An asymptotic equivalence between two frame perturbation theorems

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Abstract In this paper, two stability results regarding exponential frames are compared. The theorems, (one proven herein, and the other in [3]), each give a constant such that if $\sup_{n\in\mathbb{Z}} \|\varepsilon_n\|_\infty < C$, and $(e^{i\langle\cdot,t_n\rangle})_{n\in\mathbb{Z}^d}$ is a frame for $L_2[-\pi,\pi]^d$, then $(e^{i\langle\cdot,t_n+\varepsilon_n\rangle})_{n\in\mathbb{Z}^d}$ is a frame for $L_2[-\pi,\pi]^d$. These two constants are shown to be asymptotically equivalent for large values of d.

1 The perturbation theorems

We define a frame for a separable Hilbert space H to be a sequence $(f_n)_n \subset H$ such that for some $0 < A \leq B$,

$$A^2\|f\|^2 \leq \sum_n |\langle f, f_n\rangle|^2 \leq B^2\|f\|^2, \quad f \in H.$$

The best A^2 and B^2 satisfying the inequality above are said to be the frame bounds for the frame. If $(e_n)_n$ is an orthonormal basis for H, the synthesis operator $Le_n = f_n$ is bounded, linear, and onto, iff $(f_n)_n$ is a frame. Equivalently, $(f_n)_n$ is a frame iff the operator L^* is an isomorphic embedding, (see [2]). In this case, A and B are the best constants such that

$$A\|f\| \leq \|L^* f\| \leq B\|f\|, \quad f \in H.$$

The simplest stability result regarding exponential frames for $L_2[-\pi,\pi]$ is the theorem below, which follows immediately from [4, Theorem 13, p 160].

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Theorem 1. *Let* $(t_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ *be a sequence such that* $(h_n)_{n\in\mathbb{Z}} := \left(\frac{1}{\sqrt{2\pi}} e^{it_n x}\right)_{n\in\mathbb{Z}}$ *is a frame for* $L_2[-\pi,\pi]$ *with frame bounds* A^2 *and* B^2. *If* $(\tau_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ *and* $(f_n)_{n\in\mathbb{Z}} := \left(\frac{1}{\sqrt{2\pi}} e^{i\tau_n x}\right)_{n\in\mathbb{Z}}$ *is a sequence such that*

$$\sup_{n\in\mathbb{Z}} |\tau_n - t_n| < \frac{1}{\pi} \ln\left(1 + \frac{A}{B}\right), \tag{1}$$

then the sequence $(f_n)_{n\in\mathbb{Z}}$ *is also a frame for* $L_2[-\pi,\pi]$.

The following theorem is a very natural generalization of Theorem 1 to higher dimensions.

Theorem 2. *Let* $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ *be a sequence such that* $(h_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{\langle(\cdot),t_k\rangle}\right)_{k\in\mathbb{N}}$ *is a frame for* $L_2[-\pi,\pi]^d$ *with frame bounds* A^2 *and* B^2. *If* $(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ *and* $(f_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle(\cdot),\tau_k\rangle}\right)_{k\in\mathbb{N}}$ *is a sequence such that*

$$\sup_{k\in\mathbb{N}} \|\tau_k - t_k\|_\infty < \frac{1}{\pi d} \ln\left(1 + \frac{A}{B}\right), \tag{2}$$

then the sequence $(f_k)_{k\in\mathbb{N}}$ *is also a frame for* $L_2[-\pi,\pi]^d$.

The proof of Theorem 2 relies on the following lemma:

Lemma 1. *Choose* $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ *such that* $(h_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{\langle(\cdot),t_k\rangle}\right)_{k\in\mathbb{N}}$ *satisfies*

$$\left\| \sum_{k=1}^{n} a_k h_k \right\|_{L_2[-\pi,\pi]^d} \le B\left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2}, \quad \text{for all} \quad (a_k)_{k=1}^n \subset \mathbb{C}.$$

If $(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$, *and* $(f_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle(\cdot),\tau_k\rangle}\right)_{k\in\mathbb{N}}$, *then for all* $r,s \ge 1$ *and any finite sequence* $(a_k)_k$, *we have*

$$\left\| \sum_{k=r}^{s} a_k (h_k - f_k) \right\|_{L_2[-\pi,\pi]^d} \le B\left(e^{\pi d\left(\sup_{r\le k\le s} \|\tau_k - t_k\|_\infty\right)} - 1\right)\left(\sum_{k=r}^{s} |a_k|^2\right)^{\frac{1}{2}}.$$

This lemma is a slight generalization of Lemma 5.3, proven in [1] using simple estimates. Lemma 1 is proven similarly. Now for the proof of Theorem 2.

Proof. Define $\delta = \sup_{k\in\mathbb{N}} \|\tau_k - t_k\|_\infty$. Lemma 1 shows that the map $\tilde{L}e_n = f_n$ is bounded and linear, and that

$$\|L - \tilde{L}\| \le B\left(e^{\pi d\delta} - 1\right) := \beta A$$

for some $0 \le \beta < 1$. This implies

$$\|L^* f - \tilde{L}^* f\| \le \beta A, \quad \text{when} \quad \|f\| = 1. \tag{3}$$

Rearranging, we have

$$A(1-\beta) \leq \|\tilde{L}^* f\|, \quad \text{when} \quad \|f\| = 1.$$

By the previous remarks regarding frames, $(f_k)_{k\in\mathbb{N}}$ is a frame for $L_2[-\pi,\pi]^d$.

Theorem 3, proven in [3], is a more delicate frame perturbation result with a more complex proof:

Theorem 3. *Let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $(h_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{\langle(\cdot),t_k\rangle}\right)_{k\in\mathbb{N}}$ is a frame for $L_2[-\pi,\pi]^d$ with frame bounds A^2 and B^2. For $d \geq 1$, define*

$$D_d(x) := \left(1 - \cos\pi x + \sin\pi x + \frac{\sin\pi x}{\pi x}\right)^d - \left(\frac{\sin\pi x}{\pi x}\right)^d,$$

and let x_d be the unique number such that $0 < x_d \leq 1/4$ and $D_d(x_d) = \frac{A}{B}$. If $(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ and $(f_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle(\cdot),\tau_k\rangle}\right)_{k\in\mathbb{N}}$ is a sequence such that

$$\sup_{k\in\mathbb{N}} \|\tau_k - t_k\|_\infty < x_d, \tag{4}$$

then the sequence $(f_k)_{k\in\mathbb{N}}$ is also a frame for $L_2[-\pi,\pi]^d$.

2 An asymptotic equivalence

It is natural to ask how the constants x_d and $\frac{1}{\pi d}\ln\left(1+\frac{A}{B}\right)$ are related. Such a relationship is given in the following theorem.

Theorem 4. *If x_d is the unique number satisfying $0 < x_d < 1/4$ and $D_d(x_d) = \frac{A}{B}$, then*

$$\lim_{d\to\infty} \frac{x_d - \frac{1}{\pi d}\ln\left(1+\frac{A}{B}\right)}{\frac{\left[\ln\left(1+\frac{A}{B}\right)\right]^2}{6\pi\left(1+\frac{B}{A}\right)d^2}} = 1.$$

We prove the theorem with a sequence of propositions.

Proposition 1. *Let d be a positive integer. If*

$$f(x) := 1 - \cos(x) + \sin(x) + \operatorname{sinc}(x),$$
$$g(x) := \operatorname{sinc}(x),$$

then

1) $f'(x) + g'(x) > 0, \quad x \in (0, \pi/4),$
2) $g'(x) < 0, \quad x \in (0, \pi/4),$
3) $f''(x) > 0, \quad x \in (0, \Delta) \quad \text{for some} \quad 0 < \Delta < 1/4.$

The proof of Proposition 1 involves only elementary calculus and is omitted.

Proposition 2. *The following statements hold:*
1) For $d > 0$, $D_d(x)$ *and* $D'_d(x)$ *are positive on* $(0,1/4)$.
2) For all $d > 0$, $D''_d(x)$ *is positive on* $(0,\Delta)$.

Proof. Note $D_d(x) = f(\pi x)^d - g(\pi x)^d$ is positive. This expression yields

$$D'_d(x)/(d\pi) = f(\pi x)^{d-1} f'(\pi x) - g(\pi x)^{d-1} g'(\pi x) > 0 \quad \text{on} \quad (0,1/4)$$

by Proposition 1. Differentiating again, we obtain

$$\begin{aligned} D''_d(x)/(d\pi^2) = (d-1)\big[f(\pi x)^{d-2}(f'(\pi x))^2 - g(\pi x)^{d-2}(g'(\pi x))^2\big] + \\ + \left[f(\pi x)^{d-1} f''(\pi x) - g(\pi x)^{d-1} g''(\pi x)\right] \quad \text{on} \quad (0,1/4). \end{aligned}$$

If $g''(\pi x) \le 0$ for some $x \in (0,1/4)$, then the second bracketted term is positive. If $g''(\pi x) > 0$ for some $x \in (0,1/4)$, then the second bracketted term is positive if $f''(\pi x) - g''(\pi x) > 0$, but

$$f''(\pi x) - g''(\pi x) = \pi^2(\cos(\pi x) - \sin(\pi x))$$

is positive on $(0,1/4)$.
To show the first bracketted term is positive, it suffices to show that

$$f'(\pi x)^2 > g'(\pi x)^2 = (f'(\pi x) + g'(\pi x))(f'(\pi x) - g'(\pi x)) > 0$$

on $(0,\Delta)$. Noting $f'(\pi x) - g'(\pi x) = \pi(\cos(\pi x) + \sin(\pi x)) > 0$, it suffices to show that $f'(\pi x) + g'(\pi x) > 0$, but this is true by Proposition 1.

Note that Proposition 2 implies x_d is unique.

Corollary 1. *We have* $\lim_{d\to\infty} x_d = 0$.

Proof. Fix $n > 0$ with $1/n < \Delta$, then $\lim_{d\to\infty} D_d(1/n) = \infty$ (since f increasing implies $0 < -\cos(\pi/n) + \sin(\pi/n) + \mathrm{sinc}(\pi/n)$). For sufficiently large d, $D_d(1/n) > \frac{A}{B}$. But $\frac{A}{B} = D_d(x_d) < D_d(1/n)$, so $x_d < 1/n$ by Proposition 2.

Proposition 3. *Define* $\omega_d = \frac{1}{\pi d} \ln\left(1 + \frac{A}{B}\right)$. *We have*

$$\lim_{d\to\infty} d\left(\frac{A}{B} - D_d(\omega_d)\right) = \frac{A}{6B}\left[\ln\left(1 + \frac{A}{B}\right)\right]^2,$$

$$\lim_{d\to\infty} \frac{1}{d} D'_d(\omega_d) = \pi\left(1 + \frac{A}{B}\right),$$

$$\lim_{d\to\infty} \frac{1}{d} D'_d(x_d) = \pi\left(1 + \frac{A}{B}\right).$$

Proof. 1) For the first equality, note that

$$D_d(\omega_d) = \left[(1+h(x))^{\ln(c)/x} - g(x)^{\ln(c)/x}\right]\Big|_{x=\frac{\ln(c)}{d}} \tag{5}$$

where $h(x) = -\cos(x) + \sin(x) + \mathrm{sinc}(x)$, $g(x) = \mathrm{sinc}(x)$, and $c = 1 + \frac{A}{B}$. L'Hospital's rule implies that

$$\lim_{x\to 0}(1+h(x))^{\ln(c)/x} = c \quad \text{and} \quad \lim_{x\to 0} g(x)^{\ln(c)/x} = 1.$$

Looking at the first equality in the line above, another application of L'Hospital's rule yields

$$\lim_{x\to 0}\frac{(1+h(x))^{\ln(c)/x} - c}{x} = c\ln(c)\left[\frac{\frac{h'(x)}{1+h(x)} - 1}{x} - \frac{\ln(1+h(x)) - x}{x^2}\right]. \tag{6}$$

Observing that $h(x) = x + x^2/3 + O(x^3))$, we see that

$$\lim_{x\to 0}\frac{\frac{h'(x)}{1+h(x)} - 1}{x} = -\frac{1}{3}.$$

L'Hospital's rule applied to the second term on the right hand side of equation (6) gives

$$\lim_{x\to 0}\frac{(1+h(x))^{\ln(c)/x} - c}{x} = \frac{-c\ln(c)}{6}. \tag{7}$$

In a similar fashion,

$$\lim_{x\to 0}\frac{g(x)^{\ln(c)/x} - 1}{x} = \ln(c)\lim_{x\to 0}\left[\frac{\frac{g'(x)}{g(x)}}{x} - \frac{\ln(g(x))}{x^2}\right]. \tag{8}$$

Observing that $g(x) = 1 - x^2/6 + O(x^4)$, we see that

$$\lim_{x\to 0}\frac{\frac{g'(x)}{g(x)}}{x} = -\frac{1}{3}.$$

L'Hospital's rule applied to the second term on the right hand side of equation (8) gives

$$\lim_{x\to 0}\frac{g(x)^{\ln(c)/x} - 1}{x} = -\frac{\ln(c)}{6}. \tag{9}$$

Combining equations (5) (7), and (9), we obtain

$$\lim_{d\to\infty} d\left(\frac{A}{B} - D_d(\omega_d)\right) = \frac{A}{6B}\left[\ln\left(1 + \frac{A}{B}\right)\right]^2.$$

2) For the second equality we have, (after simplification),

$$\frac{1}{d}D_d'(\omega_d)=\pi\left[\frac{\left(1+h\left(\frac{\ln(c)}{d}\right)\right)^{\left(\ln(c)\right)/\left(\frac{\ln(c)}{d}\right)}}{1+h\left(\frac{\ln(c)}{d}\right)}-\frac{g\left(\frac{\ln(c)}{d}\right)^{\left(\ln(c)\right)/\left(\frac{\ln(c)}{d}\right)}}{g\left(\frac{\ln(c)}{d}\right)}g'\left(\frac{\ln(c)}{d}\right)\right].$$

In light of the previous work, this yields

$$\lim_{d\to\infty}\frac{1}{d}D_d'(\omega_d)=\pi\left(1+\frac{A}{B}\right).$$

3) To derive the third equality, note that $(1+h(\pi x_d))^d=\frac{A}{B}+g(\pi x_d)^d$ yields

$$\frac{1}{d}D_d'(x_d)=\pi\left[\frac{\frac{A}{B}+g(\pi x_d)^d}{1+h(\pi x_d)}h'(\pi x_d)-\frac{g(\pi x_d)^d}{g(\pi x)}g'(\pi x_d)\right]. \tag{10}$$

Also, the first inequality in propostion 3 yields that, for sufficiently large d (also large enough so that $x_d<\Delta$ and $\omega_d<\Delta$), that $D_d(\omega_d)<\frac{A}{B}=D_d(x_d)$. This implies $\omega_d<x_d$ since D_d is increasing on $(0,1/4)$. But D_d is also convex on $(0,\Delta)$, so we can conclude that

$$D_d'(\omega_d)<D_d'(x_d). \tag{11}$$

Combining this with equation (10), we obtain

$$\left[\frac{1}{d}D_d'(\omega_d)+\frac{\pi g(\pi x_d)^d}{g(\pi x_d)}g'(\pi x_d)\right]\left(\frac{1+h(\pi x_d)}{h'(\pi x_d)}\right)<\pi\left(\frac{A}{B}+g(\pi x_d)^d\right)<\pi\left(1+\frac{A}{B}\right).$$

The limit as $d\to\infty$ of the left hand side of the above inequality is $\pi\left(1+\frac{A}{B}\right)$, so

$$\lim_{d\to\infty}\pi\left(\frac{A}{B}+g(\pi x_d)^d\right)=\pi\left(1+\frac{A}{B}\right).$$

Combining this with equation (10), we obtain

$$\lim_{d\to\infty}\frac{1}{d}D_d'(x_d)=\pi\left(1+\frac{A}{B}\right).$$

Now we complete the proof of Theorem 4.
For large d, the mean value theorem implies

$$\frac{D_d(x_d)-D_d(\omega_d)}{x_d-\omega_d}=D_d'(\xi),\quad \xi\in(\omega_d,x_d),$$

so that

$$x_d-\omega_d=\frac{\frac{A}{B}-D_d(\omega_d)}{D_d'(\xi)}.$$

For large d, convexity of D_d on $(0,\Delta)$ implies

$$\frac{d\left(\frac{A}{B} - D_d(\omega_d)\right)}{\frac{1}{d}D_d'(x_d)} < d^2(x_d - \omega_d) < \frac{d\left(\frac{A}{B} - D_d(\omega_d)\right)}{\frac{1}{d}D_d'(\omega_d)}.$$

Applying Proposition 3 proves the theorem.

References

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