IP-DGFEM METHOD FOR THE $p(x)$- LAPLACIAN

LEANDRO DEL PEZZO, ARIEL LOMBARDI AND SANDRA MARTÍNEZ

Abstract. In this paper we construct an "Interior Penalty" Discontinuous Galerkin method to approximate the minimizer of a variational problem related to the $p(x)-$laplacian. The function $p : \Omega \to [p_1, p_2]$ is log- Hölder continuous and $1 < p_1 \le p_2 < \infty$. We prove the weakly convergence of the sequence of minimizers of the discrete functional to the minimizer. We also make some numerical experiments in dimension one to compare this method with the Conform Galerkin method, in the case where p_1 is next to one. This example is motivated by its applications to image processing.

1. Introduction

In this paper we study a discontinuous Galerking method to approximate the minimizer of a non homogenous functional. This functional involves the so-called $p-$Laplacian operator, i.e.,

$$\Delta_{p(x)} u = \mathrm{div}(|\nabla u(x)|^{p(x)-2}\nabla u). \tag{1.1}$$

This operator extends the classical Laplacian ($p(x) \equiv 2$) and the so-called $p-$Laplacian ($p(x) \equiv p$ with $1 < p < \infty$) and it has been recently used in image processing and in the modeling of electrorheological fluids.

In an image processing problem, the aim is to recover the real image I from an observed image ξ of the form $\xi = I + \eta$, where η is a noise.

It has been recently used variational problems for this applications. For example, L. Rudin and S. Osher propose the following model;

Minimize the functional $|Du|(\Omega)$ over all the functions in $BV(\Omega) \cap L^2(\Omega)$ such that

$$\int_\Omega u\,dx = \int_\Omega \xi\,dx \quad \text{and} \quad \int_\Omega |u-\xi|^2\,dx = \sigma^2$$

for some $\sigma > 0$.

These conditions over the space came from the assumption that η is a function that represents a white noise with mean zero and variance σ. Moreover, the authors prove that this problem is equivalent to minimize;

$$|Du|(\Omega) + \frac{\lambda}{2}\int_\Omega |u-\xi|^2\,dx$$

Key words and phrases. free boundaries, variable exponent spaces, minimization.
2000 *Mathematics Subject Classification.* 65k10, 35J20, 65N30 and 46E35.
Supported by ANPCyT PICT 2006-290, UBA X078, UBA X117 . All three authors are members of CONICET.

for some nonnegative Lagrange multiplier $\lambda = \lambda(\sigma, \xi)$. This model works when the image is piecewise constant, but not in general cases, since it has the problem that may appear false edges (*staircasing* effect). For reference of this model see [5].

Also, it is considered other method called the isotropic diffusion. This model consists on minimizing

$$\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega |u-\xi|^2\,dx.$$

This method solves the *staircasing* effect, but it has the problem that does not preserves edges.

Recently, in [2] the authors propose a new model that avoid the *staircasing* effect preserving edges. More precisely, they consider the functional

$$\int_\Omega |\nabla u|^{p(x)} + \frac{\lambda}{2}\int_\Omega |u-\xi|^2\,dx,$$

with $p:\Omega\to[1,2]$ a function such that, $p(x) = P_M(|\nabla G_\delta * \xi|(x))$ where $G_\delta(x)$ is approximation of the identity, $M >> 1$ and P_M is a function that satisfies $P_M(0) = 2$ and $P_M(x) = 1$ for all $|x| > M$.

Motivated by the mentioned applications, we study a numerical method to approximate minimizers of a functional related to the $p(x)-$ Laplacian.

More precisely, given Ω a bounded Lipschitz domain, $p : \overline{\Omega} \to [p_1, p_2]$, with $1 < p_1 \le p_2 < \infty$, $1 \le q < p^*$ and $1 \le r < p_*$, such that p is log-Hölder continuous in $\overline{\Omega}$, $r \in C^0(\partial\Omega)$, $\xi \in L^{q(\cdot)}(\Omega)$, we minimize the functional,

$$I(v) = \int_\Omega \left(|\nabla v(x)|^{p(x)} + |v(x)-\xi(x)|^{q(x)}\right)dx + \int_{\Gamma_N} |v|^{r(x)}\,dS$$

over all $v \in \mathcal{A}$, where

$$\mathcal{A} = \{v \in W^{1,p(\cdot)}(\Omega) : v = u_D \text{ in } \Gamma_D\},$$

$u_D \in W^{1,p(\cdot)}(\Omega)$ and $\partial\Omega = \Gamma_D \cup \Gamma_N$. For the definition of these spaces see Appendix A.

Here p^* and p_* are the Sobolev critical exponents for these spaces, i.e.,

$$p^*(x) := \begin{cases} \frac{p(x)N}{N-p(x)} & \text{if } p(x) < N,\\ +\infty & \text{if } p(x) \ge N,\end{cases} \quad\text{and}\quad p_*(x) := \begin{cases} \frac{p(x)(N-1)}{N-p(x)} & \text{if } p(x) < N,\\ +\infty & \text{if } p(x) \ge N.\end{cases} \tag{1.2}$$

We can observe that, for the applications, it is relevant to study the minimization problem in the case where p approaches the value 1 in some regions. We can see, making some numerical experiments, that the minimizers have grate derivative in that regions. For this reason, the Conform Finite Elements are not appropriate, since in this case, we need thin mashes to obtain good approximations (see Section 6).

The method that we considered is the so-called Discontinuous Galerkin. These type of methods are relative new in the theoretical point of view. In the paper [1], we can find a unification of all these type of methods. In all the examples of this paper, the authors take as model a linear differential equation.

Our aim is to study, in the future, the minimization problem for the case when p approaches the value 1 in some regions (where there is no weak formulation). For this reason, we think that the best way to find approximations is finding a good discretization of the minimization problem. We take a similar discretization of the one in the paper [4] where the authors study the case where p is constant.

Our discrete functional is the following,

$$I_h(v_h) = \int_\Omega \left(|\nabla v_h + R(v_h)|^{p(x)} + |u_h - \xi|^{q(x)} \right) dx + \int_{\Gamma_D} |v_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)} \, dS$$
$$+ \int_{\Gamma_{int}} |[\![v_h]\!]|^{p(x)} \mathbf{h}^{1-p(x)} \, dS + \int_{\Gamma_N} |u_h|^{r(x)} \, dS.$$

where $\mathbf{h}$ is the local mesh size, Γ_{int} is the union of the interior edges of the elements, $[\![v_h]\!]$ is the jump of the function between two edges and ∇v_h denotes the elementwise gradient of v_h. Finally, R is the lifting operator defined in Section 3.1, which represents the contributions of the jumps to the distributional gradient. Observe that, the second term of the functionals impose weakly that the minimizers satisfy the boundary condition.

Now the discrete problem is to find a minimizer u_h of I_h over all the functions that are polynomial of degree almost k in each element, denoted by $\mathcal{S}^k(\mathcal{T}_h)$. See Appendix C.

In this paper we prove in which sense the sequence u_h converges to the minimizer u of I over the space $\mathcal{A}$. In fact, we prove the following,

Theorem 1.1. *Let Ω be a bounded domain with Lipschitz boundary. Let $p : \Omega \to (1, \infty)$ be log-Hölder continuous in $\overline{\Omega}$ and $u_D \in W^{2,p_2}(\Omega)$. For each $h \in (0, 1]$, let $u_h \in S^k(\mathcal{T}_h)$ be the minimizer of I_h. Then there exist a subsequence and u the minimizer of I such that*

$$u_{h_j} \overset{*}{\rightharpoonup} u \qquad \text{weakly* in } BV(\Omega), \tag{1.3}$$

$$u_{h_j} \to u \qquad \text{in } L^{q(\cdot)}(\Omega) \quad \forall\, 1 \le q(x) < p^*(x), \tag{1.4}$$

$$u_{h_j} \to u \qquad \text{in } L^{r(\cdot)}(\partial\Omega) \quad \forall\, 1 \le r(x) < p_*(x), \tag{1.5}$$

$$\nabla u_{h_j} \rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega), \tag{1.6}$$

$$I_h(u_{h_j}) \to I(u), \tag{1.7}$$

$$\int_{\Gamma_D} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)} \, dS + \int_{\Gamma_{int}} |[\![u_h]\!]|^{p(x)} \mathbf{h}^{1-p(x)} \, dS \to 0. \tag{1.8}$$

Lastly, we want to mention where we need the regularity hypotheses over the function $p(x)$. To prove Theorem 1.1 we need to use the continuity of the imbedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$, the continuity of the Trace operator $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$ and the Poincare inequality. As we can see in Theorem A.10, Theorem A.11 and Theorem A.8 that results only covers the case where p is log-Hölder, $p \in C^0(\overline{\Omega})$ and $r \in C^0(\partial\Omega)$.

We also use strongly that p is log-Hölder in Proposition A.9, this result says that if κ is an element with diameter h_κ and p_- and p_+ are the maximum and minimum of p over κ then $h_\kappa^{p_- - p_+}$ is bounded independent of h_κ. This property is crucial to prove several results along the paper.

The hypothesis that Ω has Lipschitz boundary came from Theorem A.8 and Theorem A.11. From now on we will assume this hypothesis over the domain.

On the other hand, to prove the convergence of the sequence u_h we need a technical hypothesis under the boundary condition u_D. As we can see, Lemma 4.5 only covers the case where $u_D \in W^{2,p_2}(\Omega)$.

Outline of the paper. In Section 2 we study the reconstruction operator and we prove some error estimates that are crucial results for the rest of the paper (Corollary 2.5).

In Section 3 we prove the boundedness of the Lifting operator (Theorem 3.3).

In Section 4 we prove the Broken Poincarè inequality (Theorem 4.1), the coercivity of the functional (Theorem 4.2) and finally we give the proof of Theorem 1.1.

In Section 5 we study the convergence of the Conform Element Method.

In Section 6 we give an example in dimension one and compere both methods.

In Appendices A and B we state several properties of the Variable Exponent Sobolev Spaces and of the functions of Bonded Variation.

In Appendix C we give some definitions and properties related to the mesh and to the Broken Sobolev Spaces.

2. The operator Q_h

In many Galerkin Discontinuous problems it is used a priori bounds to prove the Poincarè inequality of the discrete space. To prove these inequalities it is required to use a reconstruction operator. In this section we define, as in [4], a family of quasi-interpolant operators and prove some error estimates depending on the mesh size. These results are more general in one sense, because we prove bounds in the variable $p-$ norm, but weaker than previous one (see [3]) in the sense that only covers the case of the finite dimensional space $\mathcal{S}^k(\mathcal{T}_h)$. This last restriction came from the fact that in Lemma C.7 we need to use the equivalence of the norms in the space of polynomials.

Now we define and study the reconstructing operator. For each $h \in (0,1]$, let

$$Q_h \colon S^k(\mathcal{T}_h) \to W^{1,\infty}(\Omega)$$

be the linear operator defined by

$$Q_h(u) = \sum_{z\in\mathcal{N}_h} \pi_z(u)\lambda_z,$$

where λ_z is the standard P^1 nodal basis function associated with the vertex z on the mesh $\mathcal{T}_h$ and π_z is the local projection operator defined in Lemma C.9.

In the next theorem, we will give some local estimations of the $L^{q(\cdot)}(\kappa)$ and $L^{q(\cdot)}(e)$ norms in terms of the $W^{1,p(\cdot)}(\mathcal{T}_h \cap T_\kappa)$ seminorm and h.

Theorem 2.1. *Let $p, q \colon \Omega \to [1,\infty)$ be* log-*Hölder continuous in $\overline{\Omega}$. Then the operator Q_h satisfies*

$$\|u - Q_h(u)\|_{L^{q(\cdot)}(\kappa)} \le C h_\kappa^{\frac{N}{q_-}-\frac{N}{p_-}+1} |u|_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)} \qquad \forall \kappa \in \mathcal{T}_h, \tag{2.9}$$

$$\|u - Q_h(u)\|_{L^{q(\cdot)}(e)} \le C h_e^{\frac{N-1}{q_-}-\frac{N}{p_-}+1} |u|_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_e)} \quad \forall e \in \mathcal{E}_h \cap \partial\Omega, \tag{2.10}$$

$$\|\nabla Q_h(u)\|_{L^{p(\cdot)}(\kappa)} \le C |u|_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)} \qquad \forall \kappa \in \mathcal{T}_h, \tag{2.11}$$

for all $u \in S^k(\mathcal{T}_h)$ *where* C *is a constant independent of* h.

Proof. We proceed in three steps.

Step 1. We will show first the inequality (2.9).

Fix $\kappa \in \mathcal{T}_h$. For $z \in \mathcal{N}_h \cap \kappa$, using Lemma C.7, Hypothesis C.2, Proposition A.9 and Proposition A.1 (6), we get

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C h_z^{\frac{N}{q_-} - N} \|u - \pi_z(u)\|_{L^1(T_z)}.$$

Thus, by Lemma C.9 and Lemma C.6, we have

$$\begin{aligned} \|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} &\leq C h_z^{\frac{N}{q_-} - N + 1} |Du|(T_z) \\ &\leq C h_z^{\frac{N}{q_-} - N + 1} \left(\|\nabla u\|_{L^1(T_z)} + \sum_{e \subset T_z} \int_e |[\![u]\!]| \, ds \right). \end{aligned}$$

Then, again using Lemma C.7, Proposition A.9 and Remark C.3 we have

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C h_z^{\frac{N}{q_-} + 1} \left(h_z^{-\frac{N}{p_-}} \|\nabla u\|_{L^{p(\cdot)}(T_z)} + h_z^{-N} \sum_{e \subset T_z} \int_e |[\![u]\!]| \, ds \right). \tag{2.12}$$

To estimate the second term, we use Hölder inequality and Proposition A.9, obtaining

$$\begin{aligned} \int_e |[\![u]\!]| \, ds &\leq 2 \| \, [\![u]\!] h_e^{-\frac{1}{p'(x)}} \|_{L^{p(\cdot)}(e)} \| h_e^{\frac{1}{p'(x)}} \|_{L^{p'(\cdot)}(e)} \\ &\leq C \| \, [\![u]\!] h_e^{-\frac{1}{p'(x)}} \|_{L^{p(\cdot)}(e)} h_e^{1 - \frac{1}{p_-}} \|1\|_{L^{p'(\cdot)}(e)}. \end{aligned} \tag{2.13}$$

Now, by Proposition A.1 (5), we have that

$$\|1\|_{L^{p'(\cdot)}(e)} \leq C h_e^{(N-1)(1 - \frac{1}{p_-})}.$$

Then, by Hypothesis C.2, we obtain

$$\int_e |[\![u]\!]| \, ds \leq C \| [\![u]\!] h_e^{-\frac{1}{p'(x)}} \|_{L^{p(\cdot)}(e)} h_z^{N(1 - \frac{1}{p_-})},$$

therefore, summing on all $e \subset T_z$ and using (2.12), we arrive to

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C h_z^{\frac{N}{q_-} - \frac{N}{p_-} + 1} |u|_{W^{1,p(\cdot)}(\mathcal{T}_h \cap T_z)}. \tag{2.14}$$

Now, as in the proof Theorem 7 in [4], we have the inequality (2.9).

Step 2. We now show the inequality (2.10).

Fix $e \in \mathcal{E}_h \cap \partial\Omega$. The set $e \cap T_z$ is a face of an element in $\mathcal{T}_h$. By the inequality (C.4),

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(e \cap T_z)} \leq C h_\kappa^{-\frac{1}{q_-}} \|u - \pi_z(u)\|_{L^{q(\cdot)}(\kappa \cap T_z)}.$$

Again, following the lines in [4] and using that p and q are log-Hölder continuous in $\overline{\Omega}$, we arrive to the inequality (2.10).

Step 3. Finally, we will show the inequality (2.11).

Fix $\kappa \in \mathcal{T}_h$. First, we observe that

$$\|\nabla Q_h u\|_{L^{p(\cdot)}(\kappa)} \leq \sum_{z \in \mathcal{N}_h \cap \kappa} \|(\pi_z(u) - u)\nabla\lambda_z\|_{L^{p(\cdot)}(\kappa)} + \sum_{z \in \mathcal{N}_h \cap \kappa} \|\nabla u \lambda_z\|_{L^{p(\cdot)}(\kappa)} + \|\nabla u\|_{L^{p(\cdot)}(\kappa)}.$$

Now, using Hypothesis (C.2), we have that there exists a constant C_1 such that $|\nabla\lambda_z| < C_1 h^{-1}$ in κ, and by (2.14) we get,

$$\begin{aligned} \|\nabla Q_h u\|_{L^{p(\cdot)}(\kappa)} &\leq C \sum_{z \in \mathcal{N}_h \cap \kappa} |u|_{W^{1,p(\cdot)}(\mathcal{T}_h \cap T_z)} + |u|_{W^{1,p(\cdot)}(\mathcal{T}_h \cap T_\kappa)} \\ &= (C+1)|u|_{W^{1,p(\cdot)}(\mathcal{T}_h \cap T_\kappa)}. \end{aligned}$$

The proof is now complete. □

Our next aim is to prove some global estimates. For this we will need some definitions.

Definition 2.2. Let $p : \Omega \to [1, \infty)$ and p^* defined by (1.2). Given $q : \Omega \to [1, \infty)$ and $q \leq p^*$ in Ω, we define

$$\gamma = \max\left\{\sup\left\{q(x)\frac{N - p(x)}{Np(x)} : x \in \Omega\right\}, 0\right\}.$$

Observe that $0 \leq \gamma \leq 1$ and $\gamma = 0$ if $p(x) \geq N$ for all $x \in \Omega$ and $\gamma = 1$ if $p(x) < N$ and $q(x) = p^*(x)$ for all $x \in \Omega$.

Definition 2.3. Let $p : \Omega \to [1, \infty)$ and p_* defined by (1.2). Given $q : \Omega \to [1, \infty)$ and $q \leq p_*$ in Ω, we define

$$\beta = \max\left\{\sup\left\{q(x)\frac{N - p(x)}{p(x)(N-1)} : x \in \Omega\right\}, 0\right\}.$$

Observe that $0 \leq \beta \leq 1$ and $\beta = 0$ if $p(x) \geq N$ for all $x \in \Omega$ and $\beta = 1$ if $p(x) < N$ and $q(x) = p_*(x)$ for all $x \in \Omega$.

Lemma 2.4. *Let $p, q : \Omega \to [1, \infty)$ be* log-*Hölder continuous in $\overline{\Omega}$. Suppose that*

$$|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \leq 1. \tag{2.15}$$

Then, for all $u \in S^k(\mathcal{T}_h)$, we have,

- *if $p \leq q \leq p^*$ in $\overline{\Omega}$, then*

$$\int_\Omega |u - Q_h(u)|^{q(x)}\, dx \leq C h^{N(1-\gamma)}, \tag{2.16}$$

$$\int_\Omega |\nabla Q_h(u)|^{p(x)}\, dx \leq C, \tag{2.17}$$

- *if $p \leq q \leq p_*$ in $\overline{\Omega}$, then*

$$\int_{\partial\Omega} |u - Q_h(u)|^{q(x)}\, dS \leq C h^{(N-1)(1-\beta)}, \tag{2.18}$$

where $C = C(p_1, p_2, \Omega, C_{log}, N)$ and γ and β are given in Definitions 2.2 and 2.3 respectively.

Proof. First observe that, by (2.9), we have

$$\int_\kappa \frac{|u-Q_h(u)|^{q(x)}}{\left(Ch_k^{\frac{N}{q_-}-\frac{N}{p_-}+1}|u|_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)}\right)^{q(x)}}dx \le 1 \quad \forall \kappa\in\mathcal{T}_h,$$

and by hypothesis (2.15), we get

$$\frac{1}{Ch_k^{N-\frac{Nq_-}{p_-}+q_-}|u|^{q_-}_{W^{1,p(\cdot)}}}\int_\kappa |u-Q_h(u)|^{q(x)}dx \le 1 \quad \forall \kappa\in\mathcal{T}_h.$$

Then, by Proposition A.9,

$$\begin{aligned}\int_\kappa |u-Q_h(u)|^{q(x)}\,dx &\le Ch_\kappa^{N-\frac{Nq_-}{p_-}+q_-}|u|^{q_-}_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)}\\ &\le Ch_\kappa^{N-\frac{Nq(x)}{p(x)}+q(x)}|u|^{q_-}_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)} \quad \forall x\in\kappa\end{aligned}$$

for any $\kappa\in\mathcal{T}_h$. Therefore,

$$\int_\kappa |u-Q_h(u)|^{q(x)}\,dx \le Ch^{N(1-\gamma)}|u|^{q_-}_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)} \quad \forall\kappa\in\mathcal{T}_h. \tag{2.19}$$

On the other hand, by Remark C.3, the number of $\kappa\subset T_\kappa$ is bounded independent of h. Using this fact and Proposition A.1 (6), we have that

$$|u|^{q_-}_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)} \le C\sum_{\kappa\subset T_\kappa}\left(\|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)} + \|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})}\right). \tag{2.20}$$

On the other hand, if we suppose that $\|\nabla u\|_{L^{p(\cdot)}(\kappa)}\ge h_\kappa^{N/q_-}$, then, by Proposition A.9 (2), we have that

$$\|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)} \le C\|\nabla u\|^{q_+}_{L^{p(\cdot)}(\kappa)}. \tag{2.21}$$

Arguing as before, if $\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})}\ge h_\kappa^{N/q_-}$, we have that

$$\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \le C\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_+}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})}. \tag{2.22}$$

Now, we take

$$A=\left\{\kappa\in\mathcal{T}_h : \|\nabla u\|_{L^{p(\cdot)}(\kappa)}\ge h_\kappa^{N/q_-}\right\},$$

and

$$B=\left\{\kappa\in\mathcal{T}_h : \|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})}\ge h_\kappa^{N/q_-}\right\}.$$

Observe that,

$$\begin{aligned}&\sum_{\kappa\in A'}\|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)} \le \sum_{\kappa\in A^c} h_\kappa^N \le C \quad \text{if } \kappa\in A^c\\ &\sum_{\kappa\in B^c}\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \le C \quad \text{if } \kappa\in B^c.\end{aligned} \tag{2.23}$$

On the other hand, by hypothesis (2.15), we have that $\|\nabla u\|_{L^{p(\cdot)}(\kappa)} \le 1$, and then for all $\kappa \in \mathcal{T}_h$

$$\begin{aligned} &\|\nabla u\|^{q_+}_{L^{p(\cdot)}(\kappa)} \le \|\nabla u\|^{p_+}_{L^{p(\cdot)}(\kappa)} \le \int_\kappa |\nabla u|^{p(x)}\,dx, \\ &\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_+}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \le \int_{\kappa\cap\Gamma_{int}} |[\![u]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\,dx. \end{aligned} \tag{2.24}$$

Since, each κ appears only in finitely many sets $T_{\kappa'}$, we have by (2.20),(2.21),(2.22),(2.23) and (2.24)

$$\begin{aligned} \sum_{\kappa\in\mathcal{T}_h} |u|^{q_-}_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)} &\le C\left(\sum_{\kappa\in A}\|\nabla u\|^{q_+}_{L^{p(\cdot)}(\kappa)} + \sum_{\kappa\in A^c}\|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)}\right) \\ &\quad + C\left(\sum_{\kappa\in B}\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_+}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} + \sum_{\kappa\in B^c}\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})}\right) \\ &\le C\left(\sum_{\kappa\in A}\int_\kappa |\nabla u|^{p(x)}\,dx + \sum_{\kappa\in A^c} h_\kappa^N + C\sum_{\kappa\in B}\int_{\kappa\cap\Gamma_{int}} |[\![u]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\,ds + \sum_{\kappa\in B^c} h_\kappa^N\right) \\ &= C\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\Gamma_{int}} |[\![u]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\,ds + C\right). \end{aligned}$$

Thus, by (2.15) and (2.19), we get

$$\begin{aligned} \int_\Omega |u - Q_h(u)|^{q(x)}\,dx &= \sum_{\kappa\in\mathcal{T}_h}\int_\kappa |u - Q_h(u)|^{q(x)}\,dx \\ &\le Ch^{N(1-\gamma)} \sum_{\kappa\in\mathcal{T}_h} |u|^{q_-}_{W^{1,p(\cdot)}(\mathcal{T}_h\cap T_\kappa)} \\ &\le Ch^{N(1-\gamma)}. \end{aligned}$$

Lastly, using the same argument, (2.10) and (2.11), we get

$$\int_{\partial\Omega} |u - Q_h(u)|^{q(x)}\,dS \le Ch^{(N-1)(1-\beta)}$$

and

$$\int_\Omega |\nabla Q_h(u)|^{p(x)}\,dx \le C,$$

where C is independent of h. □

The following corollary is immediately

Corollary 2.5. *Let $p, q : \Omega \to [1,\infty)$ be log-Hölder continuous in $\overline{\Omega}$. Then, for all $u \in S^k(\mathcal{T}_h)$, we have,*

- *if $p \le q \le p^*$ in Ω, then*

$$\int_\Omega |u - Q_h(u)|^{q(x)}\,dx \le Ch^{N(1-\gamma)} \max\left\{|u|^{q_1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}, |u|^{q_2}_{W^{1,p(\cdot)}(\mathcal{T}_h)}\right\} \tag{2.25}$$

$$\int_\Omega |\nabla Q_h(u)|^{p(x)}\,dx \le C \max\left\{|u|^{p_1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}, |u|^{p_2}_{W^{1,p(\cdot)}(\mathcal{T}_h)}\right\}, \tag{2.26}$$

- *if $p \le q \le p_*$ in Ω, then*

$$\int_{\partial\Omega} |u - Q_h(u)|^{q(x)}\, dS \le C h^{(N-1)(1-\beta)} \max\left\{ |u|^{q_1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}, |u|^{q_2}_{W^{1,p(\cdot)}(\mathcal{T}_h)} \right\}, \tag{2.27}$$

where $C = C(p_1, p_2, \Omega, C_{log}, N)$ and γ and β are given in Definitions 2.2 and 2.3.

Proof. It follows by Lemma 2.4, taking $v = u|u|^{-1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}$. □

Remark 2.6. Under the same hypothesis of the last corollary, if $1 \le q \le p^*$ in Ω, we have that, for all $u \in S^k(\mathcal{T}_h)$,

$$\begin{aligned} \|u - Q_h(u)\|_{L^{q(\cdot)}(\Omega)} \le& C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)}, \\ \|\nabla Q_h(u)\|_{L^{p(\cdot)}(\Omega)} \le& C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)}, \end{aligned}$$

where $C = C(p_1, p_2, \Omega, C_{log}, N)$.

3. The lifting operator

We begin this section by defining, as in [4], the lifting operator, i.e.,

Definition 3.1. Let $l \ge 0$ and $R\colon W^{1,p(\cdot)}(\mathcal{T}_h) \to S^l(\mathcal{T}_h)^N$ defined as,

$$\int_\Omega \langle R(u), \phi\rangle\, dx = -\int_{\Gamma_{int}} \langle [\![u]\!], \{\phi\}\rangle\, dS \quad \forall \phi \in S^l(\mathcal{T}_h)^N.$$

This operator appears in the the first term of the discretized functional I_h. As we can see from the definition, this operators represents the contribution of the jumps to the distributional gradient. That is why it is crucial to add this term to have the consistence of the method.

Now we will give a bound of the $L^{p(\cdot)}(\Omega)$-norm of $R(u)$ in terms of the jumps of u in Γ_{int}.

Since in our case, we are deling with the Orlicz norm, we can't prove the boundedness directly from the definition. When p is constant the proof follows from an inf-sup condition. But, in our case, we can prove this condition, but we can not use it to prove the result. Instead, we had to find a local characterization of R to prove a local bound, and finally prove the global bound.

We give first the local estimate.

Lemma 3.2. *There exists a constant C_1 such that, for any $\kappa \in \mathcal{T}_h$, we have*

$$\|R(u)\|_{L^{p(\cdot)}(\kappa)} \le C\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof. We proceed in two steps.

Step 1. We first want to prove that,

$$|R(u)| \le \frac{C}{h_\kappa^N} \sum_{e\subset\kappa} \int_e |[\![u]\!]|\, dS \quad \forall \kappa \in \mathcal{T}_h \tag{3.28}$$

where C is independent of κ and h.

We began by observing that, by Hypothesis C.1, there exist $m = m(k, N) \in \mathbb{N}$ such that for each $\kappa \in \mathcal{T}_h$,

$$R(u)|_\kappa \circ F_\kappa = \sum_{i=1}^m a_i \varphi_i(x),$$

where $\{\varphi_i\}$ is the standard nodal base of $\left(P^l\right)^N$ in the reference element $\hat{\kappa} := F_\kappa^{-1}(\kappa)$.

Using the definition of R we have that for each $1 \leq j \leq m$,

$$\int_\Omega R(u)\varphi_j \circ F_\kappa^{-1}(x)\,dx = \sum_{i=1}^m a_i \int_\kappa \varphi_i \circ F_\kappa^{-1}(x)\varphi_j \circ F_\kappa^{-1}(x)\,dx = -\sum_{e\in\kappa}\int_e [\![u]\!]\{\varphi_j \circ F_\kappa^{-1}(x)\}\,dS.$$

On the other hand, if we change variables and we use Hypothesis C.2 and that $|\varphi_i(x)| \leq 1$, we get,

$$\int_\kappa \varphi_i \circ F_\kappa^{-1}(x)\varphi_j \circ F_\kappa^{-1}(x)\,dx = h_\kappa^N \int_{\hat{\kappa}} \varphi_i(x)\varphi_j(x)\frac{|\det(DF_\kappa)|}{{h_\kappa}^N}\,dx = h_\kappa^N d_{ij}$$

with $d_{ij} \sim 1$.

Therefore,

$$R(u)|_\kappa \circ F_\kappa = \frac{1}{h_\kappa^N}\sum_{i=1}^m (D^{-1}b)_i\varphi_i(x)\,dx,$$

where $D = (d_{ij})$ and $b_j = -\sum_{e\in\kappa}\int_e [\![u]\!]\{\varphi_j \circ F_\kappa^{-1}(x)\}\,dS$.

Thus, using that $|\varphi_i(x)| \leq 1$, we arrive to (3.28).

Step 2. Now, we show that there exists a constant C_1 such that, for any $\kappa \in \mathcal{T}_h$, we have

$$\|R(u)\|_{L^{p(\cdot)}(\kappa)} \leq C\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

By inequality (2.13), we have

$$\int_e |[\![u]\!]|\,ds \leq Ch_e^{N(1-\frac{1}{p_-})}\|[\![u]\!]h_e^{-\frac{1}{p'(x)}}\|_{L^{p(\cdot)}(e)}.$$

Thus, by Hypothesis C.2 and (3.28), we have that

$$|R(u)| \leq \frac{C}{h_\kappa^{N/p_-}}\sum_{e\subset\kappa}\|[\![u]\!]h_e^{-\frac{1}{p'(x)}}\|_{L^{p(\cdot)}(e)}.$$

Now, take $T = \sum_{e\in\kappa}\|[\![u]\!]h_e^{-\frac{1}{p'(x)}}\|_{L^{p(\cdot)}(e)}$, then

$$\int_\kappa \left|\frac{R(u)}{T}\right|^{p(x)} dx \leq C\int_\kappa h_\kappa^{-Np(x)/p_-}\,dx \leq Ch_\kappa^{N(1-p_+/p_-)} \leq C$$

where in the last inequality we are using Proposition A.9.

The result follows now by Remark C.3. □

Lemma 3.3. *Let $p : \Omega \to: [1,\infty)$ be log-Hölder continuous in Ω, then there exist a constant C such that,*

$$\|R(u)\|_{L^{p(\cdot)}(\Omega)} \leq C\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof. First, if we assume that $\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})} \leq 1$, we can prove using Lemma 3.2 and proceeding as in Lemma 2.4 that,

$$\int_\Omega |R(u)|^{p(x)}\,dx \leq C.$$

Finally, taking $v = u(\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})})^{-1}$, we obtain the desired result.

□

4. Convergence of the method

In this section we first prove the broken Poincarè Sobolev Inequality inequality which is crucial to have compactness. We also prove the coercivity to arrive finally to the proof of Theorem 1.1.

Theorem 4.1. *Let $p : \Omega \to [1, +\infty)$ log −Hölder continuous in $\overline{\Omega}$. There exists a constant C such that,*

$$\|u - (u)_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \le C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \quad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0, 1].$$

In particular,

$$\|u\|_{L^{p^*(\cdot)}(\Omega)} \le C\Big(\|u\|_{L^1(\Omega)} + |u|_{W^{1,p(\cdot)}(\mathcal{T}_h)}\Big) \quad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0, 1].$$

Proof. We began by observing that

$$\|u - (u)_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \le \|u - Q_h(u)\|_{L^{p^*(\cdot)}(\Omega)} + \|Q_h(u) - (Q_h(u))_\Omega\|_{L^{p^*(\cdot)}(\Omega)} + C\|Q_h(u) - u\|_{L^1(\Omega)}.$$

Then, using the Remark 2.6 and Theorem A.8, we have

$$\|u - (u)_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \le C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \quad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0, 1].$$

The proof is complete. □

Theorem 4.2. *For each $h \in (0, 1]$ let $u_h \in W^{1,p(\cdot)}(\mathcal{T}_h)$. If there exist a constant C independent of h such that for all $h \in (0, 1]$, $I_h(u_h) \le C$, then*

$$\sup_{h\in(0,1]} \Big(\|u_h\|_{L^1(\Omega)} + |u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}\Big) < \infty.$$

Moreover,

$$\sup_{h\in(0,1]} \int_{\partial\Omega} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS < \infty.$$

Proof. Since $I_h(u_h) \le C$ then $\|\mathbf{h}^{-1/p'(x)}[\![u_h]\!]\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \le C$, and by Lemma 3.3 we have,

$$\|R(u_h)\|_{L^{p(\cdot)}(\Omega)} \le C,$$

hence

$$\int_\Omega |R(u_h)|^{p(x)}\, dx \le C.$$

Using the third inequality in Proposition A.5 we obtain,

$$\begin{aligned}\int_\Omega |R(u_h) + \nabla u_h|^{p(x)}\, dx &\ge 2^{1-p_2} \int_\Omega |\nabla u_h|^{p(x)}\, dx - \int_\Omega |R(u_h)|^{p(x)}\, dx \\ &\ge 2^{1-p_2} \int_\Omega |\nabla u_h|^{p(x)}\, dx - C.\end{aligned}$$

Therefore,

$$I_h(u_h) + C \ge 2^{1-p_2} \int_\Omega |\nabla u_h|^{p(x)}\, dx + \int_{\Gamma_D} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS + \int_{\Gamma_{int}} |[\![u_h]\!]|^{p(x)} \mathbf{h}^{1-p(x)}\, dS.$$

Thus, as $I_h(u_h) \le C$, we obtain that $|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}$ and $\int_{\partial\Omega} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS$ are uniformly bounded.

Finally, by Theorem B.2, Lemma C.6, Proposition A.1 and the fact that $h \leq 1$ we have,

$$\begin{aligned}\|u_h\|_{L^1(\Omega)} \leq & C\left(|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \int_{\Gamma_D} |u_h|\, dS\right)\\ \leq & C\left(|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \int_{\Gamma_D} |u_D|\, dS + \|(u_h - u_D)\mathbf{h}^{-1/p'(x)}\|_{L^{p(\cdot)}(\Gamma_D)}\|\mathbf{h}^{1/p'(x)}\|_{L^{p'(\cdot)}(\Gamma_D)}\right)\\ \leq & C\left(|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \int_{\Gamma_D} |u_D|\, dS + \|(u_h - u_D)\mathbf{h}^{-1/p'(x)}\|_{L^{p(\cdot)}(\Gamma_D)}\right).\end{aligned}$$

This completes the proof.

□

Theorem 4.3. *Let $p\colon [1,\infty) \to \mathbb{R}$ by log-Hölder in $\overline{\Omega}$. Let $u_h \in W^{1,p(\cdot)}(\mathcal{T}_h)$ be such that,*

$$\sup_{h\in(0,1]} (\|u_h\|_{L^1(\Omega)} + |u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}) < \infty.$$

Then, there exists a sequence $h_j \to 0$ and a function $u \in W^{1,p(\cdot)}(\Omega)$ such that,

$$\begin{aligned} u_{h_j} &\overset{*}{\rightharpoonup} u \quad \text{weakly* in } BV(\Omega) \text{ and}\\ \nabla u_{h_j} + R(u_{h_j}) &\rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega).\end{aligned}$$

Proof. The proof follows as in Theorem 13 in [4], using Lemma C.6, Lemma 3.2 and the Poincarè inequality (see Lemma A.7).

□

Lemma 4.4. *Let $p : \Omega \to (1,\infty)$ be log-Hölder continuous in $\overline{\Omega}$. Let $1 \leq q(x) < p^*(x)$ for all x in Ω and $r \in C^0(\partial\Omega)$ satisfying $1 \leq r(x) < p_*(x)$ for all x in $\partial\Omega$. Let $u_h \in S^k(\mathcal{T}_h)$ be under the conditions of Theorem 4.2, then there exixts a sequence $h_j \to 0$ and a function $u \in W^{1,p(\cdot)}(\Omega)$ such that*

$$u_{h_j} \to u \quad \text{in } L^{q(\cdot)}(\Omega), \tag{4.29}$$

$$u_{h_j} \to u \quad \text{in } L^{r(\cdot)}(\partial\Omega). \tag{4.30}$$

Proof. First we prove (4.29). By Theorem 4.3, there exists a sequence $h_j \to 0$ and $u \in W^{1,p(\cdot)}(\Omega)$ such that $u_{h_j} \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$ then, by the compactness of the embedding $BV(\Omega) \subset L^1(\Omega)$, there exists a subsequence of u_{h_j}, still denote u_{h_j}, such that $u_{h_j} \to u$ in $L^1(\Omega)$. Since $\|u_{h_j}\|_{L^1} + |u_{h_j}|_{W^{1,p(\cdot)}(\mathcal{T}_h)}$ is bounded, by Theorem 4.1, $\|u\|_{L^{p^*(\cdot)}(\Omega)}$ is bounded, and by Theorem 4.3 and Theorem A.10, $u \in W^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$. Therefore, using Theorem A.3, we obtain that

$$u_{h_j} \to u \quad \text{in } L^{q(\cdot)}(\Omega),$$

for all $1 \leq q(x) < p^*(x)$.

Now we prove (4.30). We began by observing that, by Corollary 2.5, $\|u_h - Q_h u_h\|_{L^{p(\cdot)}(\partial\Omega)} \to 0$. On the other hand, since the trace operator $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ is compact (see Theorem A.11) and $Q_h u_h$ is bounded in $W^{1,p(\cdot)}(\Omega)$ (see inequality (2.26)), it follows that there exists a subsequence of u_{h_j}, still denote u_{h_j}, such that $Q_{h_j} u_{h_j} \to u$ in $L^{r(\cdot)}(\partial\Omega)$. Therefore $u_{h_j} \to u$ in $L^{r(\cdot)}(\partial\Omega)$. □

Before we proving the convergence of the minimizers, we need an auxiliary lemma. In this step is where we need more regularity of the boundary data.

Lemma 4.5. *Let $h \in (0,1]$, and $p : \Omega \to (1,\infty)$ be log-Hölder continuous in $\overline{\Omega}$. Assume that $u_D \in W^{2,p_2}(\Omega)$ and let $v \in W^{2,p_2}(\Omega) \cap \mathcal{A}$ then, there exists $v_h \in S^1(\mathcal{T}_h) \cap W^{1,p(\cdot)}(\Omega)$, such that*

$$\|v_h - v\|_{W^{1,p(\cdot)}(\Omega)} \to 0 \quad \text{as } h \to 0,$$

and

$$I_h(v_h) \to I(v) \quad \text{as } h \to 0.$$

Proof. By Theorem 3.1.5 in [6], there exists $v_h \in S^1(\mathcal{T}_h) \cap W^{1,p_2}(\Omega)$ such that

$$\|v - v_h\|_{L^{p_2}(\partial\kappa)} \le C h_\kappa |v|_{2,p_2(\kappa)},$$

for each $\kappa \in \mathcal{T}_h$. Using Remark C.4 and summing over all $e \in \partial\Omega$ we have,

$$\int_{\partial\Omega} |v - v_h|^{p_2} \mathbf{h}^{1-p_2}\, ds \le Ch|v|^{p_2}_{2,p_2(\Omega)}, \tag{4.31}$$

therefore by Hölder inequality, we have

$$\int_{\partial\Omega} |v - v_h|^{p(x)} \mathbf{h}^{1-p(x)}\, ds \le C\| |v - v_h|^{p(\cdot)} \mathbf{h}^{(1-p_2)p(\cdot)/p_2} \|_{L^{p_2/p(\cdot)}(\partial\Omega)}.$$

Since,

$$\int_{\partial\Omega} (|v - v_h|^{p(x)} \mathbf{h}^{(1-p_2)p(x)/p_2})^{p_2/p(x)}\, ds = \int_{\partial\Omega} |v - v_h|^{p_2} \mathbf{h}^{(1-p_2)}\, ds \to 0 \quad \text{as } h \to 0$$

then by (4.31),

$$\int_{\partial\Omega} |v - v_h|^{p(x)} \mathbf{h}^{1-p(x)}\, ds \to 0 \quad \text{as } h \to 0.$$

Since $v_h \in W^{1,p(\cdot)}(\Omega)$ then $[\![v_h]\!] = 0$ and $R(v_h) = 0$. Finally, using Theorem A.6, we obtain de desired result.

□

Now we are in condition to prove Theorem 1.1,

Proof of Theorem 1.1. First, take $w_h \in S^1(\mathcal{T}_h) \cap W^{1,p}(\Omega)$ converging strongly to u_D in norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ and $|\cdot|_{W^{1,p(\cdot)}(\mathcal{T}_h)}$ (see Lemma C.5). By Theorem 4.5, we have that $I_h(w_h) \to I(u_D)$, therefore $I_h(u_h)$ is bounded. By Theorem 4.2, Lemma 4.2 and Theorem 4.3, there exists $u \in W^{1,p(\cdot)}(\Omega)$ such that

$$u_{h_j} \overset{*}{\rightharpoonup} u \quad \text{weakly* in } BV(\Omega) \text{ and}$$
$$\nabla u_{h_j} + R(u_{h_j}) \rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega).$$

On the other hand, by Lemma 4.4, we obtain (1.4) and (1.5). Since the penalty term,

$$\int_{\Gamma_D} \mathbf{h}^{1-p} |u_h - u_D|^p\, dS$$

is bounded, we have, by Lemma 4.4

$$\|u - u_D\|_{L^{p(\cdot)}(\Gamma_D)} \le \|u - u_{h_j}\|_{L^{p(\cdot)}(\Gamma_D)} + \|u_{h_j} - u_D\|_{L^{p(\cdot)}(\Gamma_D)} \to 0.$$

Then $u \in \mathcal{A}$.

By Lemma 4.4 and Proposition A.6 we have,

$$\begin{aligned} I(u) &\leq \liminf_{j\to\infty} \left[\int_\Omega \Big(|\nabla u_{h_j} + R(u_{h_j})|^{p(x)} + |u_{h_j} - \xi|^{q(x)} \Big)\, dx + \int_{\Gamma_N} |u_{h_j}|^{r(x)}\, dS \right] \\ &\leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}). \end{aligned} \tag{4.32}$$

Lastly, we want to prove that u is the minimizer of I. Let $v \in \mathcal{A} \cap W^{2,p_2}(\Omega)$, and let $v_h \in S^1(\mathcal{T}_h) \cap W^{1,p}(\Omega)$ as in Lemma 4.5. Then $I_h(v_h) \to I(v)$. Therefore, by (4.32)

$$I(u) \leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}) \leq \lim_{j\to\infty} I_{h_j}(v_{h_j}) = I(v). \tag{4.33}$$

Now, let $w \in W^{1,p(\cdot)}(\Omega) \cap \mathcal{A}$, then for any $\varepsilon > 0$ there exists $v \in \mathcal{A} \cap W^{2,p_2}(\Omega)$ such that $\|v - w\|_{W^{1,p(\cdot)}(\Omega)} < \varepsilon$. By Theorem A.6 we have that $I(v) < I(w) + \varepsilon$, therefore by (4.33)

$$I(u) \leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}) \leq I(w) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get

$$I(u) \leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}) \leq I(w) \quad \forall w \in W^{1,p(\cdot)}(\Omega) \cap \mathcal{A}.$$

Therefore $I(u) \leq I(w)$. Moreover, taking $w = u$, we have that $I_{h_j}(u_{h_j}) \to I(u)$ and (1.8). So we also have that $R(u_{h_j}) \to 0$ which implies (1.6). Since u is the unique minimizer of I, the whole sequence u_h converge to u. □

5. CONFORM

To make a complete study of this problem we will prove the convergence of the Continuous Galerkin finite element method for our problem. In the next section we will make a comparison of both method for an example.

For simplicity we take the following functional,

$$I(u) = \int_\Omega \Big(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u - \xi|^{q(x)}}{q(x)} \Big)\, dx$$

with $q(x) < p^*(x)$. Then, since the functional J is strictly convex and coercive in $\mathcal{A}$ there exists a unique minimizer of the problem.

We take now a partition of Ω as in Hypothesis C.1 and the usual conform subspace U_h^k of $W^{1,p(\cdot)}(\Omega)$. This subspace consists of all continuous functions such that they are polynomials functions of degree at most k in each $\kappa \in \mathcal{T}_h$. We will assume that for some h', $u_D \in U_{h'}^k$. Let now $h \leq h'$ and $V_h^k = \{v_h \in U_h^k(\mathcal{T}_h)$ such that $v_h = u_D$ in $\partial\Omega\}$. For simplicity we may assume that $h' = 1$.

Again, by the strict convexity of I, for each $h \in (0, 1]$ there exists a $u_h \in V_h^k$ such that u_h is a minimizer in V_h^k of I.

Now we prove that the sequences $u_h \to u$ in $W^{1,p(\cdot)}(\Omega)$.

Theorem 5.1. *The sequence $u_h \to u$ in $W^{1,p(\cdot)}(\Omega)$, where u is the unique minimizer of I.*

Proof. Since $I(u_h)$ is uniformly bounded, there exists a subsequence, $u_{h_j} \rightharpoonup u$ weakly in $W^{1,p(\cdot)}(\Omega)$ (for simplicity we will noted u_h). Let now $\Pi_h : C_0^\infty(\Omega) \to U_h^k$ be the interpolant mapping defined in Theorem 3.1.5 [6] and let $\phi \in C_0^\infty(\Omega)$. Using the minimality of u_h we have, $I(u_h) \leq I(\Pi_h\phi + u_D)$. Since I is convex and continuous we have,

$$I(u) \leq \liminf_{h\to 0} I(u_h) \leq \liminf_{h\to 0} I(\Pi_h\phi + u_D).$$

On the other hand, by Theorem 3.1.5 in [6] we have that

$$\|\Pi_h\phi - \phi\|_{1,p(\cdot)} \leq C\|\Pi_h\phi - \phi\|_{1,p_2} \to 0.$$

Then by the continuity of I we have that $\lim I(\Pi_h\phi+u_D) = I(\phi+u_D)$, therefore $I(u) \leq I(\phi+u_D)$. By the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ we conclude that u is a minimizer.

Now we want to prove the strong convergence. Let,

$$R(w)v = \int_\Omega |\nabla w|^{p(x)-2}\nabla w\nabla v\,dx$$
$$T(w)v = \int_\Omega |w-\xi|^{q(x)-2}(w-\xi)v\,dx$$

and $V_{h0}^k = \{v_h \in U_h^k$ such that $v_h = 0$ in $\partial\Omega\}$. Therefore, for all $v \in W_0^{1,p(\cdot)}(\Omega)$ and for all $v_h \in V_{h0}^k$ we have

$$R(u)v = T(u)v \tag{5.34}$$
$$R(u_h)v_h = T(u_h)v_h. \tag{5.35}$$

Let us prove first the case $1 < p \leq 2$. By Proposition A.5 we have that,

$$\int_\Omega |\nabla u_h - \nabla u|^2(|\nabla u_h| + |\nabla u|)^{p-2}\,dx \leq C(R(u_h)(u_h-u) - R(u)(u_h-u)). \tag{5.36}$$

We want to prove that the right hand side of the last inequality goes to zero.

Since $u_h \rightharpoonup u$ we have that $R(u)(u_h-u) \to 0$. We only have to prove that $R(u_h)(u_h-u) \to 0$ as $h \to 0$.

Now, let us take for any $h \in (0,1]$, $\phi_h \in V_{0h}^1$ such that $\|\phi_h - (u_D-u)\|_{W^{1,p(\cdot)}(\Omega)} \to 0$, then by (5.35) we have,

$$\begin{aligned}
R(u_h)(u_h-u) &= R(u_h)(u_h-u_D) + R(u_h)(u_D-u) = T(u_h)(u_h-u_D) + R(u_h)(u_D-u)\\
&= T(u_h)(\phi_h-(u_D-u_h)) - T(u_h)\phi_h + R(u_h)(u_D-u-\phi_h) + R(u_h)\phi_h\\
&= T(u_h)(\phi_h-(u_D-u_h)) + R(u_h)(u_D-u-\phi_h)\\
&= T(u_h)(\phi_h-(u_D-u)) + T(u_h)(u_h) - T(u_h)u + R(u_h)(u_D-u-\phi_h)
\end{aligned}$$

Since $u_h \rightharpoonup u$ weakly in $W^{1,p(\cdot)}(\Omega)$, we get $u_h \to u$ in $L^{q(\cdot)}(\Omega)$. Therefore $\|\nabla u_h\|_{L^{p(\cdot)}(\Omega)}$ and $\|u_h\|_{L^{q(\cdot)}(\Omega)}$ are uniformly bounded, thus

$$\begin{aligned}
|R(u_h)(u_h-u)| &\leq \||u_h-\xi|^{q-1}\|_{L^{q'(\cdot)}(\Omega)}\|u_h-u\|_{L^{q(\cdot)}(\Omega)}\\
&\quad + (\|\nabla u_h\|_{L^{p(\cdot)}(\Omega)} + \||u_h-\xi|^{q-1}\|_{L^{q'(\cdot)}(\Omega)})\|u_D-u-\phi_h\|_{L^{p(\cdot)}(\Omega)} \to 0
\end{aligned}$$

Therefore $R(u_h)(u_h-u) \to 0$ and by (5.36) we have that

$$\int_\Omega |\nabla u_h - \nabla u|^2(|\nabla u_h| + |\nabla u|)^{p-2}\,dx \to 0 \quad \text{as } h \to 0.$$

Now using Hölder inequality and Proposition A.1 we have that $\nabla u_h \to \nabla u$ in $L^{p(\cdot)}(\Omega)$.

The case $p \geq 2$ follows analogously, using Proposition A.5. If $|\{p < 2\}| \neq 0$ and $|\{p \geq 2\}| \neq 0$ we can prove that $\|\nabla u_h - u\|_{L^{p(\cdot)}(\{p<2\})} \to 0$ and $\|\nabla u_h - u\|_{L^{p(\cdot)}(\{p\geq 2\})} \to 0$ and the proof is completed.

□

6. Examples

In this section, we will give some examples in one dimension. Our idea is to compare the Continuous Galerking Finite Element Method (CGFEM) versus the Discontinuous Galerking Finite Element Method (DGFEM). We will see in an example, where the function p attains values near one, that our method converges faster to the solution.

Example. Let $\Omega = (-1, 1)$ and $p : [-1, 1] \to [1, 2]$ given by

$$p(x) = \begin{cases} -\frac{1-\varepsilon}{a}x + 1 + \varepsilon & \text{if } -a < x \leq 0, \\ -\frac{1-\varepsilon}{a}x + 1 + \varepsilon & \text{if } -a < x \leq 0, \\ \quad 2 & \text{if } \quad a \leq |x| \leq 1, \end{cases}$$

where $0 < \varepsilon, a < 1$.

For this function $p(x)$ and for a given $B > 0$, we study the following problem,

$$\begin{cases} ((u'(x))^{p(x)-1}) = 0 & \text{in } (0,1) \\ u(1) = -u(1) = B. \end{cases} \tag{6.37}$$

We began by observing that, since the operator is strictly monotone, we have an unique solution of (6.37). Moreover, the solution satisfies $(u'(x))^{p(x)-1} = C$ for some constant C and

$$u(x) = C(x+1) - B \quad \text{if } -1 \leq x \leq -a, \tag{6.38}$$

$$u(x) = C(x-1) + B \quad \text{if } \quad a \leq x \leq 1. \tag{6.39}$$

Now, if $-a < x \leq 0$ we have

$$u(x) = \int_{-a}^{x} C^{\frac{-a}{(\varepsilon-1)s - a\varepsilon}}\, ds + C(1-a) - B = \int_{-x}^{a} C^{\frac{a}{(1-\varepsilon)s + a\varepsilon}}\, ds + C(1-a) - B.$$

and if $0 < x < a$

$$u(x) = u(0) + \int_0^x C^{\frac{a}{(1-\varepsilon)s + a\varepsilon}}\, ds.$$

Therefore, by (6.39) and the last equation, we get

$$C(a-1) + B = u(a) = 2\int_0^a C^{\frac{a}{(1-\varepsilon)s + a\varepsilon}}\, ds + C(1-a) - B,$$

that is

$$B = \int_0^a C^{\frac{a}{(1-\varepsilon)s + a\varepsilon}}\, ds - C(a-1).$$

On the other hand, since the derivative of u at zero has modulus $C^{1/\varepsilon}$, if $C > 1$ we have

$$\lim_{\varepsilon \to 0} |u'(0)| = +\infty.$$

This is reasonable since we expect to have big derivative when p approaches the value one.

From now on, we take $\varepsilon = a = .01$ and since it is easer to get B from C, we impose $C = 1.3$. Then the function u has the form,

$$u(x) = \begin{cases} 1.3(x+1) - B & \text{if } -1 \leq x \leq -.01 \\ -\int_0^{-x} h(s)\, ds & \text{if } -.01 \leq x \leq 0 \\ \int_0^{x} h(s)\, ds & \text{if } \quad 0 \leq x \leq .01 \\ 1.3(x-1) + B & \text{if } \quad .01 \leq x \leq 1, \end{cases}$$

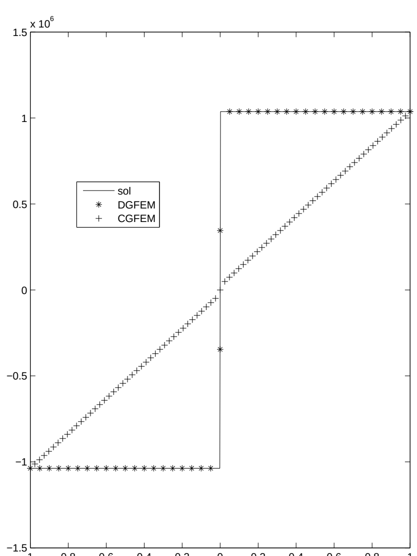
where $h(s) = 1.3^{\frac{100}{1+s999}}$ and $B \simeq 1.03\ 10^6$. Observe that in this case,

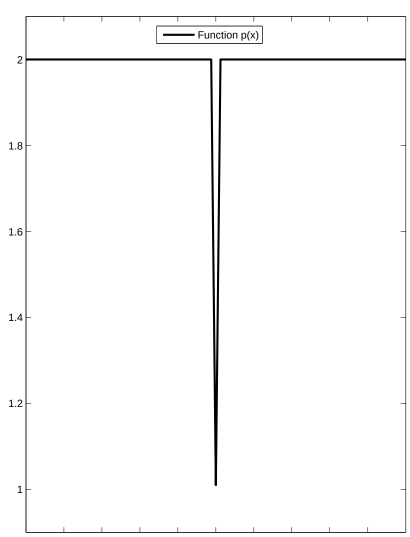
$$|u'(0)| = h(0) = 1.3^{100} \simeq 2.410^{11}.$$

Now, we find the corresponding solution for the CGFEM and the DGFEM. In both cases we take an uniform partition of $[-1, 1]$ in n subintervals with size $2/n$. Observe that for the continuous method, we impose the boundary conditions and then, the space where we find minimizers has dimension $n-2$. For the Discontinuous method, since we do not impose conditions on the boundary, and the number of nodal basis are $2n-2$, we are minimizing in a space of this dimension. Therefore, to make a true comparison between both methods, we compare the discrete problem for the DGFEM in $n-$dimension with the CGFEM in $n-2-$dimension.

We want to mention that, to find minimizers of both discrete problems, we use a BFGS Quasi-Newton method (see [15] and [19]). These methods are good in lower dimensions, but they can give bad approximations and also be very slow when the dimension is too big.

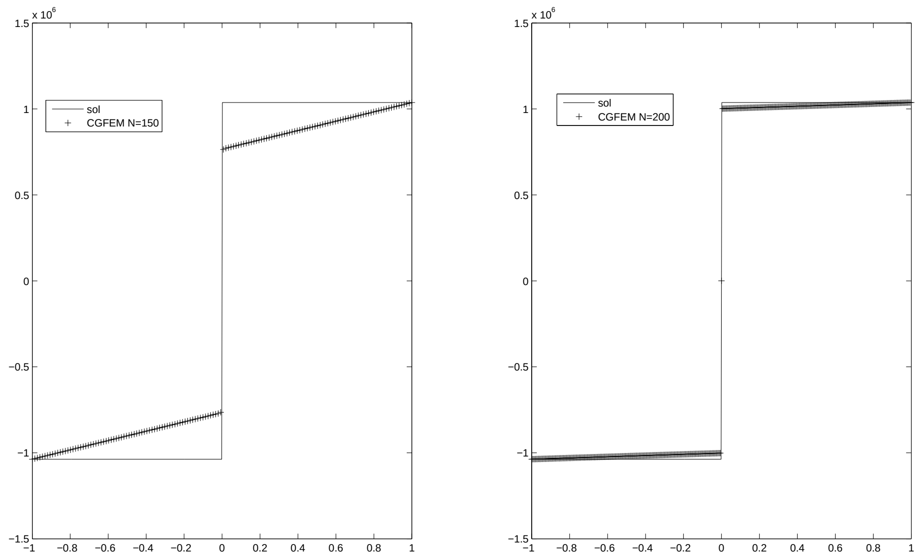
In the next two figures, we plot first the solution versus the approximation using the DGFEM and CGFEM for the case $n = 41$ and $n = 81$ respectively. The second figure is the graphic of the function $p(x)$.





Note that, when we use the CGFEM the discrete solution is close to the function $y = x$ which is a solution of (6.37) with $p \equiv 2$, that means that this method needs a smaller step to see the points where p it's closer to one.

In the following figure we can see that, the minimizers of the continuous methods are far from the solution even for $n = 150$. We need $n = 200$ to arrive to a good approximation of u.



Appendix A. The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$

We will now introduce the space $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let $p\colon \Omega \to [1, +\infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_1 = essinf\, p(x)$ and $p_2 = esssup\, p(x)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$ is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{k > 0\colon \varrho_{p(\cdot)}(u/k) \leq 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

The following Properties can be obtained directly from the definition of the norm,

Proposition A.1. *If $u \in L^{p(\cdot)}(\Omega)$, $\|u\|_{L^{p(\cdot)}(\Omega)} = \lambda$, then*

(1) $\lambda < 1$ $(= 1, > 1)$ *if only if* $\displaystyle\int_\Omega |u(x)|^{p(x)}\, dx < 1$ $(= 1, > 1)$,

(2) *if* $\lambda \geq 1$, *then* $\displaystyle\lambda^{p_1} \leq \int_\Omega |u(x)|^{p(x)}\, dx \leq \lambda^{p_2}$,

(3) *if* $\lambda \leq 1$, *then* $\displaystyle\lambda^{p_2} \leq \int_\Omega |u(x)|^{p(x)}\, dx \leq \lambda^{p_1}$,

(4) $\displaystyle\int_\Omega |u_n(x)|^{p(x)}\,dx \to 0$ *if only if* $\|u\|_{p(\cdot)} \to 0$.

(5) $\|1\|_{p(\cdot)} \leq \max\left\{|\Omega|^{\frac{1}{p_1}}, |\Omega|^{\frac{1}{p_2}}\right\}$.

(6) *If* $\Omega = \bigcup_{i=1}^{m} \Omega_i$ *where* $\Omega_i \subset \Omega$ *are open sets then there exists a constant* $C > 0$ *depending on* m *such that,*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \sum_{i=1}^{m} \|u\|_{L^{p(\cdot)}(\Omega_i)}.$$

Proof. For the proof see Theorem 1.3 and Theorem 1.4 in [13]. □

For the proofs of the following three theorems we refer the reader to [17].

Theorem A.2. *Let* $q(x) \leq p(x)$*, then* $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *continuously.*

Theorem A.3. *Let* $p, q, r : \Omega \to [1, \infty)$ *be such that* $p(x) \leq r(x) < q(x)$ *for all* $x \in \Omega$*. Then there exist constants* $C, \mu > 0$ *and* $\nu \geq 0$ *such that for every* $u \in L^{p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega)$ *the inequality*

$$\|u\|_{r(\cdot)} \leq C\|u\|^{\mu}_{p(\cdot)}\|u\|^{\nu}_{q(\cdot)}$$

holds.

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that, u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}$ a Banach space.

Theorem A.4. *Let* $p'(x)$ *such that,*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Then $L^{p'(\cdot)}(\Omega)$ *is the dual of* $L^{p(\cdot)}(\Omega)$*. Moreover, if* $p_1 > 1$*,* $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are reflexive.*

We give now some well known inequalities,

Proposition A.5. *For any* x *fixed we have the following inequalities*

$$|\eta - \xi|^{p(x)} \leq C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta - \xi) \qquad \text{if } p(x) \geq 2,$$

$$|\eta - \xi|^2\Big(|\eta| + |\xi|\Big)^{p(x)-2} \leq C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta - \xi) \qquad \text{if } p(x) < 2,$$

$$|\eta|^{p(x)} \leq 2^{p(x)-1}(|\eta - \xi|^{p(x)} + |\xi|^{p(x)}) \qquad \text{if } p(x) \geq 1.$$

These inequalities say that the function $A(x, q) = |q|^{p(x)-2}q$ is strictly monotone.

Proposition A.6. *Let* $u_n, u \in W^{1,p(\cdot)}(\Omega)$

(1) *If*

$$\nabla u_n \rightharpoonup \nabla u \text{ weakly in } L^{p(\cdot)}(\Omega),$$

then

$$\int_\Omega |\nabla u|^{p(x)}\,dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)}\,dx.$$

(2) *If*

$$u_n \to u \text{ strongly in } W^{1,p(\cdot)}(\Omega),$$

then

$$\int_\Omega |\nabla u_n|^{p(x)}\,dx \to \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Proof. For the proof of (1) see proof of Theorem 2.1 in [14].

(2) follows by Proposition A.5 and the dominate convergence Theorem. □

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have the following version of Poincare's inequity (see Theorem 3.10 in [17]).

Lemma A.7. *If $p:\Omega\to[1,+\infty)$ is continuous in $\overline{\Omega}$, there exists a constant C such that for every $u\in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

We also have the following version of the Poincaré inequality (see Lemma 2.1 in [16]),

Theorem A.8. *Let $\Omega\subset\mathbb{R}^n$ be a Lipschits domain. Let $p:\Omega\to[1,+\infty)$ and $p\le q\le p^*$. Then,*

$$\|u-(u)_\Omega\|_{L^{q(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u\in W^{1,p(\cdot)}(\Omega)$.

In order to have better properties of these spaces, we need more hypotheses on the regularity of $p(x)$.

We say that p is log-*Hölder continuous* in Ω if there exists a constant C_{log} such that

$$|p(x)-p(y)| \le \frac{C_{log}}{\log\left(e+\frac{1}{|x-y|}\right)}$$

for all $x,y\in\Omega$.

Proposition A.9. *Let $p:\Omega\to[1,\infty)$ be* log-*Hölder continuous and bounded. Let* $\alpha>0$, $D\subset\Omega$ *and $h = diam(D)$ then,*

(1) *There exist constants C independent of h such that*

$$h^{\alpha(p(x)-p(y))} \le C \quad \forall x,y\in D. \tag{A.1}$$

Moreover, if $p(x)$ is continuous in $\overline{D}$ then the inequality (A.1) *holds for all $x,y\in\overline{D}$.*

(2) *If $A\ge h^\alpha$ then $A^{p(x)}\le CA^{p(y)}$ for all $x,y\in D$ such that $p(x)\le p(y)$.*

Proof. Let $x,y\in D$. If $p(x)\ge p(y)$ or $h\ge 1$ the result follows since Ω is bounded. If $p(x)\le p(y)$ and $h<1$, since p is log-Hölder, we have

$$p(y)-p(x) \le \frac{C}{\log\left(e+\dfrac{1}{|x-y|}\right)} \le \frac{C}{\log\left(e+\dfrac{1}{h}\right)}.$$

Then, we get (A.1).

By (A.1) and as $A \geq h^\alpha$, we have

$$A^{p(x)} = A^{p(y)} \left(\frac{A}{h^\alpha}\right)^{p(x)-p(y)} h^{\alpha(p(x)-p(y))} \leq CA^{p(y)},$$

for all $x, y \in \Omega$ such that $p(x) \leq p(y)$. □

It was proved in [8], Theorem 3.7, that if one assumes that p is log-Hölder continuous then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ (see also [7], [17] and [18]). See [10] for more references on this topic.

We now state some Sobolev imbedding Theorems,

Theorem A.10. *Let Ω be a Lipschitz domain. Let $p : \Omega \to [1, \infty)$ and p log-Hölder continuous. Then the imbedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous.*

Proof. See [9]. □

Theorem A.11. *Let Ω be an open bounded domain with Lipschitz boundary. Suppose that $p \in C^0(\overline{\Omega})$ with $p_1 > 1$. If $r \in C^0(\partial\Omega)$ satisfies the condition*

$$1 \leq r(x) < p_*(x) \quad \forall x \in \partial\Omega,$$

then there is a compact boundary trace embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.

Proof. See [12, Corollary 2.4]. □

Appendix B. BV functions

In this Appendix, Ω is a bounded subset of $\mathbb{R}^N$. Let us denote by $\mathcal{M}(\Omega, \mathbb{R}^N)$ the space of all $\mathbb{R}^N-$ valued Borel measure, let $\mathcal{H}^{N-1}$ denote the $(N-1)-$dimensional Hausdorff measure and, for a set $A \subset \mathbb{R}^N$, let $\dim_H(A)$ denote the Hausdorff dimension of A.

We will now give some well-known results concerning the bounded variational functions. They can be found, for instance, in [4, 11].

Let $u \in L^1(\Omega)$. We say that u is a function of *bounded variation* on Ω if its distributional derivative is a measure, i.e., there exists $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega u \operatorname{div} \phi \, dx = -\int_\Omega \phi \cdot d\mu \quad \forall \phi \in C_c^1(\Omega, \mathbb{R}^N).$$

The measure μ will be denote by Du, and its components by $\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}$. The space of all functions of bounded variation on Ω will be denote by $BV(\Omega)$.

For $u \in BV(\Omega)$ we define the *total variation* of u on Ω as

$$\sup \left\{ \int u \operatorname{div} \phi \, dx \colon \phi \in C_c^1(\Omega, \mathbb{R}^N), \ \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

We will denote by $|Du|(\Omega)$.

The space $BV(\Omega)$ is equipped with the norm

$$\|u\|_{BV} := \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

Now we give the Trace Theorem for BV.

Theorem B.1 (Trace Theorem). *Assume Ω is open and bounded, with $\partial\Omega$ Lipschitz. There exists a bounded linear mapping $T\colon BV(\Omega) \to L^1(\partial U)$ (we write $Tu = u$). such that*

$$\int_\Omega u \operatorname{div} \phi \, dx = -\int_\Omega \phi \cdot dDu + \int_{\partial\Omega} \langle \phi, \nu \rangle u \, ds \quad \forall u \in BV(\Omega) \quad \forall \phi \in C^1(\mathbb{R}^n, \mathbb{R}^n),$$

where ν is the unit outward normal to $\partial\Omega$.

Moreover, if $u \in BV(\Omega)$ then for $\mathcal{H}^{N-1}$ − a.e. $x \in \partial\Omega$ the identity

$$Tu(x) = \lim_{r\to 0} \fint_{B(x,r)\cap\Omega} u \, dx$$

holds.

Proof. The reader interested in this proof may refer to [11]. □

Lastly we give the Friedrichs Inequality for BV.

Theorem B.2 (Friedrichs Inequality). *Let Ω be an bounded open subset of $\mathbb{R}^N$ with $\partial\Omega$ Lipschitz, let $u \in BV(\Omega)$ and let Γ be a subaset of $\partial\Omega$ with positive surface measure. Then, there exists a constant C such that*

$$\|u\|_{L^1(\Omega)} \leq C \left(|Du|(\Omega) + \int_\Gamma |u| \, ds \right).$$

Proof. See Lemma 10 of [4]. □

Appendix C. The mesh $\mathcal{T}_h$ and properties of $W^{1,p(\cdot)}(\mathcal{T}_h)$

In this appendix we describe the type of mesh that we consider in this work and we introduce the variable broken Sobolev space.

Hypothesis C.1. *We will assume that Ω is a polygonal Lipschitz domain and let $(\mathcal{T}_h)_{h\in(0,1]}$ a family of partitions of $\overline{\Omega}$ into polyhedral elements. We assume that there exist a finite number of reference polyhedra $\hat{\kappa}_1, ..., \hat{\kappa}_r$ such that for all $\kappa \in \mathcal{T}_h$ there exists an invertible affine map F_κ such that, $\kappa = F_\kappa(\hat{\kappa}_i)$. We assume that each $\kappa \in \mathcal{T}_h$ is close and that diam$(\kappa) \leq h$ for all $\kappa \in \mathcal{T}_h$.*

Now we give some notation,

$$\mathcal{E}_h = \{\kappa \cap \kappa' : \dim_H(\kappa \cap \kappa') = N-1\} \cup \{\kappa \cap \partial\Omega : \dim_H(\kappa \cap \partial\Omega) = N-1\},$$
$$\Gamma_{int} = \bigcup \{e \in \mathcal{E}_h : \dim_H(e \cap \partial\Omega) < N-1\}.$$

$\mathcal{N}_h$ is the set of nodes of $\mathcal{T}_h$. For every $z \in \mathcal{N}_h$ and $e \in \mathcal{E}_h$ we define,

$$T_z = \bigcup\{\kappa \in \mathcal{T}_h \colon z \in \kappa\}, \quad T_\kappa = \bigcup\{T_z \colon z \in \kappa\}, \quad T_e = \bigcup\{T_\kappa \colon e \in \kappa\},$$
$$h_\kappa = \operatorname{diam}(\kappa), \qquad h_z = \operatorname{diam}(T_z) \qquad \text{and} \quad h_e = \operatorname{diam}(e).$$
$$p_- = \operatorname*{ess\,inf}_{x\in\kappa} p(x) \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_{x\in\kappa} p(x).$$

We assume that the mesh satisfies the following hypotheses,

Hypothesis C.2. *The family of partitions $(\mathcal{T}_h)_{h\in(0,1]}$ satisfies the Hypotesis C.1 and*

(a) *There exist positive constants* C_1 *and* C_2*, independent of* h*, such that for each element* $\kappa \in \mathcal{T}_h$

$$C_1 h_\kappa^N \leq |\kappa| \leq C_2 h_\kappa^N.$$

(b) *There exists a constant* $C_1 > 0$ *such that for all* $h \in (0,1]$ *and for all face* $e \in \mathcal{E}_h$ *there exists a point* $x_e \in e$ *and a radius* $\rho_e \geq C_1 diam(e)$ *such that* $B_{\rho_e}(x_e) \cap A_e \subset e$*, where* A_e *is the affine hyperplane spanned by* e*. Moreover, there are positive constants such that*

$$ch_\kappa \leq h_e \leq Ch_\kappa, \quad ch_{\kappa'} \leq h_e \leq Ch_{\kappa'}$$

where $e = \kappa \cap \kappa'$*.*

We use the notation $\sim$ to compare two quantities that differ up to a constants independent on h.

From now on, we consider meshes that satisfy the Hypothesis C.2.

Remark C.3. By the regularity assumption off the mesh, me have the following,

$$\sharp\{z \in \kappa\} \sim 1, \qquad \sharp\{\kappa \subset T_z \cap \mathcal{T}_h\} \sim 1,$$

$$\sharp\{\kappa \subset T_\kappa \cap \mathcal{T}_h\} \sim 1, \quad \sharp\{e \subset T_z \cap \mathcal{E}_h\} \sim 1 \quad \text{and} \quad \sharp\{e \subset T_\kappa \cap \mathcal{E}_h\} \sim 1$$

Remark C.4. As a consequence, we have that $\dim(T_\kappa) \sim h_\kappa$ and for each $z \in \kappa$ and $e \subset \partial\kappa$, $h_z \sim h_\kappa$ and $h_e \sim h_\kappa$.

Proof. See the discussion on section 4.2 in [4]. □

Now, we introduce the finite element spaces associated with $\mathcal{T}_h$. We define the variable broken Sobolev space as

$$W^{1,p(\cdot)}(\mathcal{T}_h) = \{u \in L^1(\Omega) \colon u|_\kappa \in W^{1,p(\cdot)}(\kappa) \text{ for all } \kappa \in \mathcal{T}_h\}$$

and the subspace

$$S^k(\mathcal{T}_h) = \{u \in L^1(\Omega) \colon u|_\kappa \circ F_\kappa \in P^k \text{ for all } \kappa \in \mathcal{T}_h\}$$

where P^k is the space of polynomials functions of degree at most $k \geq 1$.

For $u \in W^{1,p(\cdot)}(\mathcal{T}_h)$, we define the jump of u, as

$$[\![u]\!] = u^+\nu^+ + u^-\nu^-,$$

and for $\phi \in (W^{1,p(\cdot)}(\mathcal{T}_h))^N$, we define the average of ϕ, as

$$\{\phi\} = \frac{\phi^+ + \phi^-}{2}.$$

Lemma C.5. *Let* $1 \leq p < \infty$ *be log-Hölder, and let* $v \in W^{1,p(\cdot)}(\Omega)$*, then for each* $h \in (0,1]$ *there exists* $v_h \in S^1(\mathcal{T}_h) \cap W^{1,p(\cdot)}(\Omega)$ *such that,*

$$\|v - v_h\|_{W^{1,p(\cdot)}(\Omega)} \to 0 \quad \text{as } h \to 0.$$

Proof. Since p is log-Hölder, we have that $C^\infty(\bar{\Omega})$ are dense in $W^{1,p(\cdot)}(\Omega)$ (see Therem 3.7 in [8]). Then the proof follows by standard approximation theory (see [6]). □

Let $\mathbf{h} : \mathbf{\Omega} \to \mathbb{R}$ a piecewise constant function define by: $\mathbf{h(x)} = \mathbf{diam}(\kappa)$ if $x \in \kappa$ and $\mathbf{h(x)} = \mathbf{diam(e)}$ if $x \in e$.

We consider the followings seminorms in $W^{1,p(\cdot)}(\mathcal{T}_h)$,

$$|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \sum_{e\in\Gamma_{int}} \|[\![u]\!]\mathbf{h}^{\frac{-1}{p'(x)}}\|_{L^{p(\cdot)}(e)},$$

and

$$|u|_{W_D^{1,p(\cdot)}(\mathcal{T}_h)} = |u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \sum_{e\in\Gamma_D} \|u\mathbf{h}^{\frac{-1}{p'(x)}}\|_{L^{p(\cdot)}(e)}.$$

Lemma C.6. *For all $p\colon [1,\infty) \to \mathbb{R}$, there exist a constant C, independent of h such that,*

$$|Du|(\Omega) \le C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h),\ \forall h \in (0,1].$$

Proof. For all $u \in W^{1,p(\cdot)}(\mathcal{T}_h)$, we have that

$$|Du|(\Omega) \le \int_\Omega |\nabla u|\,dx + \int_{\Gamma_{int}} |[\![u]\!]|\,ds = \int_\Omega |\nabla u|\,dx + \sum_{e\in\Gamma_{int}} \int_e |[\![u]\!]|\,ds.$$

Thus, by Hölder inequality, the item (5) of Proposition A.1 and the Hypothesis C.2, there exists a constant C depending only of $|\Omega|$, p_1 and p_2 such that

$$|Du|(\Omega) \le C\left(\|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \sum_{e\in\Gamma_{int}} \|\mathbf{h}^{\frac{-1}{p'(x)}}[\![u]\!]\|_{L^{p(\cdot)}(e)}\right).$$

The proof is now complete. □

Lemma C.7. *Let $(\mathcal{T}_h)_{h\in(0,1]}$ be a family of partitions of Ω. Then, for each function $p, q\colon \Omega \to [1,\infty)$, there exists a constant $C > 0$ independent of h, such that for any $\kappa \in \mathcal{T}_h$*

$$\|u\|_{L^{p(\cdot)}(\kappa)} \le C h_\kappa^{\frac{N}{p_+}-\frac{N}{q_-}} \|u\|_{L^{q(\cdot)}(\kappa)} \quad \forall u \in S^k(\mathcal{T}_h),\ \forall h \in (0,1].$$

Proof. Let $\kappa \in \mathcal{T}_h$, $\hat\kappa$ its corresponding reference element and $F_\kappa\colon \hat\kappa \to \kappa$ the associated affine mapping. We set $J = |\det(DF_\kappa)|$. Using the Hypothesis C.2, we have $C^{-1}h_\kappa^N \le J \le Ch_\kappa^N$, for some constant C which is independent of κ. Let $K > 0$, then we have

$$\int_\kappa \left(\frac{|u|}{K}\right)^{p(x)} dx = \int_{\hat\kappa} \left(\frac{|u\circ F_\kappa|}{K}\right)^{p\circ F_\kappa(x)} J\,dx \le Ch_\kappa^N \int_{\hat\kappa} \left(\frac{|u\circ F_\kappa|}{K}\right)^{p\circ F_\kappa(x)} dx.$$

Thus,

$$\|(Ch_\kappa^N)^{-1/p(x)}u\|_{L^{p(\cdot)}(\kappa)} \le \|u\circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat\kappa)}.$$

Using that $h_\kappa \ll 1$, we obtain

$$\|u\|_{L^{p(\cdot)}(\kappa)} \le (Ch_\kappa^N)^{1/p_+} \|u\circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat\kappa)}. \tag{C.1}$$

Similarly, we have

$$\|u\circ F_\kappa\|_{L^{q\circ F_\kappa(\cdot)}(\hat\kappa)} \le (Ch_\kappa^{-N})^{1/q_-} \|u\|_{L^{q(\cdot)}(\kappa)}. \tag{C.2}$$

As in a finite dimensional space, all the norms are equivalent, we have that there exist a constant $\bar{C}$ depending only on N such that,

$$\|u\circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat\kappa)} \le C\|u\circ F_\kappa\|_{L^{p_2}(\hat\kappa)} \le C\|u\circ F_\kappa\|_{L^{q_1}(\hat\kappa)} \le \bar{C}\|u\circ F\|_{L^{q(\cdot)}(\hat\kappa)}, \tag{C.3}$$

where in the first and last inequality we are using Theorem A.2.

Finally, by (C.1), (C.2) and (C.3) we arrive to the desired result. □

Lemma C.8. *If $p(x)$ is* log*-Hölder continuous then, for any $\kappa\in\mathcal{T}_h$ and $u\in S^k(\mathcal{T}_h)$ we have, for $e\in\mathcal{E}_h\cap\partial\Omega$,*

$$\|h_\kappa^{\frac{1}{p(x)}}u\|_{L^{p(\cdot)}(e\cap A)} \le C\|u\|_{L^{p(\cdot)}(\kappa\cap A)}.$$

for any $A\subset\overline{\Omega}$ such that $\mathcal{H}^{N-1}(e\cap A)>0$ where $C=C(p_1,p_2,N,\Omega)$.

In particular, we have,

$$\|u\|_{L^{p(\cdot)}(e\cap A)} \le Ch_\kappa^{-\frac{1}{p_-}}\|u\|_{L^{p(\cdot)}(\kappa\cap A)} \quad \forall u\in S^k(\mathcal{T}_h). \tag{C.4}$$

Proof. Let F_κ and $\hat\kappa$ be as in the proof of Lemma C.7 and let $\hat{e}=F_\kappa^{-1}(e)$ and $\hat{A}=F_\kappa^{-1}(A)$.

Therefore,

$$\int_{e\cap A}\left(\frac{|u(x)|}{k}\right)^{p(x)} dS \le Ch_\kappa^{N-1}\int_{\hat{e}\cap\hat{A}}\left(\frac{|u\circ F_\kappa(x)|}{k}\right)^{p\circ F_\kappa(x)} dS,$$

then

$$\|(C^{-1}h_\kappa)^{\frac{1}{p(x)}}\frac{u}{h_\kappa^{N/p(x)}}\|_{L^{p(\cdot)}(e\cap A)} \le \|u\circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{e}\cap\hat{A})},$$

Using Theorem A.2 and that all the norms are equivalent, we have

$$\|u\circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{e}\cap\hat{A})} \le C\|u\circ F_\kappa\|_{L^{p_2}(\hat{e}\cap\hat{A})} \le C\|u\circ F_\kappa\|_{L^{1}(\hat{e}\cap\hat{A})}.$$

On the other hand, by the local inverse estimation in page 13 in [4] we have,

$$\|u\circ F_\kappa\|_{L^{1}(\hat{e}\cap\hat{A})} \le C\|u\circ F_\kappa\|_{L^{1}(\hat\kappa\cap\hat{A})}.$$

Using again Theorem A.2, we obtain

$$\|u\circ F_\kappa\|_{L^{1}(\hat\kappa\cap\hat{A})} \le C\|u\circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat\kappa\cap\hat{A})}.$$

Using all the inequalities, we arrive at

$$\left\|h_\kappa{}^{\frac{1}{p(x)}}\frac{u}{h_\kappa^{N/p(x)}}\right\|_{L^{p(\cdot)}(e\cap A)} \le C\left\|\frac{u}{h_\kappa^{N/p(x)}}\right\|_{L^{p(\cdot)}(\kappa\cap A)}.$$

Finally, we obtain

$$\|h_\kappa{}^{\frac{1}{p(x)}}u\|_{L^{p(\cdot)}(e\cap A)} \le Ch_\kappa^{\frac{N(p_--p_+)}{p_-p_+}}\|u\|_{L^{p(\cdot)}(\kappa)},$$

By Remark A.9 we get

$$\|h_\kappa{}^{\frac{1}{p(x)}}u\|_{L^{p(\cdot)}(e\cap A)} \le Ce^{N\frac{C}{p_1^2}}\|u\|_{L^{p(\cdot)}(\kappa\cap A)}.$$

Now, equation (C.4) follows immediately. □

The next result establishes the existence of the local projector operator.

Lemma C.9. *For all $z \in \mathcal{N}_h$ there exists a lineal map $\pi_z : BV(\Omega) \to \mathbb{R}$ such that*

$$\|u - \pi_z(u)\|_{L^1(T_z)} \leq Ch_z|Du|(T_z) \quad \forall u \in BV(\Omega)$$

where C is a constant independent of h and z.

Proof. See Subsection 4.1 of [4]. □

References

1. D.N Arnold, F. Brezzi, B. Cockburn, and Marini, *Unified analysis of discontinuous galerkin methods for elliptic problems*, SIAM J. Num. Anal (2002), no. 39, 1749–1779.
2. Erik M. Bollt, Rick Chartrand, Selim Esedoḡlu, Pete Schultz, and Kevin R. Vixie, *Graduated adaptive image denoising: local compromise between total variation and isotropic diffusion*, Adv. Comput. Math. **31** (2009), no. 1-3, 61–85.
3. Susanne C. Brenner, *Poincaré-Friedrichs inequalities for piecewise H^1 functions*, SIAM J. Numer. Anal. **41** (2003), no. 1, 306–324 (electronic).
4. Annalisa Buffa and Christoph Ortner, *Compact embeddings of broken Sobolev spaces and applications*, IMA J. Numer. Anal. **29** (2009), no. 4, 827–855.
5. Antonin Chambolle and Pierre-Louis Lions, *Image recovery via total variation minimization and related problems*, Numer. Math. **76** (1997), no. 2, 167–188.
6. Ph. Ciarlet, *The finite element method for elliptic problems*, vol. 68, North-Holland, Amsterdam, 1978.
7. L Diening, *Theoretical and numerical results for electrorheological fluids,*, Ph.D. thesis, University of Freiburg, Germany (2002).
8. L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), no. 2, 245–253.
9. ______, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. **268** (2004), 31–43.
10. L. Diening, P. Hästö, and A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces*, Function Spaces, Differential Operators and Nonlinear Analysis, Milovy, Math. Inst. Acad. Sci. Czech Republic, Praha, 2005.
11. L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
12. Xianling Fan, *Boundary trace embedding theorems for variable exponent Sobolev spaces*, J. Math. Anal. Appl. **339** (2008), no. 2, 1395–1412.
13. Xianling Fan and Dun Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446.
14. J. Fernández Bonder, S. Martínez, and N Wolanski, *A free boundary problem for the $p(x)$-laplacian*, Nonlinear Analysis **72** (2010), no. 2, 1078–1103.
15. D. Goldfarb, *A family of variable metric updates derived by variational means*, **24** (1970), 23–26.
16. Petteri Harjulehto and Peter Hästö, *A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces*, Rev. Mat. Complut. **17** (2004), no. 1, 129–146.
17. Kováčik and Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J **41** (1991), 592–618.
18. S. Samko, *Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbf{R}^N)$*, Direct and inverse problems of mathematical physics (Newark, DE, 1997), Int. Soc. Anal. Appl. Comput., vol. 5, Kluwer Acad. Publ., Dordrecht, 2000, pp. 333–342.
19. D.F. Shanno, *Conditioning of quasi-newton methods for function minimization*, Mathematics of Computing **24** (1970), 647–656.

Departamento de Matemática, FCEyN
UBA (1428) Buenos Aires, Argentina.

E-mail address: ldpezzo@dm.uba.ar, aldoc7@dm.uba.ar, smartin@dm.uba.ar