

THE WITT GROUP OF NON-DEGENERATE BRAIDED FUSION CATEGORIES

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ABSTRACT. We give a characterization of Drinfeld centers of fusion categories as non-degenerate braided fusion categories containing a Lagrangian algebra. Further we study the quotient of the monoid of non-degenerate braided fusion categories modulo the submonoid of the Drinfeld centers and show that its formal properties are similar to those of the classical Witt group.

1. INTRODUCTION

Tensor categories are ubiquitous in many areas of mathematics and it seems worthwhile to study them deeper. The simplest class of tensor categories is formed by so called fusion categories ([ENO1], see 2.1 below for a definition). It is known ([ENO1]) that over an algebraically closed field \mathbb{k} of characteristic zero there are only countably many equivalence classes of fusion categories and that the classification of these equivalence classes is essentially independent from the field \mathbb{k} (namely, an embedding of fields $\mathbb{k} \subset \mathbb{k}'$ induces a bijection between the sets of equivalence classes of fusion categories over \mathbb{k} and over \mathbb{k}'). Thus the classification of fusion categories seems to be a natural and interesting problem. This problem is very far from its solution at the moment.

An interesting additional structure that one might impose on a tensor category is a *braiding* ([JS2]). For a fusion category \mathcal{A} , its *Drinfeld center* $\mathcal{Z}(\mathcal{A})$ is a braided fusion category, see Section 2.3. Our first main result addresses the following question: when is a braided fusion category \mathcal{C} equivalent to the Drinfeld center of some fusion category? The answer we give is as follows: \mathcal{C} should be *non-degenerate* in the sense of [DGNO] and \mathcal{C} should contain a *Lagrangian algebra*, that is, a connected étale algebra of maximal possible size, see Section 4. More precisely, we show that the 2-groupoid of fusion categories is equivalent to the 2-groupoid of *quantum Manin pairs*, where a quantum Manin pair consists of a non-degenerate braided fusion category and a Lagrangian algebra in this category. This result can be considered as (a step in) a reduction of the classification of all fusion categories to the classification of braided fusion categories.

The problem of classification of all braided fusion categories (even of non-degenerate ones) seems to be very interesting but is almost as inaccessible as a classification of all fusion categories. The second main result of this paper is an observation that there is an interesting algebraic structure in this classification. Namely, we prove that the quotient of the monoid of non-degenerate braided fusion categories by the submonoid of Drinfeld centers has formal properties similar to those of the classical Witt group of the quadratic forms over a field. Moreover, we show that the Witt group of finite abelian groups endowed with a non-degenerate quadratic forms embeds naturally into this quotient. Thus we call it the *Witt group of non-degenerate*

braided fusion categories and consider its computation as a fundamental problem in the study of fusion categories. Further we show that each Witt equivalence class contains a unique representative which is *completely anisotropic* (Theorem 5.12); this result is a counterpart of the statement that in the classical Witt group each Witt class contains a unique anisotropic quadratic form.

An interesting subgroup of the Witt group is the *unitary Witt group* (see Definition 5.23) consisting of the classes of *pseudounitary* braided fusion categories. A well known source of examples of pseudounitary braided fusion categories is the representation theory of affine Lie algebras, see, e.g., [BaKi, Chapter 7]. Namely, for any simple finite dimensional Lie algebra \mathfrak{g} and a positive integer k one constructs a pseudounitary non-degenerate braided fusion category $\mathcal{C}(\mathfrak{g}, k)$ consisting of integrable highest weight modules of level k over the affinization of \mathfrak{g} . We do not know any elements of the unitary Witt group that are not in the subgroup generated by the classes $[\mathcal{C}(\mathfrak{g}, k)]$. It would be very interesting to find out whether such elements exist. The relations between the classes $[\mathcal{C}(\mathfrak{g}, k)]$ (or, more generally, between the classes of known braided fusion categories) are of great interest. By Corollary 5.8, any such relation produces at least one fusion category; one can hope to construct new examples of fusion categories in this way (see [CMS, Appendix] for an example of this kind). In Section 6 we give examples of such relations using the theory of *conformal embeddings* and *coset models* of central charge $c < 1$. It would be interesting to see whether other relations exist. At this moment even all relations between the classes $[\mathcal{C}(sl(2), k)]$ are not completely known (see Section 6.4).

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2. PRELIMINARIES

Throughout this paper our base field \mathbb{k} is an algebraically closed field of characteristic zero.

2.1. Fusion categories. By definition (see [ENO1]), a *multi-fusion category* over \mathbb{k} is a \mathbb{k} -linear semisimple rigid tensor category with finitely many simple objects and finite dimensional spaces of morphisms. A multi-fusion category is called a *fusion category* if its unit object $\mathbf{1}$ is simple. By a *fusion subcategory* of a fusion category we always mean a full tensor subcategory. Let Vec denote the fusion category of finite dimensional vector spaces over \mathbb{k} . Any fusion category \mathcal{A} contains a trivial fusion subcategory consisting of multiples of $\mathbf{1}$. We will identify this subcategory with Vec . A fusion category \mathcal{A} is called *simple* if Vec is the only proper fusion subcategory of \mathcal{A} .

A fusion category is called *pointed* if all its simple objects are invertible. For a fusion category \mathcal{A} we denote \mathcal{A}_{pt} the maximal pointed fusion subcategory of \mathcal{A} . We say that \mathcal{A} is *unpointed* if $\mathcal{A}_{pt} = \text{Vec}$.

We will denote $\mathcal{A} \boxtimes \mathcal{B}$ the tensor product of fusion categories \mathcal{A} and \mathcal{B} .

For a fusion category \mathcal{A} we denote by $\mathcal{O}(\mathcal{A})$ the set of isomorphism classes of simple objects in \mathcal{A} .

Let \mathcal{A} be a fusion category and let $K(\mathcal{A})$ be its Grothendieck ring. There exists a unique ring homomorphism $\text{FPdim} : K(\mathcal{A}) \rightarrow \mathbb{R}$ such that $\text{FPdim}(X) > 0$ for any $0 \neq X \in \mathcal{A}$, see [ENO1, Section 8.1]. For a fusion category \mathcal{A} one defines (see [ENO1, Section 8.2]) its *Frobenius-Perron dimension*:

$$(1) \quad \text{FPdim}(\mathcal{A}) = \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X)^2.$$

For any object X in \mathcal{A} let $[X]$ denote the corresponding element of the Grothendieck ring $K(\mathcal{A})$. One defines the (virtual) *regular object* of \mathcal{A} by

$$(2) \quad R_{\mathcal{A}} = \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X) [X] \in K(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R},$$

see, e.g., [ENO1, Section 8.2]. The regular object $R_{\mathcal{A}}$ is uniquely characterized by the following properties (see *loc. cit.*):

- (1) $[X]R_{\mathcal{A}} = \text{FPdim}(X)R_{\mathcal{A}}$ for any $X \in \mathcal{A}$;
- (2) $\text{FPdim}(R_{\mathcal{A}}) = \text{FPdim}(\mathcal{A})$.

Let $\mathcal{A}_1, \mathcal{A}_2$ be fusion categories such that $\text{FPdim}(\mathcal{A}_1) = \text{FPdim}(\mathcal{A}_2)$. By [EO, Proposition 2.19] any fully faithful tensor functor $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is an equivalence.

There is another notion of dimension \mathcal{A} , the *categorical* (or *global*) dimension defined as follows (see [Mu4]). For each simple object X in \mathcal{A} pick an isomorphism $a_X : X \xrightarrow{\sim} X^{**}$ and set

$$(3) \quad \dim(\mathcal{A}) = \sum_{X \in \mathcal{O}(\mathcal{A})} |X|^2,$$

where $|X|^2 = \text{Tr}_X(a_X)\text{Tr}_{X^*}((a_X^{-1})^*)$. By [ENO1, Theorem 2.3], $\dim(\mathcal{A})$ is a non-zero element in \mathbb{k} .

A fusion category \mathcal{A} over $\mathbb{k} = \mathbb{C}$ is called *pseudo-unitary* if $\dim(\mathcal{A}) = \text{FPdim}(\mathcal{A})$, see [ENO1, Section 8.4]. A pseudo-unitary fusion category \mathcal{A} has a unique spherical structure such that the categorical dimension $\dim(X)$ of any object X in \mathcal{A} equals $\text{FPdim}(X)$, see [ENO1, Proposition 8.23]. It is easy to see that if \mathcal{A}_1 and \mathcal{A}_2 are pseudo-unitary then so is $\mathcal{A}_1 \boxtimes \mathcal{A}_2$.

2.2. Braided fusion categories. A *braided* fusion category is a fusion category \mathcal{C} endowed with a braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, see [JS2]. For a braided fusion category its *reverse* \mathcal{C}^{rev} is the same fusion category with a new braiding $\tilde{c}_{X,Y} = c_{Y,X}^{-1}$. A braided fusion category is *symmetric* if $\tilde{c} = c$.

Recall from [Mu2] that objects X and Y of a braided fusion category \mathcal{C} are said to *centralize* each other if

$$(4) \quad c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}.$$

The *centralizer* \mathcal{D}' of a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is defined to be the full subcategory of objects of \mathcal{C} that centralize each object of \mathcal{D} . It is easy to see that \mathcal{D}' is a fusion subcategory of \mathcal{C} . Clearly, \mathcal{D} is symmetric if and only if $\mathcal{D} \subset \mathcal{D}'$.

Definition 2.1. (see [DGNO, Definition 2.28 and Proposition 3.7]) We will say that a braided fusion category \mathcal{C} is *non-degenerate* if $\mathcal{C}' = \text{Vec}$.

A non-degenerate braided fusion category $\mathcal{C} \neq \text{Vec}$ is *prime* if it has no proper non-degenerate braided fusion subcategories other than Vec . Clearly, a non-trivial simple braided fusion category is prime.

For a fusion subcategory \mathcal{D} of a non-degenerate braided fusion category \mathcal{C} one has the following properties (see [DGNO, Theorem 3.10]):

$$(5) \quad \mathcal{D}' = \mathcal{D},$$

$$(6) \quad \text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C}).$$

A *pre-modular* category is a braided fusion category equipped with a spherical structure. A pre-modular category \mathcal{C} is *modular* (i.e., its S -matrix is invertible) if and only if \mathcal{C} is non-degenerate [DGNO, Proposition 3.7].

The following statement is well known. We include its proof for the reader's convenience.

Proposition 2.2. *Let $\mathcal{C} \neq \text{Vec}$ be a non-degenerate braided fusion category. Then*

$$(7) \quad \mathcal{C} = \mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n,$$

where $\mathcal{C}_1, \dots, \mathcal{C}_n$ are prime non-degenerate subcategories of \mathcal{C} . Furthermore, if \mathcal{C} is unpointed then its decomposition (7) into a tensor product of prime non-degenerate subcategories is unique up to a permutation of factors.

Proof. Existence of the decomposition (7) is established in [Mu2], so we only need to prove uniqueness. If $\mathcal{D} \subset \mathcal{C}$ is a fusion subcategory, let $\mathcal{D}_i \subset \mathcal{C}_i$ be the fusion subcategory generated by all simple objects $X_i \in \mathcal{C}_i$ such that there is a simple $X = X_1 \boxtimes \cdots \boxtimes X_i \boxtimes \cdots \boxtimes X_n \in \mathcal{D}$. Clearly we have $\mathcal{D} \subset \mathcal{D}_1 \boxtimes \cdots \boxtimes \mathcal{D}_n$, but the converse need not hold. If it does, we say that \mathcal{D} factorizes. Denoting by \mathcal{D}_{ad} the fusion subcategory of \mathcal{D} generated by $X \otimes X^*$, where X runs through simple objects of \mathcal{D} , the fact that

$$X \otimes X^* = (X_1 \otimes X_1^*) \boxtimes \cdots \boxtimes (X_n \otimes X_n^*) \succ \mathbf{1} \boxtimes \cdots \boxtimes \mathbf{1} \boxtimes (X_i \otimes X_i^*) \boxtimes \mathbf{1} \boxtimes \cdots \boxtimes \mathbf{1}$$

implies that $\mathcal{D}_{ad} \supset (\mathcal{D}_{ad})_i$, thus \mathcal{D}_{ad} factorizes. Let $\mathcal{D} \subset \mathcal{C}$ be a non-degenerate fusion subcategory. Since \mathcal{C} is unpointed, i.e., $\mathcal{C}_{pt} = \text{Vec}$, \mathcal{D} is unpointed and by [DGNO, Corollary 3.27] we have $\mathcal{D}_{ad} = (\mathcal{D}_{pt})' \cap \mathcal{D} = \mathcal{D}$. Thus \mathcal{D} factorizes, i.e. $\mathcal{D} = \mathcal{D}_1 \boxtimes \cdots \boxtimes \mathcal{D}_n$, where each \mathcal{D}_i is non-degenerate. Since \mathcal{C}_i is prime, we must have either $\mathcal{D}_i = \text{Vec}$ or $\mathcal{D}_i = \mathcal{C}_i$ for each $i = 1, \dots, n$. In particular, every prime non-degenerate fusion subcategory $\mathcal{D} \subset \mathcal{C}$ coincides with some \mathcal{C}_i . Hence, (7) is unique up to a permutation of factors. \square

Remark 2.3. The proof actually also shows the following stronger result: If $\mathcal{D} \subset \mathcal{C}$ is an unpointed and non-degenerate fusion subcategory then $\mathcal{D} = \mathcal{D}_1 \boxtimes \cdots \boxtimes \mathcal{D}_n$, where each \mathcal{D}_i is either $\mathcal{D}_i = \text{Vec}$ or $\mathcal{D}_i = \mathcal{C}_i$. This means that the prime factors \mathcal{C}_i that are unpointed appear in every prime factorization of \mathcal{C} , whether or not \mathcal{C} itself is unpointed.

2.3. Drinfeld center of a fusion category. For any fusion category \mathcal{A} its *Drinfeld center* $\mathcal{Z}(\mathcal{A})$ is defined as the category whose objects are pairs (X, γ_X) , where X is an object of \mathcal{A} and $\gamma_X : V \otimes X \simeq X \otimes V$, $V \in \mathcal{A}$ is a natural family of isomorphisms, satisfying a certain compatibility condition, see [JS1, Definition 3] or [Ka, Definition XIII.4.1]. It is known that $\mathcal{Z}(\mathcal{A})$ is a braided fusion category. We have (see [Mu5] and [ENO1, Theorem 2.15, Proposition 8.12]):

$$(8) \quad \dim(\mathcal{Z}(\mathcal{C})) = \dim(\mathcal{C})^2, \quad \text{FPdim}(\mathcal{Z}(\mathcal{C})) = \text{FPdim}(\mathcal{C})^2.$$

It is known that $\mathcal{Z}(\mathcal{A})$ is non-degenerate, see [DGNO, Corollary 3.9].

For a braided fusion category \mathcal{C} there are two braided functors

$$(9) \quad \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}) \quad : \quad X \mapsto (X, c_{-,X}),$$

$$(10) \quad \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C}) \quad : \quad X \mapsto (X, \tilde{c}_{-,X}).$$

These functors are fully faithful and so we can identify \mathcal{C} and \mathcal{C}^{rev} with their images in $\mathcal{Z}(\mathcal{C})$. These images centralize each other, i.e., $\mathcal{C}' = \mathcal{C}^{\text{rev}}$. This allows to define a braided tensor functor

$$(11) \quad G : \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C}).$$

It was shown in [Mu4] and [DGNO, Proposition 3.7] that G is a braided equivalence if and only if \mathcal{C} is non-degenerate.

Let \mathcal{C} be a braided fusion category and let \mathcal{A} be a fusion category. Let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a tensor functor.

Definition 2.4. A structure of a *central functor* on F is a functor $F' : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{A})$ whose composition with the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ equals F .

Equivalently, a structure of central functor on F is a natural family of isomorphisms $Y \otimes F(X) \xrightarrow{\sim} F(X) \otimes Y$, $X \in \mathcal{C}$, $Y \in \mathcal{A}$, satisfying certain compatibility conditions, see [Be, Section 2.1].

2.4. Separable algebras. Let \mathcal{A} be a fusion category. In this paper an *algebra* $A \in \mathcal{A}$ is an associative algebra with unit, see e.g., [O, Definition 3.1].

Definition 2.5. An algebra $A \in \mathcal{A}$ is said to be *separable* if the multiplication morphism $m : A \otimes A \rightarrow A$ splits as a morphism of A -bimodules.

- Remark 2.6.**
- (i) The morphism m is surjective (due to the existence of unit in A), so the definition makes sense.
 - (ii) Observe that if $F : \mathcal{A} \rightarrow \mathcal{B}$ is a tensor functor then $F(A) \in \mathcal{B}$ is a separable algebra for a separable algebra $A \in \mathcal{A}$.

For an algebra $A \in \mathcal{A}$ let \mathcal{A}_A , ${}_A\mathcal{A}$, ${}_A\mathcal{A}_A$ denote, respectively, abelian categories of right A -modules, left A -modules, A -bimodules, see e.g., [O, Definition 3.1].

Proposition 2.7. *For an algebra $A \in \mathcal{A}$ the following conditions are equivalent:*

- (i) A is separable;
- (ii) the category \mathcal{A}_A is semisimple;
- (iii) the category ${}_A\mathcal{A}$ is semisimple;
- (iv) the category ${}_A\mathcal{A}_A$ is semisimple.

Proof. Assume that A is separable. Note that A considered as a bimodule over itself is a direct summand of the A -bimodule $A \otimes A$. Thus any $M = M \otimes_A A \in \mathcal{A}_A$ is a direct summand of $M \otimes_A A \otimes A = M \otimes A$. The object $M \otimes A \in \mathcal{A}_A$ is projective (see e.g. [O, Section 3.1]). Thus any $M \in \mathcal{A}_A$ is projective and we have implication (i) \Rightarrow (ii). The implication (i) \Rightarrow (iii) is proved similarly.

The implications (ii) \Rightarrow (iv) and (iii) \Rightarrow (iv) follow from [ENO1, Theorem 2.16]. Finally, the implication (iv) \Rightarrow (i) is obvious. \square

Let \mathcal{C} be a braided fusion category. Recall that an algebra A in \mathcal{C} is called *commutative* if $m \circ c_{A,A} = m$, where $m : A \otimes A \rightarrow A$ is the multiplication of A , see e.g., [KiO, Definition 1.1].

Example 2.8. Let G be a finite group and let $\mathcal{A} = \text{Rep}(G)$ be the fusion category of finite dimensional representations of G . Let $A = \text{Fun}(G)$ be the algebra of \mathbb{k} -valued functions on G . The group G acts on A via left translations, so A can be considered as a commutative algebra in \mathcal{A} . The algebra A is called the *regular algebra* of the category $\mathcal{A} = \text{Rep}(G)$. It is easy to check that A is separable, see e.g. [DGNO, Proposition 2.53].

More generally we say that a braided fusion category \mathcal{E} is *Tannakian* [De] if there is a braided equivalence $F : \mathcal{E} \simeq \text{Rep}(G)$; in this case the algebra $F^{-1}(A)$ (with $A \in \text{Rep}(G)$ as above) is called a regular algebra $A_{\mathcal{E}}$ of \mathcal{E} . It is known that the algebra $A_{\mathcal{E}}$ is unique up to isomorphism. (Such an isomorphism is non-unique, in particular $\text{Aut } A_{\mathcal{E}} \cong G$.) See, e.g., [DGNO, Section 2.13].

2.5. Equivariantization and de-equivariantization. Let \mathcal{A} be a fusion category with an action of a finite group G . In this case one can define the fusion category \mathcal{A}^G of G -equivariant objects in \mathcal{A} . Objects of this category are objects X of \mathcal{A} equipped with an isomorphism $u_g : g(X) \rightarrow X$ for all $g \in G$, such that

$$u_{gh} \circ \gamma_{g,h} = u_g \circ g(u_h),$$

where $\gamma_{g,h} : g(h(X)) \rightarrow gh(X)$ is the natural isomorphism associated to the action. Morphisms and tensor product of equivariant objects are defined in an obvious way. This category is called the *G -equivariantization* of \mathcal{A} . One has $\text{FPdim}(\mathcal{A}^G) = |G|\text{FPdim}(\mathcal{A})$. See [Br, Mu3] and [DGNO, Section 4] for details.

Example 2.9. Let H be a normal subgroup of G . Then there is a natural action of G/H on \mathcal{A}^H and $(\mathcal{A}^H)^{G/H} \cong \mathcal{A}^G$.

There is a procedure opposite to equivariantization, called the *de-equivariantization*. Namely, let \mathcal{A} be a fusion category and let $\mathcal{E} = \text{Rep}(G) \subset \mathcal{Z}(\mathcal{A})$ be a Tannakian subcategory which embeds into \mathcal{A} via the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$. Let $A = \text{Fun}(G)$ be the regular algebra of \mathcal{E} . It is a commutative algebra in $\mathcal{Z}(\mathcal{A})$ and so the category \mathcal{A}_G of left A -modules in \mathcal{A} is a fusion category with the tensor product \otimes_A , called *de-equivariantization* of \mathcal{A} . One has $\text{FPdim}(\mathcal{A}_G) = \text{FPdim}(\mathcal{A})/|G|$.

The above constructions are canonically inverse to each other, i.e., there are canonical equivalences $(\mathcal{A}_G)^G \cong \mathcal{A}$ and $(\mathcal{A}^G)_G \cong \mathcal{A}$, see [DGNO, Section 4.1].

2.6. Module categories over fusion categories. Let \mathcal{A} be a fusion category. A left \mathcal{A} -module category is a finite semisimple Abelian \mathbb{k} -linear category \mathcal{M} together with a bifunctor $\otimes : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}$ and a natural family of isomorphisms

$$(X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M) \quad \text{and} \quad \mathbf{1} \otimes M \xrightarrow{\sim} M$$

for $X, Y \in \mathcal{A}$, $M \in \mathcal{M}$, satisfying certain coherence conditions. See [O] for details and for the definitions of \mathcal{A} -module functors and their natural transformations. A typical example of a left \mathcal{A} -module category is the category \mathcal{A}_A of right modules over a separable algebra A in \mathcal{A} [O]. An \mathcal{A} -module category is called *indecomposable* if it is not equivalent to a direct sum of two non-trivial \mathcal{A} -module categories.

The category of \mathcal{A} -module endofunctors of a right \mathcal{A} -module category \mathcal{M} will be denoted by $\mathcal{A}_{\mathcal{M}}^*$. It is known that $\mathcal{A}_{\mathcal{M}}^*$ is a multi-fusion category, see [ENO1, Theorem 2.18] (it is a fusion category if and only if \mathcal{M} is indecomposable).

Let \mathcal{M} be an indecomposable right \mathcal{A} -module category. We can regard \mathcal{M} as an $(\mathcal{A}_{\mathcal{M}}^*, \mathcal{A})$ -bimodule category. Its $(\mathcal{A}_{\mathcal{M}}^*, \mathcal{A})$ -bimodule endofunctors can be identified,

on the one hand, with functors of left multiplication by objects of $\mathcal{Z}(\mathcal{A}_{\mathcal{M}}^*)$, and on the other hand, with functors of right multiplication by objects of $\mathcal{Z}(\mathcal{A})$. Combined, these identifications yield a canonical equivalence of braided categories

$$(12) \quad \mathcal{Z}(\mathcal{A}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{A}_{\mathcal{M}}^*).$$

This result is due to Schauenburg, see [Sch].

3. ÉTALE ALGEBRAS AND CENTRAL FUNCTORS

3.1. Étale algebras in braided fusion categories.

Definition 3.1. An algebra $A \in \mathcal{C}$ is said to be *étale* if it is both commutative and separable. We say that an étale algebra $A \in \mathcal{C}$ is *connected* if $\dim_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$.

Remark 3.2. (i) The terminology of Definition 3.1 is justified by the fact that étale algebras in the usual sense can be characterized by the property from Definition 3.1.

(ii) Any étale algebra canonically decomposes as a direct sum of connected ones.

Example 3.3. (i) Let $\mathcal{E} \subset \mathcal{C}$ be a Tannakian subcategory. Then a regular algebra $A_{\mathcal{E}} \in \mathcal{C}$ (see Example 2.8) is connected étale.

(ii) Let \mathcal{C} be a pre-modular category. Let A be a commutative algebra in \mathcal{C} such that $\dim_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$, the pairing $A \otimes A \xrightarrow{m} A \rightarrow \mathbf{1}$ is non-degenerate, $\theta_A = \mathrm{id}_A$, and $\dim(A) \neq 0$. It is proved in [KiO, Theorem 3.3] that such an A is connected étale.

Remark 3.4. In general if $A \in \mathcal{C}$ is a connected étale algebra and $A \rightarrow \mathbf{1}$ is a nonzero homomorphism (it is unique up to a scalar) then the pairing $A \otimes A \xrightarrow{m} A \rightarrow \mathbf{1}$ is non-degenerate. Indeed the kernel of this pairing would be a non-trivial ideal of A (= non-trivial subobject in the category \mathcal{C}_A); but the category \mathcal{C}_A is semisimple and $\dim_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}_A}(A, A) = \dim_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$. In particular, this implies that any étale algebra is a self-dual object of \mathcal{C} (use Remark 3.2 (ii) for disconnected étale algebras).

3.2. From central functors to étale algebras.

Lemma 3.5. *Let \mathcal{C} be a braided fusion category, let \mathcal{A} a fusion category, and let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a central functor. Let $I : \mathcal{A} \rightarrow \mathcal{C}$ be the right adjoint functor of F . Then the object $A = I(\mathbf{1}) \in \mathcal{C}$ has a canonical structure of connected étale algebra.*

Proof. Let $\phi : \mathcal{C} \rightarrow \mathrm{Vec}$ be the contravariant representable functor corresponding to A , that is, $\phi(X) = \mathrm{Hom}_{\mathcal{C}}(X, A) \cong \mathrm{Hom}_{\mathcal{A}}(F(X), \mathbf{1})$. The linear map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(F(X_1), \mathbf{1}) \otimes \mathrm{Hom}_{\mathcal{A}}(F(X_2), \mathbf{1}) &\rightarrow \\ \mathrm{Hom}_{\mathcal{A}}(F(X_1) \otimes F(X_2), \mathbf{1} \otimes \mathbf{1}) &\cong \mathrm{Hom}_{\mathcal{A}}(F(X_1 \otimes X_2), \mathbf{1}) \end{aligned}$$

defines a natural morphism

$$(13) \quad \nu_{X_1, X_2} : \phi(X_1) \otimes \phi(X_2) \rightarrow \phi(X_1 \otimes X_2)$$

such that the compositions

$$\begin{aligned} \phi(X_1) \otimes \phi(X_2) \otimes \phi(X_3) &\rightarrow \phi(X_1 \otimes X_2) \otimes \phi(X_3) \rightarrow \phi(X_1 \otimes X_2 \otimes X_3), \\ \phi(X_1) \otimes \phi(X_2) \otimes \phi(X_3) &\rightarrow \phi(X_1) \otimes \phi(X_2 \otimes X_3) \rightarrow \phi(X_1 \otimes X_2 \otimes X_3) \end{aligned}$$

are equal. Such a morphism (13) is the same thing as an associative multiplication $m : A \otimes A \rightarrow A$ (namely, $m \in \phi(A \otimes A)$ is defined by $m := \nu_{A,A}(\text{id}_A \otimes \text{id}_A)$, where id_A is considered as an element of $\phi(A)$). By definition, $\text{Hom}_{\mathcal{C}}(\mathbf{1}, A) = \text{Hom}_{\mathcal{A}}(F(\mathbf{1}), \mathbf{1}) = \text{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) = \mathbb{k}$. It is easy to see that the image of $1 \in \mathbb{k}$ in $\text{Hom}_{\mathcal{C}}(\mathbf{1}, A)$ is a unit of the algebra A .

To prove the commutativity of $m : A \otimes A \rightarrow A$, we have to show that ν_{X_1, X_2} agrees with the braiding $c_{X_1, X_2} : X_1 \otimes X_2 \rightarrow X_2 \otimes X_1$. In other words, we have to show that if $l_i \in \text{Hom}_{\mathcal{A}}(F(X_i), \mathbf{1})$ then the diagram

$$(14) \quad \begin{array}{ccccc} F(X_1 \otimes X_2) & \xrightarrow{\sim} & F(X_1) \otimes F(X_2) & \xrightarrow{l_1 \otimes l_2} & \mathbf{1} \\ F(c_{X_1, X_2}) \downarrow & & & & \downarrow \text{id}_{\mathbf{1}} \\ F(X_2 \otimes X_1) & \xrightarrow{\sim} & F(X_2) \otimes F(X_1) & \xrightarrow{l_2 \otimes l_1} & \mathbf{1} \end{array}$$

commutes. This is an immediate consequence of the naturality of braiding with a central object.

It follows from [EO, Theorem 3.17] that the category of right A -modules in \mathcal{C} identifies with the image of F in \mathcal{A} and hence is semisimple. By Proposition 2.7 semisimplicity of the category of A -modules implies the semisimplicity of the category of A -bimodules. In particular, the morphism of A -bimodules $m : A \otimes A \rightarrow A$, i.e., A is separable. \square

- Example 3.6.** (i) Let $\mathcal{C} = \text{Rep}(G)$ and $F : \mathcal{C} \rightarrow \text{Vec}$ the forgetful functor. Then the étale algebra A from Lemma 3.5 is the regular algebra, see Example 2.8.
- (ii) Let Vec_G^ω be the fusion category of finite dimensional G -graded vector spaces with the associativity constraint twisted by a 3-cocycle $\omega \in Z^3(G, \mathbb{k}^\times)$. Let $\mathcal{C} = \mathcal{Z}(\text{Vec}_G^\omega)$ and $F : \mathcal{C} \rightarrow \text{Vec}_G^\omega$ the forgetful functor. Then the étale algebra A from Lemma 3.5 is the regular algebra of $\text{Rep}(G) \subset \mathcal{C}$.
- (iii) Let $\mathcal{C} = \mathcal{Z}(\text{Rep}(G)) \cong \mathcal{Z}(\text{Vec}_G)$ and $F : \mathcal{C} \rightarrow \text{Rep}(G)$ the forgetful functor. Then the étale algebra A from Lemma 3.5 is the group algebra of G considered as an algebra in \mathcal{C} . Notice that in this case the algebra $F(A)$ in the symmetric tensor category $\text{Rep}(G)$ is non-commutative unless G is commutative.

Remark 3.7. Lemma 3.5 fails over fields of characteristic $p > 0$. Namely the algebra $A = I(\mathbf{1})$ is still commutative (with the same proof) but it can fail to be separable. Here is a counter-example. Let G be a finite abelian group of order divisible by p . Take $\mathcal{C} = \text{Vec}_G$, i.e., \mathcal{C} is the category of finite-dimensional G -graded vector spaces with the obvious symmetric braided structure. Let $\mathcal{D} = \text{Vec}$ and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be the functor of forgetting the grading. Then A is the group algebra of G , which is not étale. In this example the category of A -bimodules identifies with $\text{Rep}(G)$ and is not semisimple.

3.3. The tensor category \mathcal{C}_A corresponding to an étale algebra A . Let \mathcal{C} be a braided fusion category and let $A \in \mathcal{C}$ be a connected étale algebra. Let \mathcal{C}_A be the category of right A -modules and let

$$(15) \quad F_A : \mathcal{C} \rightarrow \mathcal{C}_A : X \mapsto X \otimes A$$

be the free module functor. The category \mathcal{C}_A is semisimple by Proposition 2.7. Any object M of \mathcal{C}_A can be endowed with a structure of A -bimodule with the left

A -module structure given by

$$A \otimes M \xrightarrow{c_{M,A}^{-1}} M \otimes A \rightarrow M.$$

In this way the category \mathcal{C}_A gets a structure of tensor category with tensor product \otimes_A . The functor F_A has an obvious structure of tensor functor. The category \mathcal{C}_A is rigid since any object M in \mathcal{C}_A is a direct summand of the rigid object $F_A(M) = M \otimes A = M \otimes_A (A \otimes A)$. The unit object of \mathcal{C}_A is $A = F_A(\mathbf{1})$ and the connectedness of A implies that $A \in \mathcal{C}_A$ is simple. Thus, \mathcal{C}_A is a fusion category.

Remark 3.8. One can use a different structure of A -bimodule on $M \in \mathcal{C}_A$ using the composition

$$A \otimes M \xrightarrow{c_{A,M}} M \otimes A \rightarrow M.$$

The structure of tensor category on \mathcal{C}_A obtained in this way is opposite to the structure defined above.

Example 3.9. Let \mathcal{C} be a braided fusion category and let $\mathcal{E} \subset \mathcal{C}$ be a Tannakian subcategory. Let $A \in \mathcal{E}$ be the regular algebra (which is connected étale by Example 3.3 (i)). In the terminology of [DGNO, Section 4.2] the fusion category \mathcal{C}_A introduced above is the de-equivariantization of \mathcal{C} (cf. Section 2.5) viewed as a fusion category over \mathcal{E} .

3.4. The central functor $\mathcal{C} \rightarrow \mathcal{C}_A$. Observe that the free module functor (15) has a natural structure of a central functor, see Definition 2.4. Indeed, we have $F_A(X) = X \otimes A$, and, hence, $F_A(X) \otimes_A Y = X \otimes Y$. Similarly, $Y \otimes_A F_A(X) = Y \otimes X$. These two objects are isomorphic via the braiding of \mathcal{C} (one can check that the braiding gives an isomorphism of A -modules using the commutativity of A) and, hence, F_A lifts to a braided tensor functor $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}_A)$.

This construction is in a sense converse to Lemma 3.5. Namely, if we apply it to the algebra $A = I(\mathbf{1})$ then the category \mathcal{C}_A identifies with the image of \mathcal{C} in \mathcal{A} . In the other direction, the object $I(\mathbf{1})$ constructed using the functor F_A is canonically isomorphic (as an algebra) to A .

Let $\mathcal{A}_1, \mathcal{A}_2$ be fusion categories. We will say that a tensor functor $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is *surjective* if any object in \mathcal{A}_2 is a subobject of some $F(X)$, $X \in \mathcal{A}_1$.

Remark 3.10. Some authors use the term *dominant* functor for what we call a surjective functor, see [Br, BrN].

Lemma 3.11. *For a connected étale algebra A in a braided fusion category \mathcal{C} we have*

$$(16) \quad \text{FPdim}(\mathcal{C}_A) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)}.$$

Proof. The functor (15) is surjective. Using [ENO1, Proposition 8.11] we compute

$$\frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(\mathcal{C}_A)} = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{FPdim}(X) [F_A(X) : \mathbf{1}] = \text{FPdim}(I(\mathbf{1})),$$

where I is the right adjoint of F_A and $\mathcal{O}(\mathcal{C})$ denotes the set of simple objects of \mathcal{C} . Since $A = I(\mathbf{1})$, the result follows. \square

3.5. Subcategory $\mathcal{C}_A^0 \subset \mathcal{C}_A$ of dyslectic modules. Let \mathcal{C} be a braided fusion category and $A \in \mathcal{C}$ be a connected étale algebra.

Let ${}_A\mathcal{C}_A$ be the fusion category of A -bimodules in \mathcal{C} (see, e.g., [O]). We have two tensor embeddings $M \mapsto M_{\pm}$ from \mathcal{C}_A and $\mathcal{C}_A^{\text{rev}}$ to ${}_A\mathcal{C}_A$. Namely, using the braiding we can define on a right A -module M the left A -module structure by

$$(17) \quad A \otimes M \xrightarrow{c_{A,M}} M \otimes A \rightarrow M \quad \text{or by} \quad A \otimes M \xrightarrow{c_{M,A}^{-1}} M \otimes A \rightarrow M.$$

Both structures make M an A -bimodule, and we will denote the results by M_+ and M_- , respectively. Clearly, the functors $M \mapsto M_{\pm}$ are sections of the forgetful functor ${}_A\mathcal{C}_A \rightarrow \mathcal{C}_A$. Moreover,

$$(18) \quad T_A : \mathcal{C}_A \boxtimes \mathcal{C}_A^{\text{rev}} \rightarrow {}_A\mathcal{C}_A : M \boxtimes N \mapsto M_+ \otimes_A N_-$$

has a natural structure of tensor functor.

Definition 3.12. A module $M \in \mathcal{C}_A$ is *dyslectic* (or *local*, in alternative terminology) if the identity map id_M is an isomorphism of A -bimodules $M_+ \simeq M_-$.

Equivalently, a module $M \in \mathcal{C}_A$ is dyslectic if the following diagram

$$(19) \quad \begin{array}{ccc} M \otimes A & \xrightarrow{c_{A,M} \circ c_{M,A}} & M \otimes A \\ & \searrow \rho & \swarrow \rho \\ & M & \end{array}$$

commutes. Here $\rho : M \otimes A \rightarrow M$ denotes the action of A on M .

The notion of dyslectic module was introduced by Pareigis in [P]. See also [KiO].

Remark 3.13. Note that a simple $M \in \mathcal{C}_A$ is dyslectic if and only if $M_+ \simeq M_-$ as A -bimodules. Indeed, since the functors $M \mapsto M_{\pm}$ from \mathcal{C}_A to ${}_A\mathcal{C}_A$ are embeddings, for any simple $M \in \mathcal{C}_A$ any isomorphism between A -bimodules M_+ and M_- must be a multiple of id_M .

Dyslectic modules form a full subcategory of \mathcal{C}_A which will be denoted by \mathcal{C}_A^0 . It is known (see [P, Section 2] and [KiO]) that \mathcal{C}_A^0 is closed under \otimes_A and that the braiding in \mathcal{C} induces a natural braided structure in \mathcal{C}_A^0 . Thus, \mathcal{C}_A^0 is a braided fusion category.

Example 3.14. Let $\mathcal{E} \subset \mathcal{C}$ be a Tannakian subcategory and let $A \in \mathcal{E}$ be a regular algebra, see Example 2.8. Then [DGNO, Proposition 4.56(i)] says that \mathcal{C}_A^0 is equivalent to the de-equivariantization of \mathcal{E}' , cf. Section 2.5.

Lemma 3.15. *Let \mathcal{C} be a braided fusion category, let A be an étale algebra in \mathcal{C} , and let X be an object of \mathcal{C} . Then the free module $X \otimes A$ is dyslectic if and only if X centralizes A .*

Proof. Consider the following diagram, where we omit identity maps and associativity constraints:

$$(20) \quad \begin{array}{ccccc} & & A \otimes X \otimes A & & \\ & \nearrow^{c_{X \otimes A, A}} & & \searrow_{c_{A, X \otimes A}} & \\ X \otimes A \otimes A & \xrightarrow{c_{A, A}} & X \otimes A \otimes A & \xrightarrow{c_{A, A}} & X \otimes A \otimes A \\ & \searrow_{m_A} & & \nearrow_{m_A} & \\ & & X \otimes A & & \end{array}$$

The two upper triangles commute by the hexagon axioms and the two lower triangles commute since A is commutative. Therefore,

$$(\text{id}_X \otimes m_A) \circ (c_{A, X} \circ c_{X, A} \otimes \text{id}_A) = (\text{id}_X \otimes m_A) \circ c_{A, X \otimes A} \circ c_{X \otimes A, A} \circ (\text{id}_X \otimes c_{A, A}^{-1}),$$

which means that $X \otimes A$ is dyslectic if and only if

$$(21) \quad (\text{id}_X \otimes m_A) \circ (c_{A, X} \circ c_{X, A} \otimes \text{id}_A) = \text{id}_X \otimes m_A.$$

In other words, commutativity of the perimeter of the above diagram is equivalent to commutativity of the diamond in the middle. Let $u_A : \mathbf{1} \rightarrow A$ denote the unit of A . Suppose that (21) holds. We have

$$\begin{aligned} c_{A, X} \circ c_{X, A} &= (\text{id}_X \otimes m_A) \circ (\text{id}_{X \otimes A} \otimes u_A) \circ c_{A, X} \circ c_{X, A} \\ &= (\text{id}_X \otimes m_A) \circ (c_{A, X} \circ c_{X, A} \otimes \text{id}_A) \circ (\text{id}_{X \otimes A} \otimes u_A) \\ &= (\text{id}_X \otimes m_A) \circ (\text{id}_{X \otimes A} \otimes u_A) = \text{id}_{X \otimes A}. \end{aligned}$$

where the third equality holds by (21). Thus, (21) is equivalent to $c_{A, X} \circ c_{X, A} = \text{id}_{X \otimes A}$. Combining the above equivalences we get the result. \square

3.6. The category $\text{Rep}_{\mathcal{A}}(A)$ and its center. Let \mathcal{A} be a fusion category and let $F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ be the forgetful functor. Let $A \in \mathcal{Z}(\mathcal{A})$ be a connected étale algebra. Observe that any right $F(A)$ -module $M \in \mathcal{A}$ has a natural structure of left $F(A)$ -module defined as $F(A) \otimes M \xrightarrow{\sim} M \otimes F(A) \rightarrow M$. It is easy to verify that in this way M acquires a structure of $F(A)$ -bimodule.

Definition 3.16. The category $\text{Rep}_{\mathcal{A}}(A)$ is a tensor category of right $F(A)$ -modules in \mathcal{A} with tensor product $\otimes_{F(A)}$.

Remark 3.17. (i) Assume that \mathcal{C} is a braided fusion category and $A \in \mathcal{C}$ is a connected étale algebra. Then A can be considered as a connected étale algebra in $\mathcal{Z}(\mathcal{C})$ via the braided functor $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$. In this case the categories \mathcal{C}_A and $\text{Rep}_{\mathcal{C}}(A)$ are identical. Nevertheless the tensor structures on \mathcal{C}_A and $\text{Rep}_{\mathcal{C}}(A)$ are *opposite* to each other, see Remark 3.8.

(ii) The category $\text{Rep}_{\mathcal{C}}(A)$ is equivalent to the category of *left* $F(A)$ -modules.

Arguments similar to those in Section 3.3 show that $\text{Rep}_{\mathcal{A}}(A)$ is a semisimple rigid tensor category. Its unit object $F(A)$ may be reducible, so in general $\text{Rep}_{\mathcal{A}}(A)$ is not a fusion category. In general $\text{Rep}_{\mathcal{A}}(A)$ is an example of a connected multi-fusion category, see Section 2.1

Remark 3.18. Given an étale algebra $A \in \mathcal{Z}(\mathcal{A})$ there is a surjective tensor functor

$$\mathcal{A} \rightarrow \text{Rep}_{\mathcal{A}}(A) : X \mapsto X \otimes F(A).$$

Conversely, let $G : \mathcal{A} \rightarrow \mathcal{B}$ be a tensor functor and let $I : \mathcal{B} \rightarrow \mathcal{A}$ be its right adjoint. Then the object $I(\mathbf{1}) \in \mathcal{A}$ has a natural lift to $\mathcal{Z}(\mathcal{A})$. Moreover, it has a natural structure of an étale algebra in $\mathcal{Z}(\mathcal{A})$. The algebra $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{A})$ is connected if and only if the functor G is not decomposable into a non-trivial direct sum of tensor functors. Similarly to Section 3.4 these two constructions are inverse to each other. See [BrN] for details.

It is easy to see that the forgetful functor $\mathcal{Z}(\mathcal{A})_A^0 \hookrightarrow \mathcal{Z}(\mathcal{A})_A \rightarrow \text{Rep}_{\mathcal{A}}(A)$ has a canonical structure of central functor. Thus, it lifts to a braided tensor functor

$$(22) \quad \mathcal{Z}(\mathcal{A})_A^0 \rightarrow \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A)).$$

The following result was proved by Schauenburg (see [Sch, Corollary 4.5]) under much weaker assumptions on the category \mathcal{A} and commutative algebra $A \in \mathcal{Z}(\mathcal{A})$ than ours.

Theorem 3.19. *The functor (22) is a braided equivalence $\mathcal{Z}(\mathcal{A})_A^0 \cong \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A))$.*

Sketch of proof. We just sketch a construction of an inverse functor. Let $M \in \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A))$. For any $X \in \mathcal{A}$ consider $X \otimes F(A) \in \text{Rep}_{\mathcal{A}}(A)$. Then $(X \otimes F(A)) \otimes_{F(A)} M = X \otimes M$ and $M \otimes_{F(A)} (X \otimes F(A)) = M \otimes X$. It is easy to see now that the central structure of M as $F(A)$ -module defines a central structure of M as an object of \mathcal{A} . Moreover one verifies directly that $F(A)$ -module structure on M gives A -module structure on this lift of M to $\mathcal{Z}(\mathcal{A})$; the resulting object of $\mathcal{Z}(\mathcal{A})_A$ lies in $\mathcal{Z}(\mathcal{A})_A^0$. Finally, this assignment has a natural structure of tensor functor. \square

3.7. Properties of braided tensor functors.

Proposition 3.20. *Let \mathcal{C}, \mathcal{D} be braided fusion categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a surjective braided tensor functor. Let $I : \mathcal{D} \rightarrow \mathcal{C}$ be the right adjoint functor of F and let $A := I(\mathbf{1})$ be the canonical connected étale algebra constructed in Lemma 3.5. Then $A \in \mathcal{C}'$.*

Proof. Since F is a central functor, \mathcal{D} identifies with the category \mathcal{C}_A of A -modules in \mathcal{C} , cf. Section 3.4. We claim that every A -module is dyslectic, i.e., that $\mathcal{C}_A = \mathcal{C}_A^0$. Indeed, the fusion category ${}_A\mathcal{C}_A$ identifies with the category of \mathcal{C} -module endofunctors of \mathcal{D} , see [O] (the action of \mathcal{C} on \mathcal{D} is defined via $F : \mathcal{C} \rightarrow \mathcal{D}$). Under this identification, for every simple object $M \in \mathcal{D}$ the bimodules M_{\pm} correspond to endofunctors of left and right multiplication by M . But these endofunctors are isomorphic via the braiding of \mathcal{D} , i.e., M is dyslectic.

In particular, for every $X \in \mathcal{C}$ the free A -module $X \otimes A$ is dyslectic. Hence, Lemma 3.15 implies that every $X \in \mathcal{C}$ centralizes A , i.e., $A \in \mathcal{C}'$. \square

Remark 3.21. Note that the étale algebra A from Proposition 3.20 is a commutative algebra in a symmetric fusion category \mathcal{C}' . Therefore, A belongs to the maximal Tannakian subcategory $\mathcal{E} = \text{Rep}(G) \subset \mathcal{C}'$. The restriction of $F : \mathcal{C} \rightarrow \mathcal{D}$ to \mathcal{E} identifies with the restriction functor $\text{Rep}(G) \rightarrow \text{Rep}(H)$, where H is a subgroup of G . Hence, the étale algebra A identifies with the algebra $\text{Fun}(G/H)$ of functions on G invariant under translations by elements of H .

Corollary 3.22. *Let $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a surjective braided tensor functor between braided fusion categories. There exists a braided fusion category \mathcal{C} with an action of a finite group G , a subgroup $H \subset G$, and braided tensor equivalences*

$\mathcal{C}_1 \cong \mathcal{C}^G$, $\mathcal{C}_2 \cong \mathcal{C}^H$ such that the diagram

$$(23) \quad \begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{C}^G & \xrightarrow{Forg} & \mathcal{C}^H \end{array}$$

commutes. Here $Forg : \mathcal{C}^G \rightarrow \mathcal{C}^H$ is the functor of “partially forgetting equivariance”.

Proof. This is a consequence of Example 2.9, Remark 3.21, and the fact that equivariantization and de-equivariantization are mutually inverse constructions, see [DGNO, Theorem 4.4] and Section 2.5. \square

Definition 3.23. A braided fusion category \mathcal{C} is called *almost non-degenerate* if the symmetric category \mathcal{C}' is either trivial or is equivalent to the category of super vector spaces.

In other words, \mathcal{C} is almost non-degenerate if \mathcal{C}' does not contain any non-trivial Tannakian subcategories.

Corollary 3.24. *Any braided tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between braided fusion categories with \mathcal{C} almost non-degenerate is fully faithful.*

Remark 3.25. Using [EO, Theorem 2.5] and [De, Proposition 2.14] one can relax the assumptions of Corollary 3.24 on the category \mathcal{D} : it is enough to assume that \mathcal{D} is a abelian rigid braided tensor category with finite dimensional Hom spaces and finite lengths of all objects.

Let \mathcal{C} be a braided fusion category, $A \in \mathcal{C}$ be a connected étale algebra and $F_A : \mathcal{C} \rightarrow \mathcal{C}_A$ be the functor (15). It extends to a functor

$$(24) \quad \tilde{F}_A : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}_A)$$

in such a way that F_A is the composition of \tilde{F}_A and the forgetful functor $\mathcal{Z}(\mathcal{C}_A) \rightarrow \mathcal{C}_A$.

Corollary 3.26. *Assume \mathcal{C} is almost non-degenerate. Then functor (24) is fully faithful and the functor $T_A : \mathcal{C}_A \boxtimes \mathcal{C}_A^{\text{rev}} \rightarrow {}_A\mathcal{C}_A$ defined in (18) is surjective.*

Proof. The first assertion is Corollary 3.24. To prove the second assertion observe that that F_A is dual to T_A (in the sense of [ENO1, Section 5.7]) with respect to the module category \mathcal{C}_A . So the result follows from [ENO1, Proposition 5.3]. \square

3.8. Tensor complements. Let \mathcal{C} be a non-degenerate braided fusion category, see Definition 2.1. Let $A \in \mathcal{C}$ be a connected étale algebra. Then A can be considered as a connected étale algebra in \mathcal{C}^{rev} and in $\mathcal{Z}(\mathcal{C})$ via the embedding

$$\mathcal{C}^{\text{rev}} = \text{Vec} \boxtimes \mathcal{C}^{\text{rev}} \hookrightarrow \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \cong \mathcal{Z}(\mathcal{C}),$$

see (11).

Lemma 3.27. *Under the identification $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ we have*

$$\mathcal{Z}(\mathcal{C})_A = \mathcal{C} \boxtimes \mathcal{C}_A^{\text{rev}} \quad \text{and} \quad \mathcal{Z}(\mathcal{C})_A^0 = \mathcal{C} \boxtimes (\mathcal{C}^{\text{rev}})_A^0.$$

Proof. The first statement is obvious and the second one is an immediate consequence. \square

Corollary 3.28. *For a non-degenerate \mathcal{C} and a connected étale algebra $A \in \mathcal{C}$ there is a braided equivalence $\mathcal{Z}(\mathcal{C}_A) \simeq \mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\text{rev}}$. In particular the category \mathcal{C}_A^0 is non-degenerate.*

Proof. Combine Theorem 3.19 and Lemma 3.27. \square

Remark 3.29. (i) One verifies that the embedding functor

$$\mathcal{C} = \mathcal{C} \boxtimes \text{Vec} \hookrightarrow \mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\text{rev}} \cong \mathcal{Z}(\mathcal{C}_A)$$

is naturally isomorphic to the functor \tilde{F}_A from (24). This can be used for an alternative proof of Proposition 3.26.

- (ii) If we assume in addition that \mathcal{C} is modular and A is as in Example 3.3(ii) then \mathcal{C}_A^0 has a natural spherical structure, see e.g. [KiO]. In this case Corollary 3.28 gives an alternative proof of [KiO, Theorem 4.5].

Corollary 3.30. *For a non-degenerate \mathcal{C} and a connected étale algebra $A \in \mathcal{C}$ we have*

$$(25) \quad \text{FPdim}(\mathcal{C}_A^0) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)^2}.$$

Proof. This follows immediately from Corollary 3.28 and equations (8) and (16). \square

4. QUANTUM MANIN PAIRS

4.1. Definition of a quantum Manin pair. We start with the following consequence of Corollary 3.26:

Corollary 4.1. *Let \mathcal{C} be a non-degenerate braided fusion category and $A \in \mathcal{C}$ a connected étale algebra in \mathcal{C} . Assume that $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$. Then*

- (i) *The functor $\tilde{F}_A : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}_A)$ defined in (24) is a braided tensor equivalence*
(ii) *The functor $T_A : \mathcal{C}_A \boxtimes \mathcal{C}_A^{\text{rev}} \rightarrow {}_A\mathcal{C}_A$ defined in (18) is a tensor equivalence.*

Proof. By Lemma 3.11

$$\text{FPdim}(\mathcal{C}_A) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)}.$$

Hence,

$$\text{FPdim}(\mathcal{Z}(\mathcal{C}_A)) = \frac{\text{FPdim}(\mathcal{C})^2}{\text{FPdim}(A)^2} = \text{FPdim}(\mathcal{C}),$$

see (8). Since by Corollary 3.26 \tilde{F}_A is a fully faithful functor between categories of equal Frobenius-Perron dimension, it is necessarily an equivalence by [EO, Proposition 2.19]. Hence the dual functor T_A is also an equivalence. \square

Definition 4.2. A *quantum Manin pair* (\mathcal{C}, A) consists of a non-degenerate braided fusion category \mathcal{C} and a connected étale algebra $A \in \mathcal{C}$ such that $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$.

Remark 4.3. Observe that by (25) the condition $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$ is equivalent to the condition $\mathcal{C}_A^0 = \text{Vec}$.

Quantum Manin pairs form a 2-groupoid \mathcal{QM} : a 1-morphism between two such pairs (\mathcal{C}_1, A_1) and (\mathcal{C}_2, A_2) is defined to be a pair (Φ, ϕ) , where $\Phi : \mathcal{C}_1 \simeq \mathcal{C}_2$ is a braided equivalence and $\phi : \Phi(A_1) \xrightarrow{\sim} A_2$ is an isomorphism of algebras; a

2-morphism between pairs (Φ, ϕ) and (Φ', ϕ') is a natural isomorphism of tensor functors $\mu : \Phi \xrightarrow{\sim} \Phi'$ such that the following diagram commutes:

$$(26) \quad \begin{array}{ccc} \Phi(A_1) & \xrightarrow{\mu} & \Phi'(A_1) \\ & \searrow \phi & \swarrow \phi' \\ & & A_2 \end{array}$$

On the other hand, we have the 2-groupoid \mathfrak{FC} of fusion categories: objects are fusion categories, 1-morphisms are tensor equivalences, and 2-morphisms are isomorphisms of tensor functors. We have a 2-functor $\mathcal{QM} \rightarrow \mathfrak{FC}$ defined by $(\mathcal{C}, A) \mapsto \mathcal{C}_A$.

Proposition 4.4. *This 2-functor $\mathcal{QM} \rightarrow \mathfrak{FC}$ is a 2-equivalence.*

Proof. Let $\mathcal{A} \in \mathfrak{FC}$. The forgetful functor $F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ has an obvious structure of central functor. Let $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$ be its right adjoint. By Lemma 3.5, $I(\mathbf{1})$ is a connected étale algebra. It is known that $\text{FPdim}(I(\mathbf{1})) = \text{FPdim}(\mathcal{C})$, see e.g. [EO, Lemma 3.41]. So (8) implies that $(\mathcal{Z}(\mathcal{A}), I(\mathbf{1})) \in \mathcal{QM}$. Thus we get a 2-functor $\mathfrak{FC} \rightarrow \mathcal{QM}$. Using Corollary 4.1 and the results from Section 3.4 we see that it is quasi-inverse to the 2-functor $\mathcal{QM} \rightarrow \mathfrak{FC}$. \square

Remark 4.5. Proposition 4.4 can be viewed as a categorical analogue of the following reconstruction of the double of a quasi-Lie bialgebra from a Manin pair (i.e., a pair consisting of a metric Lie algebra and its Lagrangian subalgebra) in the theory of quantum groups [Dr, Section 2]:

Let \mathfrak{g} be a finite-dimensional metric Lie algebra (i.e., a Lie algebra on which a non-degenerate invariant symmetric bilinear form is given). Let \mathfrak{l} be a Lagrangian subalgebra of \mathfrak{g} . Then \mathfrak{l} has a structure of a quasi-Lie bialgebra and there is an isomorphism between \mathfrak{g} and the double $\mathfrak{D}(\mathfrak{l})$ of \mathfrak{l} . The correspondence between Lagrangian subalgebras of \mathfrak{g} and doubles isomorphic to \mathfrak{g} is bijective, see [Dr, Section 2] for details.

4.2. Lagrangian algebras and Morita 2-equivalence.

Definition 4.6. Let \mathcal{C} be a non-degenerate braided fusion category. A connected étale algebra in \mathcal{C} will be called *Lagrangian* if $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$.

Thus, A is Lagrangian if and only if (\mathcal{C}, A) is a quantum Manin pair.

Remark 4.7. Let $\mathcal{E} \subset \mathcal{C}$ be a Lagrangian subcategory of \mathcal{C} , i.e., a Tannakian subcategory such that $\mathcal{E}' = \mathcal{E}$, see [DGNO, Definition 4.57]. Then the regular algebra of \mathcal{E} is a Lagrangian algebra in \mathcal{C} .

Proposition 4.8. *Let \mathcal{A} be a fusion category and let $\mathcal{C} = \mathcal{Z}(\mathcal{A})$. There is a bijection between the sets of Lagrangian algebras in \mathcal{C} and indecomposable \mathcal{A} -module categories.*

Proof. By Corollary 4.1 every Lagrangian algebra $B \in \mathcal{C}$ determines a braided equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{B})$, where $\mathcal{B} := \mathcal{C}_B$. Conversely, any braided equivalence between \mathcal{C} and $\mathcal{Z}(\mathcal{B})$ determines a central functor $\mathcal{C} \rightarrow \mathcal{B}$ and, hence, a Lagrangian algebra in \mathcal{C} . As we observed in Section 3.4 these two constructions are inverses of each other.

Thus it suffices to prove that the set of braided equivalences between $\mathcal{Z}(\mathcal{A})$ and centers of fusion categories is in bijection with indecomposable \mathcal{A} -module categories.

This is done in [ENO2, Theorem 3.1] and [ENO3, Theorem 1.1]. Namely, the bijection is provided by assigning to an \mathcal{A} -module category \mathcal{M} braided equivalence (12). \square

Remark 4.9. Note that the bijection in Proposition 4.8 is given by the so-called full centre construction. In particular, $I(\mathbf{1})$ is the full centre of \mathcal{A} as a module category over itself. In the case when \mathcal{A} is modular the statement of the proposition was verified in [KR, Theorem 3.22]. Note also that in this case the bijection can be lifted to an equivalence of groupoids (module categories with module equivalences by one side and Lagrangian algebras and isomorphisms by the other) [DKR].

4.3. Lattice of subcategories. Let \mathcal{A} be a fusion category and let (\mathcal{C}, A) be the corresponding Manin pair. Here $\mathcal{C} = \mathcal{Z}(\mathcal{A})$ and $A = I(\mathbf{1})$, where $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$ is the induction functor.

Let $\mathfrak{L}(\mathcal{A})$ denote the lattice of fusion subcategories of \mathcal{A} and let $L(A)$ denote the lattice of étale subalgebras of A .

Theorem 4.10. *There is a canonical anti-isomorphism of lattices $\mathfrak{L}(\mathcal{A}) \simeq L(A)$. If $B \subset A$ is the subalgebra corresponding to the subcategory $\mathcal{B} \subset \mathcal{A}$ under this anti-isomorphism, then $FPdim(B)FPdim(\mathcal{B}) = FPdim(\mathcal{A})$.*

Proof. Let $\mathcal{B} \subset \mathcal{A}$ be a fusion subcategory. Define the *relative center* $\mathcal{Z}_{\mathcal{B}}(\mathcal{A})$ to be the tensor category whose objects are pairs (X, γ_X) , where X is an object of \mathcal{A} and $\gamma_X : V \otimes X \simeq X \otimes V$, $V \in \mathcal{A}$ is a natural family of isomorphisms, satisfying the same compatibility condition as in the definition of $\mathcal{Z}(\mathcal{A})$. Consider the forgetful functor

$$(27) \quad F_{\mathcal{B}} : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{Z}_{\mathcal{B}}(\mathcal{A})$$

and define $\alpha(\mathcal{B}) = I_{\mathcal{B}}(\mathbf{1})$ where $I_{\mathcal{B}}$ is the right adjoint of $F_{\mathcal{B}}$. We have

$$(28) \quad FPdim(\alpha(\mathcal{B})) = \frac{FPdim(\mathcal{A})}{FPdim(\mathcal{B})}$$

by [DGNO, Section 3.6] and [ENO1, Corollary 8.11].

In the opposite direction, given an étale subalgebra $B \subset A$ we have a tensor functor $\mathcal{C}_B \rightarrow \mathcal{C}_A$ inducing A -modules from B -modules. Let $\beta(B)$ be the full image in $\mathcal{C}_A = \mathcal{A}$ of the subcategory $\mathcal{C}_B^0 \subset \mathcal{C}_B$ under this functor.

By construction, $\mathcal{C}_{\alpha(\mathcal{B})} = \mathcal{Z}_{\mathcal{B}}(\mathcal{A})$ and $\mathcal{C}_{\alpha(\mathcal{B})}^0 = \mathcal{Z}(\mathcal{B})$. The induction functor

$$(29) \quad \mathcal{C}_{\alpha(\mathcal{B})} \rightarrow \mathcal{C}_A = \mathcal{A}$$

identifies with the forgetful functor $\mathcal{Z}_{\mathcal{B}}(\mathcal{A}) \rightarrow \mathcal{A}$ and so maps surjectively $\mathcal{Z}(\mathcal{B})$ to \mathcal{B} . Thus, $\beta(\alpha(\mathcal{B})) = \mathcal{B}$.

Conversely, we claim that there is an equivalence $\mathcal{Z}_{\beta(B)}(\mathcal{A}) \xrightarrow{\sim} \mathcal{C}_B$ such that the forgetful functor $F_{\beta(B)} : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{Z}_{\beta(B)}(\mathcal{A})$ identifies with the free module functor $\mathcal{C} \rightarrow \mathcal{C}_B$. This immediately implies that $\alpha(\beta(B)) = B$. To prove this claim, note that the braiding of \mathcal{C} allows to equip any A -module induced from \mathcal{C}_B with a morphism permuting it with A -modules induced from \mathcal{C}_B^0 . This gives rise to a tensor functor

$$(30) \quad \mathcal{C}_B \rightarrow \mathcal{Z}_{\beta(B)}(\mathcal{A}).$$

The functor restricts to an equivalence between \mathcal{C}_B^0 and $\mathcal{Z}(\beta(B)) \subset \mathcal{Z}_{\beta(B)}(\mathcal{A})$. Observe that an object of \mathcal{C}_B whose image under the functor (30) is in $\mathcal{Z}(\beta(B))$

must belong to \mathcal{C}_B^0 . Hence, (30) is an equivalence. The remaining part of the claim follows from commutativity of the diagram

$$(31) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{A \otimes -} & \mathcal{Z}(\mathcal{C}_A) \\ B \otimes - \downarrow & & \downarrow F_{\beta(B)} \\ \mathcal{C}_B & \xrightarrow{A \otimes_B -} & \mathcal{Z}_{\beta(B)}(\mathcal{A}), \end{array}$$

where the bottom arrow is the induction functor (30).

Equation (28) implies the assertion about Frobenius-Perron dimensions.

Finally, to see that α is a lattice anti-homomorphism it suffices to note that an inclusion of subcategories $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{A}$ induces a forgetful functor $\mathcal{Z}_{\mathcal{B}_2}(\mathcal{A}) \rightarrow \mathcal{Z}_{\mathcal{B}_1}(\mathcal{A})$ compatible with (27). This, in turn, yields an inclusion of étale algebras $I_{\mathcal{B}_2}(\mathbf{1}) \subset I_{\mathcal{B}_1}(\mathbf{1})$ in \mathcal{C} . \square

Example 4.11. Let us illustrate Theorem 4.10. Let G be a finite group.

- (i) Let $\mathcal{A} = \text{Rep}(G)$ be the fusion category of representations of G . Its fusion subcategories are of the form $\text{Rep}(G/N)$ where N ranges over the set of all normal subgroups of G . The étale algebra in $\mathcal{Z}(\text{Rep}(G))$ corresponding to the subcategory $\text{Rep}(G/N)$ is the group algebra $\mathbb{k}N$. As an object of $\mathcal{Z}(\text{Rep}(G))$ it has the following description. It is a G -graded algebra with non-zero graded components labelled by elements of N , the G -action on $\mathbb{k}N$ is the conjugation action (see [Da1], where étale algebras in $\mathcal{Z}(\text{Rep}(G))$ were classified).
- (ii) Let $\mathcal{A} = \text{Vec}_G^\omega$ be the fusion category of G -graded vector spaces with the associativity constraint twisted by a 3-cocycle $\omega \in Z^3(G, \mathbb{k}^\times)$. Fusion subcategories of \mathcal{A} correspond to subgroups $H \subset G$. A typical such subcategory is $\text{Vec}_H^{\omega|_H}$. The corresponding étale algebra in $\mathcal{Z}(\text{Vec}_G^\omega)$ is the algebra of \mathbb{k} -valued functions on G invariant under translations by elements of H .

4.4. Quantum Manin triples. Recall that a *Manin triple* consists of a metric Lie algebra \mathfrak{g} along with Lagrangian Lie subalgebras \mathfrak{g}_+ , \mathfrak{g}_- such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space. It was shown by Drinfeld in [Dr, Section 2] that Manin triples are in bijection with pairs of dual Lie bialgebras (cf. Remark 4.5).

Below we extend this result to the “quantum” setting.

Definition 4.12. A *quantum Manin triple* (\mathcal{C}, A, B) consists of a non-degenerate braided fusion category \mathcal{C} along with connected étale algebras A, B in \mathcal{C} such that both (\mathcal{C}, A) and (\mathcal{C}, B) are quantum Manin pairs and the category of (A, B) -bimodules in \mathcal{C} is equivalent to Vec .

Example 4.13. Let H be a semisimple Hopf algebra and let $\text{Rep}(H)$ denote the category of finite-dimensional representations of H . Let $\mathcal{C} := \mathcal{Z}(\text{Rep}(H))$. It is well known that \mathcal{C} is equivalent, as a braided fusion category, to $\text{Rep}(D(H))$ where $D(H)$ is the Drinfeld double of H . There is a canonical Hopf algebra isomorphism $D(H) \cong D((H^*)^{op})$, where H^* denotes the dual Hopf algebra and op stands for the opposite multiplication. We thus have two central functors, to wit the forgetful functors,

$$\mathcal{C} \rightarrow \text{Rep}(H) \quad \text{and} \quad \mathcal{C} \rightarrow \text{Rep}((H^*)^{op}).$$

Let A and B denote the étale algebras in \mathcal{C} corresponding to these functors constructed as in Section 3.2.

We claim that (\mathcal{C}, A, B) is a quantum Manin triple. The only thing that needs to be checked is that the category of $(A - B)$ -bimodules in \mathcal{C} is trivial. Note that $A = (H^*)^{op}$ and $B = H$ as $D(H)$ -module algebras (i.e., algebras in $\mathcal{C} = \text{Rep}(D(H))$). The category of $(H^*)^{op} \otimes H$ -bimodules in $\text{Rep}(D(H))$ is nothing but the category of $D(H)$ -Hopf modules which is equivalent to Vec by the Fundamental Theorem of Hopf modules (see [Mo] for the definition of a Hopf module and the Fundamental Theorem).

Conversely, let (\mathcal{C}, A, B) be a quantum Manin triple. Then Vec has a structure of a \mathcal{C}_A -module category and so $\mathcal{C}_A \cong \text{Rep}(H)$ for a semisimple Hopf algebra H (equivalently, \mathcal{C} has a fiber functor, i.e., a tensor functor to Vec). The dual category $(\mathcal{C}_A)_{\text{Vec}}^*$ is equivalent to \mathcal{C}_B and so $\mathcal{C}_B \cong \text{Rep}((H^*)^{op})$.

Quantum Manin triples form a 2-groupoid \mathcal{G}_1 : a 1-morphism between triples $(\mathcal{C}_1, A_1, B_1)$ and $(\mathcal{C}_2, A_2, B_2)$ is defined to be a triple (Φ, ϕ, ψ) , where $\Phi : \mathcal{C}_1 \simeq \mathcal{C}_2$ is a braided equivalence and $\phi : \Phi(A_1) \xrightarrow{\sim} A_2$, $\psi : \Phi(B_1) \xrightarrow{\sim} B_2$ are isomorphisms of algebras; a 2-morphism between triples (Φ, ϕ, ψ) and (Φ', ϕ', ψ') is a natural isomorphism of tensor functors $\mu : \Phi \simeq \Phi'$ such that $\phi = \phi' \mu_{A_1}$ and $\psi = \psi' \mu_{B_1}$ (cf. diagram (26)).

Let \mathcal{G}_2 denote the 2-groupoid whose objects are pairs (\mathcal{A}, F) where \mathcal{A} is a fusion category and $F : \mathcal{A} \rightarrow \text{Vec}$ is a fiber tensor functor; 1-morphisms between (\mathcal{A}, F) and (\mathcal{A}', F') are pairs (ι, ν) where $\iota : \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ is a tensor equivalence and $\nu : F \xrightarrow{\sim} F' \iota$ is an isomorphism of tensor functors; 2-morphisms between (ι_1, ν_1) and (ι_2, ν_2) are natural isomorphisms of tensor functors $m : \iota_1 \xrightarrow{\sim} \iota_2$ such that $\nu_2 = (F' m) \circ \nu_1$.

Proposition 4.14. *There is a 2-equivalence $(\mathcal{C}, A_1, A_2) \mapsto \mathcal{C}_{A_1}$ between \mathcal{G}_1 and \mathcal{G}_2 .*

The proof of Proposition 4.14 is similar to that of Proposition 4.4 and amounts to showing that the above constructions are inverses of each other. In fact, 2-groupoids \mathcal{G}_1 and \mathcal{G}_2 are also equivalent to the third 2-groupoid \mathcal{G}_3 which is defined in linear algebra terms: objects of \mathcal{G}_3 are semisimple Hopf algebras, 1-morphisms are twisted isomorphisms of Hopf algebras (defined in [Da]), and 2-morphisms are gauge equivalences of twists. Details of these equivalences will be given elsewhere.

Finally, we give an easy criterion which allows to recognize a quantum Manin triple. Let $R_{\mathcal{C}} \in K(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the regular object of \mathcal{C} , see Section 2.1.

Proposition 4.15. *Let \mathcal{C} be a non-degenerate braided fusion category and let $(\mathcal{C}, A), (\mathcal{C}, B)$ be quantum Manin pairs. The following conditions are equivalent:*

- (i) (\mathcal{C}, A, B) is a quantum Manin triple;
- (ii) $[A \otimes B] = R_{\mathcal{C}}$;
- (iii) $\dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes B) = 1$;
- (iv) $\dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(A, B) = 1$.

Proof. Let us prove implication (i) \Rightarrow (ii). Thus the category of (A, B) -bimodules has a unique up to isomorphism simple object M . For any $X \in \mathcal{C}$, the object $A \otimes X \otimes B$ has an obvious structure of (A, B) -bimodule. Hence $[A \otimes X \otimes B] = r_X [M]$ for some positive integer r_X . Consequently

$$[A \otimes X \otimes B] = \frac{r_X}{r_{\mathbf{1}}} [A \otimes B].$$

Computing the Frobenius-Perron dimension of both sides, we get $[A \otimes X \otimes B] = \text{FPdim}(X)[A \otimes B]$. Since the category \mathcal{C} is braided we have

$$[X][A \otimes B] = [A \otimes X \otimes B] = \text{FPdim}(X)[A \otimes B].$$

Since $\text{FPdim}(A) = \text{FPdim}(B) = \sqrt{\text{FPdim}(\mathcal{C})}$, we have $\text{FPdim}(A \otimes B) = \text{FPdim}(\mathcal{C})$. Hence, $[A \otimes B] = R_{\mathcal{C}}$.

The implication (ii) \Rightarrow (iii) is immediate and the equivalence (iii) \Leftrightarrow (iv) follows from Remark 3.4 since $\text{Hom}_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{C}}(\mathbf{1}, {}^*A \otimes B) \simeq \text{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes B)$.

Let us prove implication (iii) \Rightarrow (i). By Proposition 4.4, the central functor $F_B : \mathcal{C} \rightarrow \mathcal{C}_B$ is isomorphic to the forgetful functor $\mathcal{Z}(\mathcal{C}_B) \rightarrow \mathcal{C}_B$ (for a suitable choice of braided equivalence $\mathcal{C} \simeq \mathcal{Z}(\mathcal{C}_B)$). Consider the category $\text{Rep}_{\mathcal{C}_B}(A)$ (see Section 3.6). Notice that by Remark 3.17(ii), this category coincides with the category of (A, B) -bimodules in \mathcal{C} . Thus, we need to prove that $\text{Rep}_{\mathcal{C}_B}(A) \simeq \text{Vec}$. Recall from Section 3.6 that the category $\text{Rep}_{\mathcal{C}_B}(A)$ has a structure of multi-fusion category. On the other hand the unit object $A \otimes B$ of this category is irreducible since $\text{Hom}_{A-B}(A \otimes B, A \otimes B) = \text{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes B)$. Thus, the multi-fusion category $\text{Rep}_{\mathcal{C}_B}(A)$ is in fact a fusion category. By Theorem 3.19 and Remark 4.3 we have $\mathcal{Z}(\text{Rep}_{\mathcal{C}_B}(A)) = \mathcal{C}_A^0 = \text{Vec}$. Thus (8) implies that $\text{FPdim}(\text{Rep}_{\mathcal{C}_B}(A)) = 1$, whence $\text{Rep}_{\mathcal{C}_B}(A) = \text{Vec}$. \square

5. DEFINITION AND PROPERTIES OF THE WITT GROUP

5.1. Definition of the Witt group.

Definition 5.1. Non-degenerate braided fusion categories \mathcal{C}_1 and \mathcal{C}_2 are *Witt equivalent* if there exists a braided equivalence $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$, where $\mathcal{A}_1, \mathcal{A}_2$ are fusion categories.

It is easy to see that Witt equivalence is indeed an equivalence relation. We will denote the Witt equivalence class containing a category \mathcal{C} by $[\mathcal{C}]$. The set of Witt equivalence classes of non-degenerate braided fusion categories will be denoted \mathcal{W} . Clearly \mathcal{W} is a commutative monoid with respect to the operation \boxtimes . The unit of this monoid is $[\text{Vec}]$.

Lemma 5.2. *The monoid \mathcal{W} is a group.*

Proof. For a non-degenerate braided fusion category \mathcal{C} we have $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$, see Section 2.3. Thus $[\mathcal{C}]^{-1} = [\mathcal{C}^{\text{rev}}]$. \square

Proposition 5.3. *Let $A \in \mathcal{C}$ be an étale connected algebra. Then $[\mathcal{C}_A^0] = [\mathcal{C}]$ in \mathcal{W} .*

Proof. This is immediate from Definition 5.1 and Corollary 3.28. \square

Definition 5.4. The abelian group \mathcal{W} defined above is called *the Witt group of non-degenerate braided fusion categories*.

Remark 5.5. It is apparent from the definition that the group \mathcal{W} depends on the base field \mathbb{k} and should be denoted $\mathcal{W}(\mathbb{k})$. However it is known that any fusion category (or braided fusion category) is defined over the field of algebraic numbers \mathbb{Q} , see [ENO1, Section 2.6]. Thus an embedding $\mathbb{Q} \subset \mathbb{k}$ induces an isomorphism $\mathcal{W}(\mathbb{Q}) \simeq \mathcal{W}(\mathbb{k})$. In this sense we can talk about *the Witt group of non-degenerate braided fusion categories* (without mentioning the field \mathbb{k}). Of course this implies that the group \mathcal{W} carries a natural action of the absolute Galois group $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ and should be considered together with this action.

Remark 5.6. It follows from [ENO1, Theorems 2.28, 2.31, and Remark 2.33] that there are countably many non-equivalent braided fusion categories. In particular, the group \mathcal{W} is at most countable. We will see later that \mathcal{W} is infinite.

Proposition 5.7. *Let \mathcal{C} be a non-degenerate braided fusion category. Then $\mathcal{C} \in [\text{Vec}]$ if and only if there exist a fusion category \mathcal{A} and a braided equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{A})$.*

Proof. By definition, $\mathcal{C} \in [\text{Vec}]$ if and only if $\mathcal{C} \boxtimes \mathcal{Z}(\mathcal{B}_1) \simeq \mathcal{Z}(\mathcal{B}_2)$ with fusion categories \mathcal{B}_1 and \mathcal{B}_2 . Let $A \in \mathcal{Z}(\mathcal{B}_1)$ be a connected étale algebra such that $(\mathcal{Z}(\mathcal{B}_1), A)$ is a quantum Manin pair, see Definition 4.2. Consider the fusion category $\mathcal{A} = \text{Rep}_{\mathcal{B}_2}(A)$, see Section 3.6. By Theorem 3.19 we have $\mathcal{Z}(\mathcal{A}) \cong \mathcal{Z}(\mathcal{B}_2)_A^0$. On the other hand we have an obvious injective braided tensor functor

$$(32) \quad \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{B}_2)_A^0 : X \mapsto (X \boxtimes \mathbf{1}) \otimes A.$$

We have

$$\text{FPdim}(\mathcal{C}) = \frac{\text{FPdim}(\mathcal{Z}(\mathcal{B}_2))}{\text{FPdim}(\mathcal{Z}(\mathcal{B}_1))} = \frac{\text{FPdim}(\mathcal{Z}(\mathcal{B}_2))}{\text{FPdim}(A)^2} = \text{FPdim}(\mathcal{Z}(\mathcal{B}_2)_A^0),$$

i.e., (32) is a fully faithful tensor functor between fusion categories of equal Frobenius-Perron dimension. Therefore, it is an equivalence by [EO, Proposition 2.19]. \square

Corollary 5.8. *We have $[\mathcal{C}] = [\mathcal{D}]$ if and only if there exists a fusion category \mathcal{A} and a braided equivalence $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A})$.*

5.2. Completely anisotropic categories.

Definition 5.9. We say that a non-degenerate braided fusion category is *completely anisotropic* if the only connected étale algebra $A \in \mathcal{C}$ is $A = \mathbf{1}$.

Remark 5.10. A completely anisotropic non-degenerate braided fusion category has no Tannakian subcategories other than Vec , i.e., it is *anisotropic* in the sense of [DGNO, Definition 5.16].

Lemma 5.11. *Let \mathcal{C} be a completely anisotropic category, \mathcal{A} be a fusion category, and let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a central functor. Then F is fully faithful.*

Proof. Let $I : \mathcal{A} \rightarrow \mathcal{C}$ be the right adjoint of F . Since \mathcal{C} is completely anisotropic, Lemma 3.5 implies that $I(\mathbf{1}) = \mathbf{1}$. Thus

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\cong \text{Hom}_{\mathcal{C}}(X \otimes {}^*Y, \mathbf{1}) \cong \text{Hom}_{\mathcal{C}}(X \otimes {}^*Y, I(\mathbf{1})) \\ &\cong \text{Hom}_{\mathcal{A}}(F(X \otimes {}^*Y), \mathbf{1}) \cong \text{Hom}_{\mathcal{A}}(F(X) \otimes {}^*F(Y), \mathbf{1}) \\ &\cong \text{Hom}_{\mathcal{A}}(F(X), F(Y)). \end{aligned}$$

The result follows. \square

Theorem 5.12. *Each Witt equivalence class in \mathcal{W} contains a completely anisotropic category that is unique up to braided equivalence.*

Proof. Let \mathcal{C} be a non-degenerate braided fusion category. Let $A \in \mathcal{C}$ be a maximal étale connected algebra (which exists since by (16) the Frobenius-Perron dimensions of connected étale algebras are bounded by $\text{FPdim}(\mathcal{C})$). Any étale connected algebra in \mathcal{C}_A^0 can be considered as a connected étale algebra in \mathcal{C} , so maximality of A is equivalent to \mathcal{C}_A^0 being completely anisotropic. Thus, Proposition 5.3 implies that any Witt equivalence class contains a completely anisotropic category.

Now let \mathcal{C} and \mathcal{D} be two completely anisotropic categories such that $[\mathcal{C}] = [\mathcal{D}]$. By Corollary 5.8 there exists a fusion category \mathcal{A} and a braided equivalence $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A})$. In particular we have central functors $\mathcal{C} \rightarrow \mathcal{A}$ and $\mathcal{D}^{\text{rev}} \rightarrow \mathcal{A}$. By Lemma 5.11 these functors are fully faithful. Hence $\text{FPdim}(\mathcal{C}) \leq \text{FPdim}(\mathcal{A})$ and $\text{FPdim}(\mathcal{D}) \leq \text{FPdim}(\mathcal{A})$. Combining this with (8) we see that $\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{D}) = \text{FPdim}(\mathcal{A})$ and the functor $\mathcal{C} \rightarrow \mathcal{A}$ (and $\mathcal{D}^{\text{rev}} \rightarrow \mathcal{A}$) is an equivalence. In particular \mathcal{A} acquires a structure (in fact, two structures) of non-degenerate braided fusion category. Let \mathcal{C}' be the centralizer of \mathcal{C} in $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\mathcal{C})$. Then on one hand $\mathcal{C}' = \mathcal{D}^{\text{rev}}$ and on the other hand $\mathcal{C}' = \mathcal{C}^{\text{rev}}$, see Section 2.3. The result follows. \square

Corollary 5.13. *Let A and B be two maximal connected étale algebras in a non-degenerate braided fusion category \mathcal{C} . Then there exists a braided equivalence $\mathcal{C}_A^0 \simeq \mathcal{C}_B^0$. In particular $\text{FPdim}(A) = \text{FPdim}(B)$.*

Proof. The first statement is immediate from Theorem 5.12. The second one follows from (25). \square

The following result shows that Witt equivalence can also be understood without reference to the Drinfeld center:

Proposition 5.14. *Let $\mathcal{C}_1, \mathcal{C}_2$ be non-degenerate braided fusion categories. Then the following are equivalent:*

- (i) $[\mathcal{C}_1] = [\mathcal{C}_2]$, i.e. \mathcal{C}_1 and \mathcal{C}_2 are Witt equivalent.
- (ii) *There exist a braided fusion category \mathcal{C} , connected étale algebras $A_1, A_2 \in \mathcal{C}$ and braided equivalences $\mathcal{C}_1 \xrightarrow{\simeq} \mathcal{C}_{A_1}^0$, $\mathcal{C}_2 \xrightarrow{\simeq} \mathcal{C}_{A_2}^0$.*
- (iii) *There exist connected étale algebras $A_1 \in \mathcal{C}_1, A_2 \in \mathcal{C}_2$ and a braided equivalence $(\mathcal{C}_1)_{A_1}^0 \xrightarrow{\simeq} (\mathcal{C}_2)_{A_2}^0$.*

Proof. The implications (ii) \Rightarrow (i) and (iii) \Rightarrow (i) are immediate by Proposition 5.3.

(i) \Rightarrow (ii): By Definition 5.1, we have a braided equivalence

$$F : \mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2).$$

Thus we can define \mathcal{C} to be $\mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$, the algebra A_1 to be $F(\mathbf{1} \boxtimes I_1(\mathbf{1}))$ and the algebra A_2 to be $\mathbf{1} \boxtimes I_2(\mathbf{1})$. Here $I_i : \mathcal{A}_i \rightarrow \mathcal{Z}(\mathcal{A}_i)$ are right adjoints to the forgetful functors $\mathcal{Z}(\mathcal{A}_i) \rightarrow \mathcal{A}_i$. Finally we define the braided equivalence $\mathcal{C}_1 \rightarrow \mathcal{C}_{A_1}^0$ as

$$\mathcal{C}_1 \rightarrow \mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1)_{I_1(\mathbf{1})}^0 \xrightarrow{F} (\mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2))_{A_1}^0 = \mathcal{C}_{A_1}^0$$

and the braided equivalence $\mathcal{C}_2 \rightarrow \mathcal{C}_{A_2}^0$ as

$$\mathcal{C}_2 \rightarrow \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)_{I_2(\mathbf{1})}^0 = \mathcal{C}_{A_2}^0.$$

(i) \Rightarrow (ii) Choose étale algebras $A_i \in \mathcal{C}_i$ such that the categories $(\mathcal{C}_i)_{A_i}^0$ are completely anisotropic. Now $[(\mathcal{C}_1)_{A_1}^0] = [\mathcal{C}_1] = [\mathcal{C}_2] = [(\mathcal{C}_2)_{A_2}^0]$ together with Theorem 5.12 implies the existence of a braided equivalence $(\mathcal{C}_1)_{A_1}^0 \xrightarrow{\simeq} (\mathcal{C}_2)_{A_2}^0$. \square

Remark 5.15. 1. The proposition implies that Witt equivalence is the equivalence relation \sim on non-degenerate braided fusion categories generated by ordinary braided equivalence \simeq and the relations $\mathcal{C} \sim \mathcal{C}_A^0$, where $A \in \mathcal{C}$ is an étale algebra. But the proposition is more precise in that it says that any two Witt equivalent categories can be joined by just two invocations of $\mathcal{C} \sim \mathcal{C}_A^0$ and either one (part (iii)) or two (part (ii)) braided equivalences.

2. The proposition has applications to conformal field theory, cf. [Mu6].

5.3. The Witt group of metric groups and pointed categories. Recall that a *quadratic form* with values in \mathbb{k}^\times on a finite group A is a function $q : A \rightarrow \mathbb{k}^\times$ such that $q(-x) = q(x)$ and $b(x, y) = \frac{q(x+y)}{q(x)q(y)}$ is bilinear, see e.g. [DGNO, Section 2.11.1]. The pair (A, q) consisting of finite abelian group and quadratic form $q : A \rightarrow \mathbb{k}^\times$ is called a *pre-metric group*, see [DGNO, Section 2.11.2]. A pre-metric group (A, q) is called *metric group* if the form q is non-degenerate (i.e., the associated bilinear form $b(x, y)$ is non-degenerate).

To a pre-metric group (A, q) one assigns a unique up to a braided equivalence pointed braided fusion category $\mathcal{C}(A, q)$, where $q(a) \in \mathbb{k}^\times$ equals the braiding on the simple object $X_a \otimes X_a$ where X_a is a representative of an isomorphism class $a \in A$ (see e.g., [DGNO, Section 2.11.5]). It was shown in [JS2] that this assignment is an equivalence between the 1-categorical truncation of the 2-category of pre-metric groups and that of the 2-category of pointed braided fusion categories.

The category $\mathcal{C}(A, q)$ is non-degenerate if and only if (A, q) is a metric group.

Let (A, q) be a metric group and let $H \subset A$ be an *isotropic* subgroup (that is, $q|_H = 1$). Then $H \subset H^\perp$ where H^\perp is the orthogonal complement of H in A with respect to the bilinear form $b(x, y)$. Moreover, the restriction of q to H^\perp is the pull-back of a non-degenerate quadratic form $\tilde{q} : H^\perp/H \rightarrow \mathbb{k}^\times$. We say that $(H^\perp/H, \tilde{q})$ is an *m-subquotient* of (A, q) . Two metric groups are *Witt equivalent* if they have isomorphic m-subquotients, cf. [DGNO, Appendix A.7.1]. The set of equivalence classes has a natural structure of abelian group (with addition induced by the orthogonal direct sum) and is called the *Witt group of metric groups*, see *loc. cit.* We will denote this group \mathcal{W}_{pt} .

Proposition 5.16. *The assignment*

$$(33) \quad \mathcal{W}_{pt} \rightarrow \mathcal{W} : (A, q) \mapsto [\mathcal{C}(A, q)]$$

induces a well defined injective homomorphism $\mathcal{W}_{pt} \rightarrow \mathcal{W}$.

Proof. Let $H \subset A$ be an isotropic subgroup. Then the corresponding subcategory $\mathcal{C}(H, 1) \subset \mathcal{C}(A, q)$ is Tannakian, see e.g. [DGNO, Example 2.48]. Let $B \in \mathcal{C}(H, 1)$ be the corresponding regular algebra, see 2.8. Then the category $\mathcal{C}(A, q)_B^0$ identifies with $\mathcal{C}(H^\perp/H, \tilde{q})$. In particular, $[\mathcal{C}(A, q)] = [\mathcal{C}(H^\perp/H, \tilde{q})]$. This implies that (33) is well defined.

It is known that each class in \mathcal{W}_{pt} has a representative (A, q) which is anisotropic, that is $q(x) \neq 1$ for $A \ni x \neq 1$. It is clear that the corresponding category $\mathcal{C}(A, q)$ is completely anisotropic. Thus, (33) is injective by Theorem 5.12. \square

In what follows we will identify the group \mathcal{W}_{pt} with its image in \mathcal{W} . The group \mathcal{W}_{pt} is explicitly known, see e.g., [DGNO, Appendix A.7]. Namely,

$$\mathcal{W}_{pt} = \bigoplus_{p \text{ is prime}} \mathcal{W}_{pt}(p),$$

where $\mathcal{W}_{pt}(p) \subset \mathcal{W}_{pt}$ consists of the classes of metric p -groups.

The group $\mathcal{W}_{pt}(2)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; it is generated by two classes $[\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q_1)]$ and $[\mathcal{C}(\mathbb{Z}/4\mathbb{Z}, q_2)]$, where q_1, q_2 are any non-degenerate forms. For $p \equiv 3 \pmod{4}$ we have $\mathcal{W}_{pt}(p) \cong \mathbb{Z}/4\mathbb{Z}$ and the class $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q)]$ is a generator for any non-degenerate form q . For $p \equiv 1 \pmod{4}$ the group $\mathcal{W}_{pt}(p)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; it is generated by the two classes $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q')]$ and $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q'')]$

with $q'(l) = \zeta^{l^2}$ and $q''(l) = \zeta^{nl^2}$, where ζ is a primitive p th root of unity in k and n is any quadratic non-residue modulo p .

5.4. Property S. Let \mathcal{C} be a non-degenerate braided fusion category.

Definition 5.17. We say that \mathcal{C} has *property S* if the following conditions are satisfied:

- (S1) \mathcal{C} is completely anisotropic;
- (S2) \mathcal{C} is simple (that is, \mathcal{C} has no non-trivial fusion subcategories) and not pointed (so in particular $\mathcal{C} \not\cong \text{Vec}$).

We will also say that a class $w \in \mathcal{W}$ has property S if a completely anisotropic representative of w has property S. In Section 6.4 we will give infinitely many examples of non-degenerate braided fusion categories with property S.

Theorem 5.18. *Let $\mathcal{D} = \boxtimes_{i \in I} \mathcal{C}_i$ where \mathcal{C}_i are braided fusion categories with property S. Assume that \mathcal{D} is a Drinfeld center of a fusion category. Then there is a fixed point free involution $a : I \rightarrow I$ such that $\mathcal{C}_{a(i)} = \mathcal{C}_i^{\text{rev}}$*

Proof. Assume that $\mathcal{D} = \mathcal{Z}(\mathcal{A})$ for some fusion category \mathcal{A} . Let $F : \mathcal{D} = \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ be the forgetful functor. Choose a bijection $I = \{1, \dots, n\}$. For $1 \leq i \leq n$ let \mathcal{A}_i be the image of $\mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \dots \boxtimes \mathcal{C}_i$ under F (so \mathcal{A}_i is a fusion subcategory of \mathcal{A}).

Claim: *There is a subset $J_i \subset \{1, \dots, i\}$ such that F restricted to $\boxtimes_{j \in J_i} \mathcal{C}_j \subset \mathcal{D}$ is an equivalence $\boxtimes_{j \in J_i} \mathcal{C}_j \simeq \mathcal{A}_i$.*

Proof (of the claim). We use induction in i . For $i = 1$ we set $J_1 = \{1\}$; in this case the claim follows from Lemma 5.11. Now consider the induction step. The subcategory \mathcal{A}_{i+1} is clearly generated by \mathcal{A}_i and (the image of) $\mathcal{C}_{i+1} \subset \mathcal{A}$ (recall that by Lemma 5.11, the functor F restricted to \mathcal{C}_{i+1} is fully faithful. There are two possibilities:

- (a) the subcategories \mathcal{A}_i and \mathcal{C}_{i+1} intersect non-trivially in \mathcal{A} ; then \mathcal{A}_i contains \mathcal{C}_{i+1} since by (S2) \mathcal{C}_{i+1} has no non-trivial subcategories. In this case we set $J_{i+1} = J_i$.
- (b) \mathcal{A}_i and \mathcal{C}_{i+1} intersect trivially. Then we set $J_{i+1} = J_i \cup \{i+1\}$. We claim that the forgetful functor $\boxtimes_{j \in J_{i+1}} \mathcal{C}_j \rightarrow \mathcal{A}$ is fully faithful. As in the proof of Lemma 5.11 it is sufficient to show that for any object $Z \in \boxtimes_{j \in J_{i+1}} \mathcal{C}_j$ we have $\text{Hom}_{\mathcal{A}}(F(Z), \mathbf{1}) = \text{Hom}_{\mathcal{D}}(Z, \mathbf{1})$. Clearly, we can restrict ourselves to the case when Z is simple. In this case $Z = X \boxtimes Y$ where $X \in \boxtimes_{j \in J_i} \mathcal{C}_j$ and $Y \in \mathcal{C}_{i+1}$ are simple. Then $F(Z) = F(X) \otimes F(Y)$ where $F(X) \in \mathcal{A}_i$ and $F(Y) \in F(\mathcal{C}_{i+1})$ are simple. Then $\text{Hom}_{\mathcal{A}}(F(Z), \mathbf{1}) = \text{Hom}_{\mathcal{A}}(F(X), F(Y)^*) = 0$ unless $X = \mathbf{1}$ and $Y = \mathbf{1}$. We are done in this case and the claim is proved.

We apply now the Claim with $i = n$; we see that $\mathcal{A} = \boxtimes_{j \in J_n} \mathcal{C}_j$. Thus $\mathcal{Z}(\mathcal{A}) = \boxtimes_{j \in J_n} (\mathcal{C}_j \boxtimes \mathcal{C}_j^{\text{rev}})$ (see Section 2.3). The category \mathcal{D} does not contain non-trivial invertible objects. By Proposition 2.2 it has a unique decomposition into a product of simple categories. The result follows. \square

Corollary 5.19. *Let \mathcal{C} be a category with property S. Then $[\mathcal{C}] \in \mathcal{W}$ has order 2 if $\mathcal{C} \simeq \mathcal{C}^{\text{rev}}$ and otherwise $[\mathcal{C}] \in \mathcal{W}$ has infinite order.* \square

More precisely we have the following result. Let \mathcal{S} be the set of braided equivalence classes of categories with property S. Let $\mathcal{S}_2 \subset \mathcal{S}$ be the subset consisting of categories \mathcal{C} such that $\mathcal{C} \simeq \mathcal{C}^{\text{rev}}$ and let $\mathcal{S}_\infty = \mathcal{S} \setminus \mathcal{S}_2$. It is clear that the set \mathcal{S} is at most countable. It follows from (39) in Section 6.4 below that the set \mathcal{S}_∞ (and

hence \mathcal{S}) is infinite. Let $\mathcal{S}'_\infty \subset \mathcal{S}_\infty$ be a maximal subset such that $\mathcal{C} \in \mathcal{S}'_\infty$ implies $\mathcal{C}^{\text{rev}} \notin \mathcal{S}'_\infty$.

Corollary 5.20. *Let $\mathcal{W}_\mathcal{S} \subset \mathcal{W}$ be the subgroup generated by the categories with property \mathcal{S} . The map $(a_i)_{\mathcal{C}_i \in \mathcal{S}} \mapsto \prod_{\mathcal{C}_i \in \mathcal{S}} [\mathcal{C}_i]^{a_i}$ defines an isomorphism*

$$\bigoplus_{\mathcal{S}_2} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\mathcal{S}'_\infty} \mathbb{Z} \simeq \mathcal{W}_\mathcal{S}. \quad \square$$

Remark 5.21. 1. It is clear that the set \mathcal{S}_2 is at most countable. However we don't know whether it is empty and we don't know whether it is finite.

2. The description of the group $\mathcal{W}_\mathcal{S}$ above is non-canonical due to the choice of the set \mathcal{S}'_∞ . A better description is as follows: the set \mathcal{S} carries an involution σ which sends \mathcal{C} to \mathcal{C}^{rev} . We extend σ to the involution of the free abelian group $\mathbb{Z}[\mathcal{S}]$ generated by \mathcal{S} by linearity. Then $\mathcal{W}_\mathcal{S} \simeq \mathbb{Z}[\mathcal{S}]/\text{Image}(1 + \sigma)$.

3. An argument similar to the proof of Theorem 5.18 shows that $\mathcal{W}_\mathcal{S} \cap \mathcal{W}_{\text{pt}} = \{1\}$. Thus the subgroup of \mathcal{W} generated by $\mathcal{W}_\mathcal{S}$ and \mathcal{W}_{pt} is isomorphic to $\mathcal{W}_\mathcal{S} \times \mathcal{W}_{\text{pt}}$.

Corollary 5.22. *The \mathbb{Q} -vector space $\mathcal{W} \otimes_{\mathbb{Z}} \mathbb{Q}$ also has countable infinite dimension.*

Proof. Since \mathcal{S}_∞ is infinite, the \mathbb{Q} -vector space $\mathcal{W}_\mathcal{S} \otimes_{\mathbb{Z}} \mathbb{Q}$ has countable infinite dimension. \square

5.5. Central charge. From now on we will assume that $\mathbb{k} = \mathbb{C}$. Recall that any pseudo-unitary non-degenerate braided fusion category has a natural structure of modular tensor category (see, e.g., [DGNO, Section 2.8.2]).

Definition 5.23. Let $\mathcal{W}_{\text{un}} \subset \mathcal{W}$ be the subgroup consisting of Witt classes $[\mathcal{C}]$ of pseudo-unitary non-degenerate braided fusion categories \mathcal{C} .

Remark 5.24. Note that \mathcal{W}_{un} is *not* invariant under the Galois action from Remark 5.5 (for example class $[YL] \in \mathcal{W}_{\text{un}}$ from Section 6.4 below has a Galois conjugate not lying in \mathcal{W}_{un}). In particular, $\mathcal{W}_{\text{un}} \subsetneq \mathcal{W}$.

Now recall that for a modular tensor category \mathcal{C} one defines the *multiplicative central charge* $\xi(\mathcal{C}) \in \mathbb{C}$, see [DGNO, Section 6.2]. The following properties are well known, see, e.g., [BaKi, Section 3.1].

Lemma 5.25. (i) $\xi(\mathcal{C})$ is a root of unity;
(ii) $\xi(\mathcal{C}_1 \boxtimes \mathcal{C}_2) = \xi(\mathcal{C}_1)\xi(\mathcal{C}_2)$;
(iii) $\xi(\mathcal{C}^{\text{rev}}) = \xi(\mathcal{C})^{-1}$. \square

The statement (i) (due to Anderson, Moore and Vafa) allows us to consider the *Virasoro central charge* $c = c(\mathcal{C}) \in \mathbb{Q}/8\mathbb{Z}$, which is related to $\xi(\mathcal{C})$ by $\xi(\mathcal{C}) = e^{2\pi ic/8}$.

Lemma 5.26. *Let \mathcal{C}_1 and \mathcal{C}_2 be two pseudo-unitary non-degenerate braided fusion categories considered as modular tensor categories. Assume that \mathcal{C}_1 and \mathcal{C}_2 are Witt equivalent. Then $\xi(\mathcal{C}_1) = \xi(\mathcal{C}_2)$.*

Proof. By Corollary 5.8 $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A})$. Since the category $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}}$ is pseudo-unitary so is \mathcal{A} (use (8)). Thus, the spherical structure on $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} = \mathcal{Z}(\mathcal{A})$ is induced by the spherical structure on \mathcal{A} . In this situation [Mu5, Theorem 1.2] says that $\xi(\mathcal{Z}(\mathcal{A})) = 1$. The result follows from Lemma 5.25. \square

Now for any class $w \in \mathcal{W}_{\text{un}}$ we define $\xi(w) = \xi(\mathcal{C})$ where \mathcal{C} is a pseudo-unitary representative of the class w ; according to Lemma 5.26 this is well defined. Similarly, we set $c(w) = c(\mathcal{C})$.

Corollary 5.27. *The assignment $w \mapsto c(w)$ is a homomorphism $\mathcal{W}_{un} \rightarrow \mathbb{Q}/8\mathbb{Z}$.*

Proof. This is immediate from Lemma 5.25. \square

6. FINITE EXTENSIONS OF VERTEX ALGEBRAS

6.1. Extensions of VOAs. Let V be a rational vertex algebra, that is vertex algebra satisfying conditions 1-3 from [Hu, Section 1]. It is proved in *loc. cit.* that the category $\text{Rep}(V)$ of V -modules of finite length has a natural structure of modular tensor category; in particular $\text{Rep}(V)$ is a non-degenerate braided fusion category.

Note that a rational vertex algebra has to be *simple* (i.e., have no non-trivial ideals). This, in particular, means that VOA maps between rational vertex algebras are monomorphisms.

The category of modules $\text{Rep}(V \otimes U)$ of the tensor product of two (rational) vertex algebras is ribbon equivalent to the tensor product $\text{Rep}(V) \boxtimes \text{Rep}(U)$ of the categories of modules (see, for example [FHL]).

The relation between the central charge c_V of a rational VOA V and the central charge of the category of its modules $\text{Rep}(V)$ is given by (e.g., see [R]):

$$\xi(\text{Rep}(V)) = e^{\frac{2\pi i c_V}{8}}.$$

Now consider a finite extension of vertex algebras $V \subset W$, where V is a vertex subalgebra of W and W viewed as a V -module decomposes into a finite direct sum of irreducible V -modules¹. Then W considered as an object $A \in \text{Rep}(V)$ has a natural structure of commutative algebra; moreover this algebra satisfies the conditions from Example 3.3 (ii) and hence is étale, see [KiO, Theorem 5.2]². Furthermore, the restriction functor $\text{Rep}(W) \rightarrow \text{Rep}(V)$ induces a braided tensor equivalence $\text{Rep}(W) \simeq \text{Rep}(V)_A^0$. Thus, Proposition 5.3 implies that in this situation we have $[\text{Rep}(V)] = [\text{Rep}(W)]$. We can use this in order to construct examples of interesting relations in the group \mathcal{W} .

Example 6.1. (Chiral orbifolds.) Let G be a finite group of automorphisms of a rational vertex algebra V . The sub-VOA of invariants V^G is called the *chiral orbifold* of V . In the case when the vertex subalgebra of invariants V^G is rational we have a Witt equivalence between categories of modules $\text{Rep}(V)$, $\text{Rep}(V^G)$.

6.2. Affine Lie algebras and conformal embeddings. Let \mathfrak{g} be a finite dimensional simple Lie algebra and let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra. For any $k \in \mathbb{Z}_{>0}$ let $\mathcal{C}(\mathfrak{g}, k)$ be the category of highest weight integrable $\hat{\mathfrak{g}}$ -modules of level k , see e.g. [BaKi, Section 7.1] where this category is denoted \mathcal{O}_k^{int} . The category $\mathcal{C}(\mathfrak{g}, k)$ can be identified with the category $\text{Rep}(V(\mathfrak{g}, k))$ where $V(\mathfrak{g}, k)$ is the simple vertex algebra associated with the vacuum $\hat{\mathfrak{g}}$ -module of level k . In particular the category $\mathcal{C}(\mathfrak{g}, k)$ has a structure of modular tensor category, see [HuL], [BaKi, Chapter 7].

Example 6.2. The category $\mathcal{C}(sl(n), 1)$ is pointed. It identifies with $\mathcal{C}(\mathbb{Z}/n\mathbb{Z}, q)$ where $q(l) = e^{\pi i l^2 \frac{n-1}{n}}$, $l \in \mathbb{Z}/n\mathbb{Z}$. More generally, $\mathcal{C}(\mathfrak{g}, 1)$ (with \mathfrak{g} simply laced) is pointed [FK].

¹Note that finiteness is automatic if we assume that L_0 -eigenspaces are finite dimensional (which is standard and true e.g. for affine VOAs).

²The proof of this result in [KiO] is not complete. However for examples we are going to consider in this section the arguments from [KiO, §5.5] are sufficient.

It is known [BaKi] the categories $\mathcal{C}(\mathfrak{g}, k)$ are pseudo-unitary. In particular, we have Witt classes $[\mathcal{C}(\mathfrak{g}, k)] \in \mathcal{W}_{un} \subset \mathcal{W}$. The following formula for the central charge is very useful, see e.g. [BaKi, 7.4.5]:

$$(34) \quad c(\mathcal{C}(\mathfrak{g}, k)) = \frac{k \dim \mathfrak{g}}{k + h^\vee},$$

where h^\vee is the dual Coxeter number of the Lie algebra \mathfrak{g} .

One can construct examples of relations between the classes $[\mathcal{C}(\mathfrak{g}, k)]$ using the theory of *conformal embeddings*, see [BB, SW, KW]. Let $\bigoplus_i \mathfrak{g}^i \subset \mathfrak{g}'$ be an embedding (here \mathfrak{g}^i and \mathfrak{g}' are finite dimensional simple Lie algebras). We will symbolically write $\bigoplus_i (\mathfrak{g}^i)_{k_i} \subset \mathfrak{g}'_{k'}$ if the restriction of a $\hat{\mathfrak{g}}'$ -module of level k' to $\hat{\mathfrak{g}}^i$ has level k_i (in this case the numbers k_i are multiples of k'). Such an embedding defines an embedding of vertex algebras $\otimes_i V(\mathfrak{g}^i, k_i) \subset V(\mathfrak{g}', k')$; but in general this embedding does not preserve the Virasoro element. In the case when it does the embedding $\bigoplus_i (\mathfrak{g}^i)_{k_i} \subset \mathfrak{g}'_{k'}$ is called conformal embedding; it is known that in this case the extension of vertex algebras $\otimes_i V(\mathfrak{g}^i, k_i) \subset V(\mathfrak{g}', k')$ is finite. Thus in view of Section 6.1, we get a relation

$$(35) \quad \prod_i [\mathcal{C}(\mathfrak{g}^i, k_i)] = [\mathcal{C}(\mathfrak{g}', k')].$$

The complete classification of the conformal embeddings was done in [BB, SW] (see also [KW]) and is reproduced in the Appendix.

6.3. Cosets. Let $U \subseteq V$ be an embedding of rational vertex algebras, which does not preserve conformal vectors ω_U, ω_V (only operator products are preserved). The *centralizer* $C_V(U)$ is a vertex algebra with the conformal vector $\omega_V - \omega_U$, see [GKO]. Moreover the tensor product $U \otimes C_V(U)$ is mapped naturally to V and this map is a map of vertex algebras. In the case when V, U and $C_V(U)$ are rational we have a Witt equivalence of categories of modules:

$$\text{Rep}(U) \boxtimes \text{Rep}(C_V(U)) \simeq \text{Rep}(U \otimes C_V(U))$$

and $\text{Rep}(V)$.

Let $\bigoplus_i (\mathfrak{h}^i)_{k_i} \subset \bigoplus_j (\mathfrak{g}^j)_{k'_j}$ be an embedding of semisimple Lie algebras respecting the central charge as in Section 6.2. Let $\otimes_i V(\mathfrak{h}^i, k_i) \subset \otimes_j V(\mathfrak{g}^j, k'_j)$ be the corresponding embedding of the vertex algebras. The centralizer $C_{\otimes_j V(\mathfrak{g}^j, k'_j)}(\otimes_i V(\mathfrak{h}^i, k_i))$

is called the *coset model* and is denoted $\frac{\times_j (\mathfrak{g}^j)_{k'_j}}{\times_i (\mathfrak{h}^i)_{k_i}}$.

Sometimes coset models defined by different embeddings of semisimple Lie algebras are isomorphic. An example of such isomorphism was found by Goddard, Kent and Olive [GKO]. They observed that the following coset models³:

$$\frac{A_{1,m} \times A_{1,1}}{A_{1,m+1}}, \quad \frac{C_{m+1,1}}{C_{m,1} \times C_{1,1}}$$

are isomorphic, since they are both isomorphic to the same rational Virasoro vertex algebra Vir_{c_m} with the central charge

$$(36) \quad c_m = 1 - \frac{6}{(m+2)(m+3)}.$$

³here and in the Appendix the notation $X_{i,k}$ refers to the Lie algebra of type X_i at level k .

We can use coset models in order to construct new relations in the Witt group as follows. Assume that the central charge c of a coset model vertex algebra $\frac{\times_j(\mathfrak{g}^j)_{k'_j}}{\times_i(\mathfrak{h}^i)_{k_i}}$ is positive⁴ but less than 1⁵. It is known that in this case $c = c_m$ for some positive integer m and the vertex algebra in question contains a rational vertex subalgebra Vir_{c_m} , see [GKO]. This implies that the rational vertex algebra $\otimes_j V(\mathfrak{g}^j, k'_j)$ is a finite extension of rational vertex algebra $\otimes_i V(\mathfrak{h}^i, k_i) \otimes Vir_{c_m}$. Thus according to the results of Section 6.1 we get a relation in the Witt group

$$(37) \quad \left(\prod_i [\mathcal{C}(\mathfrak{h}^i, k_i)] \right) \cdot [Vir_{c_m}] = \prod_j [\mathcal{C}(\mathfrak{g}^j, k'_j)].$$

A special case of this relation corresponding to the coset model $\frac{A_{1,m} \times A_{1,1}}{A_{1,m+1}}$ reads

$$(38) \quad [Vir_{c_m}] = [\mathcal{C}(sl(2), m)][\mathcal{C}(sl(2), 1)][\mathcal{C}(sl(2), m+1)]^{-1}.$$

Thus combining (37) and (38) we obtain relations between the classes $[\mathcal{C}(\mathfrak{g}, k)]$.

6.4. Examples for $\mathfrak{g} = sl(2)$. We give here some examples of relations (or absence thereof) between the classes $[\mathcal{C}(sl(2), k)]$. We refer the reader to [KiO, Section 6] for more details on the categories $\mathcal{C}(sl(2), k)$. Note that all étale algebras in these categories were classified in [KiO, Theorem 6.1].

- (1) The category $\mathcal{C}(sl(2), 1)$ is pointed, moreover $\mathcal{C}(sl(2), 1) \simeq \mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q_+)$ where $q_+(1) = i$. In particular, class $[\mathcal{C}(sl(2), 1)] \in \mathcal{W}$ has order 8.
- (2) For any odd k , we have $\mathcal{C}(sl(2), k) \simeq \mathcal{C}(sl(2), k)_+ \boxtimes \mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q_{\pm})$ where $\mathcal{C}(sl(2), k)_+$ is the subcategory of “integer spin” representations and $q_{\pm}(1) = \pm i$ (see e.g. [KiO, Lemma 6.6]). The category $\mathcal{C}(sl(2), k)_+$ for an odd $k \geq 3$ has property S. Using (34) we get

$$c(\mathcal{C}(sl(2), k)_+) = \frac{3k}{k+2} + (-1)^{(k+1)/2}.$$

In particular, $2c(\mathcal{C}(sl(2), k)_+) \neq 0 \in \mathbb{Q}/8\mathbb{Z}$, so

$$(39) \quad \mathcal{C}(sl(2), k)_+ \not\cong \mathcal{C}(sl(2), k)_+^{\text{rev}}.$$

This shows that the set \mathcal{S}_{∞} from Section 5.4 is infinite.

The category $YL := \mathcal{C}(sl(2), 3)_+$ is called the Yang-Lee category. The class $[YL] \in \mathcal{W}$ is a simplest example of element of \mathcal{W} of infinite order. We will say that a braided fusion category \mathcal{C} is a *Fibonacci category* if the Grothendieck ring $K(\mathcal{C})$ is isomorphic to $K(YL)$ as a based ring. It is known that a pseudo-unitary Fibonacci category is equivalent to either YL or YL^{rev} .

- (3) The category $\mathcal{C}(sl(2), 2)$ is an example of Ising braided category, see [DGNO, Appendix B]. In particular, it follows from [DGNO, Lemma B.24] that

$$[\mathcal{C}(sl(2), 2)]^2 = [\mathcal{C}(\mathbb{Z}/4\mathbb{Z}, q)], \quad \text{where } q(l) = e^{3\pi i l^2 / 4}.$$

Thus, the order of $[\mathcal{C}(sl(2), 2)] \in \mathcal{W}$ is 16.

⁴It is known (see [GKO]) that $c \geq 0$. The case $c = 0$ corresponds exactly to the conformal embeddings discussed in Section 6.2.

⁵The list of cosets with such central charge was given in [BG] and is reproduced in the Appendix.

Using [DGNO, Lemma B.24] it is easy to see that for an odd l we have $[\mathcal{C}(sl(2), 2)]^l = [\mathcal{C}]$, where \mathcal{C} is an Ising braided category. Since there are precisely 8 equivalence classes of Ising braided categories (see [DGNO, Corollary B.16]), we get that for any Ising braided category \mathcal{C} there is a unique odd number l , $1 \leq l \leq 15$ such that $[\mathcal{C}] = [\mathcal{C}(sl(2), 2)]^l$. The number l is easy to compute from $c(\mathcal{C})$ using $c(\mathcal{C}(sl(2), 2)) = \frac{3}{2}$.

- (4) There exists a conformal embedding $sl(2)_4 \subset sl(3)_1$. Thus

$$[\mathcal{C}(sl(2), 4)] = [\mathcal{C}(sl(3), 1)] = [\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)], \quad \text{where } q(l) = e^{2\pi i l^2/3}.$$

In particular, the order of $[\mathcal{C}(sl(2), 4)] \in \mathcal{W}$ is 4.

- (5) There exists a conformal embedding $sl(2)_6 \oplus sl(2)_6 \subset so(9)_1$. Thus

$$[\mathcal{C}(sl(2), 6)]^2 = [\mathcal{C}(so(9), 1)].$$

Notice that $\mathcal{C}(so(9), 1)$ is also an example of Ising braided category. Using the central charge one computes that

$$[\mathcal{C}(sl(2), 6)]^2 = [\mathcal{C}(sl(2), 2)]^3.$$

In particular, $[\mathcal{C}(sl(2), 6)] \in \mathcal{W}$ has order 32.

- (6) The category $\mathcal{C}(sl(2), 8)$ is known to contain an étale algebra A such that $\mathcal{C}(sl(2), 8)_A^0$ is equivalent to the product of two Fibonacci categories, see e.g., [MPS, Theorem 4.1]. Using the central charge one computes that

$$[\mathcal{C}(sl(2), 8)] = [YL]^{-2}.$$

- (7) There exists a conformal embedding $sl(2)_{10} \subset sp(4)_1$. Thus,

$$[\mathcal{C}(sl(2), 10)] = [\mathcal{C}(sp(4), 1)].$$

The category $\mathcal{C}(sp(4), 1)$ is an Ising braided category. Using the central charge one computes that $[\mathcal{C}(sl(2), 10)] = [\mathcal{C}(sl(2), 2)]^7$.

- (8) Let $\mathfrak{g}(G_2)$ be a Lie algebra of type G_2 . There exists a conformal embedding $sl(2)_{28} \subset \mathfrak{g}(G_2)_1$. Thus,

$$[\mathcal{C}(sl(2), 28)] = [\mathcal{C}(\mathfrak{g}(G_2), 1)].$$

The category $\mathcal{C}(\mathfrak{g}(G_2), 1)$ is a Fibonacci category. Using the central charge one computes that

$$[\mathcal{C}(sl(2), 28)] = [YL].$$

- (9) The category $\mathcal{C}(sl(2), k)$ with k divisible by 4 is known to contain an étale algebra A of dimension 2, see [KiO, Theorem 6.1]. It is also known that in this case for $k \neq 4, 8, 28$ the category $\mathcal{C}(sl(2), k)_A^0$ has property S and is not equivalent to any category $\mathcal{C}(sl(2), k_1)_+$ with odd k_1 . Thus we get infinitely many more elements of the set \mathcal{S}_∞ . For example we see that $[\mathcal{C}(sl(2), 12)] \in \mathcal{W}$ has infinite order.

6.5. Open questions. In this section we collect some open questions about the Witt group \mathcal{W} .

Question 6.3. Is it true that \mathcal{W} is a direct sum of cyclic groups? Is there an inclusion $\mathbb{Q} \subset \mathcal{W}$?

Question 6.4. Is \mathcal{W}_{un} generated by classes $[\mathcal{C}(\mathfrak{g}, k)]$?

Remark 6.5. Notice that \mathcal{W}_{pt} is contained in the subgroup generated by $[\mathcal{C}(\mathfrak{g}, k)]$. Namely, the subgroup of \mathcal{W} generated by $[\mathcal{C}(sl(2), 1)]$ and $[\mathcal{C}(sl(2), 2)]$ contains $\mathcal{W}_{pt}(2)$. For a prime $p = 4k + 3$, the subgroup $\mathcal{W}_{pt}(p)$ is generated by $[\mathcal{C}(sl(p), 1)]$. Finally for a prime $p = 4k + 1$ choose a prime number $q < p$ which is a quadratic non-residue modulo p (it is easy to see that such a prime does exist). Then $\mathcal{W}_{pt}(p)$ is contained in the subgroup of \mathcal{W} generated by $[\mathcal{C}(sl(p), 1)]$ and $[\mathcal{C}(sl(pq), 1)]$ and $\mathcal{W}_{pt}(q)$. Thus we are done by induction.

Remark 6.6. Since the end of eighties there is a common believe among physicists that all rational conformal field theories come from lattice and WZW models via coset and orbifold (and perhaps chiral extension) constructions (see [MS]). Analogous statement for modular categories would imply that the unitary Witt group is generated by classes of affine categories $\mathcal{C}(\mathfrak{g}, k)$.

Question 6.7. What are the relations in the subgroup of \mathcal{W} generated by $[\mathcal{C}(\mathfrak{g}, k)]$? Is it true that all relations in the subgroup generated by $[\mathcal{C}(sl(2), k)]$ are described in Section 6.4? Is it possible to express some nonzero power of $[\mathcal{C}(sl(2), 12)] \in \mathcal{W}$ in terms of $[\mathcal{C}(sl(2), k)]$, $k \neq 12$? What is the order of $[\mathcal{C}(sl(2), 14)] \in \mathcal{W}$?

Question 6.8. Is there a class $w \in \mathcal{W}_S$ of order 2? Equivalently does exist a non-degenerate braided fusion category \mathcal{C} with property S and such that $\mathcal{C}^{\text{rev}} \simeq \mathcal{C}$?

Question 6.9. Is it true that torsion in \mathcal{W} is 2-primary? Is there an element of order 3 in \mathcal{W} ?

Question 6.10. What is the biggest finite order of an element of \mathcal{W} ? For example, are there elements of \mathcal{W} of order 64?

APPENDIX. CONFORMAL EMBEDDINGS AND COSETS WITH $c < 1$

Here we reproduce (from [BB, SW]) the list of maximal embeddings starting with serial embeddings (rank-level dualities, (anti-)symmetric and regular embeddings and their variants) and followed up by sporadic embeddings:

$$\begin{aligned} su(m)_n \times su(n)_m \subseteq su(mn)_1, & \quad sp(2m)_n \times sp(2n)_m \subseteq so(4mn)_1, \\ so(m)_n \times so(n)_m \subseteq so(mn)_1, & \quad so(m)_4 \times su(2)_m \subseteq sp(2m)_1, \end{aligned}$$

$$\begin{aligned} A_{n,n-1} \subseteq A_{\frac{(n-1)(n+2)}{2},1}, & \quad A_{n,n+3} \subseteq A_{\frac{n(n+3)}{2},1}, \\ A_{2n+1,2n+2} \subseteq B_{2n^2+4n+1,1}, & \quad A_{2n,2n+1} \subseteq D_{2n(n+1),1}, \\ B_{2n+1,4n+1} \subseteq B_{(4n+1)(n+1),1}, & \quad A_{2n+1,4n+5} \subseteq B_{4n^2+7n+2,1}, \\ B_{2n,4n-1} \subseteq D_{n(4n+1),1}, & \quad B_{2n,4n+3} \subseteq D_{n(4n+3),1}, \\ C_{2n,2n-1} \subseteq B_{4n^2-n-1,1}, & \quad C_{2n+1,2n+2} \subseteq B_{(4n+1)(n+1),1}, \\ C_{2n,2n+1} \subseteq D_{n(4n+1),1}, & \quad C_{2n+1,2n} \subseteq D_{n(4n+3),1}, \\ D_{2n,4n+2} \subseteq B_{4n^2+n-1,1}, & \quad D_{2n+1,4n} \subseteq B_{n(4n+3),1}, \\ D_{2n,4n-2} \subseteq D_{n(4n-1),1}, & \quad D_{2n+1,4n+4} \subseteq D_{(n+1)(4n+1),1}, \\ B_{n,2} \subseteq A_{2n,1}, & \quad D_{n,2} \subseteq A_{2n-1,1}, \\ D_{1,1} \times A_{i,1} \times A_{n-i-1} \subseteq A_{n,1}, 1 \leq i \leq n-2 & \quad D_{1,1} \times A_{n-1,1} \subseteq A_{n,1}, \\ D_{1,1} \times D_{n-1,1} \subseteq D_{n,1}, & \quad D_{1,1} \times A_{n-1,1} \subseteq D_{n,1}, \\ A_{1,1} \times A_{1,1} \times D_{n-2,1} \subseteq D_{n,1}, & \quad D_{i,1} \times D_{n-i,1} \subseteq D_{n,1}, 3 \leq i \leq n-3 \\ A_{1,1} \times A_{1,1} \times B_{n-2,1} \subseteq B_{n,1}, & \quad D_{1,1} \times B_{n-1,1} \subseteq B_{n,1}, \\ D_{i,1} \times B_{n-1,1} \subseteq B_{n,1}, 3 \leq i \leq n-2 & \quad D_{i,1} \times B_{n-i,1} \subseteq B_{n,1} \\ A_{1,2} \times D_{n-1,1} \subseteq B_{n,1}, & \quad D_{1,1} \times A_{n-1,2} \subseteq C_{n,1}, \\ D_{1,1} \times D_{5,1} \subseteq E_{6,1}, & \quad A_{1,1} \times A_{5,1} \subseteq E_{6,1}, \\ A_{2,1} \times A_{2,1} \times A_{2,1} \subseteq E_{6,1}, & \quad D_{1,1} \times E_{6,1} \subseteq E_{7,1}, \\ A_{1,1} \times D_{6,1} \subseteq E_{7,1}, & \quad A_{7,1} \subseteq E_{7,1}, \\ A_{2,1} \times A_{5,1} \subseteq E_{7,1}, & \quad D_{8,1} \subseteq E_{8,1}, \\ A_{4,1} \times A_{4,1} \subseteq A_{8,1}, & \quad A_{2,1} \times E_{6,1} \subseteq E_{8,1}, \\ A_{1,1} \times E_{7,1} \subseteq E_{8,1}, & \quad A_{8,1} \subseteq E_{8,1}, \\ A_{1,1} \times C_{3,1} \subseteq F_{4,1}, & \quad G_{2,1} \times A_{1,8} \subseteq F_{4,1}, \\ G_{2,1} \times A_{2,2} \subseteq E_{6,1}, & \quad A_{1,7} \times G_{2,2} \subseteq E_{7,1}, \\ A_{1,3} \times F_{4,1} \subseteq E_{7,1}, & \quad G_{2,1} \times C_{3,1} \subseteq E_{7,1}, \\ A_{2,6} \times A_{1,16} \subseteq E_{8,1}, & \quad G_{2,1} \times F_{4,1} \subseteq E_{8,1}, \end{aligned}$$

$$\begin{aligned} A_{1,10} \subseteq B_{2,1}, & \quad A_{1,28} \subseteq G_{2,1}, & \quad A_{2,9} \subseteq E_{6,1}, \\ A_{2,21} \subseteq E_{7,1}, & \quad A_{5,6} \subseteq C_{10,1}, & \quad A_{7,10} \subseteq D_{35,1}, \\ B_{2,12} \subseteq E_{8,1}, & \quad B_{4,2} \subseteq D_{8,1}, & \quad C_{3,5} \subseteq C_{7,1}, \\ C_{4,1} \subseteq E_{6,1}, & \quad C_{4,7} \subseteq D_{21,1}, & \quad D_{5,4} \subseteq A_{15,1}, \\ D_{6,8} \subseteq C_{16,1}, & \quad D_{8,16} \subseteq D_{64,1}, & \quad E_{6,6} \subseteq A_{26,1}, \\ E_{6,12} \subseteq D_{39,1}, & \quad E_{7,12} \subseteq C_{28,1}, & \quad E_{7,18} \subseteq B_{66,1}, \\ E_{8,30} \subseteq D_{124,1}, & \quad F_{4,3} \subseteq D_{13,1}, & \quad F_{4,9} \subseteq D_{26,1}, \\ G_{2,3} \subseteq E_{6,1}, & \quad G_{2,4} \subseteq D_{7,1}. \end{aligned}$$

Next we reproduce the list of cosets with central charge $0 < c < 1$ given in [BG]:

$$\begin{aligned} Vir_{c_n} &= \frac{A_{1,1} \times A_{1,n}}{A_{1,n+1}}, & \quad Vir_{c_n} &\subseteq \frac{A_{n+1,2}}{A_{n,2} \times u(1)}, \\ Vir_{c_n} &\subseteq \frac{C_{m+1,1}}{C_{m,1} \times C_{1,1}}, & \quad Vir_{c_1} &\subseteq \frac{so(n)_1}{so(n-1)_1}, \end{aligned}$$

$$\begin{array}{lll}
 \text{Vir}_{c_1} \subseteq \frac{A_{1,2}}{u(1)}, & \text{Vir}_{c_2} \subseteq \frac{E_{7,2}}{A_{7,2}}, & \text{Vir}_{c_3} \subseteq \frac{A_{2,1} \times A_{2,1}}{A_{2,2}}, \\
 \text{Vir}_{c_3} \subseteq \frac{A_{1,3}}{u(1)}, & \text{Vir}_{c_3} \subseteq \frac{E_{7,2}}{A_{1,2} \times D_{6,2}}, & \text{Vir}_{c_4} \subseteq \frac{E_{6,1} \times E_{6,1}}{E_{6,2}}, \\
 \text{Vir}_{c_3} \subseteq \frac{A_{2,2}}{A_{1,8}}, & \text{Vir}_{c_1} \subseteq \frac{E_{8,2}}{D_{8,2}}, & \text{Vir}_{c_2} \subseteq \frac{E_{7,1} \times E_{7,1}}{E_{7,2}}, \\
 \text{Vir}_{c_3} \subseteq \frac{A_{5,1}}{C_{3,1}}, & \text{Vir}_{c_2} \subseteq \frac{E_{8,2}}{A_{1,2} \times E_{7,1}}, & \text{Vir}_{c_1} \subseteq \frac{E_{8,1} \times E_{8,1}}{E_{8,2}}, \\
 \text{Vir}_{c_2} \subseteq \frac{B_{3,1}}{G_{2,1}}, & \text{Vir}_{c_9} \subseteq \frac{E_{8,2}}{A_{8,2}}, & \text{Vir}_{c_9} \subseteq \frac{E_{8,1} \times E_{8,2}}{E_{8,3}}, \\
 \text{Vir}_{c_3} \subseteq \frac{E_{6,1}}{F_{4,1}}, & \text{Vir}_{c_9} \subseteq \frac{F_{4,1}}{C_{3,2} \times A_{1,2}}, & \text{Vir}_{c_8} \subseteq \frac{F_{4,1} \times F_{4,1}}{F_{4,2}}, \\
 \text{Vir}_{c_4} \subseteq \frac{E_{6,2}}{C_{4,2}}, & \text{Vir}_{c_2} \subseteq \frac{F_{4,1}}{B_{4,1}}, & \text{Vir}_{c_7} \subseteq \frac{G_{2,1} \times G_{2,1}}{G_{2,2}}, \\
 \text{Vir}_{c_5} \subseteq \frac{E_{6,2}}{A_{1,2} \times A_{5,2}}, & \text{Vir}_{c_3} \subseteq \frac{G_{2,1}}{A_{2,1}}, & \text{Vir}_{c_6} \subseteq \frac{G_{2,2}}{A_{1,2} \times A_{1,6}}
 \end{array}$$

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