

Calculation of generalized Hubbell rectangular source integral

Jonathan Murley and Nasser Saad*

Department of Mathematics and Statistics, University of Prince Edward Island
Charlottetown, Prince Edward Island C1A 4P3, Canada

E-Mail: nsaad@upei.ca

September 13, 2010

Abstract

A simple formula for computing the generalized Hubbell radiation rectangular source integral

$$H \left[\begin{matrix} a, b, p, \lambda \\ \alpha, \beta, \gamma \end{matrix} \right] = \frac{\sigma a}{4\pi} \int_0^b x^\lambda (x^2 + p)^{-\alpha} {}_2F_1(\alpha, \beta; \gamma; -\frac{a^2}{x^2 + p}) dx,$$

is introduced. Tables are given to compare the numerical values derived from our approximation formula with those given earlier in the literature.

PACS: Primary 33C65, 33C05, 33C20 ; Secondary 33B15, 33C90, 03D20, 33C60, 33D60, 33D90.

keywords: Appell hypergeometric functions F_2 , Hubbell radiation rectangular source integral, Classical hypergeometric functions, Reduction formulas, Radiation field integrals.

1 Introduction

In their pioneering work, Hubbell et al. (1960) obtained a series expansion for the calculation of radiation field generated by a plane isotropic rectangular source (plaque), in which the leading term is the integral

$$I(a, b) = \frac{\sigma}{4\pi} \int_0^b \arctan \left(\frac{a}{\sqrt{x^2 + 1}} \right) \frac{dx}{\sqrt{x^2 + 1}}, \quad (1)$$

here σ is the uniform surface source strength per unit source area. In equation (1) the quantities $a = w/h$ and $b = l/h$ are defined in the range $0 < a \leq b \leq \infty$, where h is the height over the a corner of a plaque of length l and width w . For the important applications of this integral in many problems in radiation field, different methods were introduced to obtain numerical values of detector response to plaque source

$$h(a, b) = \frac{I(a, b)}{(\sigma/4\pi)} = \int_0^b \arctan \left(\frac{a}{\sqrt{x^2 + 1}} \right) \frac{dx}{\sqrt{x^2 + 1}} \quad (2)$$

*Corresponding author

For some of these methods we refer to the work of Glasser, 1984; Kalla et al., 1987; Ghose et al., 1988; Galué et al., 1988; Götze, 1995; Kalla, 1993; Timus, 1993; Michieli and Maximovic, 1996; Kalla and Khajah (1997, 2000); Prabha, 2001; Stalker, 2001; Guseinov et al., 2004; Ezure, 2005; Guseinov et al., 2005; Prabha, 2007.

Although $h(a, b)$ is not expressible in simple closed form, Glasser (1984) has evaluated it in terms of Appell's hypergeometric function F_2 (see for example Slater (1966), Ch. 8, for a study of Appell functions F_q , $q = 1, 2, 3, 4$). Indeed, elementary differentiation of (2), with respect to a , we have

$$h(a, b) = \int_0^a dx \int_0^b \frac{dy}{1+x^2+y^2} \quad (3)$$

and straightforward substitutions $x = a\sqrt{u}$ and $y = b\sqrt{v}$ allow us to write (3) as the double-integral representation of Appell's hypergeometric function F_2 (Slater (1966), Ch. 8, formula 8.2.3), consequently

$$h(a, b) = abF_2\left(1; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -a^2, -b^2\right). \quad (4)$$

Various generalizations of equation (1) have been given in the literature (see for example, Kalla et al., 1987; Galué et al., 1988; Saigo and Srivastava, 1990; Galué, 1991; Kalla, 1993; Galué et al., 1994, Kalla et al. 2002, Oner 2007). More specifically, Kalla et al. (1987) introduced a generalization defined by the integral

$$H \left[\begin{matrix} a, b, p, \lambda \\ \alpha, \beta, \gamma \end{matrix} \right] = \frac{\sigma a}{4\pi} \int_0^b x^\lambda (x^2 + p)^{-\alpha} {}_2F_1(\alpha, \beta; \gamma; -\frac{a^2}{x^2 + p}) dx \quad (5)$$

where $\gamma > \beta > 0$; $a, b, p > 0$; $-1 < \lambda < 2a - 1$; and ${}_2F_1(\alpha, \beta; \gamma; x)$ is Gauss hypergeometric function (Slater, 1966, Ch. 1). We notice that

$$H \left[\begin{matrix} a, b, 1, 0 \\ 1, \frac{1}{2}, \frac{3}{2} \end{matrix} \right] = I(a, b) \quad (6)$$

by virtue of the identity (Slater, 1966; formula 1.5.11)

$$x {}_2F_1\left(1, \frac{1}{2}; \frac{3}{2}; -x^2\right) = \arctan(x). \quad (7)$$

By selecting suitable values for the parameters α, β and γ , equation (5) can be reduced to different integrals with potential applications in radiation-field problems of specific configurations of source, barrier and detector (Kalla, 1993). Such results are also useful in illumination and heat-exchange engineering Boast, 1942; Fano et al., 1959; Hubbell, 1960. Using a simple transformation, $x = b\sqrt{u}$, equation (5) can be written as

$$H \left[\begin{matrix} a, b, p, \lambda \\ \alpha, \beta, \gamma \end{matrix} \right] = \frac{\sigma a b^{\lambda+1}}{4\pi 2p^\alpha} \int_0^1 u^{\frac{\lambda+1}{2}-1} (1-u)^{\frac{\lambda+3}{2}-\frac{\lambda+1}{2}-1} \left(1 - \frac{-b^2 u}{p}\right)^{-\alpha} {}_2F_1\left(\alpha, \beta; \gamma; \frac{-\frac{a^2}{p}}{1 - \left(-\frac{b^2}{p}\right)u}\right) du \quad (8)$$

which is easily compared with the single-integral representation of the Appell hypergeometric function F_2 (Opps et al., 2005, formula (2.6)) to yields

$$H \left[\begin{matrix} a, b, p, \lambda \\ \alpha, \beta, \gamma \end{matrix} \right] = \frac{\sigma a b^{\lambda+1}}{4\pi (\lambda+1)p^\alpha} F_2\left(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}\right) \quad (9)$$

Recently, Opps et al. (2009) establish a number of new recursion formulas for the Appell hypergeometric functions F_2 wherein some applications to the evaluation of some generalized radiation field integrals were discussed. The purpose of the present work is to continue our investigation of finding closed form and approximation formulas for effectively computing the radiation field integrals such as equations (1) and (5). In the next section, we develop a new approximation formula to evaluate precisely and to any desire degree of accuracy the generalized Hubbell radiation rectangular source integral (9). In section 3, numerical results and comparisons with previously reported values are presented.

2 The computation of $F_2(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p})$

May be one of the most important cases regarding the computations of the radiation field integrals that capture the interest of many researchers is evaluating the integral Eq. (1) effectively and precisely. Some researchers were able to evaluate $h(a, b)$ using rapidly convergent series (see the original work of Hubbell et al, 1960), further Gabutti et al. (1991) investigated $h(a, b)$ in terms of its series expansions while numerical computations of this integral have been carried out by Hanak and Cechak (1978), Götze (1995) developed an effective method for computing the Hubbell radiation rectangular source integral, Kalla and Khajah (1997) (see also Kalla and Khajah (2000)) used Tau Method to approximate $H(a, b)$, Stalker (2001) used new convergent series for evaluating $h(a, b)$ for large a and b and more recently, Ezure (2005) used Haselogrove method, Guseinov et al. (2004) used binomial expansion (see also Guseinov et al. (2005)), Prabha (2006) expressed again $h(a, b)$ using some recurrence relations (see also Prabha (2007)). For a survey of various methods in computing the Hubbell rectangular source integral Eq.(1) and its generalization, we refer to the work of Kalla et al. 2002.

In this section, we given a new approximation equation that can be used to compute $H[\frac{a,b,p,\lambda}{\alpha,\beta,\gamma}]$ to any degree of precision and, byproduct, we can, therefore, evaluate the Hubbell radiation rectangular source integral (1). Our approximation expression based on the following recurrence formula for F_2 (see Opps et al. (2009) for detailed proof.).

Theorem 1: For $|x| + |y| < 1; n \geq 0; \sigma, \alpha_1, \alpha_2 \in \mathbb{C}; \beta_1, \beta_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the Appell hypergeometric function F_2 satisfies the following identity

$$\begin{aligned} F_2(\sigma; \alpha_1, \alpha_2 - n; \beta_1, \beta_2; x, y) &= F_2(\sigma; \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) \\ &- \frac{\sigma y}{\beta_2} \sum_{k=1}^n F_2(\sigma + 1; \alpha_1, \alpha_2 - k + 1; \beta_1, \beta_2 + 1; x, y). \quad \square \quad (10) \end{aligned}$$

Writing $F_2(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p})$ as $F_2(\alpha; \beta, \frac{\lambda+3}{2} - 1; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p})$ and apply the recurrence relation (10), we obtain

$$\begin{aligned} F_2(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) &= F_2(\alpha; \beta, \frac{\lambda+3}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) \\ &- \frac{2\alpha y}{\lambda+3} F_2(\alpha+1; \beta, \frac{\lambda+3}{2}; \gamma, \frac{\lambda+5}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) \quad (11) \end{aligned}$$

By means of the identity (Slater, 1960; formula 8.3.1.3)

$$F_2(\sigma; \alpha_1, \beta_2; \beta_1, \beta_2; x, y) = (1-y)^{-\sigma} {}_2F_1(\sigma, \alpha_1; \beta_1; \frac{x}{1-y}), \quad (12)$$

we may now write Eq.(11) as

$$F_2(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) = (1 + \frac{b^2}{p})^{-\alpha} {}_2F_1(\alpha, \beta; \gamma; \frac{-a^2}{p+b^2}) - \frac{2\alpha y}{\lambda+3} F_2(\alpha+1; \beta, \frac{\lambda+3}{2}; \gamma, \frac{\lambda+5}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}). \quad (13)$$

Further, we may now regard $F_2(\alpha+1; \beta, \frac{\lambda+3}{2}; \gamma, \frac{\lambda+5}{2}; -\frac{a^2}{p}, -\frac{b^2}{p})$ as $F_2(\alpha+1; \beta, \frac{\lambda+5}{2}-1; \gamma, \frac{\lambda+5}{2}; -\frac{a^2}{p}, -\frac{b^2}{p})$ and apply the recurrence relation (10) again to obtain

$$F_2(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) = (1 + \frac{b^2}{p})^{-\alpha} {}_2F_1(\alpha, \beta; \gamma; \frac{-a^2}{p+b^2}) - \frac{2\alpha y}{\lambda+3} (1 + \frac{b^2}{p})^{-\alpha-1} {}_2F_1(\alpha+1, \beta; \gamma; \frac{-a^2}{p+b^2}) + \frac{2^2\alpha(\alpha+1)y^2}{(\lambda+3)(\lambda+5)} F_2(\alpha+2; \beta, \frac{\lambda+5}{2}; \gamma, \frac{\lambda+7}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) \quad (14)$$

After similar $n-2$ steps, we arrive at

$$F_2(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) = \sum_{k=0}^n (-1)^k \frac{(\alpha)_k}{(\frac{3+\lambda}{2})_k} \left(-\frac{b^2}{p}\right)^k \left(1 + \frac{b^2}{p}\right)^{-\alpha-k} {}_2F_1(\alpha+k, \beta; \gamma; \frac{-a^2}{p+b^2}) + \frac{(-1)^{n+1}(\alpha)_{n+1}y^{n+1}}{(\frac{3+\lambda}{2})_{n+1}} F_2(\alpha+n+1; \beta, \frac{\lambda+2n+3}{2}; \gamma, \frac{\lambda+2n+5}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) \quad (15)$$

where $(\alpha)_k$ denotes the Pochhammer symbol defined, in terms of Gamma functions, by

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} 1 & \text{if } (k=0; \alpha \in \mathbb{C} \setminus \{0\}), \\ \alpha(\alpha+1)(\alpha+2)\dots(\alpha+k-1) & \text{if } (k \in \mathbb{N}; \alpha \in \mathbb{C}), \end{cases}$$

here \mathbb{N} being the set of *positive* integers.

In principle, the computation of $F_2(\alpha+n+1; \beta, \frac{\lambda+2n+3}{2}; \gamma, \frac{\lambda+2n+5}{2}; -\frac{a^2}{p}, -\frac{b^2}{p})$ follows the same technique and consequently, for large n

$$F_2(\alpha; \beta, \frac{\lambda+1}{2}; \gamma, \frac{\lambda+3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}) \approx \sum_{k=0}^n \frac{(\alpha)_k}{(\frac{3+\lambda}{2})_k} \left(\frac{b^2}{p}\right)^k \left(1 + \frac{b^2}{p}\right)^{-\alpha-k} {}_2F_1(\alpha+k, \beta; \gamma; \frac{-a^2}{p+b^2}). \quad (16)$$

From which we now have for large n ,

$$H \left[\begin{matrix} a, b, p, \lambda \\ \alpha, \beta, \gamma \end{matrix} \right] \approx \frac{\sigma a}{4\pi} \frac{b^{\lambda+1}}{(\lambda+1)p^\alpha} \sum_{k=0}^n \frac{(\alpha)_k}{(\frac{3+\lambda}{2})_k} \left(\frac{b^2}{p}\right)^k \left(1 + \frac{b^2}{p}\right)^{-\alpha-k} {}_2F_1(\alpha+k, \beta; \gamma; \frac{-a^2}{p+b^2}) \quad (17)$$

since the two-terms asymptotic expansion of the Appell hypergeometric function F_2 developed by López and Pagola (2008) indicate that for large n , the Appell hypergeometric function

$$F_2(\alpha+n+1; \beta, \frac{\lambda+2n+3}{2}; \gamma, \frac{\lambda+2n+5}{2}; -\frac{a^2}{p}, -\frac{b^2}{p})$$

on the right-hand side of Eq.(15) approach zero.

3 Numerical results and discussion

In order to test our approximation formula (17) and to show that it indeed simplify much of the numerical complexity involved in calculating the radiation field integrals such as (1) and (5), we compare, first, our approximation formula against the exact equation obtained earlier in computing

$H \left[\begin{matrix} a, b, 1, 0 \\ \frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \right]$ (Opps et al. (2009), equation (83)), namely, for $\sigma = 1$,

$$\begin{aligned} H \left[\begin{matrix} a, b, p, 1 \\ \frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \right] &= \frac{ab^2}{8\pi\sqrt{p}} F_2\left(\frac{1}{2}; \frac{1}{2}, 1; 1, 2; -\frac{a^2}{p}, -\frac{b^2}{p}\right) \\ &= \frac{a\sqrt{p}}{4\pi} \left[\sqrt{1 + \frac{b^2}{p}} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{a^2}{p+b^2}\right) - {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{a^2}{p}\right) \right] \end{aligned} \quad (18)$$

and

$$H \left[\begin{matrix} a, b, p, 1 \\ \frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \right] = \frac{ab^2}{8\pi\sqrt{p}} \lim_{n \rightarrow \infty} \left[\sum_{k=0}^n \frac{(\frac{1}{2})_k}{(2)_k} \left(\frac{b^2}{p}\right)^k \left(1 + \frac{b^2}{p}\right)^{-\frac{1}{2}-k} {}_2F_1\left(\frac{1}{2} + k, \frac{1}{2}; 1; \frac{-a^2}{p+b^2}\right) \right]. \quad (19)$$

In Table 1, we reported our computation using equations (18) and (19) for different values of a , b and p , the calculations were performed using MATHEMATICA software version 7. It should be clear that any discrepancies may appear are due to the numerical accuracy used in computing the Gauss hypergeometric functions. In Table 2, we compared our calculated values with those

Table 1: Comparison of the values of $H\left[\begin{smallmatrix} a, b, p, 1 \\ \frac{1}{2}, \frac{1}{2}, 1 \end{smallmatrix}\right]$ for some values of a , b and p calculated using equations (18) and (19)

a	b	p	Eq. (18)	Eq. (19)
0.1	0.2	0.5	0.000 219 698 305 361 162 27	0.000 219 698 305 361 161 92
0.1	0.5	0.5	0.001 259 518 891 975 573 3	0.001 259 518 891 975 572 7
0.2	0.2	2.0	0.000 222 868 191 957 851 78	0.000 222 868 191 957 853 9
0.2	1.0	2.0	0.005 038 075 567 902 293	0.005 038 075 567 902 291
0.5	0.5	0.5	0.005 794 884 270 704 959 5	0.005 794 884 270 704 952 5
0.5	1.0	2.5	0.011 293 885 774 813 334 5	0.011 293 885 774 813 332

obtained by Guseinov and Memedov (2005) and with those obtained by Galu e et al. (1994). It should be clear that equation (17) can be used for arbitrary values of the parameters a , b , p and it is not restricted to any particular range of parameter values.

In Table 3, we report our numerical computation of the Hubbell rectangular source integral (1) for $\sigma = p = 1$ as well for some other values of p using equation (17), or simply

$$H \left[\begin{matrix} a, b, p, 0 \\ 1, \frac{1}{2}, \frac{3}{2} \end{matrix} \right] \approx \frac{\sigma ab}{4\pi p} \sum_{k=0}^n \frac{k!}{(\frac{3}{2})_k} \left(\frac{b^2}{p}\right)^k \left(1 + \frac{b^2}{p}\right)^{-1-k} {}_2F_1\left(k + 1, \frac{1}{2}; \frac{3}{2}; \frac{-a^2}{p+b^2}\right) \quad (20)$$

In the same table, we also compared our results with the earlier numerical values obtained of Guseinov and Mademov (2005) and Galu e et al (1994). It is worth noting an important feature of

Table 2: The values of $H[\frac{a,b,p,1}{\frac{1}{2},\frac{1}{2},1}]$ integrals for $\sigma = 1$ and some values of a, b and p obtained from Eq. (17), Guseinov and Mamedov (2005) and Galué et al. (1994).

a	b	p	Eq. (17)	Guseinov & Mamedov (2005)	Galué et al. (1994)
0.1	0.2	0.5	0.000 219 698 305 361 161 92	0.000 219 698 305 352 979	0.000 219 698 31
0.1	0.5	0.5	0.001 259 518 891 975 572 7	0.001 259 518 892 096 95	0.001 259 518 9
0.2	0.2	2.0	0.000 222 868 191 957 853 9	0.000 222 868 191 569 00	0.000 222 868 19
0.2	1.0	2.0	0.005 038 075 567 902 291	0.005 038 075 568 393 22	0.005 038 075 6
0.5	0.5	0.5	0.005 794 884 270 704 952 5	0.005 794 884 271 955 31	0.005 794 884 3
0.5	1.0	2.5	0.011 293 885 774 813 332	0.011 293 885 774 293 0	0.011 293 886

Table 3: The comparative values of $H[\frac{a,b,p,0}{1,0.5,1.5}]$ integral from Eq.(17), Guseinov and Mamedov (2005) and Galué et al (1994)

a	b	p	Eq. (17)	Guseinov & Mamedov (2005)	Galué et al. (1994)
0.1	0.1	0.5	0.001 570 716 369 171 686	0.001 570 716 369 157 32	0.001 570 716 4
0.1	0.5	1.0	0.003 678 199 808 681 331	0.003 678 199 808 778 47	0.003 678 199 8
0.2	0.5	2.5	0.003 067 148 756 523 266	0.003 067 148 756 507 53	0.003 067 148 8
0.2	0.8	2.0	0.005 758 600 701 534 181	0.005 758 600 701 357 22	0.005 758 600 7
0.5	0.5	1.0	0.017 188 506 077 049 23	0.017 188 506 077 717 6	0.017 188 506
0.6	0.5	1.0	0.020 067 469 440 496 68	0.020 067 469 441 718 5	0.020 067 469
0.8	0.6	2.8	0.012 248 693 964 171 94	0.012 248 693 963 979 3	
1.0	0.8	4.2	0.013 484 796 561 457 52	0.013 484 796 561 864 6	
0.5	2.0	5.4	0.012 012 547 384 013 146	0.012 012 547 385 362 9	
0.8	2.6	7.5	0.017 255 112 588 899 273	0.017 255 128 853 560 6	

the approximation formula (17) is that it is *self-adjusting*; that is, if

$$|H_{n+1} - H_n| \not\leq \epsilon \tag{21}$$

where ϵ is the desire accuracy then n should be increased to reach the required accuracy. Here, H_n refer to the right-hand side of equation (17).

Acknowledgment. The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant GP249507.

References

[1] Boast, W.B., 1942. Illuminating Engineering. McGraw-Hill, New York.

-
- [2] Ezure, H., 2005. Confirmation of accuracy of Haselgrove method using Hubbell radiation rectangular source integral, *J. Nucl. Sci. Tech.* 42, 384-389.
- [3] Fano, U., Spencer, L.V., Berger, M.J., 1959. Penetration and diffusion of x -rays, *Encyclopedia of Physics*, Vol. 38, p.671, II. Springer, Berlin.
- [4] Galué, L., 1991. Una generalización de la integral de Hubbell, *Rev. Téc. Ing. Univ. Zulia* 14, 153-160.
- [5] Galué, L., Kalla, S.L., Leubner, C., 1988. Algorithms for the approximation of the Hubbell rectangular source integral and its generalization, *Radiat. Phys. Chem.* 43, 497-502.
- [6] Galué, L., 2003. Unification of generalized radiation field integrals involving confluent hypergeometric function, *Rad. Phys. Chem.* 66, 269-273.
- [7] Galué, L., Kalla, S.L., Leubner, C., 1994. Algorithms for the approximation of the Hubbell rectangular source integral, *Radiat. Phys. Chem.* 43, 497-502.
- [8] EL-Gabali, M., Kalla, S.L., 1996. Some generalized radiation field integrals, *Computers Math. Applic* 32, 121-128.
- [9] Glasser, M.L., 1984. Solution to problem 83-6 in the problems and solutions section. *SIAM Rev.* 26, 2.
- [10] Gabutti, B., Kalla, S.L., Hubbell, J.H., 1991. Some expansions related to the Hubbell rectangular source integral. *J. Comput. Appl. Math.* 37, 273-285.
- [11] Galué, L., Kiryakova, V., 1994. Further results on a family of generalized integrals, *Radiat. Phys. Chem.* 43, 573-579.
- [12] Ghose, A.M., Bradley, D.A., Hubbell, J.H., 1988. Radiation field from a rectangular plaque source: critique of some engineering approximations. *Appl. Radiat. Isot.* 3, 421-427.
- [13] Götze, F., 1995. An effective method for computing the Hubbell rectangular integral. *Radiat. Phys. Chem.* 46, 329-331.
- [14] Guseinov, I.I., Öner, F., Mamedov, B.N., 2004. Evaluation of the Hubbell radiation rectangular source integrals using binomial coefficients, *Radiat. Phys. Chem.* 69, 109-112.
- [15] Guseinov, I.I., Mamedov, B.N., 2005. Calculation of the generalized Hubbell rectangular source integrals using binomial coefficients, *Appl. Math. Comput.* 161, 285-292.
- [16] Guseinov, I.I., Mamedov, B.N., Ekernoğlu, A.S., 2005. A new algorithm for accurate and fast evaluation of the Hubbell radiation rectangular source integral, *Radiat. Phys. Chem.* 74, 261-263.
- [17] Hanak, V., Cechak, T., 1978. Radiation field from a rectangular source, *J. Nucl. Energy Part C* 24, 94.
- [18] Hubbell, J.H., Bach, R.L., Lamkin, J.C., 1960. Radiation field from a rectangular source, *J. Res. Nat. Bur. Stand.* 64C, 121-138.

- [19] Kalla, S.L., 1993. *The Hubbell rectangular source integral and its generalizations*, Radiat. Phys. Chem. 41, 775-781.
- [20] Kalla, S.L., Al-Saqabi, B., Conde, S., 1987. Some results related to radiation field problems, Hadronic J. 10, 221-230.
- [21] Kalla, S.L., Al-Shammery, A.H., Khajah, H.G., 2002. Development of the Hubbell rectangular source integral, Acta Applicandae Mathematicae 74, 35-55.
- [22] Kalla, S.L., Khajah, H.G., 1997. Tau method approximation of some integrals related to radiation field problems, Computers Math. Applic 33, 21-27.
- [23] Kalla, S.L., Khajah, H.G., 2000. Tau method approximation of the Hubbell rectangular source integral. Radiat. Phys. Chem. 59, 17-21.
- [24] Kalla, S.L., Galué, L., Kiryakova, V., 1991. Some expansion related to a family of generalized radiation integrals, Math. Balkanic 5, 190-202.
- [25] Kalla, S.L., Leubner, C., Al-Saqabi, B., 1991. Some results for the radiation flux in a rectangular straight duct, Nucl. Sci. J. 28, 403-410.
- [26] José L. López and Pedro Pagola, *A simplification of the Laplace's method for double integrals. Application to the second Appell function*, Electronic Transactions on Numerical Analysis (ETNA), 30 (2008) 224-236.
- [27] Michieli, I., Maximovic, A., 1996. Legendre expansion related to the Hubbell rectangular source integral. Radiat. Phys. Chem. 47, 779-784.
- [28] Oner, F., 2007. On the evaluation of rectangular plane-extended sources and their associated radiation fields, Appl. Rad. Isot. 65, 1121-1124.
- [29] Opps, S.B., Saad, N., Srivastava, H.M., 2005. Some reduction and transformation formulas for the Appell hypergeometric function F_2 , J. Math. Anal. Appl. 302, 180-195.
- [30] Opps, S.B., Saad, N., Srivastava, H.M., 2009. Recursion Formulas for Appell's Hypergeometric Function F_2 with Some Applications to Radiation Field Problems, App. Math. Comput. 207, 545-558.
- [31] Prabha, H., 2001. Computation of rectangular source integral by rational parameter polynomial method. Radiat. Phys. Chem. 62, 203206.
- [32] Prabha, H., 2007. Rectangular source integral and recurrence relations. Radiat. Phys. Chem. 76, 687-690.
- [33] Saigo M., Srivastava, R., 1990. Some new results for radiation-field problems. Fukuoka Univ. Sci. Rep. 20, 1-13.
- [34] Slater, L.J., 1966. Generalized Hypergeometric Functions, Cambridge University Press, Cambridge.
- [35] Stalker, J., 2001. On Hubbell's rectangular source integral, Radiat. Phys. Chem. 62, 311-315.

- [36] Timus, D.M., 1993. Some further results on the Hubbell rectangular source integral. *J. Comput. Appl. Math.* 45, 331-336.