{i-\frac{1}{2}}s_i = s_i p_{i-\frac{1}{2}}p_i \sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}s_i$$
$$= s_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_i p_{i-\frac{1}{2}}s_i = s_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i,$$

which is the same as the right hand side of (vii). The left hand side of (viii) is

$$s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}p_i p_{i+\frac{1}{2}}s_{i-1} = s_i p_{i-\frac{1}{2}}s_i L_{i-1}s_{i-1}p_i s_{i-1}s_i p_{i-\frac{1}{2}}s_i$$
$$= s_i p_{i-\frac{1}{2}}L_{i-1}s_i s_{i-1}p_i s_{i-1}s_i p_{i-\frac{1}{2}}s_i$$
$$= s_i p_{i-\frac{1}{2}}L_{i-1}p_{i-1}p_{i-\frac{1}{2}}s_i,$$

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which is the same as the right hand side of (viii). Since the statement (ix) is evident from the relation $s_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}$, the proof of part (7) is complete.

(9) By parts (6) and (8), $L_{i+\frac{1}{2}}$ commutes with $\langle p_i, s_{i-1}\rangle$, and so with $p_{i-1} = s_{i-1}p_i s_{i-1}$. $\qquad\square$

Theorem 3.7. *If $i = 1, 2, \ldots$, then*

> *(1) L_{i+1} commutes with $\mathcal{A}_{i+\frac{1}{2}}(z)$, and σ_{i+1} commutes with $\mathcal{A}_{i-\frac{1}{2}}(z)$,*
> *(2) $L_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_i(z)$, and $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-1}(z)$.*

Consequently, the family of elements $(L_i, L_{i+\frac{1}{2}} : i = 0, 1, \ldots)$ is pairwise commutative.

Proof. (1) Observe that L_1 commutes with $\mathcal{A}_{1-\frac{1}{2}}(z)$ and L_2 commutes with $\mathcal{A}_{1+\frac{1}{2}}(z)$, while σ_2 commutes with $\mathcal{A}_{1-\frac{1}{2}}(z)$ and σ_3 commutes with $\mathcal{A}_{2-\frac{1}{2}}(z)$. Since

$$L_{i+1} \text{ commutes with } \langle s_{i-1}, p_{i-1}, p_i, p_{i-\frac{1}{2}}, p_{i+\frac{1}{2}}\rangle, \text{ if } i \geqslant 2, \text{ and}$$

$$\sigma_{i+1} \text{ commutes with } \langle s_{i-2}, p_{i-2}, p_{i-1}, p_{i-\frac{3}{2}}, p_{i-\frac{1}{2}}\rangle, \text{ if } i \geqslant 3,$$

it suffices to show that

> (i) L_{i+1} commutes with $\mathcal{A}_{i-\frac{3}{2}}(z)$, if $i \geqslant 2$, and
> (ii) σ_{i+1} commutes with $\mathcal{A}_{i-\frac{5}{2}}(z)$, if $i \geqslant 3$.

If $i \geqslant 2$, then induction on i shows that L_{i+1} commutes with $\mathcal{A}_{i-2}(z)$, while, if $i \geqslant 3$, the fact that L_{i+1} commutes with $p_{i-\frac{3}{2}}$ follows from induction on i, and the fact that σ_{i+1} commutes with $p_{i-\frac{3}{2}}$. Similarly, if $i \geqslant 3$, then induction on i shows that σ_{i+1} commutes with $\mathcal{A}_{i-3}(z)$, and that, if $i \geqslant 4$, then σ_{i+1} commutes with $p_{i-\frac{5}{2}}$.

(2) Observe that $L_{0+\frac{1}{2}}$ commutes with $\mathcal{A}_0(z)$ and $L_{1+\frac{1}{2}}$ commutes with $\mathcal{A}_1(z)$, while $\sigma_{1+\frac{1}{2}}$ commutes with $\mathcal{A}_0(z)$ and $\sigma_{2+\frac{1}{2}}$ commutes with $\mathcal{A}_1(z)$. Since

$$L_{i+\frac{1}{2}} \text{ commutes with } \langle s_{i-1}, p_{i-1}, p_i, p_{i-\frac{1}{2}}\rangle, \text{ if } i \geqslant 2, \text{ and}$$

$$\sigma_{i+\frac{1}{2}} \text{ commutes with } \langle s_{i-2}, p_{i-2}, p_{i-1}, p_{i-\frac{3}{2}}\rangle, \text{ if } i \geqslant 3,$$

it suffices to show that

> (i) $L_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-\frac{3}{2}}(z)$, if $i \geqslant 2$, and
> (ii) $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-\frac{5}{2}}(z)$, if $i \geqslant 3$.

If $i \geqslant 2$, then induction on i shows that $L_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-2}(z)$, while, if $i \geqslant 3$, the fact that $L_{i+\frac{1}{2}}$ commutes with $p_{i-\frac{3}{2}}$ follows from induction on i, and the fact that $\sigma_{i+\frac{1}{2}}$ commutes with $p_{i-\frac{3}{2}}$. Similarly, if $i \geqslant 3$, then induction on i shows that $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-3}(z)$, and that, if $i \geqslant 4$, then $\sigma_{i+\frac{1}{2}}$ commutes with $p_{i-\frac{5}{2}}$. $\qquad\square$

Proposition 3.8. *For $i = 1, 2, \ldots$, the following statements hold:*

> *(1) $(L_{i+\frac{1}{2}} + L_{i+1})p_{i+1} = p_{i+1}(L_{i+\frac{1}{2}} + L_{i+1}) = zp_{i+1}$,*
> *(2) $(L_i + L_{i+\frac{1}{2}})p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}(L_i + L_{i+\frac{1}{2}}) = zp_{i+\frac{1}{2}}$,*
> *(3) $(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1})s_i = s_i(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1})$.*

Proof. (1) It suffices to observe that:

> (i) $L_i p_{i+\frac{1}{2}} p_{i+1} = \sigma_{i+1} p_{i+1}$,
> (ii) $p_{i+\frac{1}{2}} L_i p_{i+1} = p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} p_{i+1}$,
> (iii) $p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} p_{i+1} = p_{i+\frac{1}{2}} L_i s_i p_{i+1}$,
> (iv) $s_i L_{i-\frac{1}{2}} s_i p_{i+1} + s_i L_i s_i p_{i+1} = zp_{i+1}$,
> (v) $\sigma_{i+\frac{1}{2}} p_{i+1} = s_i L_i p_{i+\frac{1}{2}} p_{i+1}$.

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With the exception of (iv), each of the statements above is evident from the defining relations or from what we have already shown. Since $(L_{\frac{1}{2}} + L_1)p_1 = zp_1$,

(3.8) $\qquad s_i L_{i-\frac{1}{2}} s_i p_{i+1} + s_i L_i s_i p_{i+1} = s_i(L_{i-\frac{1}{2}} p_i + L_i p_i)s_i = zs_i p_i s_i = zp_{i+1}$,

gives (iv) by induction.

(2) Using the expression (3.8), we have

$$
\begin{aligned}
zp_{i+\frac{1}{2}} &= zp_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
&= p_{i+\frac{1}{2}}(s_i L_{i-\frac{1}{2}} s_i p_{i+1} + s_i L_i s_i p_{i+1}) p_{i+\frac{1}{2}} \\
&= p_{i+\frac{1}{2}} L_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i s_i p_{i+1} p_{i+\frac{1}{2}} \\
&= L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} \\
&= L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}},
\end{aligned}
$$

which yields

$$
\begin{aligned}
(L_i + L_{i+\frac{1}{2}})p_{i+\frac{1}{2}} &= L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} - L_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} \\
&\quad + s_i L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + \sigma_{i+\frac{1}{2}} p_{i+\frac{1}{2}} \\
&= -p_{i+\frac{1}{2}} + zp_{i+\frac{1}{2}} - L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} \\
&= zp_{i+\frac{1}{2}}.
\end{aligned}
$$

(3) Since $\sigma_{i+1} s_i = \sigma_{i+\frac{1}{2}} = s_i \sigma_{i+1}$, we have

$$
\begin{aligned}
(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1})s_i &= L_{i-\frac{1}{2}} s_i + L_i s_i - p_{i+\frac{1}{2}} L_i s_i - L_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} \\
&\quad + s_i L_{i-\frac{1}{2}} + \sigma_{i+\frac{1}{2}} s_i - p_{i+\frac{1}{2}} L_i - s_i L_i p_{i+\frac{1}{2}} \\
&\quad + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i + \sigma_{i+1} s_i \\
&= s_i(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1}),
\end{aligned}
$$

as required. $\qquad \square$

Theorem 3.9. *If* $k = 0, 1, \ldots$, *then*

(1) *the element* $z_{k+\frac{1}{2}} = L_{\frac{1}{2}} + L_1 + L_{1+\frac{1}{2}} + \cdots + L_{k+\frac{1}{2}}$ *is central in* $\mathcal{A}_{k+\frac{1}{2}}(z)$,

(2) *the element* $z_{k+1} = L_{\frac{1}{2}} + L_1 + L_{1+\frac{1}{2}} + \cdots + L_{k+1}$ *is central in* $\mathcal{A}_{k+1}(z)$.

Proof. (1) We first show that $z_{k+\frac{1}{2}}$ commutes with $p_1, \ldots p_k$ and $p_{1+\frac{1}{2}}, \ldots, p_{k+\frac{1}{2}}$. If $i = 1, \ldots, k$, then p_i commutes with z_{i-1} and with $L_{i+\frac{1}{2}} + L_{i+1} + \cdots + L_{k+\frac{1}{2}}$. Since p_i also commutes with $L_{i-\frac{1}{2}} + L_i$, it follows that p_i commutes with $z_{k+\frac{1}{2}}$. Similarly, since $p_{i+\frac{1}{2}}$ commutes with $z_{i-\frac{1}{2}}$, $L_i + L_{i+\frac{1}{2}}$, and with $L_{i+1} + L_{i+\frac{3}{2}} + \ldots L_{k+\frac{1}{2}}$, it follows that $p_{i+\frac{1}{2}}$ commutes with $z_{k+\frac{1}{2}}$. Next, suppose that $i = 1, \ldots, k-1$. Since s_i commutes with z_{i-1}, $L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1}$, and with $L_{i+\frac{3}{2}} + L_{i+2} + \cdots + L_{k+\frac{1}{2}}$, it is evident that s_i commutes with $z_{k+\frac{1}{2}}$. As $\mathcal{A}_{k+\frac{1}{2}}(z)$ is generated by $p_1, \ldots, p_k$, $p_{1+\frac{1}{2}}, \ldots, p_{k+\frac{1}{2}}$, and $s_1, \ldots, s_{k-1}$, we conclude that $z_{k+\frac{1}{2}}$ is central in $\mathcal{A}_{k+\frac{1}{2}}(z)$.

(2) Given that $z_{k+\frac{1}{2}}$ is central in $\mathcal{A}_{k+\frac{1}{2}}(z)$, it suffices to observe that s_k commutes with z_{k+1}. $\qquad \square$

Proposition 3.10. *For* $i = 1, 2, \ldots$, *the following statements hold:*

(1) $(L_i + L_{i+\frac{1}{2}} + L_{i+1})\sigma_{i+1} = \sigma_{i+1}(L_i + L_{i+\frac{1}{2}} + L_{i+1})$,

(2) $(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}})\sigma_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}})$.

Proof. (1) Since σ_{i+1} commutes with

$$
z_{i+1} = z_{i-1} + L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1},
$$

and σ_{i+1} also commutes with $z_{i-1} + L_{i-\frac{1}{2}} \in \mathcal{A}_{i-\frac{1}{2}}(z)$, it follows that σ_{i+1} commutes with $L_i + L_{i+\frac{1}{2}} + L_{i+1}$.

(2) Given that $\sigma_{1+\frac{1}{2}} = 1$, we may suppose that $i = 2, 3, \ldots$. Since $\sigma_{i+\frac{1}{2}}$ commutes with

$$z_{i+\frac{1}{2}} = z_{i-\frac{3}{2}} + L_{i-1} + L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}},$$

and $\sigma_{i+\frac{1}{2}}$ also commutes with $z_{i-\frac{3}{2}} + L_{i-1} \in \mathcal{A}_{i-1}(z)$, it follows that $\sigma_{i+\frac{1}{2}}$ commutes with $L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}}$. $\qquad\square$

4. A Presentation for the Partition Algebras

The goal here is to rewrite the presentation for $\mathcal{A}_k(z)$ given by [HR] in Theorem 2.1 in terms of the elements $\sigma_i, \sigma_{i+\frac{1}{2}}, p_i, p_{i+\frac{1}{2}}$.

Theorem 4.1. *If $k = 1, 2, \ldots$, then $\mathcal{A}_k(z)$ is the unital associative algebra presented by the generators*

$$p_1, p_2, \ldots, p_k, p_{1+\frac{1}{2}}, \ldots, p_{k-\frac{1}{2}}, \sigma_2, \sigma_3, \ldots, \sigma_k, \sigma_{2+\frac{1}{2}}, \sigma_{3+\frac{1}{2}}, \ldots, \sigma_{k-\frac{1}{2}},$$

and the relations:

- (1) *(Involutions)*
  - (a) $\sigma_{i+\frac{1}{2}}^2 = 1$, *for* $i = 2, \ldots, k-1$.
  - (b) $\sigma_{i+1}^2 = 1$, *for* $i = 1, \ldots, k-1$.
- (2) *(Braid–like relations)*
  - (a) $\sigma_{i+1}\sigma_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}}\sigma_{i+1}$, *if* $j \neq i+1$.
  - (b) $\sigma_i\sigma_j = \sigma_j\sigma_i$, *if* $j \neq i+1$.
  - (c) $\sigma_{i+\frac{1}{2}}\sigma_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}$, *if* $j \neq i+1$.
  - (d) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, *for* $i = 1, \ldots, k-2$, *if* $s_{j-1} = \sigma_{j-\frac{1}{2}}\sigma_j$, *for* $j = 2, \ldots, k$.
- (3) *(Idempotent relations)*
  - (a) $p_i^2 = zp_i$, *for* $i = 1, \ldots, k$.
  - (b) $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}^2$, *for* $i = 1, \ldots, k-1$.
  - (c) $\sigma_{i+1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+1} = p_{i+\frac{1}{2}}$, *for* $i = 1, \ldots, k-1$.
  - (d) $\sigma_{i+\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, *for* $i = 1, \ldots, k-1$.
  - (e) $\sigma_{i+\frac{1}{2}}p_ip_{i+1} = \sigma_{i+1}p_ip_{i+1}$, *for* $i = 1, \ldots, k-1$.
  - (f) $p_ip_{i+1}\sigma_{i+\frac{1}{2}} = p_ip_{i+1}\sigma_{i+1}$, *for* $i = 1, \ldots, k-1$.
- (4) *(Commutation relations)*
  - (a) $p_ip_j = p_jp_i$, *for* $i, j = 1, \ldots, k$.
  - (b) $p_{i+\frac{1}{2}}p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}}p_{i+\frac{1}{2}}$, *for* $i, j = 1, \ldots, k-1$.
  - (c) $p_{i+\frac{1}{2}}p_j = p_jp_{i+\frac{1}{2}}$, *for* $j \neq i, i+1$.
  - (d) $\sigma_ip_j = p_j\sigma_i$ *if* $j \neq i-1, i$.
  - (e) $\sigma_ip_{j+\frac{1}{2}} = p_{j+\frac{1}{2}}\sigma_i$, *if* $j \neq i$.
  - (f) $\sigma_{i+\frac{1}{2}}p_j = p_j\sigma_{i+\frac{1}{2}}$, *if* $j \neq i, i+1$.
  - (g) $\sigma_{i+\frac{1}{2}}p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}$, *if* $j \neq i-1$.
  - (h) $\sigma_{i+\frac{1}{2}}p_i\sigma_{i+\frac{1}{2}} = \sigma_{i+1}p_{i+1}\sigma_{i+1}$, *for* $i = 1, \ldots, k-1$.
  - (i) $\sigma_{i+\frac{1}{2}}p_{i-\frac{1}{2}}\sigma_{i+\frac{1}{2}} = \sigma_ip_{i+\frac{1}{2}}\sigma_i$, *for* $i = 2, \ldots, k-1$.
- (5) *(Contraction relations)*
  - (a) $p_{i+\frac{1}{2}}p_jp_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, *for* $j = i, i+1$.
  - (b) $p_ip_{j-\frac{1}{2}}p_i = p_i$, *for* $j = i, i+1$.

Proof. We first show that the relations given in the statement above are a consequence the presentation given by by [HR] in Theorem 2.1.

(1a), (1b) Proposition 4.2 shows that $\sigma_{i+\frac{1}{2}}$ and σ_{i+1} are involutions in $\mathcal{A}_k(z)$.

(2a) To see that σ_{i+1} and $\sigma_{i+\frac{1}{2}}$ commute, we use Proposition 3.3 and Proposition 4.2 to obtain

$$(4.1) \qquad \sigma_{i+1} = \sigma_{i+\frac{1}{2}} s_i = s_i \sigma_{i+\frac{1}{2}} \qquad \text{and} \qquad \sigma_{i+\frac{1}{2}} \sigma_{i+1} = \sigma_{i+1} \sigma_{i+\frac{1}{2}} = s_i \qquad (\text{for } i = 1, \ldots, k-1)$$

where s_i is a Coxeter generator for $\mathfrak{S}_k \subset \mathcal{A}_k(z)$. Theorem 3.7 shows that σ_j commutes with $\mathcal{A}_{j-\frac{3}{2}}(z)$, and hence that σ_j commutes with $\sigma_{1+\frac{1}{2}}, \ldots, \sigma_{j-\frac{3}{2}}$. From Theorem 3.7, $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-1}(z)$, and hence with $\sigma_2, \ldots, \sigma_{i-1}$. In general however, σ_i and $\sigma_{i+\frac{1}{2}}$ do not commute.

(2b), (2c) Theorem 3.7 shows that σ_{i+1} commutes with $\sigma_2, \ldots, \sigma_{i-1}$ and that $\sigma_{i+\frac{1}{2}}$ commutes with $\sigma_{1+\frac{1}{2}}, \ldots, \sigma_{i-\frac{3}{2}}$.

(2d) Follows from (4.1), whereby for $j = 1, \ldots, k-1$, each product $\sigma_{j+\frac{1}{2}} \sigma_{j+1} = s_j$ is a Coxeter generator for $\mathfrak{S}_k \subset \mathcal{A}_k(z)$.

(3a), (3b) Are included in the set of relations given by [HR].

(3c), (3d) That $\sigma_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \sigma_{i+1} = p_{i+\frac{1}{2}}$ is given in Proposition 3.1. Proposition 3.3 shows that $\sigma_{i+\frac{1}{2}} p_{i+\frac{1}{2}} = s_i \sigma_{i+1} p_{i+\frac{1}{2}} = s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$.

(3e), (3f) Proposition 3.3 and Proposition 4.2 show that

$$p_i p_{i+1} = s_i p_i p_{i+1} = \sigma_{i+\frac{1}{2}} \sigma_{i+1} p_i p_{i+1} \qquad \text{and} \qquad \sigma_{i+\frac{1}{2}} p_i p_{i+1} = \sigma_{i+1} p_i p_{i+1}.$$

(4a)–(4c) Are included in the set of relations given by [HR].

(4d), (4e) By Theorem 3.7, σ_i commutes with $\mathcal{A}_{i-\frac{3}{2}}(z)$, and hence with $p_1, \ldots p_{i-2}$ and with $p_{1+\frac{1}{2}}, \ldots, p_{i-\frac{3}{2}}$. Proposition 3.1 shows that σ_i commutes with $p_{i-\frac{1}{2}}$.

(4f), (4g) By Theorem 3.7, $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-1}(z)$, and hence with $p_1, \ldots p_{i-1}$ and with $p_{1+\frac{1}{2}}, \ldots, p_{i-\frac{3}{2}}$. From (3d), it follows that $\sigma_{i+\frac{1}{2}}$ commutes with $p_{i+\frac{1}{2}}$.

(4h) Proposition 3.3 and Proposition 4.2 show that

$$p_i = s_i p_{i+1} s_i = \sigma_{i+\frac{1}{2}} \sigma_{i+1} p_{i+1} \sigma_{i+1} \sigma_{i+\frac{1}{2}} \qquad \text{and} \qquad \sigma_{i+\frac{1}{2}} p_i \sigma_{i+\frac{1}{2}} = \sigma_{i+1} p_{i+1} \sigma_{i+1}.$$

(4i) Proposition 3.1 shows that $p_{i-\frac{1}{2}} s_i \sigma_i = s_i \sigma_i p_{i+\frac{1}{2}}$. Proposition 3.3 and Proposition 4.2, together with the fact that σ_{i+1} commutes with $p_{i-\frac{1}{2}}$ give

$$p_{i-\frac{1}{2}} \sigma_{i+1} \sigma_{i+\frac{1}{2}} \sigma_i = \sigma_{i+1} \sigma_{i+\frac{1}{2}} \sigma_i p_{i+\frac{1}{2}} \qquad \text{and} \qquad p_{i-\frac{1}{2}} \sigma_{i+\frac{1}{2}} \sigma_i = \sigma_{i+\frac{1}{2}} \sigma_i p_{i+\frac{1}{2}}.$$

Multiplying both sides of the last expression by $\sigma_i \sigma_{i+\frac{1}{2}}$ on the left, and using Proposition 4.2 once more shows that

$$\sigma_i \sigma_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \sigma_{i+\frac{1}{2}} \sigma_i = \sigma_i \sigma_{i+\frac{1}{2}} \sigma_{i+\frac{1}{2}} \sigma_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \qquad \text{and} \qquad \sigma_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \sigma_{i+\frac{1}{2}} = \sigma_i p_{i+\frac{1}{2}} \sigma_i,$$

as required.

(5a), (5b) Are included in the set of relations given by [HR].

Next, we derive the relations given by [HR] in Theorem 2.1 from the relations (1a)–(5b) above.

(1i) By the relations (1a), (1b) and (2a),

$$\sigma_{i+1} \sigma_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}} \sigma_{i+\frac{1}{2}} \qquad \text{and} \qquad (\sigma_{i+\frac{1}{2}} \sigma_{i+1})^2 = 1, \qquad \text{for } i = 1, \ldots, k-1.$$

Thus, writing $s_i = \sigma_{i+\frac{1}{2}} \sigma_{i+1}$, for $i = 1, \ldots, k-1$, we recover (1i).

(1ii) If $j \neq i + 1$, then, by (2b) and (2c),

$$s_i s_j = \sigma_{i+\frac{1}{2}} \sigma_{i+1} \sigma_{j+\frac{1}{2}} \sigma_{j+1} = \sigma_{j+\frac{1}{2}} \sigma_{j+1} \sigma_{i+\frac{1}{2}} \sigma_{i+1} = s_j s_i,$$

as required.

(1iii) Is equivalent to (2d) with $s_i = \sigma_{i+\frac{1}{2}} \sigma_{i+1}$, for $i = 1, \ldots, k-1$.

(2i), (2ii) Are identical to the relations (3a) and (3b).

(2iii) With $s_i = \sigma_{i+1}\sigma_{i+\frac{1}{2}}$, the relations (3c) and (3d) give

$$s_i p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}\sigma_{i+1}p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$$

and

$$p_{i+\frac{1}{2}}s_i = p_{i+\frac{1}{2}}\sigma_{i+1}\sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}},$$

as required.

(2iv) With $s_i = \sigma_{i+1}\sigma_{i+\frac{1}{2}}$, the relations (2a), (3e) and (3f) give

$$s_i p_i p_{i+1} = \sigma_{i+1}\sigma_{i+\frac{1}{2}}p_i p_{i+1} = p_i p_{i+1} \qquad \text{and} \qquad p_i p_{i+1} s_i = p_i p_{i+1}\sigma_{i+1}\sigma_{i+\frac{1}{2}} = p_i p_{i+1},$$

as required.

(3i)-(3iii) Are identical to the relations (4a)-(4c).

(3iv) If $j \neq i, i+1$, then the relations (4d) and (4f) give

$$s_i p_j = \sigma_{i+\frac{1}{2}}\sigma_{i+1}p_j = \sigma_{i+\frac{1}{2}}p_j\sigma_{i+1} = p_j\sigma_{i+\frac{1}{2}}\sigma_{i+1} = p_j s_i,$$

as required.

(3v) If $j \neq i-1, i+1$, then the relations (4e) and (4f) give

$$s_i p_{j+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}\sigma_{i+1}p_{j+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}p_{j+\frac{1}{2}}\sigma_{i+1} = p_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}\sigma_{i+1} = p_{j+\frac{1}{2}}s_i,$$

as required.

(3vi) From the relations (1b) and (4h),

$$s_i p_i s_i = \sigma_{i+1}\sigma_{i+\frac{1}{2}}p_i\sigma_{i+\frac{1}{2}}\sigma_{i+1} = \sigma_{i+1}^2 p_{i+1}\sigma_{i+1}^2 = p_{i+1},$$

as required.

(3vii) From the relations (4e), (4g) and (1a), (1b),

$$s_i p_{i-\frac{1}{2}} s_i = \sigma_{i+\frac{1}{2}}\sigma_{i+1}p_{i-\frac{1}{2}}\sigma_{i+1}\sigma_{i+\frac{1}{2}} = \sigma_i\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}\sigma_i = s_{i-1}p_{i+\frac{1}{2}}s_{i-1},$$

as required.

(4i), (4ii) Are identical to the relations (5a) and (5b). $\qquad\square$

Proposition 4.2. *If $i = 0, 1, \ldots$, then $(\sigma_{i+\frac{1}{2}})^2 = 1$ and $(\sigma_{i+1})^2 = 1$.*

Proof. Given that $\sigma_{i+\frac{1}{2}}s_i = s_i\sigma_{i+\frac{1}{2}} = \sigma_{i+1}$, we obtain

$$\sigma_{i+\frac{1}{2}}^2 = (s_i\sigma_{i+1})^2 = \sigma_{i+1}^2.$$

It therefore suffices to show that $\sigma_{i+1}^2 = 1$. By definition $\sigma_1 = 1$, so we assume that $\sigma_i^2 = 1$ proceed by induction. After taking the square of the right hand side of the definition (3.2), the statement that $\sigma_{i+1}^2 = 1$ is equivalent to the following set of relations:

$$(4.2) \qquad (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)^2 = (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)(p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}});$$

$$(4.3) \quad (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)(s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}})$$
$$= (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)(s_{i-1}s_i\sigma_i s_i s_{i-1});$$

$$(4.4) \quad (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)(s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i)$$
$$= (s_{i-1}s_i\sigma_i s_i s_{i-1})(s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i);$$

(4.5) $\quad (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)$

$$= (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(s_{i-1} s_i \sigma_i s_i s_{i-1});$$

(4.6) $\quad (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^2 = (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})$

(4.7) $\quad (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) = (s_{i-1} s_i \sigma_i s_i s_{i-1})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}});$

(4.8) $\quad (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i) = (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_{i-1} s_i \sigma_i s_i s_{i-1});$

(4.9) $\quad (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i)$

$$= (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i);$$

(4.10) $\quad (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})^2 = (s_{i-1} s_i \sigma_i s_i s_{i-1})(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}});$

(4.11) $\quad (s_{i-1} s_i \sigma_i s_i s_{i-1})(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i) = (s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i)^2;$

(4.12) $\quad (s_{i-1} s_i \sigma_i s_i s_{i-1})^2 = 1;$

(4.13) $\quad (s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i)(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)$

$$= (s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i)(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}});$$

(4.14) $\quad (s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i)(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})$

$$= (s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i)(s_{i-1} s_i \sigma_i s_i s_{i-1}).$$

Considering the left hand side of (4.2), using the fact that $p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}$, together with $p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$,

$$(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)^2 = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i$$

$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}}$$

$$= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}.$$

Similarly, considering the right hand side of (4.2),

$$(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_{i+\frac{1}{2}},$$

Which demonstrates (4.2). Now consider the left hand side of (4.3)

$$(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})$$

$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i-1} \sigma_i p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}.$$

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Considering the right hand side of (4.3),

$$(p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(s_{i-1} s_i \sigma_i s_i s_{i-1}) = (p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i)(s_{i-1} s_i \sigma_i s_i s_{i-1})$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i s_i s_{i-1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1}$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1}$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1}$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_i s_i p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1}$$

$$= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1}$$

$$= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_{i-1} s_i s_{i-1}$$

$$= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} s_i s_{i-1}$$

$$= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} s_{i-1} s_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}$$

$$= (p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}).$$

Now,

$$(4.15) \qquad (p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) = p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}} p_i (\sigma_{i-\frac{1}{2}})^2 p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}$$

$$= p_{i-\frac{1}{2}},$$

which completes the proof of (4.3). For the left hand side of (4.4),

$$(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) = p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} s_i$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} s_i$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1}$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1}$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_{i-1}$$

$$= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1}$$

$$= p_{i-\frac{1}{2}} p_i (\sigma_i)^2 p_{i+\frac{1}{2}} s_{i-1}$$

$$= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1}.$$

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Considering the right hand side of (4.4),

$$
\begin{aligned}
(s_{i-1}s_i\sigma_i s_i s_{i-1})(s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i) &= s_{i-1}s_i\sigma_i p_{i+\frac{1}{2}}s_{i-1}s_i L_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
&= s_{i-1}s_i\sigma_i p_{i+\frac{1}{2}}s_{i-1}s_i L_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
&= s_{i-1}s_i s_{i-1}\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i L_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
&= p_{i-\frac{1}{2}}s_i s_{i-1}\sigma_{i-\frac{1}{2}}s_{i-1}s_i L_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
&= p_{i-\frac{1}{2}}s_i\sigma_{i-\frac{1}{2}}(s_{i-1})^2(s_i)^2 L_{i-1}p_{i-\frac{1}{2}}s_i \\
&= p_{i-\frac{1}{2}}s_i\sigma_{i-\frac{1}{2}}\sigma_i p_i p_{i-\frac{1}{2}}s_i \\
&= p_{i-\frac{1}{2}}s_i s_{i-1}p_i p_{i-\frac{1}{2}}s_i \\
&= p_{i-\frac{1}{2}}p_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
&= p_{i-\frac{1}{2}}p_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1} \\
&= p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1},
\end{aligned}
$$

which demonstrates (4.4). Considering the left hand side of (4.5),

$$
\begin{aligned}
(s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}})(q_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i) &= s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i \\
&= s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}s_i \\
&= s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
&= s_{i-1}p_{i+\frac{1}{2}}s_{i-1}\sigma_i p_i p_{i-\frac{1}{2}}p_i\sigma_i s_{i-1}p_{i+\frac{1}{2}}s_{i-1} \\
&= s_{i-1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_i\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1} \\
&= s_{i-1}(\sigma_{i-\frac{1}{2}})^2 p_{i+\frac{1}{2}}s_{i-1} \\
&= s_{i-1}p_{i+\frac{1}{2}}s_{i-1},
\end{aligned}
$$

while, for the right hand side of (4.5),

$$
\begin{aligned}
(s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}})(s_{i-1}s_i\sigma_i s_i s_{i-1}) &= s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}s_i\sigma_i s_i s_{i-1} \\
&= s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i s_i s_{i-1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_{i+1}s_{i-1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}p_i s_i p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_{i-1}p_{i+\frac{1}{2}}s_{i-1}\sigma_i p_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_{i-1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_{i-1}\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}s_i s_{i-1} \\
&= s_{i-1}p_{i+\frac{1}{2}}(\sigma_{i-\frac{1}{2}})^2 s_{i-1} \\
&= s_{i-1}p_{i+\frac{1}{2}}s_{i-1},
\end{aligned}
$$

which demonstrates (4.5). The statement (4.6) is equivalent to (4.2) which has already been verified. The right hand side of (4.7) comes to,

$$
\begin{aligned}
(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
&= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} (\sigma_{i-\frac{1}{2}})^2 p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}},
\end{aligned}
$$

while

$$
\begin{aligned}
(s_{i-1} s_i \sigma_i s_i s_{i-1})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) &= s_{i-1} s_i \sigma_i s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
&= s_{i-1} s_i \sigma_i s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} \\
&= s_{i-1} s_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} s_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} s_i p_{i+\frac{1}{2}} (\sigma_{i-\frac{1}{2}})^2 p_i p_{i-\frac{1}{2}} \\
&= s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}},
\end{aligned}
$$

as required. Since the statement (4.8) is equivalent to (4.7), we consider (4.9). Using the relation $p_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$,

$$
\begin{aligned}
(s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})&(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) \\
&= s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
&= s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} \\
&= s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
&= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} \\
&= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) &= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
&= p_{i+\frac{1}{2}} (p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}})^2 \\
&= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}}.
\end{aligned}
$$

The left hand side of (4.10) is given by (4.15), and the right hand side by

$$
\begin{aligned}
(s_{i-1}s_i\sigma_is_is_{i-1})(s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}) &= s_{i-1}s_i\sigma_is_is_{i-1}s_ip_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}p_{i-1}p_{i-\frac{1}{2}}\\
&= s_{i-1}s_i\sigma_is_is_{i-1}s_ip_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_ip_{i-1}p_{i-\frac{1}{2}}\\
&= s_{i-1}s_i\sigma_is_is_{i-1}s_ip_{i-\frac{1}{2}}s_iL_{i-1}p_{i-\frac{1}{2}}s_ip_{i-1}p_{i-\frac{1}{2}}\\
&= s_{i-1}s_i\sigma_is_is_i(s_{i-1})^2p_{i+\frac{1}{2}}s_{i-1}\sigma_ip_ip_{i-\frac{1}{2}}s_ip_{i-1}p_{i-\frac{1}{2}}\\
&= s_{i-1}s_is_{i-1}\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_ip_{i-\frac{1}{2}}p_{i-1}s_ip_{i-\frac{1}{2}}\\
&= s_{i-1}p_{i-\frac{1}{2}}s_is_{i-1}(\sigma_{i-\frac{1}{2}})^2p_ip_{i-\frac{1}{2}}p_{i-1}s_ip_{i-\frac{1}{2}}\\
&= p_{i-\frac{1}{2}}s_is_{i-1}p_ip_{i-\frac{1}{2}}p_{i-1}s_ip_{i-\frac{1}{2}}\\
&= p_{i-\frac{1}{2}},
\end{aligned}
$$

as required. Considering the left hand side of (4.11),

$$
\begin{aligned}
(s_{i-1}s_i\sigma_is_is_{i-1})(p_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i) &= s_{i-1}s_i\sigma_is_ip_{i-\frac{1}{2}}s_ip_{i+1}p_{i+\frac{1}{2}}s_{i-1}\sigma_ip_ip_{i-\frac{1}{2}}s_i\\
&= s_{i-1}s_i\sigma_is_{i-1}p_{i+\frac{1}{2}}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}s_{i-1}\sigma_ip_ip_{i-\frac{1}{2}}s_i\\
&= s_{i-1}s_i\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_ip_{i-\frac{1}{2}}s_i\\
&= s_{i-1}p_{i+\frac{1}{2}}s_{i-1}\sigma_{i-\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_ip_{i-\frac{1}{2}}s_i\\
&= s_ip_{i-\frac{1}{2}}s_ip_{i+1}p_{i+\frac{1}{2}}(\sigma_{i-\frac{1}{2}})^2p_ip_{i-\frac{1}{2}}s_i\\
&= s_ip_{i-\frac{1}{2}}s_ip_{i+1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}}s_i\\
&= s_ip_{i-\frac{1}{2}}s_i,
\end{aligned}
$$

which, by (4.15), is equal to the right hand side of (4.11). Since (4.9) is equivalent to (4.13), while (4.14) is equivalent to (4.4), the proof of the proposition is complete. $\square$

We record for later reference further consequences of the presentation given in Theorem 2.1.

Proposition 4.3. *For* $i = 1, 2, \ldots$, *the following statements hold:*

(1) $p_{i+1}\sigma_{i+1}p_{i+1} = L_ip_{i+1}$,
(2) $p_{i+1}\sigma_{i+\frac{1}{2}}p_{i+1} = (z - L_{i-\frac{1}{2}})p_{i+1}$,
(3) $p_{i+\frac{3}{2}}\sigma_{i+1}p_{i+\frac{3}{2}} = p_{i+\frac{1}{2}}p_{i+\frac{3}{2}}$.

Proof. (1) From Proposition 3.1, we obtain $p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1} = p_{i+\frac{1}{2}}L_i$. Thus

$$
p_{i+1}\sigma_{i+1} = p_{i+1}p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1} = p_{i+1}p_{i+\frac{1}{2}}L_i \quad \text{and} \quad p_{i+1}\sigma_{i+1}p_{i+1} = p_{i+1}p_{i+\frac{1}{2}}L_ip_{i+1} = L_ip_{i+1},
$$

as required.
(2) We first compute

$$
\begin{aligned}
p_{i+1}s_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i+1} &= s_ip_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i+1}\\
&= s_ip_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_ip_{i+1}\\
&= s_ip_ip_{i+\frac{1}{2}}\sigma_ip_{i-1}p_{i-\frac{1}{2}}p_is_i\\
&= s_ip_ip_{i+\frac{1}{2}}\sigma_is_{i-1}p_is_i\\
&= s_ip_ip_{i+\frac{1}{2}}\sigma_{i-\frac{1}{2}}p_is_i\\
&= \sigma_{i-\frac{1}{2}}p_{i+1}.
\end{aligned}
$$

Now observe that $p_2\sigma_{1+\frac{1}{2}}p_2 = (z - L_{\frac{1}{2}})p_2$ and hence, by induction,

$$
\begin{aligned}
p_{i+1}\sigma_{i+\frac{1}{2}}p_{i+1} &= s_i s_{i-1} p_i \sigma_{i-\frac{1}{2}} p_i s_{i-1} s_i + p_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+1} \\
&\quad + p_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+1} - p_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+1} \\
&\quad - p_{i+1} s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} \\
&= s_{i-1} s_i p_i \sigma_{i-\frac{1}{2}} p_i s_i s_{i-1} + p_{i-\frac{1}{2}} L_{i-1} p_{i+1} + L_{i-1} p_{i-\frac{1}{2}} p_{i+1} \\
&\quad - p_{i-\frac{1}{2}} L_{i-1} p_{i-1} p_{i-\frac{1}{2}} p_{i+1} - \sigma_{i-\frac{1}{2}} p_{i+1} \\
&= (z - L_{i-\frac{1}{2}}) p_{i+1}.
\end{aligned}
$$

([3]) Observe that $p_{2+\frac{1}{2}}\sigma_2 p_{2+\frac{1}{2}} = p_{2+\frac{1}{2}} s_1 p_{2+\frac{1}{2}} = p_{2+\frac{1}{2}} p_{1+\frac{1}{2}}$ and, by induction,

$$
\begin{aligned}
p_{i+\frac{3}{2}} s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i+\frac{3}{2}} &= s_{i-1} p_{i+\frac{3}{2}} s_i \sigma_i s_i p_{i+\frac{3}{2}} s_{i-1} \\
&= s_{i-1} s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} \sigma_i s_{i+1} p_{i+\frac{1}{2}} s_i s_{i+1} s_{i-1} \\
&= s_{i-1} s_i s_{i+1} p_{i+\frac{1}{2}} \sigma_i p_{i+\frac{1}{2}} s_i s_{i+1} s_{i-1} \\
&= s_{i-1} s_i s_{i+1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_i s_{i+1} s_{i-1} \\
&= p_{i+\frac{1}{2}} p_{i+\frac{3}{2}}.
\end{aligned}
$$

Therefore, using the definition ([3.2]) and the fact that $p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}$,

$$
\begin{aligned}
p_{i+\frac{3}{2}}\sigma_{i+1} p_{i+\frac{3}{2}} &= p_{i+\frac{3}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{3}{2}} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} \\
&\quad + p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} - p_{i+\frac{3}{2}} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} \\
&\quad - p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} \\
&= p_{i+\frac{3}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{3}{2}} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
&\quad + p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{3}{2}} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} - p_{i+\frac{3}{2}} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&\quad - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} s_i p_{i+\frac{3}{2}} \\
&= p_{i+\frac{3}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{3}{2}} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \\
&\quad + p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{3}{2}} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} - p_{i+\frac{3}{2}} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
&\quad - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_{i+\frac{3}{2}} \\
&= p_{i+\frac{3}{2}} p_{i+\frac{1}{2}},
\end{aligned}
$$

as required. $\qquad\square$

The next statement establishes an alternative formula for the family $(L_{i+\frac{1}{2}} : i = 0, 1 \ldots)$, for use in §[5] and [En].

Theorem 4.4. *If $i = 1, 2, \ldots$, then*

$$
L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + (z - L_{i-\frac{1}{2}}) p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}}.
$$

Proof. By Proposition [3.1] and Proposition [4.3],

$$
p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i+1} \sigma_{i+1} p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i+1} \sigma_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} = (z - L_{i-\frac{1}{2}}) p_{i+\frac{1}{2}}.
$$

Substituting the above expression into the definition of $L_{i+\frac{1}{2}}$ given in ([3.3]) provides the required statement. $\qquad\square$

5. SCHUR–WEYL DUALITY

This section uses Schur–Weyl duality to establish that the family $(L_{i+\frac{1}{2}}, L_{i+1} : i = 1, 2, \ldots)$ defined above and the Jucys–Murphy elements given by [HR] are in fact equal.

Let $n = 1, 2, \ldots$, and V be a vector space over $\mathbb{C}$ with basis $v_1, \ldots, v_n$. If $r = 1, 2, \ldots$, the tensor product

$$V^{\otimes r} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{r \text{ factors}} \qquad \text{has basis} \qquad \{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \mid 1 \leqslant i_1, \ldots, i_r \leqslant n\},$$

and is equipped, via the inclusion $\mathfrak{S}_n \subset GL_n(\mathbb{C})$, with the diagonal $\mathfrak{S}_n$–action

$$w(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r})w = v_{(i_1)w^{-1}} \otimes v_{(i_2)w^{-1}} \otimes \cdots \otimes v_{(i_r)w^{-1}}, \qquad \text{for } w \in \mathfrak{S}_n.$$

Let $A_r(n) = \mathcal{A}_r(z) \otimes_{\mathbb{Z}[z]} \mathbb{C}$, where z acts on $\mathbb{C}$ as multiplication by n. The action of $A_r(n)$ on $V^{\otimes r}$ is given (§3 of [HR]) by

$$u(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) = v_{i_{(1)u^{-1}}} \otimes v_{i_{(2)u^{-1}}} \otimes \cdots \otimes v_{i_{(r)u^{-1}}}, \qquad \text{for } u \in \mathfrak{S}_r,$$

and for $k = 1, \ldots, r-1$,

$$p_{k+\frac{1}{2}}(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) = \begin{cases} v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} & \text{if } i_k = i_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and for $k = 1, \ldots, r$,

$$p_k(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_{i_k} \otimes v_{i_{k+1}} \otimes \cdots \otimes v_{i_r}) = \sum_{j=1}^{n} v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_j \otimes v_{i_{k+1}} \otimes \cdots \otimes v_{i_r}.$$

The $A_{r+\frac{1}{2}}(n)$–action on $V^{\otimes r}$ is obtained in §3 of [HR] from the action of $A_{r+1}(n)$ on $V^{\otimes r+1}$ by restricting to the subspace $V^{\otimes r} \otimes v_n$ and identifying $V^{\otimes r}$ with $V^{\otimes r} \otimes v_n$. The next statement asserts that $\mathfrak{S}_n$ and $A_r(n)$ act as commuting operators on $V^{\otimes r}$.

Theorem 5.1 (Theorem 3.22 of [HR]). *Let $n, r \in \mathbb{Z}_{\geqslant 0}$. Let S_n^λ denote the irreducible $\mathfrak{S}_n$-module indexed by λ.*

(1) *As $(\mathbb{C}\mathfrak{S}_n, A_r(n))$-bimodules*

$$V^{\otimes r} \cong \bigoplus_{\lambda \in \hat{A}_r(n)} S_n^\lambda \otimes A_r^\lambda(n),$$

where $\lambda \in \hat{A}_r(n)$ is an indexing set for the irreducible $A_r^\lambda(n)$-modules, and the vector spaces $A_r^\lambda(n)$, for $\lambda \in \hat{A}_r(n)$, are irreducible $A_r^\lambda(n)$-modules.

(2) *As $(\mathbb{C}\mathfrak{S}_{n-1}, A_{r+\frac{1}{2}}(n))$-bimodules*

$$V^{\otimes r} \cong \bigoplus_{\lambda \in \hat{A}_{r+\frac{1}{2}}(n)} S_{n-1}^\lambda \otimes A_{r+\frac{1}{2}}^\lambda(n),$$

where $\lambda \in \hat{A}_{r+\frac{1}{2}}(n)$ is an indexing set for the irreducible $A_{r+\frac{1}{2}}^\lambda(n)$-modules, and the vector spaces $A_{r+\frac{1}{2}}^\lambda(n)$, for $\lambda \in \hat{A}_{r+\frac{1}{2}}(n)$, are irreducible $A_{r+\frac{1}{2}}^\lambda(n)$-modules.

By Theorem 3.6 of [HR], the homomorphism $A_r(n) \to \text{End}_{\mathfrak{S}_n}(V^{\otimes r})$ in Theorem 5.1 is an isomorphism whenever $n \geqslant 2r$.

If $1 \leqslant i, j \leqslant n$, let $s_{i,j} \in \mathfrak{S}_n$ denote the transposition which interchanges i and j. The next statement gives the action of the group $\langle \sigma_{i+\frac{1}{2}}, \sigma_{i+1} \mid i = 1, \ldots, r-1 \rangle$ on $V^{\otimes r}$.

Proposition 5.2. *If $k = 1, 2, \ldots,$ and $v_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \in V^{\otimes k+1}$, then*

$$(5.1) \qquad \sigma_{k+\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_{k+1}}) = \underbrace{s_{i_k,i_{k+1}} \otimes \cdots \otimes s_{i_k,i_{k+1}}}_{k-1 \ factors}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_{i_{k+1}}$$

and

$$(5.2) \qquad \sigma_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_{k+1}}) = \underbrace{s_{i_k,i_{k+1}} \otimes \cdots \otimes s_{i_k,i_{k+1}}}_{k+1 \ factors}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}}).$$

Proof. The proposition is true when $k = 1$. If $k = 2, 3, \ldots,$ observe that the linear endomorphisms defined, for $v_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \in V^{\otimes k+1}$, by

$$\theta_{k+\frac{1}{2}} : v_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \mapsto \underbrace{s_{i_k,i_{k+1}} \otimes \cdots \otimes s_{i_k,i_{k+1}}}_{k-1 \ \text{factors}}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_{i_{k+1}}$$

and

$$\theta_{k+1} : v_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \mapsto \underbrace{s_{i_k,i_{k+1}} \otimes \cdots \otimes s_{i_k,i_{k+1}}}_{k+1 \ \text{factors}}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}})$$

commute with the diagonal action of $\mathfrak{S}_n$ on $V^{\otimes k+1}$. Thus, by Theorem 5.1, $\theta_{k+\frac{1}{2}}$ and θ_{k+1}, lie in the image of the map $A_{k+1}(n) \mapsto \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k+1})$. Observe also that the action of $\theta_{k+\frac{1}{2}}$ on $V^{\otimes k+1}$ commutes with the action of $A_{k-1}(n)$ on $V^{\otimes k+1}$, and the action of θ_{k+1} on $V^{\otimes k+1}$ commutes with the action of $A_{k-\frac{1}{2}}(n)$ on $V^{\otimes k+1}$. Since $A_{k+1}(n)$ is generated by $A_{k-\frac{1}{2}}(n)$ together with $\langle \sigma_{k+\frac{1}{2}}, \sigma_{k+1}, p_k, p_{k+\frac{1}{2}} \rangle$, to show that the map $A_{k+1}(n) \to \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k+1})$ sends

$$\sigma_{k+\frac{1}{2}} \mapsto \theta_{k+\frac{1}{2}} \qquad\qquad \text{and} \qquad\qquad \sigma_{k+1} \mapsto \theta_{k+1},$$

it now suffices to verify that, as operators on $V^{\otimes k+1}$,

(1) $\theta_{k+\frac{1}{2}}^2 = \theta_{k+1}^2 = 1,$
(2) $\theta_{k+\frac{1}{2}}\theta_{k+1} = \theta_{k+1}\theta_{k+\frac{1}{2}} = s_k,$
(3) $\theta_{k+\frac{1}{2}}p_{k+\frac{1}{2}} = p_{k+\frac{1}{2}}\theta_{k+\frac{1}{2}} = p_{k+\frac{1}{2}},$
(4) $\theta_{k+\frac{1}{2}}p_kp_{k+1} = \theta_{k+1}p_kp_{k+1},$
(5) $p_kp_{k+1}\theta_{k+\frac{1}{2}} = p_kp_{k+1}\theta_{k+1},$
(6) $\theta_{k+\frac{1}{2}}p_k\theta_{k+\frac{1}{2}} = \theta_{k+1}p_{k+1}\theta_{k+1},$
(7) $\theta_{k+\frac{1}{2}}p_{k-\frac{1}{2}}\theta_{k+\frac{1}{2}} = \sigma_k p_{k+\frac{1}{2}}\sigma_k.$

Since each of the above relations can be verified by inspection, the proof of the proposition is complete. $\qquad\square$

Proposition 5.3. *If $k = 1, 2, \ldots,$ and $v_{i_1} \otimes \cdots \otimes v_{i_k} \in V^{\otimes k}$ then*

$$(5.3) \quad L_{k-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = n v_{i_1} \otimes \cdots \otimes v_{i_k} - \sum_{j=1}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k},$$

and

$$(5.4) \qquad L_k(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j=1}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_j.$$

Proof. Identify $V^{\otimes k}$ as the subspace $V^{\otimes k} \otimes \sum_{j=1}^{n} v_j \subseteq V^{\otimes k+1}$. Then

$$p_{k+1}\sigma_{k+\frac{1}{2}}p_{i+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) = p_{k+1}\sum_{j=1}^{n}\sigma_{k+\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_j)$$

$$= p_{k+1}\sum_{j=1}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_j$$

$$= \sum_{j,\ell=1}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_\ell.$$

By Proposition 4.3, as operators on $V^{\otimes k}$,

$$p_{k+1}\sigma_{k+\frac{1}{2}}p_{i+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) = (n - L_{i-\frac{1}{2}})p_{i+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n)$$

$$= \sum_{\ell=1}^{n}(n - L_{i-\frac{1}{2}})(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_\ell),$$

and the statement (5.3) follows. Next,

$$p_{k+1}\sigma_{k+1}p_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) = p_{k+1}\sum_{j=1}^{n}\sigma_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_j)$$

$$= p_{k+1}\sum_{j=1}^{n}\sigma_{k+\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_j \otimes v_{i_k})$$

$$= p_{k+1}\sum_{j=1}^{n} s_{i_k,i_{k+1}} \otimes \cdots \otimes s_{i_k,i_{k+1}}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_j) \otimes v_{i_k}$$

$$= \sum_{j,\ell=1}^{n} s_{i_k,i_{k+1}} \otimes \cdots \otimes s_{i_k,i_{k+1}}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_j) \otimes v_\ell.$$

By Proposition 4.3, as operators on $V^{\otimes k}$,

$$p_{k+1}\sigma_{k+1}p_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) = L_k p_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n)$$

$$= \sum_{\ell=1}^{n} L_k(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_\ell),$$

which yields (5.4). $\qquad\square$

As in §3 of [HR], let κ_n be the central element which is the class sum corresponding to the conjugacy class of transpositions in $\mathbb{C}\mathfrak{S}_n$,

$$\kappa_n = \sum_{1 \leqslant i < j \leqslant n} s_{i,j}.$$

For $\ell = 1, \ldots, n$, we also define

$$\kappa_{n,\ell} = \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \neq \ell}} s_{i,j},$$

so that $\kappa_{n,n} = \kappa_{n-1}$.

Proposition 5.4. *Let $n = \dim(V)$ and $r \in \mathbb{Z}_{>0}$. If $z_{r-\frac{1}{2}} \in A_{r-\frac{1}{2}}(n)$ and $z_r \in A_r(n)$ are the central elements defined by Theorem 3.9, and $v_{i_1} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$, then*

$$(5.5) \qquad z_r(v_{i_1} \otimes \cdots \otimes v_{i_r}) = \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_r}) - \left(\binom{n}{2} - rn\right)(v_{i_1} \otimes \cdots \otimes v_{i_r}),$$

and, if $r = 2, 3, \ldots$, then,

$$(5.6) \qquad z_{r-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_r}) = \kappa_{n,i_r}(v_{i_1} \otimes \cdots \otimes v_{i_r}) - \left(\binom{n}{2} - rn + 1\right)(v_{i_1} \otimes \cdots \otimes v_{i_r}).$$

Proof. The proof is by induction on r. Let $i = 1, \ldots, n$. Since $z_1 = p_1$,

$$\kappa_n v_i = \sum_{i>j} s_{j,i} v_i + \sum_{j>i} s_{i,j} v_j + \sum_{\substack{j,\ell \\ j,\ell \neq i}} s_{j,\ell} v_i$$

$$= \sum_{j=1}^{n} v_j + \left(\binom{n-1}{2} - 1\right) v_i$$

$$= z_1 v_i + \left(\binom{n}{2} - n\right) v_i,$$

which verifies (5.5) when $r = 1$. Now observe that if $v_{i_1} \otimes \cdots \otimes v_{i_k} \in V^{\otimes r}$, then the diagonal action of κ_n and κ_{n,i_r} on $V^{\otimes k}$ allows us to write

$$\kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \kappa_{n,i_k}(v_{i_1} \otimes \cdots \otimes v_{i_k}) + \sum_{\substack{j=1 \\ j \neq i_k}}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_j,$$

and

$$\kappa_{n,i_k}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k}$$
$$- \sum_{\substack{j=1 \\ j \neq i_k}}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k}.$$

Assuming that (5.5) holds for $r = 1, \ldots, k-1$, we obtain

$$z_{k-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = z_{k-1}(v_{i_1} \otimes \cdots \otimes v_{i_k}) + L_{k-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_k})$$

$$= \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} - \left(\binom{n}{2} - (k-1)n\right)(v_{i_1} \otimes \cdots \otimes v_{i_k})$$

$$+ n v_{i_1} \otimes \cdots \otimes v_{i_k} - \sum_{j=1}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k}$$

$$= \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} - \left(\binom{n}{2} - kn + 1\right)(v_{i_1} \otimes \cdots \otimes v_{i_k})$$

$$- \sum_{\substack{j=1 \\ j \neq i_k}}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k}$$

$$= \kappa_{n,i_k}(v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\binom{n}{2} - kn + 1\right)(v_{i_1} \otimes \cdots \otimes v_{i_k}),$$

which verifies (5.6) for $r = k$. Next, since (5.6) holds for $r = 2, 3, \ldots, k$, we obtain

$$
\begin{aligned}
z_k(v_{i_1} \otimes \cdots \otimes v_{i_k}) &= z_{k-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_k}) + L_k(v_{i_1} \otimes \cdots \otimes v_{i_k}) \\
&= \kappa_{n,i_k}(v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\tbinom{n}{2} - kn + 1\right)(v_{i_1} \otimes \cdots \otimes v_{i_k}) \\
&\quad + \sum_{j=1}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_j \\
&= \kappa_{n,i_k}(v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\tbinom{n}{2} - kn\right)(v_{i_1} \otimes \cdots \otimes v_{i_k}) \\
&\quad + \sum_{\substack{j=1 \\ j \neq i_k}}^{n} s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_j \\
&= \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\tbinom{n}{2} - kn\right)(v_{i_1} \otimes \cdots \otimes v_{i_k}),
\end{aligned}
$$

which verifies (5.5) for $r = k$. $\qquad\square$

Let $Z_k \in A_k(n)$ and $Z_{k+\frac{1}{2}} \in A_{k+\frac{1}{2}}(n)$ denote the central element defined by Halverson and Ram in §3 of [HR]. Then the Jucys–Murphy elements of [HR] are given by

$$
M_{k-\frac{1}{2}} = Z_{k-\frac{1}{2}} - Z_{k-1} \qquad \text{and} \qquad M_k = Z_k - Z_{k-\frac{1}{2}} \qquad \text{for } k = 1, 2, \ldots.
$$

Theorem 5.5. *if $k = 1, 2, \ldots$, then $M_{k+\frac{1}{2}} = L_{k+\frac{1}{2}}$ and $M_{k+1} = L_{k+1}$ as elements of $A_{k+1}(n)$.*

Proof. By Theorem 3.6 of [HR], the homomorphism $A_k(n) \to \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k})$ is an isomorphism whenever $n \geqslant 2k$. Since the coefficients in the expansions of $Z_k, Z_{k+\frac{1}{2}}$ and $z_k, z_{k+\frac{1}{2}}$, in terms of the basis of diagrams for $A_{k+1}(n)$, are polynomials in n, and the map $A_k(n) \to \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k})$ is an isomorphism for infinitely many values of n, to prove the theorem, it suffices to compare the action of z_k and Z_k (*resp.* $z_{k+\frac{1}{2}}$ and $Z_{k+\frac{1}{2}}$) on $V^{\otimes k}$ for an arbitrary choice of n. Identifying $V^{\otimes k}$ with the subspace $V^{\otimes k} \otimes v_n \subseteq V^{\otimes k+1}$, and κ_{n-1} with $\kappa_{n,n}$, then Theorem 3.35 of [HR] states that, as operators on $V^{\otimes k}$,

$$
(5.7) \qquad Z_k = \kappa_n - \left(\tbinom{n}{2} - kn\right) \qquad \text{and} \qquad Z_{k+\frac{1}{2}} = \kappa_{n-1} - \left(\tbinom{n}{2} - (k+1)n + 1\right).
$$

Comparing (5.7) with the action of z_k and $z_{k+\frac{1}{2}}$ on $V^{\otimes k}$ in (5.5) and (5.6) completes the proof. $\qquad\square$

Remark 5.6. Whereas $L_{\frac{1}{2}} = 0$ and $L_1 = p_1$, in [HR], the first three Jucys–Murphy elements are $M_0 = M_{\frac{1}{2}} = 1$, and $M_1 = p_1 - 1$. Thus, although $z_1 = Z_1$ as elements of $A_1(n)$, Theorem 5.5 cannot be extended to the case $k = 0$.

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