

# Gradient Estimate on the Neumann Semigroup and Applications\*

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## Abstract

We prove the following sharp upper bound for the gradient of the Neumann semigroup  $P_t$  on a  $d$ -dimensional compact domain  $\Omega$  with boundary either  $C^2$ -smooth or convex:

$$\|\nabla P_t\|_{1 \rightarrow \infty} \leq \frac{c}{t^{(d+1)/2}}, \quad t > 0,$$

where  $c > 0$  is a constant depending on the domain and  $\|\cdot\|_{1 \rightarrow \infty}$  is the operator norm from  $L^1(\Omega)$  to  $L^\infty(\Omega)$ . This estimate implies a Gaussian type point-wise upper bound for the gradient of the Neumann heat kernel, which is applied to the study of the Hardy spaces, Riesz transforms, and regularity of solutions to the inhomogeneous Neumann problem on compact convex domains.

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## 1 Introduction

Let  $\Omega$  be a  $d$ -dimensional compact Riemannian manifold with boundary  $\partial\Omega$ . Let  $N$  be the inward unit normal vector field of  $\partial\Omega$ . If  $\partial\Omega$  is  $C^2$ -smooth, then the second fundamental

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form of  $\partial\Omega$  is defined as

$$\mathbb{I}(v_1, v_2) := -\langle \nabla_{v_1} N, v_2 \rangle, \quad v_1, v_2 \in T\partial\Omega,$$

where  $T\partial\Omega$  is the tangent space of  $\partial\Omega$ . We call the boundary  $\partial\Omega$  (or the manifold  $\Omega$ ) convex if  $\mathbb{I}(v, v) \geq 0$  for all  $v \in T\partial\Omega$ .

Let  $P_t$  be the Neumann semigroup generated by  $\Delta$ , the Laplacian on  $\Omega$  with the Neumann boundary condition. Let  $p_t(x, y)$  be the Neumann heat kernel, which is the density of  $P_t$  w.r.t. the Riemannian volume measure. For any  $p, q \geq 1$ , let  $\|\cdot\|_{p \rightarrow q}$  denote the operator norm from  $L^p(\Omega)$  to  $L^q(\Omega)$ . When  $\partial\Omega$  is  $C^2$ -smooth, it is easy to prove the following uniform gradient estimate of  $P_t$ , which is important in potential analysis of the Neumann Laplacian as shown in Section 4 below.

**Theorem 1.1.** *Let  $\Omega$  be a compact Riemannian manifold with  $C^2$ -smooth boundary. Then there exists a constant  $C > 0$  such that*

$$(1.1) \quad \|\nabla P_t\|_{1 \rightarrow \infty} \leq \frac{C}{t^{(d+1)/2}}, \quad t > 0.$$

Consequently, letting  $\rho$  be the Riemannian distance on  $\Omega$ , one has

$$(1.2) \quad |\nabla p_t(\cdot, y)(x)| \leq \frac{C}{t^{(d+1)/2}} \exp\left[-\frac{\rho(x, y)^2}{ct}\right], \quad t > 0, x, y \in \Omega$$

for some constants  $C, c > 0$ .

We remark that (1.1) is sharp (for short time) since the equality holds for the classical heat semigroup on  $\mathbb{R}^d$  and some constant  $C > 0$ . Since the boundary is  $C^2$ -smooth such that the second fundamental form is bounded below, the above theorem can be proved by using the reflecting Brownian motion and exponential moments of its local time (see Section 2).

When  $\partial\Omega$  is merely Lipschitzian such that the second fundamental form is not well defined, the argument presented in Section 2 is no longer valid. Indeed, for general manifolds with Lipschitzian boundary, even the existence and uniqueness of the reflecting Brownian motion is unknown. Nevertheless, for compact convex domains on  $\mathbb{R}^d$ , the reflecting Brownian motion has been constructed by Bass and Hsu in [5, 6], which enables us to derive the above gradient estimates.

**Theorem 1.2.** *Let  $\Omega$  be a compact convex domain in  $\mathbb{R}^d$ . Then (1.1) and (1.2) hold for some constants  $C, c > 0$  and  $\rho(x, y) = |x - y|$ .*

The proofs of these two theorems will be given in the next two sections respectively. In Section 4, we introduce some applications of our results to the study of the Hardy spaces, Riesz transforms associated to the Neumann Laplacian, and regularity of solutions to the inhomogeneous Neumann problem on compact convex domains.

## 2 Proof of Theorem 1.1

We first observe that for the proof of (1.1) it suffices to consider  $t \in (0, 1]$ . Let  $dx$  denote the Riemannian volume measure on  $\Omega$ . Since  $\nabla P_t f = \nabla P_t \hat{f}$ , where  $\hat{f} := f - \int_{\Omega} f(x) dx$ , we may assume that  $f$  itself has zero integral on  $\Omega$ . Let  $\lambda_1 (> 0)$  be the first Neumann eigenvalue on  $\Omega$ . We have

$$(2.1) \quad \|P_t f^2\|_2 \leq e^{-\lambda_1 t} \|f\|_2, \quad t > 0.$$

On the other hand, by taking e.g.  $s = t$  in [19, Corollary 1.4], we obtain

$$|B(x, \sqrt{t})| \cdot |P_t f(x)| \leq C_1 \int_{B(x, \sqrt{t})} P_{2t} |f|(y) dy \leq C_2 \|f\|_1, \quad t \in (0, 1]$$

for some constant  $C_1 > 0$ , where  $B(x, \sqrt{t}) := \{y \in \Omega : \rho(x, y) \leq \sqrt{t}\}$  with volume  $|B(x, \sqrt{t})|$ . Since  $\Omega$  is compact, we have  $|B(x, \sqrt{t})| \geq ct^{d/2}$  for some constant  $c > 0$  and all  $t \geq 0, x \in \Omega$ . Therefore,

$$(2.2) \quad \|P_t\|_{1 \rightarrow \infty} \leq \frac{C_2}{t^{d/2}}, \quad t \in (0, 1]$$

for some constant  $C_2 > 0$ . Now, if (1.1) holds for  $t \leq 1$ , then (2.1) and (2.2) yield

$$\begin{aligned} \|\nabla P_t f\|_{\infty} &= \|\nabla P_{1/2} P_{t-1} P_{1/2} f\|_{\infty} \leq C_3 \|P_{t-1} P_{1/2} f\|_2 \\ &\leq C_3 e^{-(t-1)\lambda_1} \|P_{1/2} f\|_2 \leq C_4 e^{-\lambda_1 t} \|f\|_1, \quad t > 1 \end{aligned}$$

for some constants  $C_3, C_4 > 0$ . Therefore, (1.1) also holds for  $t > 1$ .

From now on, we assume that  $t \leq 1$ . Let  $K, \sigma$  be two constants such that  $\text{Ric} \geq -K$  holds on  $\Omega$  and  $\mathbb{I} \geq -\sigma$  holds on  $\partial\Omega$ . By [18, Theorem 1.1(7)], we have

$$(2.3) \quad |\nabla P_t f|^2 \leq \frac{2K^2}{(1 - e^{-2Kt})^2} (P_t f^2 - (P_t f)^2) \mathbb{E} \int_0^t e^{2\sigma l_s - 2Ks} ds, \quad f \in C(M),$$

where  $(l_s)_{s \geq 0}$  is the local time on  $\partial\Omega$  for the reflecting Brownian motion on  $\Omega$ . Next, according to [16, Proof of Lemma 2.1], there exists a constant  $c > 0$  such that

$$\mathbb{E} e^{\sigma l_s} \leq e^{c(s + \sqrt{s})}, \quad s \leq 1.$$

Therefore, it follows from (2.3) that

$$\|\nabla P_t f\|_{\infty}^2 \leq \frac{C_5}{t} \|P_t f^2\|_{\infty}, \quad f \in C(M)$$

holds for some constant  $C_5 > 0$ . Replacing  $f$  by  $P_t f$  and using (2.2), we arrive at

$$\|\nabla P_{2t}f\|_\infty \leq \frac{C_6}{t^{(d+1)/2}} \|f\|_1, \quad f \in C(M)$$

for some constant  $C_6 > 0$ . This proves (1.1) for  $t \leq 2$ .

Finally, since (1.1) is equivalent to

$$\sup_{x,y \in \Omega} |\nabla p_t(\cdot, y)(x)| \leq \frac{C}{t^{(d+1)/2}}, \quad t > 0,$$

the inequality (1.2) follows from the self-improvement property as in [10, Theorem 4.9].

### 3 Proof of Theorem 1.2

We shall make use of the reflecting Brownian motion determined as solutions to the Skorokhod equation

$$(3.1) \quad X_t = x + W_t + \int_0^t N(X_s) dl_s, \quad t \geq 0,$$

where  $x \in \Omega$ ,  $W_t$  is a  $d$ -dimensional Brownian motion on a complete filtrated probability space  $(\mathcal{E}, \mathcal{F}_t, \mathbb{P})$ ,  $X_t$  is a continuous adapted process on  $\Omega$ , and  $l_t$  is a predictable continuous increasing process with  $l_0 = 0$  which increases only when  $X_t \in \partial\Omega$ . If  $(X_t, l_t)$  solves (3.1) for some  $d$ -dimensional Brownian motion  $W_t$ , we call  $X_t$  the reflecting Brownian motion on  $\Omega$  starting from  $x$ , and call  $l_t$  its local time on  $\partial\Omega$ .

**Lemma 3.1.** *For any  $d$ -dimensional Brownian motion  $W_t$  and any  $x \in \Omega$ , (3.1) has a unique solution.*

*Proof.* (a) Uniqueness. Let  $(X_t, l_t)$  and  $(\tilde{X}_t, \tilde{l}_t)$  be two solutions to (3.1). By the Itô formula,

$$|X_t - \tilde{X}_t|^2 = 2 \int_0^t \langle X_s - \tilde{X}_s, N(X_s) \rangle dl_s + 2 \int_0^t \langle \tilde{X}_s - X_s, N(\tilde{X}_s) \rangle d\tilde{l}_s.$$

By the convexity of  $\partial\Omega$  we see that  $\langle y - z, N(y) \rangle \leq 0$  if  $y \in \partial\Omega$  and  $z \in \Omega$ . Moreover, since  $dl_s = 0$  (correspondingly,  $d\tilde{l}_s = 0$ ) for  $X_s \notin \partial\Omega$  (correspondingly,  $\tilde{X}_s \notin \partial\Omega$ ), we conclude that  $|X_t - \tilde{X}_t|^2 = 0$  for all  $t \geq 0$ .

(b) Existence. By using the regularity of the associated Dirichlet form, [6, Theorem 4.4] ensures the existence of a reflecting Brownian motion. According to [5, Theorem 1], this reflecting Brownian motion solves the Skorokhod equation (3.1) for some  $d$ -dimensional Brownian motion  $\tilde{W}_t$ , i.e. there exists  $(\tilde{X}_t, \tilde{l}_t)$  such that  $\tilde{X}_t$  is a continuous adapted process on  $\Omega$ ,  $\tilde{l}_t$  is a predictable continuous increasing process with  $\tilde{l}_0 = 0$  which increases only when  $\tilde{X}_t \in \partial\Omega$ , and

$$(3.2) \quad \tilde{X}_t = x + \tilde{W}_t + \int_0^t N(\tilde{X}_s) d\tilde{l}_s, \quad t \geq 0$$

holds. We aim to prove that for any  $d$ -dimensional Brownian motion  $W_t$ , the equation has a solution. Due to (a), this follows from the Yamada-Watanabe criterion [20] (cf. the proof of [7, Theorem 5.9]). For readers' convenience, we present below a brief proof. By the uniqueness of solutions to (3.2),  $(\tilde{X}, \tilde{l})$  is determined by  $\tilde{W}$ ; that is, there exists a measurable function

$$F = (F_1, F_2) : C([0, \infty); \mathbb{R}^d) \rightarrow C([0, \infty); \Omega) \times C_I([0, \infty); [0, \infty))$$

for  $\sigma$ -fields induced by the topology of locally uniform convergence, where  $C_I([0, \infty); [0, \infty))$  is the space of all non-negative continuous increasing functions on  $[0, \infty)$  with initial data 0, such that

$$(\tilde{X}, \tilde{l}) = F(\tilde{W}) = (F_1(\tilde{W}), F_2(\tilde{W})).$$

So, letting  $\mu$  be the Wiener measure on  $C([0, \infty); \mathbb{R}^d)$  (i.e. the distribution of  $\tilde{W}$ ), we have

$$F_1(\omega)_t = x + \omega_t + \int_0^t N(F_1(\omega)_s) dF_2(\omega)_s, \quad t \geq 0$$

for  $\mu$ -a.e.  $\omega \in C([0, \infty); \mathbb{R}^d)$ . Therefore, for any  $d$ -dimensional Brownian motion  $W$ , which has the same distribution  $\mu$ ,

$$F_1(W)_t = x + W_t + \int_0^t N(F_1(W)_s) dF_2(W)_s, \quad t \geq 0$$

holds a.s. This means that  $(X, l) := (F_1(W), F_2(W))$  solves the equation (3.1).  $\square$

**Lemma 3.2.** *Let  $X_t^x$  be the unique solution to (3.1) for  $X_0 = x \in \Omega$ . For  $f \in \mathcal{B}_b(\Omega)$ , the class of all bounded measurable functions on  $\Omega$ , let  $P_t f(x) := u(t, x)$  solve the Neumann heat equation*

$$(3.3) \quad \partial_t u(\cdot, x)(t) = \Delta u(t, \cdot)(x), \quad Nu(t, \cdot)|_{\partial\Omega} = 0, \quad u(0, \cdot) = f.$$

Then

$$P_t f(x) = \mathbb{E}f(X_{2t}^x), \quad t \geq 0, x \in \Omega.$$

*Proof.* Since  $u(t, x) := P_t f(x)$  satisfies (3.3) and  $X_t^x$  satisfies (3.1), by the Itô formula, the process

$$\Phi(s) := u(t - s, X_{2s}^x) = P_t f(x) + \int_0^s 2 \langle \nabla u(t - s, \cdot)(X_{2s}^x), dW_s \rangle, \quad s \in [0, t]$$

is a martingale. In particular,

$$P_t f(x) = \mathbb{E}\Phi(0) = \mathbb{E}\Phi(t) = \mathbb{E}f(X_{2t}^x).$$

□

*Proof of Theorem 1.2.* As explained in the proof of Theorem 1.1, it suffices to prove (1.1) for  $t \in (0, 1]$ . To this end, we shall make use of the reflecting Brownian motion introduced above.

Let  $P_t^0 f(x) = \mathbb{E}f(X_t^x)$ . To estimate the gradient of  $P_t f$ , let  $y \neq x$  be two points in  $\Omega$ , and let  $(X_t^x, l_t^x)$  and  $(X_t^y, l_t^y)$  solve (3.1) with  $X_0 = x$  and  $y$  respectively. Then, as explained in the proof of Lemma 3.1, the convexity of  $\Omega$  implies that

$$|X_t^x - X_t^y| \leq |x - y|, \quad t \geq 0.$$

Thus,

$$\frac{|P_t^0 f(y) - P_t^0 f(x)|}{|x - y|} \leq \frac{\mathbb{E}|f(X_t^x) - f(X_t^y)|}{|x - y|} \leq \mathbb{E}\left(\frac{|f(X_t^x) - f(X_t^y)|}{|X_t^x - X_t^y|}\right).$$

Letting  $y \rightarrow x$  we arrive at

$$(3.4) \quad |\nabla P_t^0 f| \leq P_t^0 |\nabla f|, \quad t \geq 0.$$

Due to an argument of Bakry-Emery [4], this implies that

$$(3.5) \quad t|\nabla P_t^0 f|^2 \leq P_t^0 f^2 - (P_t^0 f)^2, \quad t > 0, f \in \mathcal{B}_b(\Omega).$$

Indeed, for a smooth function  $g$  on  $\Omega$  satisfying the Neumann boundary condition, (3.1) and the Itô formula for  $g(X_t^x)$  imply that

$$(3.6) \quad P_t^0 g = \frac{1}{2} \int_0^t P_s \Delta g ds.$$

Since  $P_t = P_{2t}^0$ , by Lemma 3.2 we have

$$\frac{d}{dt} P_{2t}^0 g = \Delta P_{2t}^0 g.$$

Combining this with (3.6) we obtain

$$(3.7) \quad \frac{d}{ds} P_s^0 g = \frac{1}{2} \Delta P_s^0 g = \frac{1}{2} P_s^0 \Delta g.$$

Hence, it follows from (3.4) and the Jensen inequality that

$$(3.8) \quad t |\nabla P_t^0 f|^2 = \int_0^t |\nabla P_s^0 (P_{t-s}^0 f)|^2 ds \leq \int_0^t P_s^0 |\nabla P_{t-s}^0 f|^2 ds.$$

(3.7) also implies that  $\frac{d}{ds} (P_s^0 (P_{t-s}^0 f)^2) = P_s^0 |\nabla P_{t-s}^0 f|^2$ . This, together with (3.8), shows that

$$P_t^0 f^2 - (P_t^0 f)^2 = \int_0^t \frac{d}{ds} P_s^0 (P_{t-s}^0 f)^2 ds = \int_0^t P_s^0 |\nabla P_{t-s}^0 f|^2 ds \geq t |\nabla P_t^0 f|^2.$$

So, (3.5) holds. Combining (3.5) with the known uniform upper bound of the Neumann heat kernel on convex domains (see e.g. [11, Theorem 3.2.9])

$$(3.9) \quad \|P_t f\|_\infty \leq C t^{-d/2} \int_\Omega |f|(x) dx, \quad t \in (0, 1], f \in \mathcal{B}_b(\Omega),$$

we conclude that

$$\begin{aligned} \|\nabla P_t f\|_\infty &= \|\nabla P_t^0 (P_{t/2} f)\|_\infty \leq \frac{1}{\sqrt{t}} \sqrt{P_t^0 [(P_{t/2} f)^2]} \\ &\leq \frac{C}{\sqrt{t}} \|P_{t/2} f\|_\infty \leq C t^{-(d+1)/2} \int_\Omega |f|(x) dx, \quad f \in \mathcal{B}_b(\Omega) \end{aligned}$$

holds for some  $C > 0$  and all  $t \in (0, 1]$ . This implies the desired gradient estimate for  $t \in (0, 1]$ .  $\square$

## 4 Applications

Throughout this section, we let  $\Omega$  be a compact convex domain in  $\mathbb{R}^d$ , and let  $\Omega^\circ$  be the interior of  $\Omega$ . It is well known that the generator  $(\Delta, \mathcal{D}(\Delta))$  of  $P_t$  in  $L^2(\Omega)$  is a negatively definite self-adjoint operator with discrete spectrum. Let  $\Delta_N = -\Delta$ , which is thus a positive definite self-adjoint operator such that  $P_t = e^{-\Delta_N t}$ ,  $t \geq 0$ .

## 4.1 The Hardy spaces on compact convex domains

For  $0 < p \leq 1$  we let  $h^p(\mathbb{R}^d)$  denote the classical (local) Hardy space in  $\mathbb{R}^d$ . We consider two versions of this space adapted to  $\Omega$ . Let  $\mathcal{D}(\Omega)$  denote the space of  $C^\infty$  functions with compact support in  $\Omega^\circ$ , and let  $\mathcal{D}'(\Omega)$  denote its dual, the space of distributions on  $\Omega^\circ$ . The first adaptation of the local Hardy space to  $\Omega$ , denoted by  $h_r^p(\Omega)$ , consists of elements of  $\mathcal{D}'(\Omega)$  which are the restrictions to  $\Omega^\circ$  of elements of  $h^p(\mathbb{R}^d)$ . That is, for  $0 < p \leq 1$  we set

$$\begin{aligned} h_r^p(\Omega) &:= \{f \in \mathcal{D}'(\mathbb{R}^d) : \text{there exists } F \in h^p(\mathbb{R}^d) \text{ such that } F|_{\Omega^\circ} = f\} \\ &= h^p(\mathbb{R}^d) / \{F \in h^p(\mathbb{R}^d) : F = 0 \text{ in } \Omega^\circ\}, \end{aligned}$$

which is equipped with the quasi-norm

$$\|f\|_{h_r^p(\Omega)} := \inf \{ \|F\|_{h^p(\mathbb{R}^d)} : F \in h^p(\mathbb{R}^d) \text{ such that } F|_{\Omega^\circ} = f \}.$$

The second adaptation of the local Hardy space to  $\Omega$ , denoted by  $h_z^p(\Omega)$ , consists of distributions in  $\Omega^\circ$  with extension by zero to  $\mathbb{R}^d$  belonging to  $h^p(\mathbb{R}^d)$ . More specifically, for  $0 < p \leq 1$ ,

$$h_z^p(\Omega) := h^p(\mathbb{R}^d) \cap \{f \in h^p(\mathbb{R}^d) : f = 0 \text{ in } \Omega^c\} / \{f \in h^p(\mathbb{R}^d) : f = 0 \text{ in } \Omega^\circ\},$$

where  $\Omega^c := \mathbb{R}^d \setminus \Omega$ . We can identify  $h_z^p(\Omega)$  with a set of distributions in  $\mathcal{D}'(\Omega)$  which, when equipped with the natural quotient norm, becomes a subspace of  $h_r^p(\Omega)$  (see [8] for more details). Obviously, we have that  $h_z^p(\Omega) \subseteq h_r^p(\Omega)$  whenever  $d/(d+1) < p \leq 1$ .

Given a function  $f \in L^2(\Omega)$ , consider the following local version of the non-tangential maximal operator associated with the heat semigroup generated by the operator  $\Delta_N$ :

$$N_{\text{loc}, \Delta_N} f(x) := \sup_{y \in \Omega, |y-x| < t \leq 1} |e^{-t^2 \Delta_N} f(y)|, \quad x \in \Omega.$$

For  $0 < p \leq 1$ , the space  $h_{\Delta_N}^p(\Omega)$  is then defined as the completion of  $L^2(\Omega)$  in the quasi-norm

$$(4.1) \quad \|f\|_{h_{\Delta_N}^p(\Omega)} := \|N_{\text{loc}, \Delta_N} f\|_{L^p(\Omega)}.$$

Based on the gradient estimate (1.2) in Theorem 1.2 and the adapted atomic theory of Hardy spaces  $h_{\Delta_N}^p(\Omega)$ , the following result is obtained in [12], see also [3, 8, 14, 12] and references within for discussions on the Hardy spaces  $h_r^p(\Omega)$ .

**Theorem 4.1** ([12]). *Let  $\Omega$  be a compact convex domain in  $\mathbb{R}^d$ . Then  $h_z^p(\Omega) = h_{\Delta_N}^p(\Omega)$  for  $d/(d+1) < p \leq 1$ .*



## 4.2 Riesz transforms on bounded convex domains

Consider the generalised Riesz transform  $T = \nabla \Delta_N^{-1/2}$  associated to the Neumann Laplacian  $\Delta_N$ , defined by

$$Tf = \frac{2}{\sqrt{\pi}} \int_0^\infty \nabla e^{-s\Delta_N} f \frac{ds}{\sqrt{s}}.$$

The operator  $\nabla \Delta_N^{-1/2}$  is bounded on  $L^2(\Omega)$ . In [9], it is shown that the operator  $\nabla \Delta_N^{-1/2}$  is of weak (1,1) by making use of (3.9), hence by interpolation, is bounded on  $L^p(\Omega)$  for  $1 < p \leq 2$ . For the case  $p > 2$ , according to [2], the following assertions are equivalent:

(1) For all  $p \in (2, \infty)$ , there exists  $C_p$  such that

$$\|\|\nabla e^{-t\Delta_N}\|\|_{p \rightarrow p} \leq \frac{C}{\sqrt{t}}, \quad \forall t > 0.$$

(2) The Riesz transform  $\nabla \Delta_N^{-1/2}$  is bounded on  $L^p(\Omega)$  for  $p \in (2, \infty)$ .

In terms of the gradient estimate (1.2) in Theorem 1.2, it deduces the following theorem.

**Theorem 4.2.** *Let  $\Omega$  be a compact convex domain in  $\mathbb{R}^d$ . Let  $T = \nabla \Delta_N^{-1/2}$  be the Riesz transform associated to the Neumann Laplacian  $\Delta_N$  on  $\Omega$ . Then the operator  $\nabla \Delta_N^{-1/2}$  is bounded on  $L^p(\Omega)$  for all  $1 < p < \infty$ .*

*If  $p = 1$ , then the operator  $\nabla \Delta_N^{-1/2}$  is also of weak type (1, 1).*

By (1.2) and (3.9), we can also extend the Riesz transform  $\nabla \Delta_N^{-1/2}$  to a bounded operator from  $h_{\Delta_N}^p(\Omega)$  into  $L^p(\Omega)$  for  $0 < p \leq 1$ . Hence by Theorem 4.1, it can be extended to a bounded operator from  $h_z^p(\Omega)$  into  $L^p(\Omega)$  for  $d/(d+1) < p \leq 1$ . The proof is similar to that of [12, Theorem 4.2].

## 4.3 Regularity of solutions to the Neumann problem

Define the Neumann Green operator  $\mathbb{G}_N$  as the solution operator  $C^\infty(\Omega) \ni f \mapsto u = \mathbb{G}_N(f) \in W^{1,2}(\Omega)$  for the Neumann problem

$$(4.2) \quad \begin{cases} \Delta u = f & \text{in } \Omega^\circ \\ Nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where it is also assumed that  $\int_\Omega f = 0$  and the solution is normalized by requiring that  $\int_\Omega u = 0$ .

### 4.3.1 Estimate for the gradient of Green potential

Let  $W^{s,p}(\Omega)$  stand for the Sobolev space of functions in  $L^p(\Omega)$  with distributional derivatives of order  $s$  in  $L^p(\Omega)$ . By  $L^p_{\perp}(\Omega)$  and  $W^{s,p}_{\perp}(\Omega)$  we denote the subspaces of functions  $f$  in  $L^p(\Omega)$  and  $W^{s,p}(\Omega)$  subject to  $\int_{\Omega} f(x)dx = 0$ .

Recently, V. Maz'ya proved the following result.

**Theorem 4.3** ([15]). *Let  $\Omega$  be a convex domain in  $\mathbb{R}^d$ . Let  $f \in L^q_{\perp}(\Omega)$  with a certain  $q > d$ . Then there exists a constant  $c$  depending only on  $d$  and  $q$  such that the solution  $u \in W^{1,2}_{\perp}(\Omega)$  of the problem (4.2) satisfies the estimate*

$$(4.3) \quad \|\nabla u\|_{L^{\infty}(\Omega)} \leq c(d, q)C_{\Omega}^{-1}|\Omega|^{(q-d)/qd}\|f\|_{L^q(\Omega)}.$$

Note that by Theorem 1.2, we can give a simple proof of Theorem 4.3. Indeed, based on the gradient estimate (1.2) of  $p_t$  in Theorem 1.2, we have

$$|\nabla G_N(x, y)| \leq \frac{C}{|x - y|^{d-1}}, \quad \forall x, y \in \Omega,$$

where  $G_N$  is the Green function for the Neumann semigroup on  $\Omega$ , and hence by a standard argument, estimate (4.3) follows readily.

### 4.3.2 Estimate for the second-order derivatives for Green potential

Let  $\mathbb{G}_N$  be the Neumann Green operator for the Neumann problem in (4.2). Note that  $L^2$ -boundedness of the mappings

$$f \mapsto \frac{\partial^2 \mathbb{G}_N(f)}{\partial x_i \partial x_j}, \quad 1 \leq i, j \leq d,$$

has been known since the mid 1970's ([13]), but optimal  $L^p$  estimates, valid in the range  $1 < p \leq 2$ , have only been proved in the 1990's by Adolfsson and D. Jerison [1]. It should be mentioned that the aforementioned  $L^p$  continuity of two derivatives on Green potentials may fail in the class of Lipschitz domains for any  $p \in (1, \infty)$  and in the class of convex domains for any  $p \in (2, \infty)$  (see [1] for counterexamples; recall that every convex domain is Lipschitz).

A natural question is to study the regularity of the Neumann Green operator when the  $L^p$ -scale is replaced by the scale of Hardy spaces,  $H^p$ , for  $0 < p \leq 1$ . Recently, X. Duong, S. Hofmann, D. Mitrea, M. Mitrea and the second named author of this article gave a solution to the conjecture made by D.-C. Chang, S. Krantz and E.M. Stein([8]) regarding the regularity of Green operators for the Neumann problems on  $h_r^p(\Omega)$  and  $h_z^p(\Omega)$ , respectively, for all  $\frac{d}{d+1} < p \leq 1$ , and this range of  $p$ 's is sharp (see [12]).

**Theorem 4.4** ([12]). *Let  $\Omega$  be a compact, simply connected, convex domain in  $\mathbb{R}^d$  and recall that  $\mathbb{G}_N$  stands for the Green operator associated with the inhomogeneous Neumann problem (4.2). Then the operators*

$$(4.4) \quad \frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, d,$$

*originally defined on  $\{f \in C^\infty(\Omega) : \int_\Omega f \, dx = 0\}$ , extend as bounded linear mappings from  $h_z^p(\Omega)$  to  $h_r^p(\Omega)$  whenever  $d/(d+1) < p \leq 1$ .*

*If  $p = 1$ , then the operators  $\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}$ ,  $i, j = 1, \dots, d$ , are also of weak type  $(1, 1)$ . Hence by interpolation, they can be extended to bounded operators on  $L^p(\Omega)$  for  $1 < p \leq 2$ .*

The proof of this theorem is obtained by using suitable estimates for singular integrals with non-smooth kernels and an optimal on-diagonal heat kernel estimate. We should mention that the gradient estimate (1.2) played a major role in the proof of this theorem, regarding the regularity of Green operators for the Neumann problems on  $h_r^p(\Omega)$  and  $h_z^p(\Omega)$ , respectively, for all  $\frac{d}{d+1} < p \leq 1$ . For the detail of the proof, we refer the reader to [12].

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