

The Kadison-Singer problem for the direct sum of matrix algebras

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Abstract

Let M_n denote the algebra of complex $n \times n$ matrices and write M for the direct sum of the M_n . So a typical element of M has the form

$$x = x_1 \oplus x_2 \cdots \oplus x_n \oplus \cdots ,$$

where $x_n \in M_n$ and $\|x\| = \sup_n \|x_n\|$. We set $D = \{\{x_n\} \in M : x_n \text{ is diagonal for all } N\}$. We conjecture (contra Kadison and Singer (1959)) that every pure state of D extends uniquely to a pure state of M . This is known for the normal pure states of D , and we show that this is true for a (weak*) open, dense subset of all the singular pure states of D . We also show that (assuming the Continuum hypothesis) M has pure states that are not multiplicative on any maximal abelian *-subalgebra of M .

Let M_n denote the algebra of complex $n \times n$ matrices and write M for the direct sum of the M_n . So a typical element of M has the form

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where $x_n \in M_n$ and $\|x\| = \sup_n \|x_n\|$. We embed M_n in M in the natural way and M in $B(H)$ (the algebra of all bounded operators on a Hilbert space) in the natural way by viewing elements of M as block diagonal matrices. We use D to denote the diagonal elements in M (D also comprises the diagonal elements of $B(H)$ in this embedding). We write \mathcal{P} for the unique conditional

expectation of $B(H)$ (or M) onto D . Our purpose here is to study the Kadison-Singer problem in this context. Our conjecture in the context of M is:

CONJECTURE: Each pure state of D has a unique state extension to M .

This problem has many equivalent formulations. In their original paper Kadison and Singer “inclined to the view” that our conjecture had a negative answer. We take the opposite position here

In what follows, we could take the sizes of the matrix summands of M to be any finite numbers that tend to infinity with n . Our choice has been made purely for notational convenience. It is well known that all of the pure states on D extend uniquely to M if and only if they all extend uniquely to $B(H)$. In certain ways M is simpler than $B(H)$. For example M does not contain a non-atomic MASA (i.e. maximal abelian $*$ subalgebra). In other ways M is more complicated than $B(H)$. For example it has a large center and a complicated ideal structure. M is also an appealing venue for study of this problem since it may be possible to avoid the set-theoretic issues that arise in $B(H)$.

Let’s address the last point first. We begin by introducing some notation. Recall that an element $b \in B(H)$ (or $b \in M$) is called *paveable* if for all $\epsilon > 0$ there exist an integer $m > 1$ and projections $p_1, \dots, p_m \in D$ with $\sum_1^m p_j = 1$ such that

$$\left\| \sum_1^m p_j(b - \mathcal{P}(b))p_j \right\| < \epsilon.$$

There are two natural approaches to this problem. A well known result of the second author [6] states that our conjecture is true iff each element of M (or $B(H)$) is paveable. One can either show that more and more elements of $B(H)$ or M are paveable, or one can show that more and more singular pure states of D have unique extension to all of M or $B(H)$. In $B(H)$ only the first has been successful without the use of the continuum hypothesis or similar set theoretic assumptions. In particular, **no** singular pure states of D have been shown to have unique state extensions to $B(H)$ without using the continuum hypothesis or something similar.

As we show in Section 3, the situation is very different for M . In M there is a weak $*$ relatively open, unitarily invariant, dense subset of the singular

pure states of D for which pure state extensions are unique. In particular, the set of singular pure states of D that fail to have unique state extensions to M (if any) is of first category in the set of all singular pure states of D with the weak* topology. At present, even using additional set theoretic assumptions, there is no result this strong for $B(H)$. We shall also show that every element of the ideal I_{ab} of M generated by the abelian projections is paveable.

In the final section, we show that, assuming the Continuum Hypothesis, there are pure states of M that are not pure on any MASA of M .

1 Ideals in M

Let us begin by presenting some facts about ideals in M . **In this paper the term ideal will always refer to a norm-closed two-sided ideal.** The ideal structure of M is quite rich and our discussion is not complete. Some of the results presented here are known. Nevertheless it seems worthwhile to collect them and to point out connections to the Kadison-Singer problem. Let us now introduce some more notation. Write τ_n for the normalized trace on M_n and

$$\|a\|_2 = (\tau_n(a^*a))^{1/2}.$$

Ideals in M were studied by Wright in [20]. He called a subset L of projections in M a *p-ideal* if:

- (1) it is closed under equivalence and
- (2) it is closed under the formation of sups and infs

and he proved the following theorem in [20]. We write $Z(M)$ for the center of M .

Theorem 1.1. *The following statements hold.*

- (1) *The p-ideals in the set of projections in M are in one-to-one correspondence with the ideals in M .*
- (2) *Each ideal in M is generated by its projections.*
- (3) *There is a one-to-one correspondence between the maximal ideals of M and the maximal ideals in $Z(M)$.*

The center of M consists of elements of the form $\bigoplus_{n=1}^{\infty} \lambda_n I_n$, where $\{\lambda_n\}$ is a bounded complex sequence and I_n is the identity matrix in M_n . In other words the center of M can be identified with ℓ^∞ (which may also be identified with $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ denotes the Stone-Cech compactification of the natural numbers).

The first conclusion of the following Lemma is based on an idea of Sorin Popa. Details of the proof were provided by David Sherman. The second conclusion is based on the proof of Theorem 1 of [11].

Lemma 1.2. *If $x \in M_n$ with $\mathcal{P}(x) = 0$ (i.e. x has zero diagonal) and G denotes the group of symmetries (i.e. self-adjoint unitary operators) in D , then*

(1) *There is $d \in G$ such that*

$$\|xd - dx\|_2 \geq \sqrt{2}\|x\|_2.$$

(2) *If $x = x^*$, there is $u \in G$ such that*

$$\|xu - ux\| \geq \|x\|.$$

Proof. Direct calculation (using $\mathcal{P}(x) = 0$) shows that for any diagonal element d with diagonal entries $(d_1, \dots, d_n) \in G$,

$$\|xd - dx\|_2^2 = (1/n) \sum_{i \neq j} |x_{ij}|^2 |d_j - d_i|^2.$$

Choose Haar measure μ on G to have total mass 1. Because each d_i takes the values $+1, -1$ with equal probability and independently, $|d_j - d_i|^2 = 4$ or 0 with equal probability for $i \neq j$, so $i \neq j$, $\int_G |d_j - d_i|^2 d\mu(d) = 2$. Thus

$$\begin{aligned} \int_G \|xd - dx\|_2^2 d\mu(d) &= \|xd - dx\|_2^2 = (1/n) \sum_{i \neq j} |x_{ij}|^2 \int_G |d_j - d_i|^2 d\mu(d) \\ &= (1/n) \sum_{i \neq j} 2|x_{ij}|^2 d\mu = 2\|x\|_2^2. \end{aligned}$$

Since $\mu(G) = 1$, for some $d \in G$, $\|xd - dx\|_2^2 \geq 2\|x\|_2^2$.

For the second assertion we may assume that $\|x\| = 1$. Since $ux - xu$ has zero diagonal for any $u \in G$, $u(ux - xu) = x - uxu$ also has zero diagonal. Thus $\int_G (x - uxu) d\mu(u)$ also has zero diagonal, as does uxu . Next observe that $\int_G uxu d\mu(u)$ is a diagonal matrix (by direct calculation). Since $\int_G uxu d\mu(u)$ also has zero diagonal (as above), it is 0. This means that

$$\int_G u(ux - xu) d\mu(u) = \int_G x d\mu(u) - \int_G uxu d\mu(u) = x.$$

Thus $1 = \|x\| \leq \int_G \|ux - xu\| d\mu(u)$, so (since $\mu(G) = 1$) $\|ux - xu\| \geq 1$, for some $u \in G$.

□

Notation 1.3. Since the center Z of M may be identified with $\ell^\infty = C(\beta\mathbb{N})$, a closed ideal Γ in the center is determined a closed subset of $\beta\mathbb{N}$. Let Ω denote such a closed subset and define two ideals in M as follows. Write

$$I_2(\Omega) = \{x = \oplus x_n : \lim_{\omega} \|x_n\|_2 = 0, \omega \in \Omega\} \text{ and}$$

$$I_\infty(\Omega) = \{x = \oplus x_n : \lim_{\omega} \|x_n\| = 0, \omega \in \Omega\}.$$

It is clear that $I_2(\Omega) \cap Z(M) = I_\infty(\Omega) \cap Z(M) = \Gamma$.

Proposition 1.4. *Incorporating the notation above, $I_2(\Omega)$ is the maximal ideal J such that $J \cap Z(M) = \Gamma$ and $I_\infty(\Omega)$ is the minimal such ideal.*

Proof. If $x \in M$, it is clear that $\|x_n\|_2 \leq \|x_n\| \quad \forall n$, hence $I_\infty(\Omega) \subset I_2(\Omega)$. Now fix an ideal J with

$$J \cap Z(M) = \Gamma.$$

Since Γ contains an approximate unit for $I_\infty(\Omega)$, J must contain $I_\infty(\Omega)$.

For the maximality claim, by contradiction suppose that there is a self-adjoint element $a \in J$ and $\omega \in \Omega$ such that $\lim_{\omega} \|a_n\|_2 \neq 0$. Using spectral theory, we may replace a by a projection p . Since the 2-norm is continuous on $\beta\mathbb{N}$, there are a central projection $r \in M$ and a natural number k such that $\tau_n(rp) \geq (1/k)\tau_n(r) \quad \forall n$, and $\lim_{\omega} \|r_n\|_2 = 1$. Since J is an ideal, there are partial isometries $v_1, \dots, v_k \in M$ such that $\sum_1^k v_j^* p v_j \geq r$, hence $r \in J$. This means that $J \cap Z(M)$ is strictly larger than Γ which gives the desired contradiction. □

Notation: For any ideal J of M , x_J denotes the support projection of J in M^{**} , y_J denotes the supremum in M^{**} of projections in $D \cap J$, and z_J denotes the support projection in M^{**} of $J \cap Z(M)$. Clearly $x_J \geq y_J \geq z_J$. Also, if a is a self-adjoint element of M and σ is a measurable subset of \mathbb{R} , we write $\chi_\sigma(a)$ for the spectral projection of a determined by σ .

Proposition 1.5. *The following statements hold for an ideal J of M .*

- (1) $a \in (J)_{sa}$ if and only if $a \in M_{sa}$ and $1 - \chi_{(-1/n, 1/n)}(a) \in J$ for all n .
- (2) $J \cap D$ is a MASA in J .
- (3) y_J is dense in x_J .
- (4) y_J is regular in J^{**} .
- (5) $\mathcal{P}(J) \subset J \cap D$.
- (6) If $J = I_2(\Omega)$ or $I_\infty(\Omega)$, $(D + J)/J$ is a MASA in M/J .

Proof. Fix $a = a^*$ in J . Since ideals are hereditary and

$$na \geq 1 - \chi_{(-1/n, 1/n)}(a),$$

it follows that $1 - \chi_{(-1/n, 1/n)}(a) \in J$.

Next suppose a is a self adjoint element of M and $p_n = 1 - \chi_{(-1/n, 1/n)}(a)$ is in J for all n . Since J is norm closed, ap_n is in J and $\|a - ap_n\| \leq 1/n$, it follows that $a \in J$. So the assertion in part (1) is true.

Now fix a self-adjoint element a in J that commutes with elements of $J \cap D$. Since J contains the operators of finite rank, a must be diagonal, so $a \in J \cap D$ and therefore assertion (2) is true.

Next if the assertion in (3) is false, then there is a self adjoint $a \in J$ that is orthogonal to all the projections in $D \cap J$. This contradicts part (2).

(4) follows immediately from [3] Lemma 4.1(iii), because y_J dominates the central cover of $J \cap I_{\beta(N) \setminus N}$.

For (5), let $p \in J$ be a projection. Fix an integer $k > 0$, and write

$$q = \chi_{(1/k, 1]}(\mathcal{P}(p)).$$

Then it suffices to show that $q \in J$ since $\|q\mathcal{P}(p) - \mathcal{P}(p)\| < 1/k$ and J is norm closed. Now observe that

$$\text{rank}(p_n) = n\tau_n(p_n) = n\tau_n(\mathcal{P}(p)_n) \geq (n/k)\tau_n(q_n) \geq (1/k)\text{rank}(q_n) \quad (*)$$

and define a projection $r \in M$ as follows. If $\text{rank}(p_n) \geq n/k$, then set $r_n = 1_n$. If $\text{rank}(p_n) < n/k$, then choose unitaries $\{v_j\}_1^k \in M$ such that for each n , $r_n := \sum_{j=1}^k v_j p_n v_j^*$ is a projection; necessarily $\text{rank}(r_n) = k \text{rank}(p_n)$.

We claim that $r \in J$. Let $s \in M$ be the central projection defined by $s_n = r_n$ if $r_n = 1_n$, and $s_n = 0$ if not. Clearly $s \in J$ since there exist unitaries w_1, \dots, w_k in M such that $\sum_1^k w_n p_n w_n^* \geq s$. Thus $sr \in J$ and $(1-s)r \in J$ by its construction as a finite sum of projections in J . This establishes our claim.

Now note that $\text{rank}(r_n) \geq \text{rank}(q_n)$ for all n by $(*)$ above. Thus there is a unitary $U \in M$ such that $UrU^* \geq q$, and hence $q \in J$. Since k was arbitrary and J is closed, it follows that we also have $P(p) \in J$, as desired.

For (6), suppose first that $J = I_\infty(\Omega)$, then fix x in M such that x has 0 diagonal and $xd - dx = 0$ in the quotient M/J for all $d \in D$. Thus $\lim_\omega \|dx - xd\|_\infty = 0$ for all $\omega \in \Omega$ and for all $d \in D$. By Lemma 1.2, $\lim_\omega \|x\|_\infty = 0$ for every ω in Ω , i.e. $x \in I_\infty(\Omega)$. This gives the result.

If $J = I_2(\Omega)$, then fix an element x in M such that x has 0 diagonal and $xd - dx \in J$ for all $d \in D$. Thus $\lim_\omega \|xd - dx\| = 0$ for each ω in Ω and for all $d \in D$. By Lemma 1.2, $\lim_\omega \|x\|_2 = 0$ for each ω in Ω , so x is in $I_2(\Omega)$. \square

Remarks.

(a) We know by Theorem 1.5.7 and Corollary 1.5.8 in [14], that for any ideal J of M , the algebra $J+D$ is norm closed and that its quotient $(J+D)/J$ is a C^* -subalgebra of M/J . It follows that if the quotient $(D+J)/J$ is not a MASA of M/J , then our Kadison - Singer Conjecture would be false. For this reason we take assertion (6) of the last theorem as more evidence for our conjecture.

(b) If an ideal J has the form I_∞ , then $y_J = x_J = z_J$.

2 Some General C*-Algebra Results

Recall that a MASA is a maximal abelian *-subalgebra of a C*- algebra, and, if f is a state on a C*- algebra A , then its *hereditary kernel* is by definition

$$K_f = \{x \in A : f(xx^* + x^*x) = 0\}.$$

Recall that as defined in [2] a net $\{a_\alpha\}$ of positive norm one elements in a C*-algebra A **excises** a state f of A if

$$\lim_{\alpha} \|a_\alpha a a_\alpha - f(a) a_\alpha^2\| = 0 \quad \text{for every } a \text{ in } A.$$

We are interested in pure states, so we recall the basic facts. First by Proposition 2.2 of [2], if f is a pure state of a C*-algebra A , then its hereditary kernel is a maximal hereditary C*-subalgebra. Further, the covering projection p of this hereditary kernel is the limit in A^{**} of its approximate unit, and $1 - p$ (i.e. the support projection of f) has rank 1 ([14], 3.13).

Theorem 2.1. *If f is a pure state of the C*-subalgebra B of C and $\{b_\alpha\}$ is any excising net for f in B , then f has unique state extension to C if and only if $\{b_\alpha\}$ converges to a rank one projection in C^{**} .*

Proof. Suppose $\{b_\alpha\}$ converges to a projection p of rank one in the double dual of C and suppose g and h are states on C that extend f . If we fix $c \in C$ we have

$$\lim_{\alpha} g(c) = \lim_{\alpha} g(b_\alpha c b_\alpha) = g(p c p) = g(\lambda p) = \lambda = \lim_{\alpha} h(b_\alpha c b_\alpha) = h(c)$$

For the converse suppose that f has a unique state extension g to all of C . It follows that $\{b_\alpha\}$ must converge to a projection of rank one in C^{**} , else the extension would not be unique. □

Proposition 2.2. *Let A be a unital C*-algebra and f a pure state on A . There there is a MASA of A on which f is pure and from which f has unique state extension if and only if the hereditary kernel of f has an abelian approximate unit.*

Proof. Suppose that the hereditary kernel of f has an abelian approximate unit $\{b_\alpha\}$ and let B be any MASA of A that contains $\{b_\alpha\}$. Write p in A^{**} for the limit of the b_α 's. Since f is pure, p has codimension 1 as mentioned in the introduction to section 2. Since p also lies in B^{**} , it has codimension 1 there also. Since f is supported on $1 - p$ which has dimension 1 in B^{**} , its restriction to B is pure and it must have a unique state extension.

Now suppose that there there is a MASA B of A on which f is pure and from which f has unique state extension. Then the approximate unit of $\{x \in B : f(xx^* + x^*x) = 0\}$ must converge in A^{**} to a projection of codimension 1, else state extensions of f would not be unique. □

Lemma 2.3. *Let A be a C^* -algebra and x be an open, dense (i.e. $\bar{x} = 1$), central projection in A^{**} . Let $S(A)$ (resp., $PS(A)$) denote the states (resp., pure states) of A and $S(A)_x$ (resp., $PS(A)_x$) denote those states (resp., pure states) that take the value 1 on x . Then $S(A)_x$ (resp. $PS(A)_x$) is weak* dense in $S(A)$ (resp. $PS(A)$).*

Proof. Let $a_\alpha \uparrow x, a_\alpha \in A$. The complement of $S(A)_x$ (resp. $PS(A)_x$) in $S(A)$ (resp. $PS(A)$) consists of those f such that $f(a_\alpha) = 0$ for all α , and that is clearly closed.

That $S(A) = \overline{S(A)_x}$ follows since the central projection x is regular. The conclusion $PS(A) = \overline{PS(A)_x}$ follows from $S(A) = \overline{S(A)_x}$ by [8, Appendix B14]. □

3 The Ideal I_{ab} Generated by the Abelian Projections

Let I_{ab} denote the ideal generated by the abelian projections in M . It is clear that a projection $p = \oplus p_n \in M$ is abelian if and only if the rank of p_n is 0 or 1 for each n . In some sense, the existence the ideal I_{ab} is what makes M

so different from $B(H)$. We define the projection $p_B \in I_{ab}$ by the condition that $(p_B)_n$ has $1/n$ in each of its entries. Recall that

$$I_2(\beta\mathbb{N} \setminus \mathbb{N}) = \{\oplus x_n : \lim_{\omega} \|x_n\|_2 = 0, \omega \in \beta\mathbb{N} \setminus \mathbb{N}\},$$

and let K denote the compact operators in M , i.e. $K = I_{\infty}(\beta\mathbb{N} \setminus \mathbb{N})$.

Proposition 3.1. *The following statements hold.*

- (1) $I_{ab} \subset I_2(\beta\mathbb{N} \setminus \mathbb{N})$.
- (2) Any ideal J that is strictly larger than K contains an infinite dimensional abelian projection.
- (3) $K \neq I_{ab} \neq I_2(\beta\mathbb{N} \setminus \mathbb{N})$.

Proof. For the first assertion, it suffices to show that every abelian projection lies in $I_{\beta\mathbb{N} \setminus \mathbb{N}}$. This is clear since for any abelian projection $p = \oplus p_n \in M$, where $\|p_n\|_2 \leq 1/\sqrt{n}$.

For the second assertion, suppose the ideal J contains a non-compact operator. It follows that J contains a projection of infinite rank which, in turn, must dominate an infinite dimensional abelian projection.

For the third assertion, if p is an abelian projection such that $\text{rank}(p_n) = 1$ for all n , then $p \in I_{ab} \setminus K$.

If q is a projection in M such that $\text{rank}(q_{2^n}) = n$ and $\text{rank}(q_k) = 0$ when k is not a power of 2, then $q \in I_2(\beta\mathbb{N} \setminus \mathbb{N}) \setminus I_{ab}$ as follows.

$$\|q_{2^n}\|_2 = \sqrt{n}/2^{n/2} \rightarrow_{n \rightarrow \infty} 0.$$

Thus $q \in I_2(\beta\mathbb{N} \setminus \mathbb{N})$. If b is a self-adjoint linear combination of t abelian projections, then for all n , $\text{rank}(b_n) \leq t$. Since $\text{rank}(q_n) \rightarrow \infty$, $\|(b - q)_n\| \geq 1$ when $\text{rank}(q_n) > t$. Thus q can't be in I_{ab} . □

Proposition 3.2. *If ω is a free ultrafilter and $q = \oplus q_n$ is a projection in $I_2(\{\omega\}) \cap D$, then*

$$\lim_{\omega} \|(p_B q)_n\| = 0.$$

Proof. Since $\lim_{\omega} \|q_n\|_2 = 0$, if $\epsilon > 0$ we may select a set $\sigma \in \omega$ such that $\|q_n\|_2 < \epsilon$ for each n in σ . Since $q \in D$, $(qp_B q)_n$ can be viewed as a block

matrix in M_n for each $n \in \sigma$. The size of the block is $\text{rank}(q_n) \times \text{rank}(q_n)$ and the entries in the block are all $1/n$. I.e. this block is a scalar multiple of a rank 1 projection, and the multiple is $\text{rank}(q_n)/n \leq \epsilon^2$. Thus

$$\lim_{\omega} \|(p_B q)_n\| = 0.$$

□

Corollary 3.3. *If p is a projection in $D \cap I_2(\beta\mathbb{N} \setminus \mathbb{N})$, then pp_B is compact.*

Proof. Apply Proposition 3.2 to each $\omega \in (\beta\mathbb{N} \setminus \mathbb{N})$. □

Corollary 3.4. *For any proper ideal J that contains p_B , $y_J \neq x_J$.*

Proof. If $y_J = x_J$, then the projection approximate unit for $D \cap J$, which converges to y_J , would be an approximate unit for p_B , and that is false by the last proposition. □

Proposition 3.5. *If f is a pure state of M , p is an abelian projection in M such that $f(p) = 1$, and B is a MASA of M that contains p , then $f|_B$ is pure and f is its unique state extension.*

Proof. Since every MASA of M is unitarily equivalent to D , we may assume $B = D$.

Let f' be any state extension of $f|_D$ and recall that $f' = pf'p$ by Schwarz inequality. Now observe that $pMp \subset D$ because p is abelian. So, for any $a \in M$,

$$f'(a) = f'(pap) = f(pap) = f(\mathcal{P}(pap)) = f(p\mathcal{P}(a)p) = f(\mathcal{P}(a)).$$

Thus f is the unique state extension of $f|_D$.

If $f|_D$ is not pure, there would be a projection $q \in D$ such that $0 < f(q) < 1$ and this would mean that for any $a \in M$,

$$\begin{aligned} f(a) &= f(\mathcal{P}(qaq + qa(1-q) + (1-q)aq + (1-q)a(1-q))) \\ &= f(\mathcal{P}(qaq + (1-q)a(1-q))) \\ &= f(qaq + (1-q)a(1-q)) \\ &= ((qfq) + (1-q)f(1-q))(a), \end{aligned}$$

so $f = qfq + (1-q)f(1-q)$, contradicting the assumption that f is pure on M . So f is a pure state on D . □

Theorem 3.6. *The set of all singular pure states of M that take a nonzero value on I_{ab} is relatively weak* open and dense in the set of all singular pure states of M .*

Proof. The central cover x of the ideal I_{ab}/K in $(M/K)^{**}$ is an open projection. If x were not dense for M/K , there would be a non-zero hereditary subalgebra J_0 of M/K that is orthogonal to I_{ab}/K . Lift J_0 to a closed hereditary subalgebra J of M . Since J is generated by its projections, it contains an infinite rank projection p . Thus by the form of M there is an infinite rank abelian projection q in I_{ab} such that $pq=q$. However this contradicts the assumption that J is orthogonal to I_{ab}/K . Thus x is open and dense. Apply Lemma 2.3 to M/K to get that $PS(M/K)x$ is weak* dense in $PS(M/K)$. Since x is central and open, $PS(M/K)_x = \{f \in PS(M/K) : f(x) > 0\}$, $PS(M/K)_x$ must be weak* open in $PS(M/K)$. \square

Theorem 3.7. *The set of all singular pure states of D that take a nonzero value on $I_{ab} \cap D$ is relatively weak* open and dense in the set of all singular pure states of D . All of these pure states of D have unique state extension to all of M .*

Proof. The first sentence is proved in the same way as the last theorem. The second sentence follows from Theorem 3.5. \square

Theorem 3.8. *Every element of I_{ab} can be paved.*

Proof. Let $\epsilon > 0$ be given, let q be an abelian projection in M and choose $m > 2/\epsilon$. By Proposition 1.2 of [17], it will suffice to prove that q is paveable. The rank of each q_n is 1 or 0, so there are abelian projections $\{p_\alpha\}_{\alpha=1}^m$ such that if p is a rank 1 projection in D such that $pp_\alpha = 0$ for all $\alpha = 1, \dots, m$, then $\|pqp\| < \epsilon/2$. Set $r = 1 - \sum_{\alpha=1}^m p_\alpha$. Since each p_α is abelian and in D , $p_\alpha(q - \mathcal{P}(q))p_\alpha = 0$.

Define the map $\Psi : rD \rightarrow qMq$ by $\Psi(b) = qbq$. Since q is abelian, the range of Ψ intersects each M_n in either $\{0\}$ or the 1-dimensional space of multiples of q_n . By the definition of r , for each n and each abelian projection $s \in rD$, $\|(sqs)_n\| = \|(qsq)_n\| = \|(\Psi(s))_n\|_1 \leq \epsilon/2$. Choose k as a power of 2 such that $1/k < \epsilon$. Thus by Theorem 5.7 of [1] (applied inside each M_n), there are projections $\{p_\alpha\}_{\alpha=m+1}^{m+k} \in rD$ such that $\sum_{\alpha=m+1}^{m+k} p_\alpha = r$ and, for $\alpha = 1, \dots, k$, $\|\Psi(p_{m+\alpha}) - (1/k)qrq\| \leq \epsilon$. Thus $\|\Psi(p_{m+\alpha})\| = \|qp_{m+\alpha}q\| = \|p_{m+\alpha}qp_{m+\alpha}\| < 2\epsilon$. Thus $\|p_\alpha qp_\alpha\| \leq 2\epsilon$ for all $\alpha = 1, \dots, m+k$. I.e. q is paveable. \square

Theorem 3.9. *If J is any ideal of M that is not contained in I_{ab} , then there is a projection $p \in J \cap D$ and a *-isomorphism of pMp onto M .*

Proof. Since J is not contained in I_{ab} , there is a projection $q \in J \cap D$ such that $\text{rank}(q_n) \rightarrow \infty$. Thus by a routine recursive construction, there is a subsequence $\{n_k\}$ and projections $p_{n_k} \in D_{n_k}$ such that $\text{rank}(p_{n_k}) = k$ and $p_{n_k} \leq q_{n_k}$ for each $k = 1, 2, 3, \dots$. Set $p = \sum_k p_{n_k}$. Since $p \leq q$, then $p \in J \cap D$. Since $p_{n_k}Mp_{n_k}$ is *-isomorphic to M_k , there is a *-isomorphism of pMp onto M . \square

Theorem 3.10. *If J is any ideal of M that is not contained in I_{ab} , and if our Kadison Singer Conjecture is false, then there is a pure state of $D \cap J$ that does not have unique state extension to J .*

Proof. By assumption there is a pure state h of D that does not have unique state extension to M . By the last theorem there is a *-isomorphism ψ of pMp onto M for some projection $p \in J \cap D$. Note that $p \in D$ implies that $\psi|_{pDp}$ is a *-isomorphism onto D . If f_1, f_2 are distinct state extensions of h to M , then $\psi^*(f_1), \psi^*(f_2)$ are distinct states of pMp . If $a \in D$, then there exists unique $b \in D$ such that $\psi^{-1}(b) = pap$.

Now define $\theta : M \rightarrow pMp$ by $\theta(c) = pc p$. Remark that θ^* is a positive isometry from $(pMp)^*$ onto pM^*p . Thus the states $g_i = \theta^* \circ \psi^*(f_i)$ for $i=1,2$, are also distinct states. Compute:

$$g_i(a) = \theta^* \circ \psi^*(f_i)(a) = \psi^*(f_i)(pap) = \psi^*(f_i)(\psi^{-1}(b)) = f_i(b) = h(b) =$$

$$h(\psi^{-1}(pap)) = \psi^{-1*}(h)(\theta(a)) = \theta^* \circ \psi^{-1*}(h)(a)$$

Thus g_1, g_2 are distinct states of M that live on J and have the same pure restriction $\theta^* \circ \psi^{-1*}(h)$ to D . \square

Theorem 3.11. *If J is any ideal of M that is not contained in I_{ab} , and if our Kadison Singer Conjecture is false, then there is an element of J that cannot be paved.*

Proof. As in the proof of the last theorem, there is a *-isomorphism ψ of pMp onto M for some projection $p \in J \cap D$. Note that $p \in D$ implies that $\psi|_{pDp}$ is a *-isomorphism onto D . Thus if our conjecture is false, there is an element of M that can't be paved, so the image in J of that element under ψ^{-1} can't be paved. \square

We have already seen (Theorem 3.8) that every element of I_{ab} is paveable. We now compare I_{ab} with a large C^* -subalgebra of M that has the same property. We will say that $b \in M$ is an *almost permutation* if each b_n has at most one non-zero entry in each row and column, that entry being 1. The C^* -algebra \mathcal{A} generated by D and the almost permutations will be called the *permutation algebra*. It was shown in [18] that every element of \mathcal{A} is paveable.

We shall now give a characterization of the elements of \mathcal{A} . Say that $b \in M$ is *d-empty* if there is a natural number d such that each b_n has at most d non-zero entries in each row and column. Let \mathcal{F} be the $*$ -subalgebra of operators that are d -empty for some $d \in \mathbb{N}$. It was shown [19] that an operator b is in \mathcal{F} iff there are a $m \in \mathbb{N}$, $c_1, \dots, c_m \in D$ and near permutations b_1, \dots, b_m such that $b = \sum_{i=1}^m c_i b_i$; thus \mathcal{A} is the norm-closure of \mathcal{F} . \mathcal{A} also contains each element $b \in M$ whose non-zero entries have modulus larger than some positive number δ , since b_n is $(\frac{\delta}{\|b_n\|})^2$ -empty.

Remark 3.12. $\mathcal{A} + I_{ab}$ is a C^* -subalgebra of M by Corollary 1.5.8 of [14], and every element in this subalgebra is paveable. We don't know of a larger C^* -subalgebra of M with this property. It is known ([19]) that neither $\mathcal{A} \subset I_{ab}$ nor $I_{ab} \subset \mathcal{A}$ holds. The question left hanging in [19] is whether p_B is in \mathcal{A} . The answer, which surprised us, is "yes". Our intuition was that if b is any d -empty matrix, and n is much larger than d , then subtracting b from p_B affects only a "small" fraction of the entries of $(p_B)_n$, and hence the resulting norm can't be all that small. A graph theoretic technique shows that this intuition is wrong. We thank Michel de la Salle and Gilles Pisier for pointing us to a theorem from graph theory (Theorem 1.1 of [10]), which is due to Friedman and which is exactly what is needed here.

Theorem 3.13. $p_b \in \mathcal{A}$.

Proof. The range of $(p_B)_n$ is the 1-dimensional subspace spanned by $(1, \dots, 1) = v$. If b is any permutation matrix in M_n , then $bv = v$. This suggests that if we take an even integer d (much smaller than n), we might approximate $(p_B)_n$ by a d -empty element $s \in \mathcal{A}$ of the form

$$s_n = (1/d) \sum_{\alpha=1}^{d/2} (b_\alpha + b_\alpha^*),$$

where each b_α is a permutation matrix in M_n .

To estimate $\|s_n - (p_B)_n\|$ we note that $(s_n - (p_B)_n)(v) = 0$, so it suffices to carry out our estimate on the orthogonal complement of v . Since $(p_B)_n$ vanishes there, we are actually concerned with estimating the modulus of the second largest eigenvalue of the self adjoint operator s_n .

At this point Friedman's paper [10] suggests that we should show that s_n is the type of matrix that he is dealing with in his Theorem 1.1. That is, we need to show that s_n is the adjacency matrix of a d -regular undirected graph on n vertices (times $1/d$). If the matrix b_α corresponds to the permutation π_α of $1, \dots, n$. such a graph G_n is formed by defining the edges ([10] p. 2) as follows:

$$E = \{(i, \pi_\alpha(i)), (i, \pi_\alpha^{-1}(i)) : \alpha = 1, \dots, d/2; i = 1, \dots, n\}.$$

The adjacency matrix of G is exactly ds_n .

Now we can apply Theorem 1.1 of [10] to conclude that there are positive constants η, r , **independent of n** , such that with probability at least $1 - \eta/n^r$, the second eigenvalue of a random matrix of the same form as s_n will be less than $(2\sqrt{d-1}+1)/d$. Of course that expression can be made as small as we like by taking a large d , and this is independent of n .

If $\delta > 0$ is given, we choose even d so that $(2\sqrt{d-1}+1)/d < \delta$. Let n_0 be large enough so that $1 - \eta/n_0^r > 0$. Since \mathcal{A} clearly contains all of the finite rank operators in M , we may define an operator $c \in \mathcal{A}$ by letting $c_n = (p_B)_n, n < n_0$. For $n \geq n_0$ we choose c_n to one of those elements in M_n of the form s_n (for the value of d chosen above) whose second eigenvalue is less than δ . These choices ensure that $c \in \mathcal{A}$ and that $\|p_B - c\| < \delta$.

□

This proof is a classic use of the probabilistic method in combinatorics. One shows the existence of something by showing that a random choice has a positive probability of being a right choice.

We call a projection $q \in M$ a **Hadamard projection** if $\text{rank}(q_n)$ is 0 or 1 for each n , and if $\text{rank}(q_n) = 1$, then q_n has a eigenvector of the form $(e^{i\theta_1}, \dots, e^{i\theta_n})$, for real $\theta_1, \dots, \theta_n$.

Corollary 3.14. *If q is a Hadamard projection in M , then $q \in \mathcal{A}$.*

Proof. Using the notation above, let w be a unitary in D such that $w_n(e^{i\theta_1}, \dots, e^{i\theta_n}) = (1, \dots, 1)$. Then $w^*p_Bw = q \in \mathcal{A}$.

□

4 Pure States of M

Proposition 4.1. *If f_0 is a pure state of D and if f is defined on M by $f = f_0 \circ P$, then f is a pure state of M .*

Proof. The proof of [5, Corollary 3] applies without change. \square

Theorem 4.2. *If*

- (1) \mathcal{N} is a type II_1 von Neumann algebra factor that has the cardinality of the continuum,
- (2) τ denotes the faithful normal tracial state on \mathcal{N} ,
- (3) $C(\mathcal{N})$ is a collection of MASAs of \mathcal{N} that has the cardinality of the continuum, and
- (4) the continuum hypothesis holds,

then there is a pure state of \mathcal{N} that is not multiplicative when restricted to any MASA in $C(\mathcal{N})$.

Proof. The proof is essentially given in Lemma 0.5-Theorem 0.7 of [4]. However, those results assume that \mathcal{N} acts on a separable Hilbert space. Thus we need to be sure that each time that hypothesis is used in [4], we can show here that it is not needed under the hypotheses of the present theorem.

1. To modify the proof of Lemma 0.5 of [4] to handle the present theorem, we note that in a II_1 factor, any two projections with the same trace are unitarily equivalent.

2. To modify the proof of Lemma 0.6 of [4] to handle the present theorem, we need only show that for any projection $p \in \mathcal{N}$ and any singular state f on \mathcal{N} such that $f(p) = 1$, there is a sequence $\{p_n\}$ of projections in \mathcal{N} such that $f(p_n) = 1$ for all n and $p_n \downarrow 0$ strongly. Let q_j be a maximal orthogonal family of projections under p for which $f(q_j) = 0$. Since τ is faithful, this family must be at most countable, and by [15] the set of $\{q_j\}$ must be infinite. Set $p_j = \sum_{i=j}^{\infty} q_i$. This works since $\tau(p_j) \rightarrow 0$ because τ is normal.

3. To modify the proof of Theorem 0.7 of [4] to handle the present theorem, we need only note that both \mathcal{N} and $C(\mathcal{N})$ have cardinality $c = \aleph_1$ under CH. \square

Corollary 4.3. *There is a pure state g of M that is not multiplicative on any MASA of M .*

Proof. If J is any maximal ideal of M and Q_J is the quotient map of M onto M/J , then by [9], $Q_J(M)$ is a II_1 factor that has the cardinality of the continuum and the set of images of the MASAs of M also has the cardinality of the continuum. Thus the Corollary follows by taking the pure state f on $Q_J(M)$ that is non-multiplicative on each of the images of the MASAs of M from the last theorem and defining $g = f \circ Q_J$. \square

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