Value sharing of meromorphic functions and Fang's problem

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Abstract

In this paper, we shall study the uniqueness problems on meromorphic functions sharing a polynomial. We give a complete answer to a problem posed by Fang Mingliang. Our results improve or generalize those given by Fang and Hua, Yang and Hua, Fang, Fang and Qiu, Lin and Yi, Zhang, Xu, et al.

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1 Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$. We assume the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a)=S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$, $b(z)$ be small functions of $f(z)$ and $g(z)$. We say that $f(z)$, $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$, $g(z)-a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $N_{k)}(r,\frac{1}{f-a})$ (or $\overline{N}_{k)}(r,\frac{1}{f-a})$) the counting function for zeros of $f-a$ with multiplicity $\leq k$ (ignoring multiplicities), and by $N_{(k}(r,\frac{1}{f-a})$ (or $\overline{N}_{(k}(r,\frac{1}{f-a})$) the counting function for zeros of $f-a$ with multiplicity $\geq k$ (ignoring multiplicities). Moreover we set $N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+\overline{N}_{(3}(r,\frac{1}{f-a})+\cdots+\overline{N}_{(k}(r,\frac{1}{f-a})$.

We say that a finite value z_0 is called a fixed point of f if $f(z_0)=z_0$ or z_0 is a zero of $f(z)-z$.

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The following well known theorem in value distribution theory was posed by Hayman and settled by several authors almost at the same time [1, 3].

Theorem A: *Let $f(z)$ be a transcendental meromorphic function, $n \geq 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [4], Yang and Hua [15] got a unicity theorem respectively corresponding to Theorem A.

Theorem B: *Let f and g be two non-constant entire (meromorphic) functions, $n \geq 6(n \geq 11)$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Note that $f^n(z)f'(z) = \frac{1}{n+1}(f^{n+1}(z))'$, Fang [5] considered the case of kth derivative and proved

Theorem C: *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem D: *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 8$. If $(f^n(z)(f(z) - 1))^{(k)}$ and $(g^n(z)(g(z) - 1))^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Corresponding to uniqueness of entire or meromorphic functions sharing fixed points, Fang and Qiu [6] obtained the following result.

Theorem E: *Let f and g be two non-constant meromorphic (entire) functions, $n \geq 11(n \geq 6)$ a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Lin and Yi [8] obtained:

Theorem F: *Let f and g be two non-constant entire functions, and let $n \geq 7$ be a positive integer . If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share z CM, then $f \equiv g$.*

Zhang [18] extended Theorems E and F as follows.

Theorem G: *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then either*
(1) $k = 1$, $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$, or

(2) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^n = 1$.

Theorem H: *Let* f *and* g *be two non-constant entire functions, and let* n, k *be two positive integers with* $n > 2k+6$*. If* $(f^n(f-1))^{(k)}$ *and* $(g^n(g-1))^{(k)}$ *share* z *CM, then* $f \equiv g$.

Regarding Theorems G and H, Xu et al. [13] considered the case of meromorphic functions. They got

Theorem I: *Let* f *and* g *be two non-constant meromorphic functions, and let* n, k *be two positive integers with* $n > 3k+10$*. If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* z *CM,* f *and* g *share* ∞ *IM, then either* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and* c *are three constants satisfying* $4(c_1c_2)^n(nc)^2 = -1$*, or* $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^n = 1$.

Theorem J: *Let* f *and* g *be two non-constant meromorphic functions satisfying* $\Theta(\infty, f) > 2/n$*, and let* n, k *be two positive integers with* $n > 3k+12$*. If* $(f^n(f-1))^{(k)}$ *and* $(g^n(g-1))^{(k)}$ *share* z *CM,* f *and* g *share* ∞ *IM, then* $f \equiv g$.

Corresponding to Theorems C and D, Professor Fang Mingliang posed the following problem in a conference at Shanghai in 2009.

Problem 1.1. *Does Theorem C or D hold if* f *and* g *are meromorphic functions?*

Remark 1.1. Problem 1.1 seems to have been solved by Bhoosnurmath and Dyavanal [2], but their proofs contain some gaps that were pointed out by Zhang [18, Annex remarks], Xu et al [13, Remark 2], respectively. The gaps have not been filled as far as we know. Here we use different methods from theirs to fill these gaps and thus give a complete answer to Problem 1.1.

Considering Theorems I and J, one can also ask the following

Problem 1.2. *Does Theorem I or J hold without the condition "*f *and* g *share* ∞ *IM" ?*

Actually, in this paper, we consider some problems that are more general than the above two. Now we state our results as follows.

Theorem 1.2. *Let* f *and* g *be two transcendental meromorphic functions,* $p(z)$ *be a nonzero polynomial with* $\deg(p) = l \leq 5$*. Let* n, k *be two positive integers with* $n > 3k+8$*. If* $[f^n]^{(k)}$ *and* $[g^n]^{(k)}$ *share* p *CM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^n = 1$*;*
(2) if $p(z)$ *is not a constant, then* $f = c_1 e^{cQ(z)}$, $g = c_2 e^{-cQ(z)}$*, where* $Q(z) = \int_0^z p(z)\,dz$, c_1, c_2 *and* c *are constants such that* $(c_1c_2)^n(nc)^2 = -1$*,*
if $p(z)$ *is a nonzero constant* b*, the transcendental restriction on* f *and* g *can be removed, and then* $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$*, where* c_3, c_4 *and* d *are constants such that* $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

Remark 1.3. Theorem 1.2 affirmatively answered Problems 1.1 and 1.2 that Theorems C and I hold for the case of meromorphic functions. But unfortunately, Theorems D and J fail if f and g are meromorphic functions without the condition $\Theta(\infty, f) > 2/n$, even if f and g share ∞ CM. We give the following counterexample.

Example 1.1. *Let*

$$f(z) = \frac{h(z)(1-h^n(z))}{1-h^{n+1}(z)}, \quad g(z) = \frac{1-h^n(z)}{1-h^{n+1}(z)}, \tag{1.1}$$

where n is a positive integer and $h(z)$ is a non-constant meromorphic function.

We deduce from (1.1) that $f^n(f-1) = g^n(g-1)$, thus f and g satisfy the conditions of Theorem D or J, but $f \not\equiv g$.
Note that

$$T(r,f) = T(r,gh) = nT(r,h) + S(r,f).$$

By the second fundamental theorem, we deduce

$$\overline{N}(r,f) = \sum_{j=1}^{n} \overline{N}(\frac{1}{h-a_j}) \geq (n-2)T(r,h) = nT(r,h) + S(r,f),$$

where $a_j(\neq 1)$ $(j = 1, 2, \cdots, n)$ are distinct roots of the algebraic equation $h^{n+1} = 1$. Therefore,

$$\Theta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \leq 2/n.$$

Thus Theorem J is the best possible in some sense, at least for the case $\Theta(\infty, f) > 2/n$.

Corresponding to Theorem 1.2, one may pose the following problem.

Problem 1.3. *Can the condition "transcendental" be removed in Theorem 1.2 when $p(z)$ is a nonconstant polynomial with* $\deg(p) = l \leq 5$*?*

We give an affirmative answer to Problem 1.3 and get

Theorem 1.4. *Let f and g be two non-constant meromorphic functions, $p(z)$ be a nonzero polynomial with* $\deg(p) = l \leq 5$*. Let n, k be two positive integers with $n > 3k + 3l + 8$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share p CM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)}$, $g = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)\,dz$, c_1, c_2 and c are constants such that $(c_1c_2)^n(nc)^2 = -1$,
if $p(z)$ is a nonzero constant b, then $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

It's easy to obtain a uniqueness theorem of meromorphic functions concerning fixed points.

Corollary 1.5. *Let f and g be two non-constant meromorphic functions, n, k be two positive integers with $n > 3k + 11$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share z CM, then*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$; or
(2) $f = c_1e^{cz^2}$, $g = c_2e^{-cz^2}$, where c_1, c_2 and c are constants such that $(c_1c_2)^n(nc)^2 = -1$.

Remark 1.6. By using Theorem 1.2, one can improve Theorem 1 of [10] by removing the conditions "$f \neq \infty$ and $g \neq \infty$" in (ii).

Remark 1.7. In Theorem 1.2, if $p(z)$ is replaced by a small function of f, one can not easily get the representation of $f(z)$ and $g(z)$ like (2). Wang and Gao [12, Remark 3.1, Examples 3.2–3.4] gave some examples at the end of their paper to discuss the problem.

Remark 1.8. From the proof of Theorem 1.2 or 1.4, one can see that the computation will be very complicated when $\deg(p)$ becomes large, so we are not sure whether Theorem 1.2 or 1.4 holds for the general polynomial $p(z)$. Nevertheless, Theorems 1.2 and 1.4 improve or generalize the previous results such as Theorems B, C, E, G and I.

2 Preliminary lemmas and a main proposition

Lemma 2.1. *[14] Let $f(z)$ be a non-constant meromorphic function and let $a_0(z), a_1(z), \cdots, a_n(z)(\not\equiv 0)$ be small functions of f. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. *[7, 17, 16] Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f^{(k)}-c}) - N(r,\frac{1}{f^{(k+1)}}) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f^{(k)}-c}) - N_0(r,\frac{1}{f^{(k+1)}}) + S(r,f), \end{aligned}$$

where $N_0(r, \frac{1}{f^{(k+1)}})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.3. *[9] Let $f(z)$ be a non-constant meromorphic function, s, k be two positive integers. Then*

$$N_s(r,\frac{1}{f^{(k)}}) \leq T(r,f^{(k)}) - T(r,f) + N_{s+k}(r,\frac{1}{f}) + S(r,f),$$

$$N_s(r,\frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{s+k}(r,\frac{1}{f}) + S(r,f).$$

Lemma 2.4. *[17] Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r,\frac{1}{f^{(k)}}) \leq N(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.5. *[15] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and n, k be two positive integers, a be a finite nonzero constant. If f and g share a CM, then one of the following cases holds:*
(i) $T(r,f) \leq N_2(r,1/f) + N_2(r,1/g) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$, the same inequality holding for $T(r,g)$;
(ii) $fg \equiv a^2$; (iii) $f \equiv g$.

Lemma 2.6. *Let f, g be non-constant meromorphic functions, n, k be two positive integers with $n > k+2$, $a(z)(\not\equiv 0, \infty)$ be a small function of f. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a IM, then $T(r,f) = O(T(r,g))$, $\qquad T(r,g) = O(T(r,f))$.*

Proof. Let $F = f^n$. By the second fundamental theorem for small functions, we have

$$T(r, F^{(k)}) \leq \overline{N}(r,f) + \overline{N}(r, \frac{1}{F^{(k)}}) + \overline{N}(r, \frac{1}{F^{(k)} - a}) + S(r,F). \tag{2.1}$$

By (2.1) and Lemma 2.1 and Lemma 2.3 with $s = 1$ applied to F, we have

$$\begin{aligned} nT(r,f) &\leq N_{k+1}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F^{(k)} - a}) + \overline{N}(r,f) + S(r,F) \\ &\leq (k+1)\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{[f^n]^{(k)} - a}) + \overline{N}(r,f) + S(r.f) \\ &\leq (k+2)T(r,f) + \overline{N}(r, \frac{1}{[g^n]^{(k)} - a}) + S(r,f). \end{aligned}$$

Namely,

$$\begin{aligned} (n-k-2)T(r,f) &\leq \overline{N}(r, \frac{1}{[g^n]^{(k)} - a}) + S(r,f) \\ &\leq n(k+1)T(r,g) + S(r,f). \end{aligned}$$

Since $n > k+2$, we have $T(r,f) = O(T(r,g))$. Similarly we have $T(r,g) = O(T(r,f))$. This completes the proof of Lemma 2.6.

By the arguments similar to the proof of Lemma 2.6, we get the following proposition.

Proposition 2.1. *Let f be a transcendental meromorphic function, n, k be two positive integers with $n > k+2$, $a(z)(\not\equiv 0, \infty)$ be a small function of f. Then $[f^n]^{(k)} - a(z)$ has infinitely many zeros.*

Lemma 2.7. *[13] Let f and g be two non-constant meromorphic functions, $k, n > 2k+1$ be two positive integers. If $[f^n]^{(k)} = [g^n]^{(k)}$, then $f = tg$ for a constant t such that $t^n = 1$.*

Lemma 2.8. *[17, Theorem 4.8] Let F and G be two distinct nonconstant meromorphic functions, and let c be a complex number such that $c \neq 0$, 1. If F and G share 1 and c IM, and if $\overline{N}(r, 1/F) + \overline{N}(r,F) = S(r,F)$ and $\overline{N}(r, 1/G) + \overline{N}(r,G) = S(r,G)$, then F and G share 0, 1, c, ∞ CM.*

Lemma 2.9. *[11] If f and g are distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3, a_4 CM, then f is a Möbius transformation of g; two of the shared values, say a_1 and a_2 are Picard exceptional values, and the cross ratio $(a_1, a_2, a_3, a_4) = 1$.*

Lemma 2.10. *[7, Theorem 3.10] Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N(r, 1/f) + N(r, 1/f^{(k)}) = S(r, f'/f),$$

then $f = e^{az+b}$, where $a \neq 0$, b are constants.

Lemma 2.11. *Let $p(z), q(z), r(z)$ be three polynomials satisfying*

$$p^2(z) - q^2(z) = r^2(z). \tag{2.2}$$

If $\deg(p) = \deg(r) > 2\deg(q)$, *then* $q(z) \equiv 0$.

Proof. Suppose, to the contrary, that $q(z) \not\equiv 0$, then $p^2(z) \not\equiv r^2(z)$, namely, $p(z) + r(z) \not\equiv 0$ and $p(z) - r(z) \not\equiv 0$. Rewrite (2.2) as

$$q^2(z) = p^2(z) - r^2(z) = (p(z) + r(z))(p(z) - r(z)). \tag{2.3}$$

It's easy to obtain from (2.3) that $2\deg(q) = \deg(q^2) \geq \deg(p) > 2\deg(q)$, which is a contradiction.
This completes the proof of Lemma 2.11.

Lemma 2.12. *Let f, g be two transcendental meromorphic functions, $p(z)$ be a nonzero polynomial with* $\deg(p) = l \leq 5$, *n, k be two positive integers with $n > 3k + 8$. If* $[f^n]^{(k)}[g^n]^{(k)} = p^2$,
(i) if $p(z)$ is not a constant, then $f = c_1 e^{cQ(z)}$, $g = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)\,dz$, c_1, c_2 and c are constants such that $(c_1 c_2)^n (nc)^2 = -1$,
(ii) if $p(z)$ is a nonzero constant b, the transcendental restriction on f and g can be removed, and then $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$.

Proof.
Case 1. $p(z)$ is not a constant. First, we prove

$$N(r, 1/f) + N(r, 1/g) = O(\log r). \tag{2.4}$$

Suppose that z_0 is a zero of f with multiplicity s, if z_0 is a pole of g with multiplicity t, but not a zero of $p(z)$, then z_0 is a zero of $[f^n]^{(k)}$ with multiplicity $ns - k$, a pole of $[g^n]^{(k)}$ with multiplicity $nt + k$, thus we have

$$ns - k = nt + k,$$

namely

$$n(s - t) = 2k. \tag{2.5}$$

Note that $n > 3k + 8$ and we get a contradiction from (2.5). Thus z_0 is a zero of $p(z)$ and we have $N(r, 1/f) = O(\log r)$. Similarly, we get $N(r, 1/g) = O(\log r)$. Thus (2.4) holds.
Next we prove

$$N(r, f) = S(r, f), \quad N(r, g) = S(r, g). \tag{2.6}$$

Rewrite $[f^n]^{(k)}[g^n]^{(k)} = p^2$ as

$$[f^n]^{(k)} = \frac{p^2}{[g^n]^{(k)}}. \tag{2.7}$$

We deduce from (2.7) that

$$N(r, [f^n]^{(k)}) = N(r, \frac{1}{[g^n]^{(k)}}). \tag{2.8}$$

As $N(r, [f^n]^{(k)}) = nN(r, f) + k\overline{N}(r, f)$, this together with (2.4), (2.8) and Lemma 2.4 implies that

$$nN(r, f) + k\overline{N}(r, f) \le k\overline{N}(r, g) + O(\log r) + S(r, g). \tag{2.9}$$

Similarly we get

$$nN(r, g) + k\overline{N}(r, g) \le k\overline{N}(r, f) + O(\log r) + S(r, f). \tag{2.10}$$

Note that f and g are transcendental, combining (2.9) and (2.10) yields

$$N(r, f) + N(r, g) = S(r, f) + S(r, g). \tag{2.11}$$

By Lemma 2.6 we have $S(r, f) = S(r, g)$, thus we obtain (2.6). Let

$$F_1 = \frac{[f^n]^{(k)}}{p}, \quad G_1 = \frac{[g^n]^{(k)}}{p}. \tag{2.12}$$

Then

$$S(r, F_1) = S(r, f), \quad S(r, G_1) = S(r, g), \tag{2.13}$$

and

$$F_1 G_1 = 1. \tag{2.14}$$

Obviously, $F_1 \not\equiv G_1$, or else we get that F_1 is a constant, thus f is a polynomial, which contradicts our assumption.
By (2.6), (2.12), (2.13) and Lemma 2.4 we get

$$N(r, 1/F_1) \le nN(r, 1/f) + k\overline{N}(r, f) + S(r, f) \le S(r, F_1). \tag{2.15}$$

Similarly we have

$$N(r, 1/G_1) \le nN(r, 1/g) + k\overline{N}(r, g) + S(r, g) \le S(r, G_1). \tag{2.16}$$

Moreover, we have

$$N(r, F_1) = S(r, F_1), \quad N(r, G_1) = S(r, G_1). \tag{2.17}$$

It follows from (2.15)–(2.17) that

$$N(r, 1/F_1) + N(r, F_1) = S(r, F_1), \quad N(r, 1/G_1) + N(r, G_1) = S(r, G_1). \tag{2.18}$$

In view of (2.14), we know that F_1 and G_1 share 1 and -1 IM, this together with (2.18) and Lemma 2.8 implies that F_1 and G_1 share 1, -1, 0, ∞ CM, thus by Lemma 2.9 we get that 0 and ∞ are Picard values of F_1 and G_1. We deduce from (2.12) that both f and g are transcendental entire functions, by (2.4) we have

$$f(z) = P_1(z)e^{\alpha(z)}, g(z) = Q_1(z)e^{\beta(z)}, \tag{2.19}$$

where $P_1(z)$, $Q_1(z)$ are nonzero polynomials, $\alpha(z)$, $\beta(z)$ are non-constant entire functions. If $P_1(z)$ is not a constant, suppose that z_1 is a zero of f with multiplicity m, then z_1 is a zero of $[f^n]^{(k)}$ with multiplicity $nm - k(> m(3k+8) - k \geq 2k + 8 \geq 10)$, and is a zero of $p^2(z)$ with multiplicity no greater than 10 since $l \leq 5$, which leads to a contradiction. Thus $P_1(z)$ is a constant. Similarly, $Q_1(z)$ is a constant. Without loss of generality, rewrite f and g as follows.

$$f(z) = e^{\alpha(z)}, \quad g(z) = e^{\beta(z)}. \tag{2.20}$$

Then

$$T(r, \frac{(f^n)'}{f^n}) = T(r, n\alpha').$$

We claim that $\alpha + \beta \equiv C$, where C is a constant.
We deduce from (2.20) that either both α and β are transcendental functions or both α and β are polynomials. Moreover, we have

$$N(r, 1/[f^n]^{(k)}) \leq N(r, 1/p^2(z)) = O(\log r).$$

From this and (2.20) we get

$$N(r, f^n) + N(r, 1/f^n) + N(r, 1/[f^n]^{(k)}) = O(\log r).$$

If $k \geq 2$, suppose that α is a transcendental entire function. We deduce from Lemma 2.10 that α is a polynomial, which is a contradiction.
Thus α is a polynomial and so is β.
We deduce from (2.20) that

$$(f^n)^{(k)} = A[(\alpha')^k + P_{k-1}(\alpha')]e^{n\alpha}, \quad (g^n)^{(k)} = B[(\beta')^k + Q_{k-1}(\beta')]e^{n\beta},$$

where A, B are nonzero constants, $P_{k-1}(\alpha')$ and $Q_{k-1}(\beta')$ are differential polynomials in α' and β' of degree at most $k-1$ respectively. Thus we obtain

$$AB[(\alpha')^k + P_{k-1}(\alpha')][(\beta')^k + Q_{k-1}(\beta')]e^{n(\alpha+\beta)} = p^2(z). \tag{2.21}$$

We deduce from (2.21) that $\alpha(z) + \beta(z) \equiv C$ for a constant C.
If $k = 1$, from (2.21) we get

$$AB\alpha'\beta' e^{n(\alpha+\beta)} = p^2(z). \tag{2.22}$$

Let $\alpha + \beta = \gamma$. If α and β are transcendental entire functions, obviously γ is not a constant, then (2.22) implies that

$$AB\alpha'(\gamma' - \alpha')e^{n\gamma} = p^2(z). \tag{2.23}$$

Since $T(r,\gamma') = m(r,\gamma') \leq m(r, \frac{(e^{n\gamma})'}{e^{n\gamma}}) + O(1) = S(r, e^{n\gamma})$. Thus (2.23) implies that

$$\begin{aligned} T(r, e^{n\gamma}) &\leq T(r, \frac{p^2}{\alpha'(\gamma' - \alpha')}) + O(1) \\ &\leq (2 + o(1))T(r, \alpha') + S(r, e^{n\gamma}), \end{aligned}$$

which implies that

$$T(r, e^{n\gamma}) = O(T(r, \alpha')),$$

similarly we have

$$T(r, \alpha') = O(T(r, e^{n\gamma})).$$

Thus $T(r,\gamma') = S(r, e^{n\gamma}) + O(1) = S(r,\alpha')$.
In view of (2.23) and by the second fundamental theorem for small functions, we get

$$T(r,\alpha') \leq \overline{N}(r, \frac{1}{\alpha'}) + \overline{N}(r, \frac{1}{\alpha' - \gamma'}) + S(r,\alpha') \leq O(\log r) + S(r,\alpha').$$

Thus α' is a polynomial, which contradicts that α is a transcendental entire function.
Thus α and β are both polynomials and $\alpha(z) + \beta(z) \equiv C$ for a constant C.
Hence from (2.21) we get

$$C_1(\alpha')^{2k} = p^2 + \widetilde{P}_{2k-1}(\alpha'), \tag{2.24}$$

where C_1 is a nonzero constant and $\widetilde{P}_{2k-1}$ is a differential polynomial in α' of degree at most $2k-1$. Since $p(z)$ is not a constant. thus α' is a non-constant polynomial.
If $k \geq 2$, next we distinguish into five subcases below.
Subcase 1. $l = 1$. Since α' is not a constant, $\deg(\alpha') \geq 1$, by (2.24) we immediately get a contradiction.
Subcase 2. $l = 2$. Since $k \geq 2$, by (2.24) we get $\deg(\alpha') = 1$ and $k = 2$, thus α'' is a nonzero constant. From (2.21) we get

$$K[(n\alpha')^2 + n\alpha''][(n\beta')^2 + n\beta''] = p^2, \tag{2.25}$$

where K is a nonzero constant. Note that $\alpha + \beta \equiv C$, then $\alpha' + \beta' \equiv 0$ and $\alpha'' + \beta'' \equiv 0$. From (2.25) we obtain

$$K[((n\alpha')^2)^2 - (n\alpha'')^2] = p^2. \tag{2.26}$$

By Lemma 2.11, we derive $\alpha'' \equiv 0$ from (2.26), which is a contradiction.
Subcase 3. $l = 3$. Similarly as above, we get $\deg(\alpha') = 1$ and $k = 3$, thus α'' is a nonzero constant. From (2.21) we get

$$K_1[n^3(\alpha')^3 + 3n^2\alpha'\alpha''][n^3(\beta')^3 + 3n^2\beta'\beta''] = p^2, \tag{2.27}$$

where K_1 is a nonzero constant. Thus we have

$$-K_1[((n\alpha')^3)^2 - (3n^2\alpha'\alpha'')^2] = p^2. \tag{2.28}$$

By Lemma 2.11, we arrive at the same contradiction.
Subcase 4. $l = 4$. Similarly as above, we get either $\deg(\alpha') = 1$ and $k = 4$ or $\deg(\alpha') = 2$ and $k = 2$.
If $\deg(\alpha') = 1$ and $k = 4$, then α'' is a nonzero constant. From (2.21) we get

$$[(n\alpha')^4 + 3(n\alpha'')^2]^2 - [6n^3(\alpha')^2\alpha'']^2 = p^2. \tag{2.29}$$

Without loss of generality, suppose that $\alpha' = z$, or else, we only need to do a transformation of $p(z)$. We deduce from (2.29) that

$$(nz)^8 - 30n^6z^4 + 9n^4 = p^2(z), \tag{2.30}$$

which implies $p^2(z) = p^2(-z)$, thus $p(z) = p(-z)$ or $p(z) = -p(-z)$. Note that $l = 4$, thus $p(z) = p(-z)$. Suppose that $p(z) = a_4z^4 + a_2z^2 + a_0$, where $a_4 \neq 0, a_2, a_0$ are constants. Compare the coefficients at both sides of (2.30), we get $a_2 = 0$, at last we derive a contradiction by calculation.
If $\deg(\alpha') = 2$ and $k = 2$, then we get (2.26). By Lemma 2.11, we arrive at a contradiction.
Subcase 5. $l = 5$. Similarly as above, we get $\deg(\alpha') = 1$ and $k = 5$.
From (2.21) we get

$$[10n^4(\alpha')^3\alpha'' + 12n^3\alpha'\alpha'']^2 - [(n\alpha')^5 + 3n^3\alpha'(\alpha'')^2]^2 = p^2. \tag{2.31}$$

With similar discussion as in Subcase 4, we get a contradiction.
Hence $k = 1$. by induction we get

$$\begin{aligned} &\alpha' + \beta' \equiv 0, \\ &n^2e^{nC}\alpha'\beta' = p^2(z). \end{aligned}$$

By computation we get

$$\alpha' = cp(z),\ \beta' = -cp(z), \tag{2.32}$$

Hence

$$\alpha = cQ(z) + l_1,\ \beta = -cQ(z) + l_2, \tag{2.33}$$

where $Q(z)$ is defined as in Theorem 1.2, and l_1, l_2 are constants. We can rewrite f and g as

$$f = c_1e^{cQ(z)}, \qquad g = c_2e^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $(c_1c_2)^n(nc)^2 = -1$.
Case 2. If $p(z)$ is a nonzero constant b, similarly to the proof in Case 1, we deduce that α' is a nonzero constant, thus $\alpha = dz + l_3$, $\beta = -dz + l_4$.
Rewrite f and g as

$$f = c_3e^{dz}, \qquad g = c_4e^{-dz},$$

where c_3, c_4 and d are nonzero constants. We deduce that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

This completes the proof of Lemma 2.12.

Lemma 2.13. *Let f, g be two non-constant rational functions, $p(z)$ be a nonzero polynomial with $\deg(p) = l$, n, k be two positive integers with $n > 3k+3l+8$. Then there are no solutions of the functional differential equation of the following form*

$$[f^n]^{(k)}[g^n]^{(k)} = p^2. \tag{2.34}$$

Proof.
Suppose, to the contrary, that there exist non-constant rational solutions of Equation (2.34). Suppose that z_2 is a zero of f with multiplicity p_2, then z_2 is a zero of $[f^n]^{(k)}$ with multiplicity $np_2 - k$, if z_2 is not a pole of g, since $n > 3k + 3l + 8$, we get that z_2 must be a zero of p^2. Since $np_2 - k > 2k + 8 + 3l > 2l$, we get a contradiction. Therefore, z_2 must be a pole of g with multiplicity q_2, and is a pole of $[g^n]^{(k)}$ with multiplicity $nq_2 + k$, obviously $p_2 > q_2$, or else, z_2 is a pole of p, which is a contradiction since p is a polynomial. Note that $n(p_2 - q_2) - 2k > k + 3l + 8 > 2l$, we get that z_2 is a zero of p^2 with multiplicity greater than $2l$, which is a contradiction, thus f has no zero. Similarly, g has no zero. Set

$$f(z) = 1/R(z), \quad g(z) = 1/K(z), \tag{2.35}$$

where $R(z)$ and $K(z)$ are non-constant polynomials. We deduce from (2.35) that

$$[f^n(z)]^{(k)} = R_1(z)/R_2(z), \quad [g^n(z)]^{(k)} = K_1(z)/K_2(z), \tag{2.36}$$

where $R_1(z)$, $R_2(z)$, $K_1(z)$ and $K_2(z)$ are non-constant polynomials such that $\deg(R_2) > \deg(R_1)$, $\deg(K_2) > \deg(K_1)$, combining this with (2.34) leads to a contradiction.
This completes the proof of Lemma 2.13.

3 Proof of Theorem 1.2

Let $F = [f^n]^{(k)}$, $G = [g^n]^{(k)}$, $F^* = f^n$, $G^* = g^n$, $F^\star = F/p$, $G^\star = G/p$, then $F^\star$ and $G^\star$ share 1 CM.
Since p is a small function of f. by Lemma 2.8, p is a small function of g. Thus by Lemma 2.5, one of the following cases holds:
(i) $T(r, F^\star) \le N_2(r, 1/F^\star) + N_2(r, 1/G^\star) + N_2(r, F^\star) + N_2(r, G^\star) + S(r, F^\star) + S(r, G^\star)$, the same inequality holding for $T(r, G^\star)$;
(ii) $FG \equiv p^2$; (iii) $F \equiv G$.

Case (i)**.** by Lemma 2.1 and Lemma 2.3 with $s = 2$, we obtain

$$\begin{aligned} T(r, F^*) &\le N_{k+2}(r, 1/F^*) + N_{k+2}(r, 1/G^*) + (k+2)\overline{N}(r, g) + 2\overline{N}(r, f) \\ &\quad + S(r, f) + S(r, g) \\ &\le (k+2)\overline{N}(r, 1/f) + (k+2)\overline{N}(r, 1/g) + (k+2)\overline{N}(r, g) + 2\overline{N}(r, f) \\ &\quad + S(r, f) + S(r, g) \\ &\le (2k+4)T(r, g) + (k+4)T(r, f) + S(r, f) + S(r, g), \end{aligned}$$

namely

$$nT(r, f) \le (2k+4)T(r, g) + (k+4)T(r, f) + S(r, f) + S(r, g). \tag{3.1}$$

Similarly we have

$$nT(r,g) \le (2k+4)T(r,f) + (k+4)T(r,g) + S(r,f) + S(r,g). \tag{3.2}$$

From (3.1) and (3.2) we deduce that

$$(n-3k-8)(T(r,f)+T(r,g)) \le S(r,f) + S(r,g), \tag{3.3}$$

which is a contradiction since $n > 3k+8$.

Case (ii)**.** We have $[f^n]^{(k)}[g^n]^{(k)} = p^2$. By Lemma 2.12 we get the conclusion (2) of Theorem 1.2.

Case (iii)**.** We have $[f^n]^{(k)} \equiv [g^n]^{(k)}$. By Lemma 2.7 we get the conclusion (1) of Theorem 1.2.

This completes the proof of Theorem 1.2.

4 Proof of Theorem 1.4

Let F, G, F^*, G^*, $F^\star$, $G^\star$ be defined as in Section 3, then $F^\star$ and $G^\star$ share 1 CM. By Lemma 2.5, we consider three cases

Case 1. Note that $T(r,F) \le T(r,F^\star) + l\log r$, By the arguments similar to the proof of Case (i) in Theorem 1.2, we get

$$nT(r,f) \le (2k+4)T(r,g) + (k+4)T(r,f) + 3l\log r + S(r,f) + S(r,g). \tag{4.1}$$

and

$$nT(r,g) \le (2k+4)T(r,f) + (k+4)T(r,g) + 3l\log r + S(r,f) + S(r,g). \tag{4.2}$$

Since $T(r,f) \ge \log r + O(1)$ and $T(r,g) \ge \log r + O(1)$, combining this with (4.1) and (4.2) yields

$$(n-3k-3l-8)(T(r,f)+T(r,g)) \le S(r,f) + S(r,g), \tag{4.3}$$

which is a contradiction since $n > 3k+3l+8$.

Case 2. We have $[f^n]^{(k)}[g^n]^{(k)} = p^2$. It follows from Lemma 2.13 that f and g are both transcendental meromorphic functions, by Lemma 2.12, we get the conclusion (2) of Theorem 1.4.

Case 3. We have $[f^n]^{(k)} \equiv [g^n]^{(k)}$. By Lemma 2.7 we get the conclusion (1) of Theorem 1.4.

This completes the proof of Theorem 1.4.

Annex remarks

In this section, we would like to point out another gap that appears in the proof of Theorem 4 of [2]. In [2, P. 1203], on the first line below formula (6.8), the authors said:
"Let z_1 be a zero of $f-1$ of order p_1, then z_1 is zero of $[f^n(f-1)]^{(k)}$ of order p_1-k. Therefore from (6.7), we obtain

$$p_1 - k = nq_1 + q_1 + k,$$

since z_1 is a pole of g of order q_1".
A question arises:
Question: If $p_1 \leq k$, then z_1 is not a zero of $[f^n(f-1)]^{(k)}$, and thus not a pole of g. How to deal with this case?

Open problem

Look forward this paper, there are two problems unsolved. For further study, we state them as follows.

Problem 4.1. *Does Theorem J hold without the condition "f and g share ∞ IM" ?*

Problem 4.2. *Does Theorem 1.4 hold for the general polynomial $p(z)$?*

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