Cone Normed Linear Spaces

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Abstract

In this paper, we introduce cone normed linear space, study the cone convergence with respect to cone norm. Finally, we prove the completeness of a finite dimensional cone normed linear space.

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1 Introduction

Let V be a real vector space and $(\| \cdot \|)$ be a norm defined on V. Our aim is to study cone normed linear spaces. First we go through the definition of Huang and Zhang [1];
Let E be a Banach space and C be a non empty subset of E. C is called a cone if and only if
(i) C is closed.
(ii) $a,\, b \in \mathbf{R},\, a, b \geq 0,\; x,\, y \in C \;\; \Rightarrow \; ax + by \in C$
(iii) $x \in C \; or \; - x \in C$ for every $x \in E$.

Now we define the order relation $' \leq '$ on E. For any given cone $C \subseteq E$, we define linearly ordered set C with respect to $' \leq '$ by $x \leq y$ if and only if $y - x \in C$. $x < y$ will indicate $x \leq y$ and $x \neq y$; while $x \ll y$ will stands for $y - x \in intC$, $intC$ denotes the interior of C.
The cone C is called normal cone if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \;\Leftrightarrow\; \|x\| \leq K \|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of C.

2 Convergence on Cone normed spaces

Unless otherwise stated throughout this paper we shall denote θ as the null element of E.

Definition 2.1 *Let V be a vector space over the field $\mathbf{R}$. The mapping $\|\cdot\|_c : V \longrightarrow E$ is said to be a cone norm if it satisfies the following conditions:*
(i) $\|x\|_c \geq \theta \ \ \forall x \in V,$
(ii) $\|x\|_c = \theta$ *if and only if* $x = \theta_V,$
(iii) $\|\alpha x\|_c = |\alpha| \|x\| \ \ \forall x \in V, \ \alpha \in \mathbf{R},$
(iv) $\|x + y\|_c \leq \|x\|_c + \|y\|_c \ \ \ \forall x, y \in V.$

Definition 2.2 *Let V be a vector space over the field $\mathbf{R}$. $\|\cdot\|_c$ is a cone norm on V. Then $(V, \|\cdot\|_c)$ is called a cone normed linear space.*

Definition 2.3 *A sequence $\{x_n\}_n$ in V is said to***convergence** *to a point $x \in V$ if for every $\epsilon \in E$ with $\epsilon \gg \theta$ there is a positive integer n_0 such that $\|x_n - x\|_c \ll \epsilon, \quad \forall\, n \geq n_0$. It will be denoted by $\lim_{n \to \infty} x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$*

Definition 2.4 *A sequence $\{x_n\}_n$ in V is said to be* **Cauchy sequence** *if for every $\epsilon \in E$ with $\epsilon \gg \theta$ there is a positive integer n_0 such that $\|x_n - x_m\|_c \ll \epsilon, \quad \forall\, m, n \geq n_0$..*

Theorem 2.5 *Let $(V, \|\cdot\|_c)$ be a cone normed linear space with normal constant K. Let $\{x_n\}_n$ be a sequence in V. Then x_n converges to x if and only if $\|x_n - x\|_c \longrightarrow \theta$.*

Proof. Let $\{x_n\}_n$ converges to x. For every real $p > 0$, choose $\epsilon \in E$ with $\epsilon \gg 0$ such that $K\|\epsilon\| < p$. Then there exist a positive integer n_0 such that $\|x_n - x\|_c \ll \epsilon, \ \forall\, n \geq n_0$.
So, $\|\|x_n - x\|_c\| \leq K\|\epsilon\| < p, \ \ \forall\, n \geq n_0$.
Thus $\|x_n - x\|_c \longrightarrow \theta$.
Conversely, suppose $\|x_n - x\|_c \longrightarrow \theta$. For $\epsilon \in E$ with $\epsilon \gg 0$, there is a positive number $K\|\epsilon\|$ such that $\|\|x_n - x\|_c\| \leq K\|\epsilon\|, \quad \forall\, n \geq n_0(\epsilon)$
$\Rightarrow \|x_n - x\|_c \leq \epsilon, \ \ \forall\, n \geq n_0$. Hence x_n converges to x.

Definition 2.6 *The set $B_c(x, r) = \{y \in C : \|y - x\| < r\}$ $, r \in \mathbf{R}$ is called an open ball in C with center at x and radius r.*

Lemma 2.7 *Let $x \in C$. Then for every $z \in C$, $B_c(x,r) + \{z\}$ is an open ball, Where $B_c(x,r) = \{y \in C : \|y - x\| < r\}$, $r \in \mathbf{R}$.*

Proof. $B_c(x,r) + \{z\} = \{z\} + \{y \in C : \|y - x\| < r\}$

$= \{z+y : \|y-x\| < r\} = \{p : \|p-z-x\| < r\}$

$= \{p : \|p-(z+x)\| < r\} = B_c(z+x, r)$

Hence the proof.

Corollary 2.8 *If C be a cone normed linear space then $C + intC \subseteq intC$.*

Lemma 2.9 *Let $x, y, \epsilon_1, \epsilon_2 \in E$ such that $x \ll \epsilon_1$ and $y \ll \epsilon_2$ then $x + y \ll \epsilon_1 + \epsilon_2$.*

Proof. Since $x \ll \epsilon_1 \Rightarrow \epsilon_1 - x \in intC$ and $y \ll \epsilon_2 \Rightarrow \epsilon_2 - y \in intC$
Now, $\epsilon_1 + \epsilon_2 - (x + y) = (\epsilon_1 - x) + (\epsilon_2 - y) \in intC + intC \subseteq C + intC$.
Hence by corollary 2.8 we have $\epsilon_1 + \epsilon_2 - (x + y) \in intC$.
Thus $x + y \ll \epsilon_1 + \epsilon_2$.

Theorem 2.10 *In a cone normed linear space $(V, \|\cdot\|_c)$, if $x_n \longrightarrow x$ and $y_n \longrightarrow y$ then $x_n + y_n \longrightarrow x + y$*

Proof. By lemma 2.9 the theorem directly follows.

Theorem 2.11 *In a cone normed linear space $(V, \|\cdot\|_c)$, if $x_n \longrightarrow x$ and real $\lambda_n \longrightarrow \lambda$ then $\lambda_n x_n \longrightarrow \lambda x$.*

Proof. Obvious.

Theorem 2.12 *In a cone normed linear space $(V, \|\cdot\|_c)$, if $\{x_n\}_n$ and $\{y_n\}_n$ are cauchy sequences then $\{x_n + y_n\}_n$ is a cauchy sequence.*

Proof. By lemma 2.9 the theorem directly follows.

Theorem 2.13 *In a cone normed linear space $(V, \|\cdot\|_c)$, if $\{x_n\}_n$ and $\{\lambda_n\}_n \in \mathbf{R}$ are cauchy sequences then $\{\lambda_n x_n\}_n$ is a cauchy sequence.*

Proof. Obvious.

Definition 2.14 *Let $(U, \|\cdot\|_c)$ and $(V, \|\cdot\|_c)$ be two cone normed linear spaces and $f : U \longrightarrow V$ be a function, then f is said to be* **cone continuous** *at a point $x_0 \in U$ if for any given $\epsilon \in E$ with $\epsilon \gg \theta$ there exists $\delta \in E$ with $\delta \gg \theta$ such that $\|x - x_0\|_c \ll \delta \Rightarrow \|f(x) - f(x_0)\|_c \ll \epsilon$.*

Lemma 2.15 *Let V be a cone normed linear space and $x, y \in V$ then $\|x\|_c - \|y\|_c \leq \|x - y\|_c$ and $\|y\|_c - \|x\|_c \leq \|x - y\|_c$.*

Proof. $\|x\|_c = \|x - y + y\|_c \leq \|x - y\|_c + \|y\|_c$.
$\Rightarrow \|x - y\|_c + \|y\|_c - \|x\|_c \in C$.
$\Rightarrow \|x - y\|_c - (\|x\|_c - \|y\|_c) \in C$.
$\Rightarrow \|x\|_c - \|y\|_c \leq \|x - y\|_c$.
Similarly, $\|y\|_c - \|x\|_c \leq \|x - y\|_c$.

Lemma 2.16 *Let $x, y, z \in E$ and $x \leq y \ll z$ then $x \ll z$*

Proof. Since $x \leq y \Rightarrow y - x \in C$ and $y \ll z \Rightarrow z - y \in intC$.
Now, $z-x = z-y+y-x \in C+intC$. Then there exist $c_1 \in C$ and $c_2 \in intC$ such that $z - x = c_1 + c_2$. Since $c_1 \in intC$ then there exists an open ball $B_c(c_1, r)$ such that $c_1 \in B_c(c_1, r) \subseteq C$. Therefore $c_1 + c_2 \in c_2 + B_c(c_1, r)$. Then by lemma 2.7, $c_1 + c_2 \in B_c(c_1 + c_2, r)$. That is $z - x \in B_c(c_1 + c_2, r)$. Hence $z - x \in intC$. That is, $x \ll z$.

Theorem 2.17 *The cone norm function $f : (U, \| \cdot \|_c) \to E$ is cone continuous.*

Proof. Let $f(x) = \|x\|_c \; \forall x \in U$ and let $x_0 \in U$. Since $\|x\|_c, \|x_0\|_c \in E$,
So, either $\|x\|_c \geq \|x_0\|_c$ or $\|x\|_c \leq \|x_0\|_c$.
So, either $\|x\|_c - \|x_0\|_c \in C$ or $\|x\|_c - \|x_0\|_c \in C$.
So, either $\|\|x\|_c - \|x_0\|_c\|_c = \|x\|_c - \|x_0\|_c$ or $\|\|x\|_c - \|x_0\|_c\|_c = \|x_0\|_c - \|x\|_c$.
So, by lemma 2.15 we have $\|\|x\|_c - \|x_0\|_c\|_c \leq \|x - x_0\|_c$.
Let us choose $\epsilon \in E$ with $\epsilon \gg \theta$.
Then, $\|f(x) - f(x_0)\|_c \ll \epsilon \Rightarrow \|\|x\|_c - \|x_0\|_c\|_c \leq \|x - x_0\|_c \ll \epsilon$.
That is, $\|x - x_0\| \ll \delta \Rightarrow \|f(x) - f(x_0)\|_c \ll \epsilon$ whenever $\delta \in E$ with $\delta \gg \theta$ and $\delta = \epsilon$.
Hence cone norm function is cone continuous.

Definition 2.18 *A cone normed linear space $(V, \| \cdot \|_c)$ is said to be cone complete if every cauchy sequence in V converges to a point of V.*

Theorem 2.19 *Let $(V, \| \cdot \|_c)$ be a cone normed linear space such that every cauchy sequence in V has a convergent subsequence then V is cone complete.*

Proof. Let $\{x_n\}_n$ be a cauchy sequence in V and $\{x_{n_k}\}_k$ be a convergent subsequence of $\{x_n\}_n$. Since $\{x_n\}_n$ is a cauchy sequence then for any $\epsilon_1 \in E$ with $\epsilon_1 \gg \theta$ there exists a positive integer n_1 such that

$$\|x_n - x_m\|_c \ll \epsilon_1 \quad \forall\, m, n \geq n_1$$

Let $\{x_{n_k}\}_k$ converges to x. then for any given $\epsilon_2 \in E$ with $\epsilon_2 \gg \theta$ there exists a positive integer n_2 such that

$$\|x_{n_k} - x\|_c \ll \epsilon_2 \quad \forall\, n_k \geq n_2.$$

Let $n_0 = max\{n_1, n_2\}$.
$\|x_n - x\|_c = \|x_n - x_{n_k} + x_{n_k} - x\|_c \leq \|x_n - x_{n_k}\|_c + \|x_{n_k} - x\|_c \ll \epsilon_1 + \epsilon_2 \quad \forall n, n_k \geq n_0$ (by lemma 2.9).
Hence the proof.

Theorem 2.20 *Let $(V, \|\cdot\|_c)$ be a cone normed linear space and C be a normal cone with normal constant K. Then every subsequence of a convergent sequence is convergent to the same limit.*

Proof. Let $\{x_n\}_n$ be a convergent sequence in V and converges to the point $x \in V$. Let $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$.
Let $\delta \in \mathbf{R}$ then there exist $\epsilon \in E$ with $\epsilon \gg \theta$ such that $K\|2\epsilon\| < \delta$.

Since x_n converges to x, then for this $\epsilon \in E$ $\exists$ a positive integer n_0 such that

$$\|x_n - x\|_c \ll \epsilon \quad \forall n \geq n_0$$

i.e., $\|x_{n_k} - x\|_c \ll \epsilon \quad \forall n_k \geq n_0$. Hence $x_{n_k} \to x \quad \forall n_k \geq n_0$.
If possible let $\{x_{n_k}\}_k$ converges to y also. Then $\exists$ a positive integer n_1 such that

$$\|x_{n_k} - y\|_c \ll \epsilon \quad \forall n_k \geq n_1.$$

Let $n_2 = \max\{n_0, n_1\}$. Now

$$\|x-y\|_c = \|x-x_{n_k}+x_{n_k}-y\|_c \leq \|x-x_{n_k}\|_c+\|x_{n_k}-y\|_c \ll \epsilon+\epsilon = 2\epsilon\ , \forall n_k \geq n_2$$

Therefore $\|\|x - y\|_c\| \leq K\|2\epsilon\| < \delta$. This implies that $\|\|x - y\|_c\| = 0$.
Hence the proof.

3 Finite dimensional cone normed linear spaces

Lemma 3.1 *Let $\{x_1, x_2, ..., x_n\}$ be a linearly independent subset of a cone normed linear space $(V, \|\cdot\|_c)$. C be a normal cone with normal constant K, then there exist an element $c \in intC$ such that for every set of real scalars $\alpha_1, \alpha_2, ..., \alpha_n$ we have*
$\|\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n\|_c \geq c(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|) \cdots\cdots\cdots (1)$

Proof: Let $\alpha = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$. If $\alpha = 0$ then each α_i is zero and hence (1) is true.
So we now assume that $\alpha > 0$. Then (1) becomes

$\|\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n\|_c \geq c \cdots\cdots\cdots (2)$
Where $\beta_i = \frac{\alpha_i}{\alpha}$ and $\sum_{i=1}^{n} |\beta_i| = 1$.
It is sufficient to prove that there exists an element $c \in intC$ such that (2) is true for any set of scalars $\beta_1, \beta_2, ..., \beta_n$ with $\sum_{i=1}^{n} |\beta_i| = 1$
If possible let this is not true. Then there exists a sequence $\{y_m\}_m \in V$ where
$y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + ... + \beta_n^{(m)} x_n$, with $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1$, m=1, 2, ...
such that $\|y_m\|_c \to \theta$ as $m \to \infty$
Since $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1$ for m= 1, 2, ... We have $|\beta_i^{(m)}| \leq 1$ for i = 1, 2, ... , n; m= 1, 2, ...
Hence for a fixed $i = 1, 2, ..., n$; the sequence $\{\beta_i^{(m)}\}_m$ is bounded. Therefore by Bolzano-weierstrass theorem $\{\beta_1^{(m)}\}_m$ has a subsequence converging to β_1(say), and let $\{y_{1,m}\}_m$ denote the corresponding subsequence of $\{y_m\}_m$. By the same reason the sequence $\{y_{1,m}\}_m$ has a subsequence $\{y_{2,m}\}_m$(say), for which the corresponding subsequence of real scalars $\{\beta_2^{(m)}\}_m$ converges to β_2(say). We continuing this process up to n-th stage. At the n-th stage, we obtain a sequence $\{y_{n,m}\} = \{y_{n,1}, y_{n,2}, ...\}$ of $\{y_m\}_m$ whose terms are of the form $y_{n,m} = \sum_{i=1}^{n} \delta_i^{(m)} x_i$, with $\sum_{i=1}^{n} |\delta_i^{(m)}| = 1$, m=1, 2, ...
where $\delta_i^m \to \beta_i$ as $m \to \infty, \ i = 1, 2, \cdots, n$.
So as $m \to \infty, y_{n,m} \to \sum_{i=1}^{n} \beta_i x_i = y(say)$ where $\sum_{i=1}^{n} |\beta_i| = 1$ by theorem 2.10 and theorem 2.11.
This implies that not all β_i can be zero. Since $\{x_1, x_2, ..., x_n\}$ is linearly independent. Therefore $y \neq \theta_V$. Now we show that $y_{n,m} \to y$ implies $\|y_{n,m}\|_c \to \|y\|_c$
For every real $\epsilon > 0$, choose $c \in E$ with $c \gg \theta$ and $K^2\|c\| < \epsilon$. Since $y_{n,m} \to y$ as $m \to \infty$,then for this element c we can find a positive integer n_0 such that $\|y_{n,m} - y\|_c \ll c, \forall m \geq n_0$.
Therefore $\| \|y_{n,m} - y\|_c\| \leq K\|c\|, \cdots\cdots\cdots (3)$
since $x \ll y \Rightarrow y - x \in intC \subseteq C \Rightarrow x \leq y$
By lemma 2.15 we have $\|y_{n,m}\|_c - \|y\|_c \leq \|y_{n,m} - y\|_c$

$$\Rightarrow \| \|y_{n,m}\|_c - \|y\|_c\| \leq K\| \|y_{n,m} - y\|_c\| \leq K.K\|c\| \quad \text{[by (3)]}$$
$$= K^2\|c\| < \epsilon, \quad \forall m \geq n_0$$

Hence $\|y_{n,m}\|_c \to \|y\|_c$ as $m \to \infty$
As $\{y_{n,m}\}_m$ is a subsequence of $\{y_m\}_m$ and $\|y_m\|_c \to \theta$ as $m \to \infty$
Therefore $\|y_{n,m}\|_c \to \theta$ as $m \to \infty$ and so $\|y\|_c = \theta$ which gives $y = \theta_V$.
This contradiction proves the lemma.

Theorem 3.2 *Every finite dimensional cone normed linear space with normal constant K is cone complete.*

Proof: Let $(V, \|\cdot\|_c)$ be a cone normed linear space and C be a normal cone with normal constant K. Let $\{x_n\}$ be an arbitrary cauchy sequence in V.

We should show that $\{x_n\}$ converges to some element $x \in V$. Suppose that the dimension of V is m and let $\{e_1, e_2, ..., e_m\}$ be a basis of V. Then each $\{x_n\}$ has a unique representation as $x_n = \alpha_1^{(n)}.e_1 + \alpha_2^{(n)}.e_2 + ... + \alpha_m^{(n)}.e_m$ where $\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_m^{(n)} \in \mathbf{R}$ and $n = 1, 2, \cdots$
Let $\delta \in \mathbf{R}$. Then there exist $\epsilon \in E$ with $\epsilon \gg \theta$ such that $\frac{\|\epsilon\|}{K\|c\|} < \delta$. Since $\{x_n\}$ is a cauchy sequence, then for this $\epsilon \in E$ there exist a positive integer n_0 such that $\|x_n - x_r\|_c \ll \epsilon$ for all $n, r \geq n_0$
By the above lemma it follows that there exist $c \in intC$ such that
$\epsilon \gg \|x_n - x_r\|_c = \| \sum_{i=1}^{m}(\alpha_i^{(n)} - \alpha_i^{(r)})e_j\|_c \geq c \sum_{i=1}^{m} |\alpha_i^{(n)} - \alpha_i^{(r)}|$
Therefore $\epsilon \gg c \sum_{i=1}^{m} |\alpha_i^{(n)} - \alpha_i^{(r)}|$
Therefore $\|\epsilon\| \geq K\|c \sum_{i=1}^{m} |\alpha_i^{(n)} - \alpha_i^{(r)}|\| = K\|c\|. \sum_{i=1}^{m} |\alpha_i^{(n)} - \alpha_i^{(r)}|$
Therefore $\sum_{i=1}^{m} |\alpha_i^{(n)} - \alpha_i^{(r)}| \leq \frac{\|\epsilon\|}{K\|c\|}$
$\Rightarrow |\alpha_i^{(n)} - \alpha_i^{(r)}| \leq \sum_{i=1}^{m} |\alpha_i^{(n)} - \alpha_i^{(r)}| \leq \frac{\|\epsilon\|}{K\|c\|} < \delta$
Therefore $\{\alpha_i^{(n)}\}_n$ is a cauchy sequence in $\mathbf{R}$ and therefore converges to a real number α_i, $i = 1, 2, ..., m$
We now define the element $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_m e_m$, which is clearly an element of V. Moreover, since $\alpha_i^n \to \alpha_i$ as $n \to \infty$ and $i = 1, 2, ..., m$
We have
$$\begin{aligned} \|x_n - x\|_c &= \| \textstyle\sum_{i=1}^{m}(\alpha_i^{(n)} - \alpha_i)e_i\|_c \\ &\leq \textstyle\sum_{i=1}^{m} |\alpha_i^{(n)} - \alpha_i|\|e_i\|_c \\ &\to \textstyle\sum_{i=1}^{m} 0 \cdot \|e_i\|_c \\ &\to \theta. \end{aligned}$$

This completes the proof.

Definition 3.3 *Let $p \in E$ with $\theta \ll p$ and $b \in V$. Define*

$$B_p(b) = \{x \in V : \|x - b\|_c \ll p\}.$$

Definition 3.4 *Let V be a cone normed linear space and P be a subset of V. P is said to be closed if for any sequence $\{x_n\}_n$ in P converges to $x \in P$.*

Definition 3.5 *Let V be a cone normed linear space. A subset Q of V is said to be the closure of $P(\subseteq V)$ if for any $x \in Q$ there exist a sequence $\{x_n\}_n$ in P such that $x_n \to x$ as $n \to \infty$ with respect to the cone norm.*

Definition 3.6 *A subset P of a cone normed linear space V is said to be bounded if $P \subseteq B_p(b)$ for some $b \in V$ and $p \in E$ with $\theta \ll p$.*

Definition 3.7 *A subset P of a cone normed linear space V is said to be compact if any sequence $\{x_n\}_n$ in P, a subsequence can be selected which is convergent to some point of P.*

Theorem 3.8 *Let V be a cone normed linear space then every cauchy sequence in V is bounded.*

Proof. Let $\{x_n\}_n$ be a cauchy sequence in V.
$\Rightarrow$ For every $\epsilon \in E$ with $\epsilon \gg \theta$ there exists $n_0 \in \mathbf{N}$ such that

$$\|x_n - x_m\|_c \ll \epsilon, \quad \forall m, n \geq n_0.$$

$\Rightarrow$ In particular, for $\epsilon = p \in E$ with $p \gg \theta$, $\exists\, k \in \mathbf{N}$ such that

$$\|x_n - x_m\|_c \ll p, \quad \forall m, n \geq k.$$

$\Rightarrow \quad \|x_n - x_k\|_c \ll p, \quad \forall n \geq k.$
Let $m = maximal\, \{\|x_1 - x_k\|_c, \|x_2 - x_k\|_c, \cdots\cdots, \|x_{k-1} - x_k\|_c\}$.
Then $\|x_n - x_k\|_c \ll p + m, \quad \forall n \in \mathbf{N} \quad \Rightarrow \; x_n \in B_{p+m}(x_k)$.
Hence $\{x_n\}_n$ is bounded.

Theorem 3.9 *In a finite dimensional normal cone normed linear space with normal constant K, a subset M of V is compact if and only if M is closed and bounded.*

Proof: Let M be compact subset of V. Then by our formal verification it is easy to see that M is closed.
Next we show that M is bounded. If possible let M is not bounded. Let $x_0 \in V$ be a fixed element. Then there exist a point $x_1 \in M$ such that $\|x_1 - x_0\|_c \geq \epsilon$ for a chosen $\epsilon \in E$ with $\epsilon \gg \theta$. By the same reason there exists a point $x_2 \in M$ such that $\|x_2 - x_0\|_c \geq \|x_1 - x_0\|_c + \epsilon$
Continue this process we obtain a sequence $x_1, x_2, \cdots$ of the set M such that $\|x_n - x_0\|_c \geq \|x_1 - x_0\|_c + \|x_2 - x_0\|_c + \cdots + \|x_{n-1} - x_0\|_c + \epsilon \geq \|x_m - x_0\|_c + \epsilon$ for all $m < n$ by lemma 2.16
Therefore $\|x_n - x_0\|_c - \|x_m - x_0\|_c \geq \epsilon. \cdots\cdots\cdots (i)$
Now $\|x_n - x_0\|_c \leq \|x_n - x_m\|_c + \|x_m - x_0\|_c$
Therefore $\|x_n - x_0\|_c + \|x_m - x_0\|_c \leq \|x_n - x_m\|_c \cdots\cdots\cdots (ii)$
Using lemma 2.16 we get from (i) and (ii) $\epsilon \leq \|x_n - x_m\|_c$ for all $m < n$.
This shows that neither the sequence nor any subsequence of $\{x_n\}$ can converge. This contradiction proves that M is bounded.
Conversely, let M is closed and bounded and the dimension of M be n. Let $\{e_1, e_2, ..., e_n\}$ be a basis of M. Let $\{x_m\}$ be a sequence in M. Since M is bounded, then there exist $p \in E$ such that $x_n \in B_p(b)$ for some $b \in V$ for all $n \in \mathbf{N}$. Therefore $\|x_n - b\|_c \ll p$ for all $n \in \mathbf{N}$.
Now $\|x_n\|_c = \|x_n - b + b\|_c \leq \|x_n - b\|_c + \|b\|_c \ll p + \|b\|_c$. [by lemma 2.8]
Thus $\|x_n\|_c \ll p + \|b\|_c$ for all $n \in \mathbf{N}$.
Let $x_m = \alpha_1^m e_1 + \alpha_2^m e_2 + \cdots + \alpha_n^m e_n$. Where $\alpha_j^m \in \mathbf{R}$, the set of all real numbers for $m = 1, 2, ...$ and $j = 1, 2, ..., n$. Therefore by lemma 3.1, there

exists an element $c \in intC$ such that
$p + \|b\|_c \gg \|x_m\|_c = \| \sum_{j=1}^{n} \alpha_j^m e_j \|_c \geq c \sum_{j=1}^{n} |\alpha_j^m|$ for some $c \in intC$ [by lemma 3.1]
Therefore $\|p + \|b\|_c\| \geq K\|c\| \sum_{j=1}^{n} |\alpha_j^m| \Rightarrow \sum_{j=1}^{n} |\alpha_j^m| \leq \frac{\|p+\|b\|_c\|}{K\|c\|}$ [as $c \neq \theta$]
Therefore the sequence of numbers $\{\alpha_j^m\}, m = 1, 2, ...$ and $j = 1, 2, ..., n$ is bounded. So by Bolzano-Weierstrass theorem there exist a convergent subsequence of $\{\alpha_j^m\}$. Then by the calculation of the lemma 3.1, there exists a subsequence of $\{x_m\}$ that converges. Therefore M is compact. Hence the proof.

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