FUNK, COSINE, AND SINE TRANSFORMS ON STIEFEL AND GRASSMANN MANIFOLDS

B. RUBIN

Abstract. The Funk, cosine, and sine transforms on the unit sphere are indispensable tools in integral geometry. They are also known to be interesting objects in harmonic analysis. The aim of the paper is to extend basic facts about these transforms to the more general context for Stiefel or Grassmann manifolds. The main topics are composition formulas, the Fourier functional relations for the corresponding homogeneous distributions, analytic continuation, and explicit inversion formulas.

CONTENTS

1. Introduction

1.1. **History and motivation.** Our consideration has several sources.

1. For a function Φ on the unit sphere S^2 in $\mathbb{R}^3$ P. Funk [F11, F13] defined a *circle-integral function* (*die Kreisintegral-Funktion*) χ on the set of great circles as an integral of Φ over the corresponding great circle. He suggested two inversion algorithms; see [F13, pp. 285-288]. The first one relies on expansion in spherical harmonics and the second reduces the problem to Abel's integral equation. From these results Funk derived the celebrated Minkowski's theorem [Min], [Hel10,

2000 *Mathematics Subject Classification.* Primary 44A12; Secondary 47G10.

Key words and phrases. The Funk transform, the Radon transform, Fourier analysis, the cosine transform, the sine transform.

The work was supported by the NSF grants PFUND-137 (Louisiana Board of Regents), DMS-0556157, and the Hebrew University of Jerusalem.

p. 137], stating that bodies of constant circumference are bodies of constant width.

2. Funk's Kreisintegral-Funktion is now called *the Funk transform.* This concept extends to higher dimensions, when great circles on S^2 are substituted by cross-sections $S^{n-1} \cap \xi$, ξ being a k-dimensional linear subspace of $\mathbb{R}^n$, $1 \le k \le n-1$. The set of all such subspaces forms a Grassmann manifold $G_{n,k} = O(n)/(O(k) \times O(n-k))$. These transformations were extensively studied in Gelfand's school, by S. Helgason, and other authors; see [GGG, Hel90, Hel00, Hel10, Ru98, Ru02, Ru03] and references therein. Further generalization connects functions on two different Grassmannians by inclusion, namely,

$$(1.1) \quad (R_{k,\ell} f)(\eta) = \int_{\xi \subset \eta} f(\xi)\, d_\eta \xi, \qquad \xi \in G_{n,k}, \quad \eta \in G_{n,\ell}, \quad k < \ell,$$

where $d_\eta \xi$ denotes the probability measure on the Grassmann manifold $G_k(\eta)$ of all k-dimensional linear subspaces of η. Operators (1.1) are also known as Radon transforms for a pair of Grassmann manifolds. They were studied by Petrov [P67], Gelfand and his collaborators [GGŠ70, GGR], Grinberg [Gri], Kakehi [Ka], Grinberg and Rubin [GR], Zhang [Zh1, Zh2]. Affine versions of (1.1) were considered in [GK1, GK2, Ru04, Shi].

In spite of the elegance and ingenuity of the inversion methods in these publications, the resulting formulas are pretty involved. It is a challenging problem to find new simple inversion formulas and develop a theory which is parallel to that for the unit sphere. The present paper is devoted to this problem. The main idea of our approach is to treat operators (1.1) as members of the analytic family of the higher-rank cosine transforms, which will be introduced below.

3. The name *cosine transform* was given by Erwin Lutwak [Lu, p. 385] in 1990 to the integral operator

$$(1.2) \qquad (\mathfrak{C} f)(u) = \int_{S^{n-1}} f(v) |u \cdot v|\, dv, \qquad u \in S^{n-1}.$$

Since then, this term is widely used in integral and convex geometry (in parallel with its traditional meaning in the Fourier analysis). Operator (1.2) and its generalization with the kernel $|u \cdot v|^\lambda$ were studied without naming by many authors in geometry and analysis since Blaschke [Bla], Levy [Lev], Aleksandrov [Al]; see [Ga, K97, Ru03] for references. A remarkable fact, that amounts to the results of Gelfand-Shapiro [GSha] and Semyanistyi [Se63], is that cosine transforms are restrictions to S^{n-1} of the Fourier transforms of homogeneous distributions on $\mathbb{R}^n$.

Specifically, if we set

$$(\mathfrak{C}^\alpha f)(u)=\int_{S^{n-1}} f(v)|u\cdot v|^{\alpha-1}\,dv, \quad (\mathcal{F}\phi)(y)=\int_{\mathbb{R}^n}\phi(x)e^{ix\cdot y}\,dx,$$

$f\in L^1(S^{n-1})$, $\phi\in S(\mathbb{R}^n)$, then

$$\frac{1}{\Gamma(\alpha/2)}\int_{\mathbb{R}^n}\frac{(\mathfrak{C}^\alpha f)(y/|y|)}{|y|^{1-\alpha}}\,\overline{(\mathcal{F}\phi)(y)}\,dy \tag{1.3}$$
$$=\frac{c}{\Gamma((1-\alpha)/2)}\int_{\mathbb{R}^n}\frac{f(x/|x|)}{|x|^{n+\alpha-1}}\,\overline{\phi(x)}\,dx, \qquad c=2^{n+\alpha}\pi^{n-1/2}.$$

Integrals in this equality converge absolutely when $0<Re\,\alpha<1$ and extend by analyticity to all $\alpha\in\mathbb{C}$. An important observation due to Semyanistyi is that the Funk transform and its inverse are members of the analytic family of suitably normalized cosine transforms $\mathfrak{C}^\alpha$. This result was extended in [Ru02] to the Funk-Radon transforms over totally geodesic submanifolds of arbitrary dimension. The corresponding cosine transforms have found application in convex geometry [Ru08, Ru10a, RZ]. Some ideas from [Se63] were rediscovered by Koldobsky, who found remarkable application of the relevant Fourier transform technique to geometric problems; see [K05] and references therein.

4. In the last two decades a considerable attention was attracted to generalization of the cosine transform for functions on the Stiefel and Grassmann manifolds. The impetus was given in stochastic geometry by Matheron [Mat, p. 189], who conjectured that the higher-rank analogue of (1.2) (the precise definition is given later) is injective. Matheron's conjecture was disproved by Goodey and Howard [GH1], using topological results of Gluck and Warner [GW]. A self-contained Fourier analytic proof, versus [GH1], was suggested by Ournycheva and Rubin [OR05a, OR06]. Goodey and Zhang [GZ] applied higher-rank cosine transforms to the study of lower dimensional projections of convex bodies; see also [Goo, Spo]. Interesting connections to group representations can be found in the papers by Alesker and Bernstein [AB], Alesker [A], and Zhang [Zh2]. One should also mention fundamental publications by Blind, Herz, Faraut, Khekalo, Raïs, Petrov, Ricci and Stein, Stein, and others. They are devoted to analysis of homogeneous distributions on matrix spaces; see [Bli1, Bli2, FK, Herz, Kh1a, Kh2, OR05a, OR06, P70, Rai, RS, St2] and references therein.

1.2. **Plan of the paper and main results.** Below we give a brief account of main results and ideas of the paper. Further details and more results can be found in respective sections. Section 2 contains preliminaries. We recall basic facts about matrix spaces, Radon transforms over matrix planes, Riesz distributions, and the composite power function associated to the cone Ω of positive definite symmetric $m\times m$ matrices.

In Section 3 we define the higher-rank Funk transform $F_{m,k}$ as an operator (3.2) that sends functions on the Stiefel manifold $V_{n,m}$ of orthonormal m-frames in $\mathbb{R}^n$ to functions on $V_{n,k}$, where m and k may be different. We also define the dual transform $\overset{*}{F}_{m,k}$ and establish connection between our transforms and Radon transforms (1.1).

Theorem 1.1. *Let $1 \le m \le k \le n-m$. The mapping $f \to F_{m,k}f$ has a kernel*

$$\ker F_{m,k} = \{f \in L^1(V_{n,m}) : \int_{O(m)} f(v\gamma)\, d\gamma = 0 \ a.e.\}.$$

This theorem generalizes the well-known result in [Hel10], where the case $m=1$ is considered for C^∞ functions.

In Section 4 we introduce analytic families of non-normalized cosine and sine transforms for a pair of Stiefel manifolds; see (4.1)-(4.4). Necessary and sufficient conditions are obtained for these transforms to be represented by absolutely convergent integrals. Useful composition formulas, which generalize to $m>1$ the corresponding equalities for $m=1$ in [Ru02, Ru08], are derived. One of them is

$$\overset{*}{F}_{m,k}F_{m,k}f = \tilde{c}\, Q^{n-k-m} f, \tag{1.4}$$

where $Q^\alpha f$, $\alpha = n-k-m$, is the sine transform of f and a constant $\tilde{c}$ is explicitly evaluated. This formula is well-known for Radon transforms of different kinds, where the operator on the right-hand side stands for the relevant version of the Riesz potential; see, e.g., [Rad, Fug, Hel10, Ru06]. In the higher-rank case an analogue of the composition $\overset{*}{F}_{m,k}F_{m,k}$ played a crucial role in [Gri], however, it was not explicitly computed and presented only in the spectral form on highest weight vectors.

Section 5 is devoted to the Fourier analysis and analytic continuation of the cosine transform. Given $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$, we introduce *the composite cosine transform*

$$(T^{\boldsymbol{\lambda}}_{k,m}\varphi)(v) = \int_{V_{n,k}} \varphi(u)\, (v'uu'v)^{\boldsymbol{\lambda}}\, d_*u, \qquad v \in V_{n,m}, \tag{1.5}$$

where $(\cdot)^{\boldsymbol{\lambda}}$ denotes the power function of the cone Ω, "$\,'\,$" stands for the transposed matrix, and integration is performed against the invariant probability measure on $V_{n,k}$. Injectivity of such operators in the case

$k = m$ was studied in [OR05a, OR06]. For (1.5) and for the dual Funk transform $\overset{*}{F}_{m,k}$ we obtain an analogue of the Fourier functional equation (1.3); see Theorems 5.4 and 5.10. An important new feature is that a function f on the right-hand side must be replaced by a certain *complementary Radon transform* of f. The latter boils down to the identity operator, when $k = m$. The Fourier functional equation for the operator (1.5) provides complete information about meromorphic structure of distributions of the form

$$(1.6) \qquad \alpha \to (T^{\alpha}_{m,k} f, \omega), \qquad \omega \in C^{\infty}(V_{n,k}), \quad f \in L^1(V_{n,m}),$$

where $T^{\alpha}_{m,k}$ stands for the cosine or sine transform under consideration. We conjecture that if $f \in C^{\infty}(V_{n,m})$ then the polar set of the distribution (1.6) coincides with that of its pointwise counterpart

$$\alpha \to (T^{\alpha}_{m,k} f)(u) \quad \text{for each} \quad u \in V_{m,k}.$$

To the best of our knowledge, a proof of this fact represents an open problem.

In Section 6 we define normalized versions of the cosine and sine transforms. One of the main results of this section is Theorem 6.6, according to which an integrable right $O(m)$-invariant function f on $V_{n,m}$, can be reconstructed in the sense of distributions from its Funk transform $\varphi = F_{m,k} f$ by the formula

$$(1.7) \qquad \underset{\alpha = m+k-n}{a.c.} (\overset{*}{\mathscr{C}}{}^{\alpha}_{m,k} \varphi, \omega) = \varkappa_k (f, \omega), \qquad \omega \in C^{\infty}(V_{n,m}),$$

where

$$(\overset{*}{\mathscr{C}}{}^{\alpha}_{m,k} \varphi)(v) = \delta_{n,m,k}(\alpha) \int_{V_{n,k}} \varphi(u) \, (\det(v'uu'v))^{(\alpha-k)/2} \, d_* u, \quad v \in V_{n,m},$$

is the normalized dual cosine transform and the constant $\varkappa_k$ is explicitly evaluated. Thus, an inverse Funk transform is actually a member of the analytic family of suitably normalized dual cosine transforms, and all possible inversion formulas for the Radon transform (1.1) on Grassmannians can be regarded as different realizations of the analytic continuation in (1.7)[1]. Since $(f, \omega) = 0 \; \forall \omega \in C^{\infty}(V_{n,m})$ implies $f = 0$ a.e., the validity of Theorem 1.1 follows; see also Remark 3.1.

Similar inversion results are obtained for the cosine transforms. An interesting observation is that, *if $k = m$, then the Funk transform, the cosine transforms, and their inverses are (up to normalization) members of the same analytic family of operators.* This fact was known

[1] This statement does not work for the restricted Radon transform as in [GGŠ70, GGR], because the latter is, in fact, another operator.

before only for $m = 1$ and established using decomposition in spherical harmonics.

In Section 7 we suggest new realization of the inverse sine and Funk transforms in terms of powers of the Cayley-Laplace operator $\Delta = \det(\partial/\partial x_{i,j})$ in the space $\mathfrak{M}_{n,m}$ of $n \times m$ real matrices. These powers are associated with the Riesz potential. The resulting formulas differ in principle from those in [GGR, Gri, GR, Ka, P67]. They look much simpler, however, must be interpreted in the sense of distributions. For instance, if we write $x \in \mathfrak{M}_{n,m}$ in polar coordinates $x = vr^{1/2}$, $r \in \Omega$, $v \in V_{n,m}$, and denote $(E_\lambda f)(x) = (\det(r))^{\lambda/2} f(v)$, then, in the case of $n-k-m$ even, we have

$$f(v) = c_{m,k} \, (\Delta^\ell \, E_{-k} \, \overset{*}{F}_{m,k} \varphi)(v), \qquad \varphi = F_{m,k} f, \quad \ell = (n-k-m)/2,$$

where the constant $c_{m,k}$ is explicitly evaluated. If $n-k-m$ is an odd number, then fractional powers of Δ are implemented, and the resulting inversion formula is non-local; see Theorems 7.2 and 7.7. In the case $m = 1$ inversion formulas of this kind were suggested by Semyanistyi [Se63] and used in [Str81, Ru02]. The reasoning from those papers is unapplicable when $m > 1$. To get around this difficulty, we apply the relevant tools of harmonic analysis and several complex variables.

We conclude the paper by Appendix containing evaluation of an auxiliary integral and some comments on the celebrated paper [GGR] by Gelfand, Graev, and Rosu. The latter is devoted to the Radon transform (1.1) for a pair of Grassmannians. A keen reader can recognize our higher-rank cosine transform in [GGR, pp. 367, 368]. However, some important details in that paper are skipped. We reproduce the relevant calculations and explain basic difficulties.

Acknowledgements. I am indebted to many people for useful and pleasant discussions related to this paper. My special thanks go to Professors Tomoyuki Kakehi, Gestur Òlafsson, Elena Ournycheva, Angela Pasquale, Fulvio Ricci, Genkai Zhang. I am grateful to Susanna Dann who carefully read the original Funk papers. Her help was very valuable.

2. Preliminaries

In this section we establish our notation and recall some basic facts; see [OR04, OR05a, OR06] for more details.

2.1. **Notation and conventions.** Given a square matrix a, $|a|$ stands for the absolute value of $\det(a)$. We use standard notation $O(n)$ and $SO(n)$ for the orthogonal group and the special orthogonal group of

$\mathbb{R}^n$, respectively, with the normalized invariant measure of total mass 1. The abbreviation "*a.c.*" denotes analytic continuation.

Let $\mathfrak{M}_{n,m} \sim \mathbb{R}^{nm}$ be the space of real matrices $x = (x_{i,j})$ having n rows and m columns; $dx = \prod_{i=1}^{n}\prod_{j=1}^{m} dx_{i,j}$; x' is the transpose of x, $|x|_m = \det(x'x)^{1/2}$, I_m is the identity $m \times m$ matrix, and 0 stands for zero entries.

The Fourier transform of a function $\varphi \in L^1(\mathfrak{M}_{n,m})$ is defined by

$$(2.1) \qquad \hat{\varphi}(y) = (\mathcal{F}\varphi)(y) = \int_{\mathfrak{M}_{n,m}} e^{\mathrm{tr}(\mathrm{i}y'x)} \varphi(x)\, dx, \qquad y \in \mathfrak{M}_{n,m}\,.$$

The relevant Parseval equality has the form

$$(2.2) \qquad (\hat{\varphi}, \hat{\phi}) = (2\pi)^{nm}\,(\varphi, \phi), \qquad (\varphi, \phi) = \int_{\mathfrak{M}_{n,m}} \varphi(x)\overline{\phi(x)}\, dx.$$

This equality with ϕ in the Schwartz class $S(\mathfrak{M}_{n,m})$ of rapidly decreasing smooth functions is used to define the Fourier transform of the corresponding distributions. If $\phi \in S(\mathfrak{M}_{n,m})$ and $\check{\phi} = \mathcal{F}^{-1}\phi$ is the inverse Fourier transform of ϕ, then, clearly,

$$(2.3) \qquad \check{\phi}(x) = (2\pi)^{-nm}\hat{\phi}(-x), \qquad [\hat{\phi}]^{\wedge}(x) = (2\pi)^{nm}\phi(-x).$$

Let $\mathcal{S}_m \sim \mathbb{R}^{m(m+1)/2}$ be the space of $m \times m$ real symmetric matrices $s = (s_{i,j})$; $ds = \prod_{i \le j} ds_{i,j}$. We denote by $\Omega = \mathcal{P}_m$ the cone of positive definite matrices in $\mathcal{S}_m$; $\bar{\Omega}$ is the closure of Ω. For $r \in \Omega$ ($r \in \bar{\Omega}$), we write $r > 0$ ($r \ge 0$). Given a and b in S_m, the inequality $a > b$ means $a - b \in \Omega$ and the symbol $\int_a^b f(s)ds$ denotes the integral over the set $(a+\Omega)\cap(b-\Omega)$. The group $G = GL(m, \mathbb{R})$ of real non-singular $m \times m$ matrices g acts transitively on Ω by the rule $r \to grg'$. The corresponding G-invariant measure on Ω is [T, p. 18]

$$(2.4) \qquad d_*r = |r|^{-d}dr, \qquad d = (m+1)/2.$$

If T_m is the group of upper triangular $m \times m$ real matrices $t = (t_{i,j})$ with positive diagonal elements, then each $r \in \Omega$ has a unique representation $r = t't$.

The Siegel gamma function of Ω is defined by

$$(2.5) \qquad \Gamma_m(\alpha) = \int_{\Omega} \exp(-\mathrm{tr}(r))|r|^{\alpha} d_*r = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2).$$

This integral is absolutely convergent if and only if $Re\,\alpha > d - 1 = (m-1)/2$, and extends meromorphically with the polar set

$$(2.6) \qquad \{(m-1-j)/2 : j = 0, 1, 2, \dots\};$$

see [Gi], [FK], [T]. For the corresponding beta integral we have

$$\int_a^b |r-a|^{\alpha-d}|b-r|^{\beta-d}dr = B_m(\alpha,\beta)\,|b-a|^{\alpha+\beta-d}, \tag{2.7}$$

$$B_m(\alpha,\beta)=\frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\alpha+\beta)}, \qquad Re\,\alpha>d-1,\ Re\,\beta>d-1.$$

For $n \ge m$, let $V_{n,m}=\{v\in\mathfrak{M}_{n,m} : v'v=I_m\}$ be the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^n$. This is a homogeneous space with respect to the action $V_{n,m}\ni v\to\gamma v$, $\gamma\in O(n)$, so that $V_{n,m}=O(n)/O(n-m)$. We fix a measure dv on $V_{n,m}$, which is left $O(n)$-invariant, right $O(m)$-invariant, and normalized by

$$\sigma_{n,m}\equiv\int_{V_{n,m}} dv=\frac{2^m\pi^{nm/2}}{\Gamma_m(n/2)}, \tag{2.8}$$

[Mu, p. 70]. The notation $d_*v=\sigma_{n,m}^{-1}dv$ is used for the corresponding probability measure.

We denote by $G_{n,m}$ the Grassmann manifold of m-dimensional linear subspaces ξ of $\mathbb{R}^n$ equipped with the $O(n)$-invariant probability measure $d_*\xi$. Every right $O(m)$-invariant function $f(v)$ on $V_{n,m}$ can be identified with a function $\tilde f(\xi)$ by the formula $\tilde f(\{v\})=f(v)$, $\{v\}=v\mathbb{R}^m\in G_{n,m}$, so that $\int_{G_{n,m}}\tilde f(\xi)d_*\xi=\int_{V_{n,m}}f(v)d_*v$. Another identification is also possible, namely, $\overset{*}{f}(\{v\}^\perp)=f(v)$, $\{v\}^\perp\in G_{n,n-m}$.

We will be dealing with several coordinate systems on $\mathfrak{M}_{n,m}$ and $V_{n,m}$.

Lemma 2.1. (The polar decomposition). *Let* $x\in\mathfrak{M}_{n,m}$, $n\ge m$. *If* $\text{rank}(x)=m$, *then*

$$x=vr^{1/2}, \qquad v\in V_{n,m}, \qquad r=x'x\in\mathcal{P}_m, \tag{2.9}$$

and $dx=2^{-m}|r|^{(n-m-1)/2}drdv$.

For this statement see, e.g., [Herz], [Mu], [FK]. Decomposition (2.9) (but with $r\in\bar\Omega$) is valid for *any* matrix $x\in\mathfrak{M}_{n,m}$, cf. [Mu, p. 589].

Lemma 2.2. ([P70], [Ru06]) *If* $x\in\mathfrak{M}_{n,m}$, $\text{rank}(x)=m$, $n\ge m$, *then*

$$x=vt, \qquad v\in V_{n,m}, \qquad t\in T_m,$$

and

$$dx=\prod_{j=1}^m t_{j,j}^{n-j}\,dt_{j,j}\,dt_*dv, \qquad dt_*=\prod_{i<j}dt_{i,j}.$$

Lemma 2.3. (The bi-Stiefel decomposition). *Let k, m, and n be positive integers satisfying*

$$1 \le k \le n-1, \qquad 1 \le m \le n-1, \qquad k+m \le n.$$

(i) *Almost all matrices $v \in V_{n,m}$ can be represented in the form*

$$v = \begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix}, \qquad a \in \mathfrak{M}_{k,m}, \quad u \in V_{n-k,m}, \tag{2.10}$$

so that

(2.11)

$$\int\limits_{V_{n,m}} f(v)dv = \int\limits_{0<a'a<I_m} d\mu(a) \int\limits_{V_{n-k,m}} f\left(\begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix}\right) du,$$

$$d\mu(a) = |I_m - a'a|^{\delta} da, \quad \delta = (n-k)/2 - d, \quad d = (m+1)/2.$$

(ii) *If, moreover, $k \ge m$, then*

$$\int\limits_{V_{n,m}} f(v)dv = \int\limits_0^{I_m} d\nu(r) \int\limits_{V_{k,m}} dw \int\limits_{V_{n-k,m}} f\left(\begin{bmatrix} wr^{1/2} \\ u(I_m - r)^{1/2} \end{bmatrix}\right) du, \tag{2.12}$$

$$d\nu(r) = 2^{-m}|r|^{\gamma}|I_m - r|^{\delta} dr, \qquad \gamma = k/2 - d.$$

For $k = m$, this statement is due to [Herz, p. 495]. The proof of Herz was extended in [GR] to all $k+m \le n$ and simplified in [Zh1]; see also [OlR].

Lemma 2.4. *Let $u \in V_{n,k}$, $v \in V_{n,m}$; $1 \le k, m \le n$. If f is a function of $m \times k$ matrices, then*

$$\int\limits_{V_{n,k}} f(v'u)\, d_*u = \int\limits_{V_{n,m}} f(v'u)\, d_*v. \tag{2.13}$$

Proof. We should observe that formally the left-hand side is a function of v, while the right-hand side is a function of u. In fact, both are constant. To prove (2.13), let $G = O(n)$, $g \in G$, $g_1 = g^{-1}$. The left-hand side is

$$\int_G f(v'gu)\, dg = \int_G f((g_1 v)'u)\, dg_1,$$

which equals the right-hand side. □

The following statement is a particular case of Lemma 2.5 from [GR].

Lemma 2.5. *Let* $A \in \mathfrak{M}_{k,\ell},\ S = A'A \in \mathcal{P}_\ell,\ \ell \le m < k,\ m+\ell \le k,$

$$\delta = (\ell+1)/2, \qquad c = 2^{-\ell}\sigma_{k-m,\ell}\sigma_{m,\ell}/\sigma_{k,\ell}.$$

Then

$$\int_{V_{k,m}} f(A'\omega)\, d_*\omega \tag{2.14}$$
$$= c\, |S|^{\delta-k/2} \int_0^S |S-s|^{(k-m)/2-\delta}\, |s|^{m/2-\delta} ds \int_{V_{m,\ell}} f(s^{1/2}\theta)\, d_*\theta.$$

2.2. **The composite power function.** Let Ω be the cone of positive definite symmetric $m\times m$ matrices. Given $r=(r_{i,j})\in\Omega$, let $\Delta_0(r)=1$, $\Delta_1(r)=r_{1,1},\ \Delta_2(r),\ \ldots,\ \Delta_m(r)=|r|$ be the corresponding principal minors, which are strictly positive. For $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_m)\in\mathbb{C}^m$, the composite power function of the cone Ω is defined by

$$\begin{aligned} r^{\boldsymbol{\lambda}} &= \prod_{i=1}^m \left[\frac{\Delta_i(r)}{\Delta_{i-1}(r)}\right]^{\lambda_i/2} \\ &= \Delta_1(r)^{\frac{\lambda_1-\lambda_2}{2}} \ldots \Delta_{m-1}(r)^{\frac{\lambda_{m-1}-\lambda_m}{2}} \Delta_m(r)^{\frac{\lambda_m}{2}}. \end{aligned} \tag{2.15}$$

If $r=t't,\ t=(t_{i,j})\in T_m$, then

$$r^{\boldsymbol{\lambda}} = \prod_{j=1}^m t_{j,j}^{\lambda_j}. \tag{2.16}$$

This implies the following equalities:

$$r^{\boldsymbol{\lambda}+\boldsymbol{\mu}} = r^{\boldsymbol{\lambda}}\, r^{\boldsymbol{\mu}}, \quad r^{\boldsymbol{\lambda}+\boldsymbol{\alpha}_0} = r^{\boldsymbol{\lambda}}|r|^{\alpha/2}, \quad \boldsymbol{\alpha}_0=(\alpha,\ldots,\alpha); \tag{2.17}$$
$$(t'rt)^{\boldsymbol{\lambda}} = (t't)^{\boldsymbol{\lambda}}\, r^{\boldsymbol{\lambda}}, \quad t\in T_m. \tag{2.18}$$

The reverses of $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_m)$ and $r=(r_{i,j})\in\Omega$ are defined by

$$\boldsymbol{\lambda}_* = (\lambda_m,\ldots,\lambda_1); \qquad r_* = \omega r \omega, \qquad \omega = \begin{bmatrix} 0 & & 1 \\ & \cdot & \\ & \cdot & \\ 1 & & 0 \end{bmatrix}, \tag{2.19}$$

so that

$$(\boldsymbol{\lambda}_*)_j = \lambda_{m-j+1}, \qquad (r_*)_{i,j} = r_{m-i+1,m-j+1}.$$

We have

$$r^{\boldsymbol{\lambda}_*} = (r^{-1})_*^{-\boldsymbol{\lambda}}, \qquad (r^{-1})^{\boldsymbol{\lambda}} = r_*^{-\boldsymbol{\lambda}_*}. \tag{2.20}$$

The relevant gamma function is defined by

$$\Gamma_\Omega(\boldsymbol{\lambda}) = \int_\Omega r^{\boldsymbol{\lambda}} e^{-\mathrm{tr}(r)} d_*r = \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma((\lambda_j - j+1)/2); \tag{2.21}$$

see, e.g., [FK, p. 123]. The integral in (2.21) converges absolutely if and only if $Re\,\lambda_j > j-1$ for all $j = 1, \dots, m$, and extends meromorphically to all $\lambda \in \mathbb{C}^m$. The following relation holds:

$$\int\limits_{\Omega} r^{\boldsymbol{\lambda}} e^{-\mathrm{tr}(rs)} d_* r = \Gamma_\Omega(\boldsymbol{\lambda})\, s_*^{-\boldsymbol{\lambda}_*}, \qquad s \in \Omega. \tag{2.22}$$

If $\boldsymbol{\lambda}_0 = (\lambda, \dots, \lambda)$ ($\in \mathbb{C}^m$), then (cf. (2.5))

$$r^{\boldsymbol{\lambda}_0} = |r|^{\lambda/2}, \qquad \Gamma_\Omega(\boldsymbol{\lambda}_0) = \Gamma_m(\lambda/2). \tag{2.23}$$

2.3. **The Radon transform on the space of matrices.** The main references for this topic are [OR04], [OR08a], [OR08b], [P70], [Sh1], [Sh2]. Close results can be found in [GK3, Gra]. We fix positive integers k, n, and m, $0 < k < n$, and let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. For $u \in V_{n,k}$ and $t \in \mathfrak{M}_{k,m}$, the linear manifold

$$\tau = \tau(u,t) = \{x \in \mathfrak{M}_{n,m} : u'x = t\} \tag{2.24}$$

is called a *matrix* $(n-k)$*-plane* in $\mathfrak{M}_{n,m}$. We denote by $\mathfrak{T}$ the set of all such planes and consider the Radon transform

$$(\mathcal{R}_k f)(\tau) = \int_{x \in \tau} f(x),$$

that sends a function f on $\mathfrak{M}_{n,m}$ to a function $\mathcal{R}_k f$ on $\mathfrak{T}$. Precise meaning of this integral is the following:

$$(\mathcal{R}_k f)(u,t) = \int_{\mathfrak{M}_{n-k,m}} f\left(g_u \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega, \tag{2.25}$$

where $g_u \in SO(n)$ is a rotation satisfying

$$g_u u_0 = u, \qquad u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \in V_{n,k}. \tag{2.26}$$

The next statement is a matrix generalization of the projection-slice theorem. It links together the Fourier transform (2.1) and the Radon transform (2.25). In the case $m = 1$, this theorem can be found in [Na, p. 11] ($k = 1$) and [Ke, p. 283] (any $1 \le k < n$).

Theorem 2.6. ([OR05b]) *For* $f \in L^1(\mathfrak{M}_{n,m})$ *and* $1 \le m \le k$,

$$(\mathcal{F} f)(\xi b) = [\tilde{\mathcal{F}}(\mathcal{R}_k f)(u, \cdot)](b), \quad u \in V_{n,k}, \quad b \in \mathfrak{M}_{k,m} \tag{2.27}$$

where $\tilde{\mathcal{F}}$ *stands for the Fourier transform on* $\mathfrak{M}_{k,m}$.

2.4. **Riesz potentials and the Cayley-Laplace operator.** The *Riesz distribution* h_α on $\mathfrak{M}_{n,m}$ is defined by

$$(h_\alpha, f) = a.c. \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} |x|_m^{\alpha-n} f(x) dx, \qquad f \in \mathcal{S}(\mathfrak{M}_{n,m}), \tag{2.28}$$

$$\gamma_{n,m}(\alpha) = \frac{2^{\alpha m} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \qquad \alpha \neq n-m+1, \, n-m+2, \dots . \tag{2.29}$$

For $Re\,\alpha > m-1$, the distribution h_α is regular and agrees with the ordinary function $h_\alpha(x) = |x|_m^{\alpha-n}/\gamma_{n,m}(\alpha)$. The *Riesz potential* of a function $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ is defined by

$$(I^\alpha f)(x) = (h_\alpha, f_x), \qquad f_x(\cdot) = f(x - \cdot). \tag{2.30}$$

For $Re\,\alpha > m-1$, $\alpha \neq n-m+1, \, n-m+2, \dots$, (2.30) is represented in the classical form by the absolutely convergent integral

$$(I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} f(x-y) |y|_m^{\alpha-n} dy. \tag{2.31}$$

This integral operator is well known in the rank-one case $m = 1$ [Ru96, SKM, Sa2, St1].

The *Cayley-Laplace operator* Δ on the space $\mathfrak{M}_{n,m}$ is defined by

$$\Delta = \det(\partial' \partial). \tag{2.32}$$

Here ∂ is an $n \times m$ matrix whose entries are partial derivatives $\partial/\partial x_{i,j}$. In the Fourier transform terms, the action of Δ represents a multiplication by the polynomial $(-1)^m |y|_m^2$.

Theorem 2.7. [Ru06, Theorem 5.2]
Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\alpha \in \mathbb{C}$, $\alpha \neq n-m+1, \, n-m+2, \dots$. Then

$$(h_\alpha, f) = (2\pi)^{-nm} (|y|_m^{-\alpha}, (\mathcal{F}f)(y)), \tag{2.33}$$

$$(-1)^{mk} \Delta^k h_{\alpha+2k} = h_\alpha, \quad I^{-2k} = (-1)^{mk} \Delta^k; \quad k = 0, 1, 2, \dots . \tag{2.34}$$

Thus, one can formally write

$$I^\alpha = [(-1)^m \Delta]^{-\alpha/2}. \tag{2.35}$$

The next statement generalizes (2.33) to the case of composite power functions.

Lemma 2.8. *Let $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$. Then for all $\boldsymbol{\lambda} \in \mathbb{C}^m$,*

$$\int\limits_{\mathfrak{M}_{n,m}} \frac{(y'y)^{\boldsymbol{\lambda}}}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{n}_0)} \overline{(\mathcal{F}\phi)(y)} \, dy = c_{\boldsymbol{\lambda}} \int\limits_{\mathfrak{M}_{n,m}} \frac{(x'x)_*^{-\boldsymbol{\lambda}_* - \mathbf{n}_0}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} \overline{\phi(x)} \, dx, \tag{2.36}$$

where $c_{\boldsymbol{\lambda}} = 2^{nm+|\boldsymbol{\lambda}|} \pi^{nm/2}$, $|\boldsymbol{\lambda}| = \lambda_1 + \ldots + \lambda_m$.

Remark 2.9. Both sides of (2.33) and (2.36) are understood in the sense of analytic continuation. Integrals in (2.36) converge simultaneously when $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ belongs to the set $\mathfrak{L} = \check{\mathfrak{L}} \cap \tilde{\mathfrak{L}}$, where

$$\check{\mathfrak{L}} = \{\boldsymbol{\lambda} : \; Re\, \lambda_j > j - n - 1 \quad \text{for each } j = 1, \dots, m\},$$
$$\tilde{\mathfrak{L}} = \{\boldsymbol{\lambda} : \; Re\, \lambda_j < j - m \quad \text{for each } j = 1, \dots, m\}.$$

The diagonal $\lambda_1 = \ldots = \lambda_m = \lambda$ does not belong to $\mathfrak{L}$. This explains essential difficulties when one tries to prove (2.33) directly as in [Ru06]. Formula (2.36) was established by Khekalo [Kh1a], who extended the argument from [St1, Chapter III, Sec. 3.4] to functions of matrix argument. Khekalo's proof was reproduced in [OR05a, p. 61].

3. The Higher-Rank Funk Transform

3.1. **Definitions and duality.** The classical Funk transform on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is defined by

$$(3.1) \qquad (Ff)(u) = \int_{\{v \in S^{n-1} : \, u \cdot v = 0\}} f(v)\, d_u v, \qquad u \in S^{n-1};$$

see, e.g., [GGG, Hel10]. We suggest the following generalization of (3.1), in which $u \in V_{n,k}$ and $v \in V_{n,m}$ are elements of respective Stiefel manifolds, $1 \le k, m \le n-1$. The *higher-rank Funk transform* sends a function f on $V_{n,m}$ to a function $F_{m,k} f$ on $V_{n,k}$ by the formula

$$(3.2) \qquad (F_{m,k} f)(u) = \int_{\{v \in V_{n,m} : \, u'v = 0\}} f(v)\, d_u v, \qquad u \in V_{n,k}.$$

The corresponding dual transform

$$(3.3) \qquad (\overset{*}{F}_{m,k} \varphi)(v) = \int_{\{u \in V_{n,k} : \, v'u = 0\}} \varphi(u)\, d_v u, \qquad v \in V_{n,m},$$

acts in the opposite direction. The condition $u'v = 0$ means that subspaces $u\mathbb{R}^k \in G_{n,k}$ and $v\mathbb{R}^m \in G_{n,m}$ are mutually orthogonal. Hence, necessarily, $k + m \le n$. The case $k = m$, when both f and its Funk transform live on the same manifold, is of particular importance and coincides with (3.1) when $k = m = 1$. We denote $F_m = F_{m,m}$.

To give the new transforms precise meaning, we set $G = O(n)$,

$$(3.4) \qquad K_0 \;=\; \left\{ \tau \in G : \tau = \begin{bmatrix} \gamma & 0 \\ 0 & I_k \end{bmatrix}, \quad \gamma \in O(n-k) \right\},$$

$$(3.5) \qquad \check{K}_0 \;=\; \left\{ \rho \in G : \rho = \begin{bmatrix} \delta & 0 \\ 0 & I_m \end{bmatrix}, \quad \delta \in O(n-m) \right\},$$

$$(3.6)\quad u_0=\left[\begin{array}{c}0\\ I_k\end{array}\right],\quad \check{u}_0=\left[\begin{array}{c}I_k\\ 0\end{array}\right];\qquad v_0=\left[\begin{array}{c}0\\ I_m\end{array}\right],\quad \check{v}_0=\left[\begin{array}{c}I_m\\ 0\end{array}\right],$$

$$u_0,\check{u}_0\in V_{n,k};\qquad v_0,\check{v}_0\in V_{n,m}.$$

Then (3.2) and (3.3) can be explicitly written as

$$(F_{m,k}f)(u)=\int_{V_{n-k,m}} f\left(g_u\left[\begin{array}{c}\omega\\ 0\end{array}\right]\right)d_*\omega=\int_{K_0} f(g_u\tau\check{v}_0)\,d\tau, \tag{3.7}$$

$$(\overset{*}{F}_{m,k}\varphi)(v)=\int_{V_{n-m,k}} \varphi\left(g_v\left[\begin{array}{c}\theta\\ 0\end{array}\right]\right)d_*\theta=\int_{\check{K}_0} \varphi(g_v\rho\check{u}_0)\,d\rho, \tag{3.8}$$

where g_u and g_v are orthogonal transformations satisfying $g_u u_0 = u$ and $g_v v_0 = v$, respectively.

Remark 3.1. Since the measure $d_*\omega$ is right $O(m)$-invariant, then, for all $u\in V_{n,k}$,

$$(F_{m,k}f)(u)=(F_{m,k}\tilde{f})(u),\qquad \tilde{f}(v)=\int_{O(m)} f(v\gamma)\,d\gamma \tag{3.9}$$

(similarly for $(\overset{*}{F}_{m,k}\varphi)(v)$). Hence, the set of all $f\in L^1(V_{n,m})$, for which $\tilde{f}(v)=0$ a.e., is a subset of $\ker F_{m,k}$; cf. Theorem 1.1.

Lemma 3.2. *Let $1\le k,m\le n-1;\ k+m\le n$. Then*

$$\int_{V_{n,k}} (F_{m,k}f)(u)\,\varphi(u)\,d_*u=\int_{V_{n,m}} f(v)\,(\overset{*}{F}_{m,k}\varphi)(v)\,d_*v \tag{3.10}$$

provided that at least one of these integrals is finite when f and φ are replaced by $|f|$ and $|\varphi|$, respectively.

Proof. We write the left-hand side as

$$\int_G (F_{m,k}f)(gu_0)\varphi(gu_0)\,dg=\int_G \varphi(gu_0)\,dg\int_{K_0} f(g\tau\check{v}_0)\,d\tau.$$

Let $\varkappa\in O(n)$ be such that $u_0=\varkappa\check{u}_0$ and denote

$$\zeta=\left[\begin{array}{ccc}0&0&I_m\\ 0&I_{n-m-k}&0\\ I_k&0&0\end{array}\right]. \tag{3.11}$$

Keeping in mind that $\zeta v_0 = \check{v}_0$, $\zeta' u_0 = \check{u}_0$, and $\check{K}_0$ is a stabilizer of v_0, we continue:

$$
\begin{aligned}
l.h.s &= \int_G \varphi(g\varkappa\check{u}_0)\,dg \int_{\check{K}_0} d\rho \int_{K_0} f(g\tau\zeta\rho' v_0)\,d\tau \\
&= \int_G f(\lambda v_0)\,d\lambda \int_{\check{K}_0} d\rho \int_{K_0} \varphi(\lambda\rho\zeta'\tau'\varkappa\check{u}_0)\,d\tau \\
&\quad \text{(note that } \zeta'\tau'\varkappa\check{u}_0 = \zeta'\tau' u_0 = \zeta' u_0 = \check{u}_0) \\
&= \int_G f(\lambda v_0)\,d\lambda \int_{\check{K}_0} \varphi(\lambda\rho\check{u}_0)\,d\rho \\
&= \int_{V_{n,m}} f(v)\,(\overset{*}{F}_{m,k}\varphi)(v)\,d_*v,
\end{aligned}
$$

as desired. □

Corollary 3.3. *If $f \in L^1(V_{n,m})$, then the Funk transform $(F_{m,k}f)(u)$ exists as an absolutely convergent integral for almost all $u \in V_{n,k}$. Moreover,*

$$\int_{V_{n,k}} (F_{m,k}f)(u)\,d_*u = \int_{V_{n,m}} f(v)\,d_*v.$$

A similar statement holds for the dual transform $\overset{*}{F}_{m,k}\varphi$.

Example 3.4. *Let $\varphi(u) = |v_0' u u' v_0|^{(\alpha-k)/2}$, $v_0 = \left[\begin{array}{c} 0 \\ I_m \end{array}\right]$, $u \in V_{n,k}$. If $1 \le m \le k \le n-m$, $Re\,\alpha > m-1$, then*

$$(\overset{*}{F}_{m,k}\varphi)(v) = c_\alpha\,|I_m - v_0' v v' v_0|^{(\alpha-k)/2}, \qquad v \in V_{n,m}, \tag{3.12}$$

$$c_\alpha = \frac{\Gamma_m((n-m)/2)}{\Gamma_m(k/2)}\,\frac{\Gamma_m(\alpha/2)}{\Gamma_m((\alpha+n-m-k)/2)}.$$

Proof. The condition $m \le k$ has a simple explanation: if $k < m$ then $|v_0' u u' v_0| \equiv 0$. Let us prove (3.12). By (3.8),

$$(\overset{*}{F}_{m,k}\varphi)(v) = \int_{V_{n-m,k}} |z_\theta z_\theta'|^{(\alpha-k)/2}\,d_*\theta, \qquad z_\theta = v_0' g_v \left[\begin{array}{c} \theta \\ 0 \end{array}\right].$$

We write

$$g_v' v_0 = \left[\begin{array}{c} A \\ B \end{array}\right], \qquad A \in \mathfrak{M}_{n-m,m}, \quad B \in \mathfrak{M}_{m,m}, \tag{3.13}$$

so that $z_\theta = A'\theta$. Then we represent A in polar coordinates $A = wS^{1/2}$, $S = A'A \in \mathcal{P}_m$, $w \in V_{n-m,m}$. This gives

$$(\overset{*}{F}_{m,k}\varphi)(v) = \int_{V_{n-m,k}} |A'\theta\theta' A|^{(\alpha-k)/2}\,d_*\theta = c_\alpha\,|S|^{(\alpha-k)/2},$$

where

$$c_\alpha = \int_{V_{n-m,k}} |w'\theta\theta' w|^{(\alpha-k)/2}\, d_*\theta$$

can be computed using formula (8.16) (with n replaced by $n-m$ and λ by $\alpha - k$). Let $\varkappa = \left[\begin{array}{c} I_{n-m} \\ 0 \end{array} \right] \in V_{n,n-m}$. Then, by (3.13), $A = \varkappa' g'_v v_0$ and

$$S = A'A = v'_0 g_v \varkappa\varkappa' g'_v v_0 = v'_0 g_v (I_n - v_0 v'_0) g'_v v_0 = I_m - v'_0 v v' v_0.$$

This gives the result. □

Example 3.4 and duality (3.10) yield

$$\int_{V_{n,k}} (F_{m,k} f)(u)\, |v'_0 u u' v_0|^{(\alpha-k)/2}\, d_*u = c_\alpha \int_{V_{n,m}} f(v)\, |I_m - v'_0 v v' v_0|^{(\alpha-k)/2}\, d_*v.$$

Owing to $O(n)$-invariance, one can replace v_0 in this formula by an arbitrary $w \in V_{n,m}$. This gives the following statement.

Lemma 3.5. *Let $w \in V_{n,m}$, $Re\, \alpha > m-1$,*

$$1 \le m \le k \le n-m. \tag{3.14}$$

Then

$$\int_{V_{n,k}} (F_{m,k} f)(u)\, |w' u u' w|^{(\alpha-k)/2}\, d_*u = c_\alpha \int_{V_{n,m}} f(v)\, |I_m - w' v v' w|^{(\alpha-k)/2}\, d_*v,$$

provided that the integral on the right-hand side is absolutely convergent.

3.2. **Connection with Radon transforms on Grassmannians.** There is an intimate connection between the higher-rank Funk transform and the well-known Radon transforms for a pair of Grassmann manifolds by inclusion. The latter were studied by several authors, who used different methods; see, e.g., [GGR, GGŠ70, Gri, GR, Ka, P67, Zh1, Zh2].

Suppose that $f(v)$ and $\varphi(u)$ are right $O(m)$-invariant and $O(k)$-invariant functions on $V_{n,m}$ and $V_{n,k}$, respectively. We define the corresponding functions on Grassmannians by setting

$$\tilde f(\xi) = \overset{*}{f}(\tau) = f(v), \qquad \xi = \{v\} \in G_{n,m}, \quad \tau = \{v\}^\perp \in G_{n,n-m}, \tag{3.15}$$

$$\tilde\varphi(\zeta) = \overset{*}{\varphi}(\eta) = \varphi(u), \qquad \zeta = \{u\} \in G_{n,k}, \quad \eta = \{u\}^\perp \in G_{n,n-k}, \tag{3.16}$$

and consider the Radon transforms

$$(R_{m,n-k} \tilde f)(\eta) = \int_{\xi \subset \eta} \tilde f(\xi)\, d_\eta \xi, \quad (\overset{*}{R}_{m,n-k} \overset{*}{\varphi})(\xi) = \int_{\eta \supset \xi} \overset{*}{\varphi}(\eta)\, d_\xi \eta; \tag{3.17}$$

$$(3.18)\qquad (R_{k,n-m}\tilde{\varphi})(\tau)=\int\limits_{\zeta\subset\tau}\tilde{\varphi}(\zeta)\,d_\tau\zeta,\quad (\overset{*}{R}_{k,n-m}\overset{*}{f})(\zeta)=\int\limits_{\tau\supset\zeta}\overset{*}{f}(\tau)\,d_\zeta\tau.$$

Here $d_\eta\xi,\ d_\xi\eta,\ d_\tau\zeta,\ d_\zeta\tau$ denote the relevant probability measures. Then (3.7) and (3.8) imply

$$(3.19)\qquad (F_{m,k}f)(u)=(R_{m,n-k}\tilde{f})(\eta)=(\overset{*}{R}_{k,n-m}\overset{*}{f})(\zeta),$$

$$(3.20)\qquad (\overset{*}{F}_{m,k}\varphi)(v)=(R_{k,n-m}\tilde{\varphi})(\tau)=(\overset{*}{R}_{m,n-k}\overset{*}{\varphi})(\xi).$$

These equalities hold under the standard assumption

$$\dim G_{n,m}\le\dim G_{n,n-k}\quad\text{or}\quad\dim G_{n,k}\le\dim G_{n,n-m},$$

according to which it is usually assumed $1\le m\le k\le n-m$.

4. Cosine and Sine Transforms. Composition formulas

Lemma 3.5 suggests to introduce the following integral operators:

$$(4.1)\qquad (\mathcal{C}^\alpha_{m,k}f)(u)\ =\ \int_{V_{n,m}}f(v)\,|v'uu'v|^{(\alpha-k)/2}\,d_*v,$$

$$(4.2)\qquad (\overset{*}{\mathcal{C}}{}^\alpha_{m,k}\varphi)(v)\ =\ \int_{V_{n,k}}\varphi(u)\,|v'uu'v|^{(\alpha-k)/2}\,d_*u,$$

$$(4.3)\qquad (\mathcal{S}^\alpha_{m,k}f)(u)\ =\ \int_{V_{n,m}}f(v)\,|I_m-v'uu'v|^{(\alpha+k-n)/2}\,d_*v,$$

$$(4.4)\qquad (\overset{*}{\mathcal{S}}{}^\alpha_{m,k}\varphi)(v)\ =\ \int_{V_{n,k}}\varphi(u)\,|I_m-v'uu'v|^{(\alpha+k-n)/2}\,d_*u.$$

$$u\in V_{n,k},\qquad v\in V_{n,m},\qquad 1\le m,k\le n-1.$$

We call $\mathcal{C}^\alpha_{m,k}f$ and $\mathcal{S}^\alpha_{m,k}f$ the *cosine transform* and the *sine transform* of f, respectively. Integrals $\overset{*}{\mathcal{C}}{}^\alpha_{m,k}\varphi$ and $\overset{*}{\mathcal{S}}{}^\alpha_{m,k}\varphi$ are called the *dual cosine transform* and the *dual sine transform*. The terminology stems from the fact that, in the case $k=m=1$, when u and v are unit vectors,

$$|v'uu'v|=(u\cdot v)^2=\cos^2\omega,\quad |I_m-v'uu'v|=1-(u\cdot v)^2=\sin^2\omega,$$

where ω is the angle between u and v; see also [A, AB, GR, OR06, OR05a, Zh2], regarding higher-rank analogues of the cosine transform in the language of Grassmannians.

Remark 4.1. When dealing with operators (4.1) and (4.2), we restrict our consideration to the case $m\le k$, because, if $m>k$, then $|v'uu'v|=$

0 for all $v \in V_{n,m}$ and all $u \in V_{n,k}$. Similarly, for (4.3) and (4.4), we assume $m \le n-k$, because, if $m > n-k$, then

$$|I_m - v'uu'v| = |I_m - v'\mathrm{Pr}_{\{u\}}v| = |v'\mathrm{Pr}_{\{u\}^\perp}v| = |v'\tilde{u}\tilde{u}'v| = 0$$

(here $\tilde{u}$ is an arbitrary $(n-k)$-frame orthogonal to $\{u\}$). The case $k=n$, when $v'uu'v \equiv I_m$, is also not interesting. Clearly,

$$(\mathcal{S}^\alpha_{m,k}f)(u) = (\mathcal{C}^\alpha_{m,n-k}f)(\tilde{u}) = \int_{V_{n,m}} f(v)\, |v'\tilde{u}\tilde{u}'v|^{(\alpha-(n-k))/2}\, d_*v. \tag{4.5}$$

The case $k=m$, when $\mathcal{C}^\alpha_{m,k}$ and $\mathcal{S}^\alpha_{m,k}$ coincide with their duals, are of particular importance. In this case we denote

$$(M^\alpha f)(u) = \int_{V_{n,m}} f(v)\, |v'uu'v|^{(\alpha-m)/2}\, d_*v, \quad 1 \le m \le n-1, \tag{4.6}$$

$$(Q^\alpha f)(u) = \int_{V_{n,m}} f(v)\, |I_m - v'uu'v|^{(\alpha+m-n)/2}\, d_*v, \quad 2m \le n, \tag{4.7}$$

where $u \in V_{n,m}$. As we shall see below, Q^α serves (after suitable normalization) as a substitute for the Riesz potential operator in the framework of the corresponding Radon theory.

The following statement gives precise information about convergence of integrals (4.1)-(4.7).

Theorem 4.2. *Let $f \in L^1(V_{n,m})$, $\varphi \in L^1(V_{n,k})$, $1 \le m,k \le n-1$.*
(i) *Integrals (4.1)-(4.4) converge absolutely almost everywhere if and only if $Re\,\alpha > m-1$.*
(ii) *If $1 \le m \le k \le n-1$, then*

$$\int_{V_{n,k}} (\mathcal{C}^\alpha_{m,k}f)(u)\, d_*u = c_1 \int_{V_{n,m}} f(v)\, d_*v, \tag{4.8}$$

$$\int_{V_{n,m}} (\overset{*}{\mathcal{C}}{}^\alpha_{m,k}\varphi)(v)\, d_*v = c_1 \int_{V_{n,k}} \varphi(u)\, d_*u, \tag{4.9}$$

$$c_1 = \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)}{\Gamma_m(k/2)\,\Gamma_m((\alpha+n-k)/2)}.$$

(iii) *If $1 \le m \le n-k$, then*

$$\int_{V_{n,k}} (\mathcal{S}^\alpha_{m,k}f)(u)\, d_*u = c_2 \int_{V_{n,m}} f(v)\, d_*v, \tag{4.10}$$

$$\int_{V_{n,m}} (\overset{*}{\mathcal{S}}{}^\alpha_{m,k}\varphi)(v)\, d_*v = c_2 \int_{V_{n,k}} \varphi(u)\, d_*u, \tag{4.11}$$

$$c_2 = \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((n-k)/2)\,\Gamma_m((\alpha+k)/2)}.$$

Proof. Equalities (4.8) and (4.9) hold by Fubini's theorem, owing to Lemma 2.4 and (8.16). The proof of (4.10) and (4.11) is similar. It suffices to note that

$$\int_{V_{n,k}} |I_m - v'uu'v|^{(\alpha+k-n)/2} \, d_*u = \int_{V_{n,m}} |v'\tilde{u}\tilde{u}'v|^{(\alpha+k-n)/2} \, d_*v$$

for any $\tilde{u} \in V_{n,n-k}$, so that (8.16) is applicable. The validity of (i) follows from the proof of (4.8)-(4.11). □

Functions $(\mathcal{C}^\alpha_{m,k} f)(u)$ and $(\mathcal{S}^\alpha_{m,k} f)(u)$ are right $O(k)$-invariant. Similarly, $(\overset{*}{\mathcal{C}}{}^\alpha_{m,k}\varphi)(v)$ and $(\overset{*}{\mathcal{S}}{}^\alpha_{m,k}\varphi)(v)$ are right $O(m)$-invariant. Hence, transformations (4.1)-(4.4) actually take functions on the Stiefel manifolds to functions on the corresponding Grassmann manifolds. If f and φ are right-invariant in the suitable sense, our operators actually act from one Grassmannian to another.

Below we derive a series of formulas connecting cosine, sine, and Funk transforms. Similar formulas and their applications in the case $m = 1$ can be found in [Ru02, Ru08].

Theorem 4.3. *Let $f \in L^1(V_{n,m})$, $1 \le m \le k \le n-m$. If $Re\, \alpha > m-1$, then*

$$\overset{*}{\mathcal{C}}{}^\alpha_{m,k} F_{m,k} f = \overset{*}{F}_{m,k} \mathcal{C}^\alpha_{m,k} f = c_\alpha \, Q^{\alpha+n-k-m} f, \tag{4.12}$$

$$c_\alpha = \frac{\Gamma_m((n-m)/2)\, \Gamma_m(\alpha/2)}{\Gamma_m(k/2)\, \Gamma_m((\alpha+n-m-k)/2)}.$$

If $Re\, \alpha > k-1$, then

$$\overset{*}{\mathcal{C}}{}^\alpha_{m,k} F_{m,k} f = \overset{*}{F}_{m,k} \mathcal{C}^\alpha_{m,k} f = \tilde{c}_\alpha \, M^{\alpha+m-k} F_m f, \tag{4.13}$$

$$\tilde{c}_\alpha = \frac{\Gamma_m(m/2)\, \Gamma_m(\alpha/2)}{\Gamma_m(k/2)\, \Gamma_m((\alpha+m-k)/2)}.$$

Proof. The equality $\overset{*}{\mathcal{C}}{}^\alpha_{m,k} F_{m,k} f = c_\alpha \, Q^{\alpha+n-k-m} f$ in (4.12) mimics Lemma 3.5. The equality $\overset{*}{\mathcal{C}}{}^\alpha_{m,k} F_{m,k} f = \overset{*}{F}_{m,k} \mathcal{C}^\alpha_{m,k} f$ holds by duality:

$$\begin{aligned}(\overset{*}{F}_{m,k} \mathcal{C}^\alpha_{m,k} f, \omega) &= (f, \overset{*}{\mathcal{C}}{}^\alpha_{m,k} F_{m,k} \omega) = c_\alpha \, (f, Q^{\alpha+n-k-m}\omega) \\ &= c_\alpha \, (Q^{\alpha+n-k-m} f, \omega), \qquad \omega \in C^\infty(V_{n,m}).\end{aligned}$$

Note that by Theorem 4.2, $Q^{\alpha+n-k-m} f \in L^1(V_{n,m})$, because the conditions $k \le n-m$ and $Re\, \alpha > m-1$ imply $Re\, \alpha + n - k - m > m-1$.

To obtain (4.13), we first write (4.12) with $k = m$. This gives

$$M^\alpha F_m f = \frac{\Gamma_m((n-m)/2)\, \Gamma_m(\alpha/2)}{\Gamma_m(m/2)\, \Gamma_m((\alpha+n-2m)/2)} \, Q^{\alpha+n-2m} f. \tag{4.14}$$

Then we replace α by $\alpha-k+m$ in (4.14) to get $Q^{\alpha+n-k-m}f$ on the right-hand side and compare the result with (4.12). □

Theorem 4.4. *Let* $f \in L^1(V_{n,m})$*,* $Re\,\alpha > m-1$*. If* $k \le n-m$*,* $u \in V_{n,k}$*, then*

$$(F_{m,k}M^\alpha f)(u) = c_{n,k,m}(\alpha)\,(\mathcal{S}^{\alpha+n-k-m}_{m,k}f)(u), \tag{4.15}$$

$$c_{n,k,m}(\alpha) = \frac{\Gamma_m((n-k)/2)\,\Gamma_m(\alpha/2)}{\Gamma_m(m/2)\,\Gamma_m((\alpha+n-k-m)/2)}.$$

If $2m \le n$*,* $u \in V_{n,m}$*, then*

$$(F_m M^\alpha f)(u) = (M^\alpha F_m f)(u) = c_{n,m}(\alpha)\,(Q^{\alpha+n-2m}f)(u), \tag{4.16}$$

$$c_{n,m}(\alpha) = \frac{\Gamma_m((n-m)/2)\,\Gamma_m(\alpha/2)}{\Gamma_m(m/2)\,\Gamma_m((\alpha+n-2m)/2)}.$$

Proof. By (3.7), denoting $f_u = f \circ g_u$, we obtain

$$\begin{aligned}(F_{m,k}M^\alpha f)(u) &= \int_{V_{n-k,m}} (M^\alpha f_u)\left(\left[\begin{array}{c}\omega\\0\end{array}\right]\right) d_*\omega \\ &= \int_{V_{n-k,m}} d_*\omega \int_{V_{n,m}} f_u(v)\,\left|\left[\begin{array}{c}\omega\\0\end{array}\right]' v\right|^{\alpha-m} d_*v.\end{aligned}$$

Set $v = \left[\begin{array}{c}a\\b\end{array}\right]$, $a \in \mathfrak{M}_{n-k,m}$, $b \in \mathfrak{M}_{k,m}$ and write a in polar coordinates

$$a = zs^{1/2}, \quad z \in V_{n-k,m}, \quad s = a'a = v'\sigma_0\sigma_0'v, \quad \sigma_0 = \left[\begin{array}{c}I_{n-k}\\0\end{array}\right] \in V_{n,n-k}.$$

Changing the order of integration, we have

$$(F_{m,k}M^\alpha f)(u) = \int_{V_{n,m}} f_u(v)\,d_*v \int_{V_{n-k,m}} |\omega' a|^{\alpha-m}\,d_*\omega.$$

The inner integral equals $c\,|s|^{(\alpha-m)/2}$, where

$$c = \int_{V_{n-k,m}} |\omega' z|^{\alpha-m}\,d_*\omega = \frac{\Gamma_m((n-k)/2)\,\Gamma_m(\alpha/2)}{\Gamma_m(m/2)\,\Gamma_m((\alpha+n-k-m)/2)};$$

see (8.16). Hence,

$$\begin{aligned}(F_{m,k}M^\alpha f)(u) &= c\int_{V_{n,m}} f_u(v)\,|v'\sigma_0\sigma_0'v|^{(\alpha-m)/2}\,d_*v \\ &= c\int_{V_{n,m}} f(v)\,|I_m - v'uu'v|^{(\alpha-m)/2}\,d_*v,\end{aligned}$$

as desired. Equality (4.16) follows from (4.12) and (4.15). □

The following useful factorizations hold.

Corollary 4.5. *Let* $f \in L^1(V_{n,m})$.
(i) *If* $2m \le n$, $Re\,\alpha > n-m-1$, *then*

$$Q^\alpha f = d_\alpha\, M^{\alpha+2m-n} F_m f = d_\alpha\, F_m M^{\alpha+2m-n} f, \tag{4.17}$$

$$d_\alpha = \frac{\Gamma_m(m/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((n-m)/2)\,\Gamma_m((\alpha+2m-n)/2)}.$$

(ii) *If* $m \le k$, $Re\,\alpha > k-1$, *and* $u \in V_{n,k}$, *then for any* $\tilde{u} \in \{u\}^\perp$,

$$(\mathcal{C}^\alpha_{m,k} f)(u) = \tilde{d}_\alpha\, (F_{m,n-k} M^{\alpha+m-k} f)(\tilde{u}), \tag{4.18}$$

$$\tilde{d}_\alpha = \frac{\Gamma_m(m/2)\,\Gamma_m(\alpha/2)}{\Gamma_m(k/2)\,\Gamma_m((\alpha+m-k)/2)}.$$

Proof. (i) can be obtained from (4.16) if we replace α by $\alpha+2m-n$. To prove (ii), we re-write (4.15) using (4.5). Then we replace k by $n-k$ and $\tilde{u}$ by u. This gives

$$(\mathcal{C}^{\alpha+k-m}_{m,k} f)(u) = \check{d}_\alpha\, (F_{m,n-k} M^\alpha f)(\tilde{u}), \qquad \tilde{u} \in \{u\}^\perp,$$

$$\check{d}_\alpha = \frac{\Gamma_m(m/2)\,\Gamma_m((\alpha+k-m)/2)}{\Gamma_m(k/2)\,\Gamma_m(\alpha/2)}.$$

Now we replace $\alpha+k-m$ by α, and we are done. □

Formula (4.18) was obtained by the author several years ago and reported to S. Alesker, who gave another proof of it; cf. [A, Proposition 1.2]. A similar factorization in the language of Grassmannians is presented in [Zh2, Lemma 3.4], however, without proof and without explicit constant.

The following statement extends Theorem 4.3 to $\alpha = 0$; cf. [Gri, Theorem 2.4], where the close result was obtained in the language of Grassmannians in the spectral form.

Theorem 4.6. *Let* $1 \le k, m \le n-1$; $2m \le n-k$. *Then*

$$\overset{*}{F}_{m,k} F_{m,k} f = \tilde{c}\, Q^{n-k-m} f, \qquad \tilde{c} = \frac{2^m \pi^{(n-m)m/2}\, \Gamma_m((n-k)/2)}{\Gamma_m(n/2)\,\Gamma_m((n-k-m)/2)}. \tag{4.19}$$

Proof. Let g_v be an orthogonal transformation which sends $v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ to $v \in V_{n,m}$. We denote $f_v(w) = f(g_v w)$. By (3.2) and (3.3),

$$\begin{aligned}
(\overset{*}{F}_{m,k} F_{m,k} f)(v) &= \int_{\check{K}_0} (F_{m,k} f)(g_v \rho \check{u}_0)\, d\rho \\
&= \int_{\check{K}_0} d\rho \int_{O(n-k)} f_v \left(\rho \zeta' \begin{bmatrix} \gamma & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} I_m \\ 0 \end{bmatrix} \right) d\gamma,
\end{aligned}$$

where $\zeta \in O(n)$ is defined by (3.11) and satisfies $\zeta' u_0 = \check{u}_0$. Hence,

$$(\overset{*}{F}_{m,k} F_{m,k} f)(v) = \int_{O(n-m)} d\delta \int_{V_{n-k,m}} f_v \left(\begin{bmatrix} \delta & 0 \\ 0 & I_m \end{bmatrix} \zeta' \begin{bmatrix} w \\ 0 \end{bmatrix} \right) d_* w.$$

Using the bi-Stiefel decomposition (2.12) (replace n by $n-k$, and k by m), the last expression can be written as

$$\begin{aligned}
&\frac{1}{\sigma_{n-k,m}} \int_{O(n-m)} d\delta \int_0^{I_m} d\nu(r) \int_{V_{m,m}} d\gamma \\
&\times \int_{V_{n-k-m,m}} f_v \left(\begin{bmatrix} \delta & 0 \\ 0 & I_m \end{bmatrix} \zeta' \begin{bmatrix} \gamma r^{1/2} \\ u(I_m - r)^{1/2} \\ 0 \end{bmatrix} \right) du. \\
&= \frac{1}{\sigma_{n-k,m}} \int_0^{I_m} d\nu(r) \int_{V_{m,m}} d\gamma \int_{V_{n-k-m,m}} du \\
&\times \int_{O(n-m)} f_v \left(\begin{bmatrix} \delta \begin{bmatrix} 0 \\ u \end{bmatrix} (I_m - r)^{1/2} \\ \gamma r^{1/2} \end{bmatrix} \right) d\delta \\
&= \frac{\sigma_{n-k-m,m}}{\sigma_{n-k,m}} \int_0^{I_m} d\nu(r) \int_{V_{m,m}} d\gamma \\
&\times \int_{V_{n-m,m}} f_v \left(\begin{bmatrix} \theta (I_m - r)^{1/2} \\ \gamma r^{1/2} \end{bmatrix} \right) d\theta.
\end{aligned}$$

Here

$$d\nu(r) = 2^{-m} |r|^{m/2-d} |I_m - r|^{(n-k-m)/2-d}\, dr, \qquad 2m \le n-k.$$

Now we change variables $r \to I_m - r$ and use (2.12) (with k replaced by $n-m$) in the opposite direction. We obtain

$$\begin{aligned}
(\overset{*}{F}_{m,k} F_{m,k} f)(v) &= \frac{\sigma_{n-k-m,m}}{\sigma_{n-k,m}} \int_{V_{n,m}} f_v(w) |w'(I_n - v_0 v_0')w|^{-k/2}\, dw \\
&= \frac{\sigma_{n-k-m,m}\, \sigma_{n,m}}{\sigma_{n-k,m}} \int_{V_{n,m}} f(w) |I_m - w' v v' w|^{-k/2}\, d_* w.
\end{aligned}$$

Owing to (4.7) and (2.8), this is exactly what we need. □

Remark 4.7. The assumption $2m \le n-k$ in Theorem 4.6 is necessary for absolute convergence of the integral on the right-hand side in (4.19), whereas the left hand-side is finite a.e. under the weaker assumption $m \le n-k$, which is sharp. By Lemma 5.13 below, the condition $2m \le n-k$ can be eliminated if we interpret (4.19) in the $\mathcal{S}'$-sense.

5. Cosine Transforms via the Fourier Analysis

In this section we proceed to develop the theory of the Funk, cosine and sine transforms using the Fourier analysis in the ambient matrix space. The consideration essentially relies on the idea of analytic continuation, when the one-dimensional exponent α is replaced by a vector-valued complex parameter $\boldsymbol{\lambda} \in \mathbb{C}^m$.

5.1. **The composite cosine transform.** Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ and let $(\cdot)^{\boldsymbol{\lambda}}$ be the corresponding composite power function associated with the cone Ω; see Section 2.2. We consider *the composite cosine transform*

$$(5.1) \qquad (T^{\boldsymbol{\lambda}}_{k,m}\varphi)(v) = \int_{V_{n,k}} \varphi(u)\,(v'uu'v)^{\boldsymbol{\lambda}}\,d_*u, \qquad v \in V_{n,m},$$

which was studied in [OR06, OR05a] when $k = m$. The dual cosine transform (4.2) is a particular case of (5.1), corresponding to $\lambda_1 = \ldots = \lambda_m = \alpha - k$.

Lemma 5.1. *Let $1 \le m \le k \le n-1$, $\varphi \in L^1(V_{n,k})$,*

$$(5.2) \qquad \mathfrak{L} = \{\boldsymbol{\lambda} \in \mathbb{C}^m : Re\,\lambda_j > j-k-1 \quad \text{for each} \quad j = 1, \dots, m\}.$$

The integral $(T^{\boldsymbol{\lambda}}_{k,m}\varphi)(v)$ converges absolutely for almost all $v \in V_{n,m}$ if and only if $\boldsymbol{\lambda} \in \mathfrak{L}$ and represents an analytic function of $\boldsymbol{\lambda}$ in this domain. Moreover,

$$(5.3) \qquad \int_{V_{n,m}} (T^{\boldsymbol{\lambda}}_{k,m}\varphi)(v)\,d_*v = \frac{\Gamma_m(m/2)\,\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{k}_0)}{\Gamma_m(k/2)\,\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)} \int_{V_{n,k}} \varphi(u)\,d_*u.$$

This statement follows immediately from Lemma 8.4 in Appendix.

5.2. **The complementary Radon transform.** This is a new transformation that does not occur in [OR05a, OR06] in the case $k = m$. It takes a function $\varphi(u)$ on $V_{n,k}$ to a function $(A_{k,m}\varphi)(v)$ on $V_{n,m}$ by the formula

$$(5.4) \qquad \begin{aligned} (A_{k,m}\varphi)(v) &= \int_{V_{n-m,k-m}} \varphi\left(g_v \begin{bmatrix} a & 0 \\ 0 & I_m \end{bmatrix}\right) d_*a \\ &= (F_{k-m,m}\varphi([\cdot, v]))(v), \qquad 1 \le m \le k \le n-1, \end{aligned}$$

where g_v is an orthogonal transformation that sends $v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ to $v \in V_{n,m}$. This can be regarded as a "partial Funk transform" (cf. (3.7)), where φ is integrated over all orthonormal k-frames u in $\mathbb{R}^n$, the last m columns of which are replaced by v, and the first $k-m$

columns are orthogonal to v. If $k = m$, then $A_{k,m}$ reduces to the identity operator.

Lemma 5.2. *Let* $1 \le m \le k \le n-1$, $\varphi \in L^1(V_{n,k})$. *Then*

$$\int_{V_{n,m}} (A_{k,m}\varphi)(v)\, d_*v = \int_{V_{n,k}} \varphi(u)\, d_*u.$$

Proof.

$$\begin{aligned} l.h.s &= \int_{O(n)} dg \int_{O(n-m)} \varphi\left(g\begin{bmatrix} \delta & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} 0 \\ I_k \end{bmatrix}\right) d\delta \\ &= \int_{O(n)} \varphi\left(g\begin{bmatrix} 0 \\ I_k \end{bmatrix}\right) dg = r.h.s. \end{aligned}$$

□

5.3. **The basic functional equation.** The following lemma from [OR06] will be needed. We present it with proof in view of its fundamental importance. We denote

$$(5.5)\quad \mathbf{\Lambda} = \{\boldsymbol{\lambda}\in\mathbb{C}^m : Re\,\lambda_j > j-n-1 \quad \textit{for each} \quad j \in \{1,\dots,m\},$$

$$(5.6)\quad \mathbf{\Lambda}_0 = \{\boldsymbol{\lambda}\in\mathbb{C}^m : \lambda_j = j-n-l \quad \textit{for some} \quad j \in \{1,\dots,m\} \; \textit{and some} \quad l \in \{1,3,5,\dots\}\}.$$

Lemma 5.3. *Let* $f \in L^1(V_{n,m})$, $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$. *The integrals*

$$(5.7)\quad (r^{\boldsymbol{\lambda}} f,\phi) = \int\limits_{\mathfrak{M}_{n,m}} r^{\boldsymbol{\lambda}} f(v)\,\overline{\phi(x)}\,dx, \quad (r_*^{\boldsymbol{\lambda}} f,\phi) = \int\limits_{\mathfrak{M}_{n,m}} r_*^{\boldsymbol{\lambda}} f(v)\,\overline{\phi(x)}\,dx$$

are absolutely convergent if and only if $\boldsymbol{\lambda} \in \mathbf{\Lambda}$ *and extend as meromorphic functions of* $\boldsymbol{\lambda}$ *with the polar set* $\mathbf{\Lambda}_0$. *The normalized integrals*

$$(5.8)\qquad \frac{(r^{\boldsymbol{\lambda}} f,\phi)}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)}, \qquad \frac{(r_*^{\boldsymbol{\lambda}} f,\phi)}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)},$$

$\mathbf{n}_0 = (n,\dots,n)$, *extend as entire functions of* $\boldsymbol{\lambda}$.

Proof. Consider the first integral. We set $x = vt$, $v \in V_{n,m}$, $t \in T_m$, and make use of Lemma 2.2. By taking into account that $x'x = t't$ and $t(t't)^{-1/2} \in O(m)$, owing to (2.16), we obtain

$$(5.9)\qquad (r^{\boldsymbol{\lambda}} f,\phi) = \int\limits_{\mathbb{R}_+^m} F(t_{1,1},\dots,t_{m,m}) \prod_{j=1}^m t_{j,j}^{\lambda_j+n-j} dt_{j,j}\,,$$

where

$$F(t_{1,1},\dots,t_{m,m})=\int\limits_{\mathbb{R}^{m(m-1)/2}} dt_* \int\limits_{V_{n,m}} f(v)\overline{\phi(vt)}\,dv, \quad dt_*=\prod_{i<j} dt_{i,j}.$$

Since F extends as an even Schwartz function in each argument, it can be written as $F(t_{1,1},\dots,t_{m,m})=F_0(t^2_{1,1},\dots,t^2_{m,m})$, where $F_0\in\mathcal{S}(\mathbb{R}^m)$ (use, e.g., Lemma 5.4 from [Tr, p. 56]). Replacing $t^2_{j,j}$ by $s_{j,j}$, we represent (5.9) as a direct product of one-dimensional distributions

$$(5.10)\qquad (r^{\boldsymbol{\lambda}}f,\phi)=(\prod_{j=1}^m (s_{j,j})_+^{(\lambda_j+n-j-1)/2},\ F_0(s_{1,1},\dots,s_{m,m})).$$

It follows that the the first integral in (5.7) is absolutely convergent provided $Re\,\lambda_j>j-n-1$, i.e., $\boldsymbol{\lambda}\in\boldsymbol{\Lambda}$. The condition $\boldsymbol{\lambda}\in\boldsymbol{\Lambda}$ is strict. To see this, we choose $f\equiv 1$ and $\phi(x)=e^{-\mathrm{tr}(x'x)}$ so that

$$(5.11)\qquad (r^{\boldsymbol{\lambda}}f,\phi)=\int\limits_{\mathfrak{M}_{n,m}} (x'x)^{\boldsymbol{\lambda}}\,e^{-\mathrm{tr}(x'x)}\,dx=2^{-m}\sigma_{n,m}\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0).$$

Furthermore, since $(s_{j,j})_+^{(\lambda_j+n-j-1)/2}$ extends as a meromorphic distribution with the only poles $\lambda_j=j-n-1, j-n-3,\dots$, then, by the fundamental Hartogs theorem [Sha], the function $\boldsymbol{\lambda}\to(r^{\boldsymbol{\lambda}}f,\phi)$ extends meromorphically with the polar set $\boldsymbol{\Lambda}_0$. By the same reason, a direct product of the normalized distributions

$$(s_{j,j})_+^{(\lambda_j+n-j-1)/2}/\Gamma((\lambda_j+n-j+1)/2)$$

is an entire function of $\boldsymbol{\lambda}$.

Let us consider the second integral in (5.7). Changing variable $x=y\omega$, where ω is a matrix from (2.19), we obtain $r_*=(x'x)_*=(\omega y'y\omega)_*=y'y$. Hence (set $y=u\tau,\ u\in V_{n,m},\ \tau\in T_m$),

$$\begin{aligned}(r_*^{\boldsymbol{\lambda}}f,\phi) &= \int\limits_{\mathfrak{M}_{n,m}} (y'y)^{\boldsymbol{\lambda}}\, f(y\omega(\omega y'y\omega)^{-1/2})\,\overline{\phi(y\omega)}\,dy\\ &= \int\limits_{\mathbb{R}^m_+} \Phi(\tau_{1,1},\dots,\tau_{m,m})\prod_{j=1}^m \tau_{j,j}^{\lambda_j+n-j}d\tau_{j,j}\end{aligned}$$

where, as above (note that $\tau\omega\,(\omega\tau'\tau\omega)^{-1/2}\in O(m)$),

$$\begin{aligned}\Phi(\tau_{1,1},\dots,\tau_{m,m}) &= \int\limits_{\mathbb{R}^{m(m-1)/2}} d\tau_* \int\limits_{V_{n,m}} f(u)\,\overline{\phi(u\tau\omega)}\,du\\ &= \Phi_0(\tau^2_{1,1},\dots,\tau^2_{m,m}), \qquad \Phi_0\in\mathcal{S}(\mathbb{R}^m).\end{aligned}$$

This gives

$$(r_*^{\boldsymbol{\lambda}} f, \phi) = (\prod_{j=1}^{m} (s_{j,j})_+^{(\lambda_j+n-j-1)/2}, \ \Phi_0(s_{1,1}, \dots, s_{m,m})),$$

and the result follows as in the previous case. □

We introduce the following normalized extensions of $(A_{k,m}\varphi)(v)$ and $(T^{\boldsymbol{\lambda}}_{k,m}\varphi)(v)$ from $V_{n,m}$ to the ambient matrix space $\mathfrak{M}_{n,m}$:

$$(\tilde{A}^{\boldsymbol{\lambda}}_{k,m}\varphi)(x) = \frac{r_*^{-\boldsymbol{\lambda}_*-\mathbf{n}_0}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} (A_{k,m}\varphi)(v), \tag{5.12}$$

$$(\tilde{T}^{\boldsymbol{\lambda}}_{k,m}\varphi)(x) = \frac{r^{\boldsymbol{\lambda}}}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{k}_0)} (T^{\boldsymbol{\lambda}}_{k,m}\varphi)(v), \tag{5.13}$$

where $x \in \mathfrak{M}_{n,m}, \ x = vr^{1/2}, \ r = x'x \in \Omega, \ v \in V_{n,m}$.

The next theorem plays a key role in the whole paper.

Theorem 5.4. *Let φ be an integrable right $O(m)$-invariant function on $V_{n,m}$, $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$, $1 \le m \le k \le n-1$. Then for every $\boldsymbol{\lambda} \in \mathbb{C}^m$,*

$$(\tilde{T}^{\boldsymbol{\lambda}}_{k,m}\varphi, \mathcal{F}\phi) = \check{c}\, (\tilde{A}^{\boldsymbol{\lambda}}_{k,m}\varphi, \phi), \quad \check{c} = \frac{2^{nm+|\boldsymbol{\lambda}|}\, \pi^{nm/2}\, \Gamma_m(n/2)}{\Gamma_m(k/2)}, \tag{5.14}$$

where both sides are understood in the sense of analytic continuation.

Proof. It suffices to show the following.

(a) Equality (5.14) holds for all $\boldsymbol{\lambda}$ in a certain domain $\mathfrak{D} \subset \mathbb{C}^m$, where both sides of (5.14) are analytic functions of $\boldsymbol{\lambda}$;

(b) The right-hand side of (5.14) extends from $\mathfrak{D}$ to all $\boldsymbol{\lambda} \in \mathbb{C}^m$ as an entire function.

Let $\mathfrak{D} = \mathfrak{L} \cap \tilde{\mathfrak{L}}$, where

$$\begin{aligned} \mathfrak{L} &= \{\boldsymbol{\lambda} \in \mathbb{C}^m : Re\,\lambda_j > j-k-1 \quad \textit{for each} \quad j = 1, \dots, m\}, \\ \tilde{\mathfrak{L}} &= \{\boldsymbol{\lambda} \in \mathbb{C}^m : Re\,\lambda_j < j-m \quad \textit{for each} \quad j = 1, \dots, m\} \end{aligned}$$

(note that $\mathfrak{D}$ is nonempty !). Absolute convergence of the integral

$$(\tilde{A}^{\boldsymbol{\lambda}}_{k,m}\varphi, \, \phi) = \int_{\mathfrak{M}_{n,m}} (\tilde{A}^{\boldsymbol{\lambda}}_{k,m}\varphi)(x)\overline{\phi(x)}\, dx$$

for all $\boldsymbol{\lambda} \in \tilde{\mathfrak{L}}$ and extendability of this expression by analyticity to all $\boldsymbol{\lambda} \in \mathbb{C}^m$ follow from Lemma 5.3, owing to Lemma 5.2: just change the notation in the second integral in (5.7) and make use of the equality $(\boldsymbol{\lambda}_*)_j = \lambda_{m-j+1}$ (the latter gives $Re\,\lambda_j < j-m$). Absolute convergence and analyticity of the left-hand side of (5.14) for all $\boldsymbol{\lambda} \in \mathfrak{L}$ follow from Lemma 5.3 too, thanks to Lemma 4.2 (note that $\mathfrak{L} \subset \boldsymbol{\Lambda}$, cf. (5.5)). Thus, it remains to prove (5.14) for $\boldsymbol{\lambda} \in \mathfrak{D}$.

Assuming $\boldsymbol{\lambda} \in \mathfrak{L}$, by (2.18) we have

$$I \equiv (\tilde{T}^{\boldsymbol{\lambda}}_{k,m}\varphi, \mathcal{F}\phi) = \int\limits_{V_{n,k}} \varphi(u) J(u)\, d_*u, \tag{5.15}$$

$$J(u) = \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{k}_0)} \int\limits_{\mathfrak{M}_{n,m}} (x'uu'x)^{\boldsymbol{\lambda}}\, \overline{(\mathcal{F}\phi)(x)}\, dx, \qquad u \in V_{n,k}.$$

Let g_u be an orthogonal transformation that sends $u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$ to $u \in V_{n,k}$. We set $x = g_u \begin{bmatrix} a \\ b \end{bmatrix}$, $a \in \mathfrak{M}_{n-k,m}$, $b \in \mathfrak{M}_{k,m}$. Then, by (2.25),

$$\begin{aligned} J(u) &= \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{k}_0)} \int\limits_{\mathfrak{M}_{k,m}} (b'b)^{\boldsymbol{\lambda}} db \int\limits_{\mathfrak{M}_{n-k,m}} \overline{(\mathcal{F}\phi)\left(g_u \begin{bmatrix} a \\ b \end{bmatrix}\right)}\, da \\ &= \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{k}_0)} \int\limits_{\mathfrak{M}_{k,m}} (b'b)^{\boldsymbol{\lambda}}\, \overline{(\mathcal{R}_k\mathcal{F}\phi)(u,b)}\, db. \end{aligned}$$

Owing to (2.36) and Remark 2.9 (we recall that $\boldsymbol{\lambda} \in \mathfrak{D}$!), the last expression can be written as

$$\begin{aligned} &\frac{2^{km+|\boldsymbol{\lambda}|}\pi^{km/2}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} \int\limits_{\mathfrak{M}_{k,m}} (b'b)_*^{-\boldsymbol{\lambda}_*-\mathbf{k}_0}\, \overline{(\tilde{\mathcal{F}}^{-1}(\mathcal{R}_k\mathcal{F}\phi)(u,\cdot))(b)}\, db \\ &= \frac{2^{|\boldsymbol{\lambda}|}\pi^{-km/2}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} \int\limits_{\mathfrak{M}_{k,m}} (b'b)_*^{-\boldsymbol{\lambda}_*-\mathbf{k}_0}\, \overline{(\tilde{\mathcal{F}}(\mathcal{R}_k\mathcal{F}\phi)(u,\cdot))(b)}\, db. \end{aligned}$$

By the Projection-Slice Theorem (see (2.27)),

$$(\tilde{\mathcal{F}}(\mathcal{R}_k\mathcal{F}\phi)(u,\cdot))(b) = (\mathcal{F}\mathcal{F}\phi)(ub) = (2\pi)^{nm}\phi(-ub). \tag{5.16}$$

Hence,

$$J(u) = c_\lambda \int\limits_{\mathfrak{M}_{k,m}} (b'b)_*^{-\boldsymbol{\lambda}_*-\mathbf{k}_0}\, \overline{\phi(ub)}\, db, \qquad c_\lambda = \frac{2^{nm+|\boldsymbol{\lambda}|}\pi^{m(n-k/2)}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)}.$$

This calculation enables us to transform (5.15) as follows:

$$\begin{aligned} I &= c_\lambda \int\limits_{V_{n,k}} \varphi(u) d_*u \int\limits_{\mathfrak{M}_{k,m}} (b'b)_*^{-\boldsymbol{\lambda}_*-\mathbf{k}_0}\, \overline{\phi(ub)}\, db \\ &= 2^{-m} c_\lambda \int\limits_\Omega r_*^{-\boldsymbol{\lambda}_*-\mathbf{k}_0} |r|^{(k-m-1)/2} \psi(r)\, dr, \end{aligned}$$

where

$$\begin{aligned}\psi(r) &= \int\limits_{V_{k,m}} d\eta \int\limits_{V_{n,k}} \varphi(u)\,\overline{\phi(u\eta r^{1/2})}\,d_*u \quad (\text{set } \eta_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{k,m})\\
&= \sigma_{k,m} \int\limits_{O(k)} d\gamma \int\limits_{V_{n,k}} \varphi(u)\overline{\phi(u\gamma\eta_0 r^{1/2})}\,d_*u\\
&= \sigma_{k,m} \int\limits_{V_{n,k}} \varphi(u)\,\overline{\phi(u\eta_0 r^{1/2})}\,d_*u.\end{aligned}$$

Using notation from (3.5) and (3.6), we continue

$$\begin{aligned}\psi(r) &= \sigma_{k,m} \int\limits_{O(n)} \varphi(gu_0)\,\overline{\phi(gu_0\eta_0 r^{1/2})}\,dg\\
&\qquad (\text{note that } u_0\eta_0 = v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n,m} \text{ and } \check{K}_0 v_0 = v_0)\\
&= \sigma_{k,m} \int\limits_{O(n)} \varphi(gu_0)\,dg \int\limits_{\check{K}_0} \overline{\phi(g\rho' v_0 r^{1/2})}\,d\rho\\
&= \sigma_{k,m} \int\limits_{\check{K}_0} d\rho \int\limits_{O(n)} \varphi(gu_0)\,\overline{\phi(g\rho' v_0 r^{1/2})}\,dg\\
&= \sigma_{k,m} \int\limits_{O(n)} \overline{\phi(\lambda v_0 r^{1/2})}\,d\lambda \int\limits_{\check{K}_0} \varphi(\lambda\rho u_0)\,d\rho\\
&= \frac{\sigma_{k,m}}{\sigma_{n,m}} \int\limits_{V_{n,m}} \overline{\phi(vr^{1/2})}\,dv \int\limits_{O(n-m)} \varphi\left(g_v \begin{bmatrix} \delta & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0 \\ I_k \end{bmatrix}\right) d\delta\\
&= \frac{\sigma_{k,m}}{\sigma_{n,m}} \int\limits_{V_{n,m}} \overline{\phi(vr^{1/2})}\,(A_{k,m}\varphi)(v)\,dv. \qquad (5.17)\end{aligned}$$

Hence (use (2.17) and note that $|r| = |r_*|$),

$$\begin{aligned}I &= \frac{c_\lambda \sigma_{k,m}}{2^m \sigma_{n,m}} \int\limits_{\Omega} r_*^{-\boldsymbol{\lambda}_*} |r|^{-(m+1)/2}\,dr \int\limits_{V_{n,m}} \overline{\phi(vr^{1/2})}\,(A_{k,m}\varphi)(v)\,dv\\
&= \check{c} \int\limits_{\mathfrak{M}_{n,m}} (\tilde{A}^{\boldsymbol{\lambda}}_{k,m}\varphi)(x)\,\overline{\phi(x)}\,dx,\end{aligned}$$

as desired. □

The following corollaries hold in the case $\lambda_1 = \ldots = \lambda_m = \alpha - k$. We denote

$$(E_\lambda f)(x) = |r|^{\lambda/2} f(v), \qquad x = vr^{1/2} \in \mathfrak{M}_{n,m}. \tag{5.18}$$

Corollary 5.5. *Let φ be an integrable right $O(k)$-invariant function on $V_{n,k}$, $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$, $1 \le m \le k \le n-1$. Then for every $\alpha \in \mathbb{C}$,*

$$\left(\frac{E_{\alpha-k} \overset{*}{\mathcal{C}}{}^{\alpha}_{m,k} \varphi}{\Gamma_m(\alpha/2)}, \mathcal{F}\phi \right) = \check{c}_1 \left(\frac{E_{k-\alpha-n} A_{k,m} \varphi}{\Gamma_m((k-\alpha)/2)}, \phi \right), \tag{5.19}$$

$$\check{c}_1 = \frac{2^{m(n+\alpha-k)}\, \pi^{nm/2}\, \Gamma_m(n/2)}{\Gamma_m(k/2)},$$

where both sides are understood in the sense of analytic continuation.

In particular, if $k = m$, the following statement holds for the cosine transform

$$(M^\alpha f)(u) = \int_{V_{n,m}} f(v)\, |v'uu'v|^{(\alpha-m)/2}\, d_*v.$$

Corollary 5.6. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$, $1 \le m \le n-1$. Then for every $\alpha \in \mathbb{C}$,*

$$\left(\frac{E_{\alpha-m} M^\alpha f}{\Gamma_m(\alpha/2)}, \mathcal{F}\phi \right) = \check{c}_2 \left(\frac{E_{m-\alpha-n} f}{\Gamma_m((m-\alpha)/2)}, \phi \right), \tag{5.20}$$

$$\check{c}_2 = \frac{2^{m(n+\alpha-m)}\, \pi^{nm/2}\, \Gamma_m(n/2)}{\Gamma_m(m/2)},$$

where both sides are understood in the sense of analytic continuation.

Remark 5.7. For better understanding of (5.19) (and also (5.20)) we note that the domains, where the left-hand side and the right-hand side of this equality exist as absolutely convergent integrals, have no points in common, when $m > 1$. To implement analytic continuation we had to switch from $\alpha \in \mathbb{C}$ to $\boldsymbol{\lambda} \in \mathbb{C}^m$.

Corollary 5.8. *Let φ be an integrable right $O(k)$-invariant function on $V_{n,k}$, $1 \le m \le k \le n-1$. Then for any $\omega \in C^\infty(V_{n,m})$, the function*

$$\zeta_*(\alpha) = \left(\frac{1}{\Gamma_m(\alpha/2)}\ \overset{*}{\mathcal{C}}{}^{\alpha}_{m,k} \varphi, \omega \right)$$

extends to all $\alpha \in \mathbb{C}$ as an entire function.

Proof. Let $\psi(r)$ be a nonnegative C^∞ function with compact support away from the boundary of the cone Ω. Choose $\mathcal{F}\phi$ in (5.19) so that

$(\mathcal{F}\phi)(x) = \psi(r)\omega(v)$, $x = vr^{1/2}$, $v \in V_{n,m}$. Then for $Re\,\alpha > m-1$, the left-hand side of (5.19) becomes $\zeta_*(\alpha) h_\psi(\alpha)$,

$$h_\psi(\alpha) = 2^{-m} \int_\Omega |r|^{(n+\alpha-m-k-1)/2} \overline{\psi(r)}\, dr.$$

This gives

$$a.c.\, \zeta_*(\alpha) = a.c.\, \frac{1}{h_\psi(\alpha)} \left(\frac{E_{\alpha-k} \overset{*}{\mathcal{C}}{}^{\alpha}_{m,k} \varphi}{\Gamma_m(\alpha/2)}, \psi\omega \right). \tag{5.21}$$

Owing to uniqueness of analytic continuation and Corollary 5.5, the right-hand side of (5.21) is well defined and independent of ψ. □

The following statement holds by duality.

Corollary 5.9. *Let f be an infinitely differentiable right $O(m)$-invariant function on $V_{n,m}$, $1 \le m \le k \le n-1$. Then for any $\omega \in C^\infty(V_{n,m})$, the function*

$$\zeta(\alpha) = \Big(\frac{1}{\Gamma_m(\alpha/2)} \mathcal{C}^\alpha_{m,k} f, \omega \Big)$$

extends to all $\alpha \in \mathbb{C}$ as an entire function.

5.3.1. *Remarks and conjectures.*

1. We believe that Corollary 5.9 holds for every $f \in L^1(V_{n,m})$. This would follow from our next conjecture.

2. It is natural to expect that for infinitely differentiable f and φ, the normalized functions

$$\alpha \to \frac{1}{\Gamma_m(\alpha/2)} (\mathcal{C}^\alpha_{m,k} f)(u), \qquad \alpha \to \frac{1}{\Gamma_m(\alpha/2)} (\overset{*}{\mathcal{C}}{}^{\alpha}_{m,k} \varphi)(v) \tag{5.22}$$

extend as entire functions of α pointwise, that is, for every u and v, and, moreover, these extensions are C^∞ functions of u and v, respectively. We suppose that the proof of this fact can be given by using decomposition in Stiefel/Grassmann harmonics (see, e.g, [Ge, Str75, Str86, TT] for this theory).

3. Analytic continuation of integrals, like $\mathcal{C}^\alpha_{m,k} f$, was briefly discussed in [GGR, p. 368], where it was suggested to replace such integrals by those over the matrix space $\mathfrak{M}_{m,n-m}$, so that $\dim \mathfrak{M}_{m,n-m} = m(n-m) = \dim G_{n,m}$. However, the corresponding calculations were skipped in that paper. Our impression is that the actual procedure is more complicated; see Section 8.1 for further discussion.

5.4. **A functional equation for the dual Funk transform.** The next theorem specifies Corollary 5.5 for $\alpha = 0$. According to this theorem, the dual Funk transform $\overset{*}{F}_{m,k}$ can be regarded as a member of the analytic family $\{\overset{*}{\mathcal{C}}{}^{\alpha}_{m,k}\}$ if the latter is suitably normalized.

Theorem 5.10. *Let φ be an integrable right $O(m)$-invariant function on $V_{n,m}$, $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$, $1 \le m \le k \le n-1$. Then*

$$(E_{-k} \overset{*}{F}_{m,k}\varphi, \mathcal{F}\phi) = \tilde{c}\,(E_{k-n}A_{k,m}\varphi, \phi), \tag{5.23}$$

where $A_{k,m}$ is the operator (5.4),

$$\tilde{c} = \frac{2^{m(n-k)}\,\pi^{nm/2}\,\Gamma_m((n-k)/2)}{\Gamma_m(k/2)}.$$

Proof. Passing to polar coordinates, we obtain

$$(E_{-k} \overset{*}{F}_{m,k}\varphi, \mathcal{F}\phi) = 2^{-m}\sigma_{n,m} \int\limits_{V_{n,m}} (\overset{*}{F}_{m,k}\varphi)(v)\,h(v)\,d_*v,$$

$$h(v) = \int\limits_{\Omega} \overline{(\mathcal{F}\phi)(vr^{1/2})}\,|r|^{(n-k-m-1)/2}\,dr.$$

Hence, by duality (3.10),

$$(E_{-k} \overset{*}{F}_{m,k}\varphi, \mathcal{F}\phi) = 2^{-m}\sigma_{n,m} \int\limits_{V_{n,k}} \varphi(u)(F_{m,k}h)(u)\,d_*u, \tag{5.24}$$

where, by (3.7),

$$\begin{aligned}
(F_{m,k}h)(u) &= \int\limits_{V_{n-k,m}} h\left(g_u\begin{bmatrix}\omega\\0\end{bmatrix}\right) d_*\omega \\
&= \int\limits_{V_{n-k,m}} d_*\omega \int\limits_{\Omega} \overline{(\mathcal{F}\phi)\left(g_u\begin{bmatrix}\omega\\0\end{bmatrix} r^{1/2}\right)}\,|r|^{(n-k-m-1)/2}\,dr \\
&= \frac{2^m}{\sigma_{n-k,m}} \int\limits_{\mathfrak{M}_{n-k,m}} \overline{(\mathcal{F}\phi)\left(g_u\begin{bmatrix}y\\0\end{bmatrix}\right)}\,dy \\
&= \frac{2^m}{\sigma_{n-k,m}}\,\overline{(\mathcal{R}_k\mathcal{F}\phi)(u,0)}.
\end{aligned}$$

The last expression can be regarded as analytic continuation of the Riesz distribution (2.28), so that

$$(F_{m,k}h)(u) = \frac{2^m}{\sigma_{n-k,m}} \underset{\alpha=0}{a.c.} \frac{1}{\gamma_{k,m}(\alpha)} \int\limits_{\mathfrak{M}_{k,m}} \overline{(\mathcal{R}_k \mathcal{F}\phi)(u,z)}\, |z|_m^{\alpha-k}\, dz.$$

Now (2.33) yields

$$(F_{m,k}h)(u) = \frac{2^{m-km}\pi^{-km}}{\sigma_{n-k,m}} \underset{\alpha=0}{a.c.} \int\limits_{\mathfrak{M}_{k,m}} \overline{\tilde{\mathcal{F}}[(R_k\mathcal{F}\phi)(u,\cdot)](y)}\, |y|_m^{-\alpha}\, dy.$$

Hence, by (5.16),

$$(F_{m,k}h)(u) = \frac{2^{m(1-k+n)}\pi^{m(n-k)}}{\sigma_{n-k,m}} \int\limits_{\mathfrak{M}_{k,m}} \overline{\phi(uy)}\, dy.$$

Using (5.24) and passing to polar coordinates, we obtain

$$\begin{aligned}(E_{-k} \overset{*}{F}_{m,k}\varphi, \mathcal{F}\phi) &= \frac{(2\pi)^{nm-km}\,\sigma_{n,m}}{\sigma_{n-k,m}} \int\limits_{V_{n,k}} \varphi(u) d_*u \int\limits_{\mathfrak{M}_{k,m}} \overline{\phi(uy)}\, dy \\ &= \frac{(2\pi)^{nm-km}\sigma_{n,m}}{2^m\, \sigma_{n-k,m}} \int\limits_{\Omega} |r|^{(k-m-1)/2}\, \psi(r)\, dr,\end{aligned}$$

where, by (5.17),

$$\psi(r) = \frac{\sigma_{k,m}}{\sigma_{n,m}} \int\limits_{V_{n,m}} \overline{\phi(vr^{1/2})}\, (A_{k,m}\varphi)(v)\, dv.$$

Setting $x = vr^{1/2}$, we arrive at (5.23), as desired. □

Corollary 5.11. *Let φ be an integrable right $O(k)$-invariant function on $V_{n,k}$, $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$, $1 \le m \le k \le n-1$. Then*

$$(E_{-k} \overset{*}{F}_{m,k}\varphi, \phi) = d_0 \underset{\alpha=0}{a.c.} \left(\frac{E_{\alpha-k} \overset{*}{\mathcal{C}}{}^{\alpha}_{m,k}\varphi}{\Gamma_m(\alpha/2)}, \phi \right), \tag{5.25}$$

$$\tilde{d}_0 = \frac{\Gamma_m((n-k)/2)\Gamma_m(k/2)}{\Gamma_m(n/2)}.$$

Proof. The statement follows immediately from (5.23) and (5.19). □

In the case $k = m$ we have the following

Corollary 5.12. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$, $1 \le m \le n-1$. Then*

$$(E_{-m}F_m f, \mathcal{F}\phi) = \tilde{c}_0\,(E_{m-n}f, \phi), \tag{5.26}$$

$$\tilde{c}_0 = \frac{2^{m(n-m)}\,\pi^{nm/2}\,\Gamma_m((n-m)/2)}{\Gamma_m(m/2)}.$$

5.5. **Analytic properties of the sine transform.** Corollary 4.5, combined with analytic properties of the cosine transform, enables us to study analytic continuation of the sine transform

$$(Q^\alpha f)(u) = \int_{V_{n,m}} f(v)\,|I_m - v'uu'v|^{(\alpha+m-n)/2}\,d_*v, \qquad u \in V_{n,m}.$$

Lemma 5.13. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$; $2m \le n$, $\omega \in C^\infty(V_{n,m})$. Then $(Q^\alpha f, \omega)$ extends as a meromorphic function of α with the polar set $\{m-1, m-2, \dots\}$ by the formula*

$$(Q^\alpha f, \omega) = d_\alpha\,(M^{\alpha+2m-n}F_m f, \omega), \tag{5.27}$$

$$d_\alpha = \frac{\Gamma_m(m/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((n-m)/2)\,\Gamma_m((\alpha+2m-n)/2)}.$$

The normalized function

$$\eta(\alpha) = \frac{1}{\Gamma_m(\alpha/2)}\,(Q^\alpha f, \omega) \tag{5.28}$$

extends as an entire function of α.

Proof. By (4.17), for $Re\,\alpha > n-m-1$ we have $Q^\alpha f = d_\alpha\,M^{\alpha+2m-n}F_m f$. Hence,

$$(Q^\alpha f, \omega) = \frac{c\,(M^{\alpha+2m-n}F_m f, \omega)}{\Gamma_m((\alpha+2m-n)/2)}, \qquad c = \frac{\Gamma_m(m/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((n-m)/2)}.$$

Applying Corollary 5.8 with $\varphi = F_m f$ and $k = m$, we get the result. □

Remark 5.14. We conjecture that $\eta(\alpha)$ is an entire function even if f is not right $O(m)$-invariant, as in the case $m = 1$; cf. [Ru02, p. 474].

6. Normalized Cosine and Sine Transforms

For the following it is convenient to suitably normalize cosine and sine transforms. The normalized transforms will be denoted by the corresponding calligraphic letters. Assuming $1 \le m \le k \le n-1$, we set

A. For $u \in V_{n,k}$, $v \in V_{n,m}$:

$$(6.1)\qquad (\mathscr{C}^{\alpha}_{m,k} f)(u) = \delta_{n,m,k}(\alpha) \int_{V_{n,m}} f(v)\, |v'uu'v|^{(\alpha-k)/2}\, d_* v,$$

$$(6.2)\qquad (\overset{*}{\mathscr{C}}{}^{\alpha}_{m,k} \varphi)(v) = \delta_{n,m,k}(\alpha) \int_{V_{n,k}} \varphi(u)\, |v'uu'v|^{(\alpha-k)/2}\, d_* u,$$

$$(6.3)\qquad (\mathscr{S}^{\alpha}_{m,k} f)(u) = d_{n,m,k}(\alpha) \int_{V_{n,m}} f(v)\, |I_m - v'uu'v|^{(\alpha+k-n)/2}\, d_* v,$$

$$(6.4)\qquad (\overset{*}{\mathscr{S}}{}^{\alpha}_{m,k} \varphi)(v) = d_{n,m,k}(\alpha) \int_{V_{n,k}} \varphi(u)\, |I_m - v'uu'v|^{(\alpha+k-n)/2}\, d_* u;$$

$$\delta_{n,m,k}(\alpha) = \frac{\Gamma_m(m/2)}{\Gamma_m(n/2)} \frac{\Gamma_m((k-\alpha)/2)}{\Gamma_m(\alpha/2)}, \qquad \alpha + m - k \neq 1, 2, \dots;$$

$$d_{n,m,k}(\alpha) = \frac{\Gamma_m(k/2)}{\Gamma_m(n/2)} \frac{\Gamma_m((n-k-\alpha)/2)}{\Gamma_m(\alpha/2)}, \qquad \alpha + k + m - n \neq 1, 2, \dots;$$

B. For $u \in V_{n,m},\ v \in V_{n,m}$:

$$(6.5)\qquad (\mathscr{M}^{\alpha} f)(u) = \delta_{n,m}(\alpha) \int_{V_{n,m}} f(v)\, |u'v|^{\alpha-m}\, d_* v,$$

$$(6.6)\quad (\mathscr{Q}^{\alpha} f)(u) = d_{n,m}(\alpha) \int_{V_{n,m}} f(v)\, |I_m - v'uu'v|^{(\alpha+m-n)/2}\, d_* v, \quad 2m \le n;$$

$$\delta_{n,m}(\alpha) = \frac{\Gamma_m(m/2)}{\Gamma_m(n/2)} \frac{\Gamma_m((m-\alpha)/2)}{\Gamma_m(\alpha/2)}, \qquad \alpha \neq 1, 2, \dots;$$

$$d_{n,m}(\alpha) = \frac{\Gamma_m(m/2)\, \Gamma_m((n-m-\alpha)/2)}{\Gamma_m(n/2)\, \Gamma_m(\alpha/2)}, \qquad \alpha + 2m - n \neq 1, 2, \dots;$$

cf. Remark 4.1. By Theorem 4.2, all these integrals are absolutely convergent if $Re\, \alpha > m-1$. Excluded values of α belong to the polar set of the corresponding gamma function in the numerator; cf. (2.6).

Theorem 6.1. (i) *Let φ be an integrable right $O(k)$-invariant function on $V_{n,k}$, $\omega \in C^\infty(V_{n,m})$, $1 \le m \le k \le n-1$. Then*

$$(6.7)\qquad \underset{\alpha=0}{a.c.}\, (\overset{*}{\mathscr{C}}{}^{\alpha}_{m,k} \varphi, \omega) = \mu_k\, (\overset{*}{F}_{m,k} \varphi, \omega), \quad \mu_k = \frac{\Gamma_m(m/2)}{\Gamma_m((n-k)/2)}.$$

In particular, for any integrable right $O(m)$-invariant function f on $V_{n,m}$, $2m \le n$, we have

$$(6.8)\qquad \underset{\alpha=0}{a.c.}\, (\mathscr{M}^{\alpha} f, \omega) = \mu_m\, (F_m f, \omega), \quad \mu_m = \frac{\Gamma_m(m/2)}{\Gamma_m((n-m)/2)}.$$

Proof. Equality (6.7) follows from (5.19), (5.23), and (6.2); (6.8) is a consequence of (6.7). □

The following statement holds by duality.

Corollary 6.2. *Let f be an infinitely differentiable right $O(m)$-invariant function on $V_{n,m}$, $1 \le m \le k \le n-1$. Then for any $\omega \in C^\infty(V_{n,m})$,*

$$\underset{\alpha=0}{a.c.} (\mathscr{C}^\alpha_{m,k} f, \omega) = \mu_k \, (F_{m,k} f, \omega). \tag{6.9}$$

Theorem 6.1 and Corollary 6.2 show that the Funk transform $F_{m,k}$ and its dual can be regarded (up to a constant multiple) as members of the corresponding analytic families of normalized cosine transforms. If $m = 1$, then (6.7)-(6.9) agree with [Ru08, Lemma 3.1].

Lemma 6.3. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $1 \le m \le k \le n-m$,*

$$Re\, \alpha > m-1, \qquad \alpha \ne k-m+1, k-m+2, \dots .$$

Then

$$\overset{*}{\mathscr{C}}{}^\alpha_{m,k} F_{m,k} f = \varkappa_k \mathscr{Q}^{\alpha+n-k-m} f, \qquad \varkappa_k = \frac{\Gamma_m((n-m)/2)}{\Gamma_m(k/2)}. \tag{6.10}$$

Proof. Write (4.12) in terms of normalized operators and you are done. □

The next theorem contains inversion formulas for the Funk transform F_m and the cosine transform $\mathscr{M}^\alpha$ in terms of distributions. The corresponding results for $m = 1$ are due to Semyanistyi [Se63].

Theorem 6.4. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $2m \le n$. If $Re\, \alpha > m-1$, $\alpha \ne m, m+1, m+2, \dots$, then*

$$\underset{\beta=2m-\alpha-n}{a.c.} (\mathscr{M}^\beta \mathscr{M}^\alpha f, \omega) = (f, \omega), \qquad \omega \in C^\infty(V_{n,m}). \tag{6.11}$$

Moreover (cf. (6.8)),

$$\underset{\alpha=2m-n}{a.c.} (\mathscr{M}^\alpha F_m f, \omega) = \varkappa_m (f, \omega), \qquad \varkappa_m = \frac{\Gamma_m((n-m)/2)}{\Gamma_m(m/2)}. \tag{6.12}$$

Proof. We write (5.20) in the form

$$\left(\frac{E_{\beta-m} M^\beta g}{\Gamma_m(\beta/2)}, \mathcal{F}\phi \right) = \check{c}_2 \left(\frac{E_{m-\beta-n} g}{\Gamma_m((m-\beta)/2)}, \phi \right), \tag{6.13}$$

$$\check{c}_2 = \frac{2^{m(n+\beta-m)}\, \pi^{nm/2}\, \Gamma_m(n/2)}{\Gamma_m(m/2)},$$

and set $g = M^\alpha f$. By Theorem 4.2, g is integrable and right $O(m)$-invariant on $V_{n,m}$. Then we compute analytic continuation at the point $\beta = 2m - \alpha - n$ and obtain

$$\left(\frac{E_{m-\alpha-n} M^{2m-\alpha-n} M^\alpha f}{\Gamma_m((2m-\alpha-n)/2)}, \mathcal{F}\phi\right) = \tilde{c}_2 \left(\frac{E_{\alpha-m} M^\alpha f}{\Gamma_m((\alpha+n-m)/2)}, \phi\right), \tag{6.14}$$

$$\tilde{c}_2 = \frac{2^{m(m-\alpha)}\, \pi^{nm/2}\, \Gamma_m(n/2)}{\Gamma_m(m/2)}.$$

By (5.20) the right-hand side of (6.14) can be written as

$$\frac{\tilde{c}_2\, \Gamma_m(\alpha/2)}{\Gamma_m((\alpha+n-m)/2)} \left(\frac{E_{\alpha-m} M^\alpha f}{\Gamma_m(\alpha/2)}, \phi\right) = c_3 \left(\frac{E_{m-\alpha-n} f}{\Gamma_m((m-\alpha)/2)}, \mathcal{F}^{-1}\phi\right),$$

where

$$c_3 = \frac{(2\pi)^{mn}\, \Gamma_m^2(n/2)\, \Gamma_m(\alpha/2)}{\Gamma_m^2(m/2)\, \Gamma_m((\alpha+n-m)/2)}.$$

To finalize calculations, we set $(\mathcal{F}^{-1}\phi)(x) = (2\pi)^{-mn}(\mathcal{F}\phi)(-x)$ and invoke normalizing factors according to (6.5). This yields

$$(E_{m-\alpha-n} \mathscr{M}^{2m-\alpha-n} \mathscr{M}^\alpha f, \mathcal{F}\phi) = (E_{m-\alpha-n} f, \mathcal{F}\phi), \tag{6.15}$$

where both sides are understood as analytic continuations from the respective domains. As in the proof of Corollary 5.8, we choose $\mathcal{F}\phi$ in (6.15) so that $(\mathcal{F}\phi)(x) = \psi(r)\omega(v)$, $x = vr^{1/2}$, $v \in V_{n,m}$. We recall that $\omega \in C^\infty(V_{n,m})$ and $\psi(r)$ is a nonnegative C^∞ function on the cone Ω with compact support away from the boundary $\partial\Omega$. Then the left-hand side of (6.15) becomes

$$\underset{\beta=2m-\alpha-n}{a.c.} (E_{\beta-m} \mathscr{M}^\beta \mathscr{M}^\alpha f, \psi\omega) = h_\psi(\alpha) \underset{\beta=2m-\alpha-n}{a.c.} (\mathscr{M}^\beta \mathscr{M}^\alpha f, \omega),$$

$$h_\psi(\alpha) = 2^{-m} \int_\Omega |r|^{-(\alpha+1)/2} \overline{\psi(r)}\, dr,$$

and the right-hand side equals $h_\psi(\alpha)\,(f, \omega)$. Owing to uniqueness of analytic continuation, this gives the result.

Let us prove (6.12). We set $g = F_m f$ in (6.13). Clearly, g is integrable and right $O(m)$-invariant; cf. Corollary 3.3. Then we compute analytic continuation at the point $\beta = 2m - n$ and make use of Corollary 5.12. This gives

$$\underset{\beta=2m-n}{a.c.} \left(\frac{E_{\beta-m} M^\beta F_m f}{\Gamma_m(\beta/2)}, \mathcal{F}\phi\right) = \check{c}_0\, (E_{-m} F_m f, \phi) = \check{c}_{00}\, (E_{m-n} f, \mathcal{F}\phi),$$

$$\check{c}_0 = \frac{2^{m^2}\, \pi^{nm/2}\, \Gamma_m(n/2)}{\Gamma_m(m/2)\, \Gamma_m((n-m)/2)}, \qquad \check{c}_{00} = \frac{\Gamma_m(n/2)}{\Gamma_m^2(m/2)}.$$

Hence, after normalization,

$$\underset{\beta=2m-n}{a.c.}(E_{\beta-m}\mathscr{M}^{\beta}F_m f,\mathcal{F}\phi)=\varkappa_m(E_{m-n}f,\mathcal{F}\phi),\qquad \varkappa_m=\frac{\Gamma_m((n-m)/2)}{\Gamma_m(m/2)}.$$

As above, this implies (6.12). □

The following important statement shows that the identity operator is a member of the analytic family $\{\mathscr{Q}^\alpha\}$ of the normalized sine transforms, corresponding to $\alpha=0$.

Theorem 6.5. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$; $2m\le n$, $\omega\in C^\infty(V_{n,m})$. Then $(\mathscr{Q}^\alpha f,\omega)$ extends as a meromorphic function of α with the polar set $\{n-2m+1,n-2m+2,\dots\}$. Moreover,*

$$\underset{\alpha=0}{a.c.}(\mathscr{Q}^\alpha f,\omega)=(f,\omega). \tag{6.16}$$

Proof. The statement follows from (5.27) and (6.12) if we use normalization (6.6) and (6.5):

$$\begin{aligned}\underset{\alpha=0}{a.c.}(\mathscr{Q}^\alpha f,\omega) &= \frac{\Gamma_m^2(m/2)}{\Gamma_m(n/2)}\underset{\beta=2m-n}{a.c.}\left(\frac{M^\beta F_m f}{\Gamma_m(\beta/2)},\omega\right)\\ &= \frac{\Gamma_m(m/2)}{\Gamma_m((n-m)/2)}\underset{\beta=2m-n}{a.c.}(\mathscr{M}^\beta F_m f,\omega)=(f,\omega).\end{aligned}$$

□

Equalities (6.10) and (6.16) imply the following inversion result for the Funk transform $F_{m,k}$.

Theorem 6.6. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $\omega\in C^\infty(V_{n,m})$, $1\le m\le k\le n-m$. Then*

$$\underset{\alpha=m+k-n}{a.c.}(\overset{*}{\mathscr{C}}{}^{\alpha}_{m,k}F_{m,k}f,\omega)=\varkappa_k(f,\omega),\qquad \varkappa_k=\frac{\Gamma_m((n-m)/2)}{\Gamma_m(k/2)}. \tag{6.17}$$

It would be natural to find an inversion formula for the cosine transform $\mathscr{C}^\alpha_{m,k}$, $k>m$, similar to (6.11). Regretfully it is not available, however, the following lemma enables us to reduce inversion of $\mathscr{C}^\alpha_{m,k}$ to the already known procedure for F_m; cf. (6.12).

Lemma 6.7. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $\omega\in C^\infty(V_{n,m})$, $1\le m\le k\le n-m$,*

$$Re\,\alpha>k-1,\qquad \alpha\ne k,k+1,k+2,\dots .$$

Then

$$\underset{\beta=-\alpha}{a.c.}(\overset{*}{\mathscr{S}}{}^{\beta}_{m,k}\mathscr{C}^\alpha_{m,k}f,\omega)=\mu_k\,(F_m f,\omega),\qquad \mu_k=\frac{\Gamma_m(m/2)}{\Gamma_m((n-k)/2)}. \tag{6.18}$$

Proof. By (4.13) and (6.5),

$$\mathscr{M}^{\alpha+m-k} F_m f = c \, \overset{*}{F}_{m,k} \mathcal{C}^\alpha_{m,k} f, \qquad c = \delta_{n,m}(\alpha+m-k)/\tilde{c}_\alpha.$$

Hence, by (6.11),

$$\begin{aligned}(F_m f, \omega) &= \underset{\beta=m-n+k-\alpha}{a.c.} c\, (\mathscr{M}^\beta \, \overset{*}{F}_{m,k} \mathcal{C}^\alpha_{m,k} f, \omega) \\ &= \underset{\beta=m-n+k-\alpha}{a.c.} c\, \delta_{n,m}(\beta)\, ((F_{m,k} M^\beta)^* \mathcal{C}^\alpha_{m,k} f, \omega),\end{aligned}$$

and (4.15) yields

$$(F_m f, \omega) = \underset{\beta=m-n+k-\alpha}{a.c.} c\, \delta_{n,m}(\beta)\, c_{n,k,m}(\beta)\, (\overset{*}{\mathcal{S}}{}^{\beta+n-k-m}_{m,k} \mathcal{C}^\alpha_{m,k} f, \omega).$$

Now we replace operators on the right-hand side by their normalizations (6.4) and (6.1). A simple calculation completes the proof. □

The following "pointwise" conjecture looks natural.

Conjecture 6.8. *Let $f(v)$ be an infinitely differentiable $O(m)$-invariant function on $V_{n,m}$.*
(a) *For every $v \in V_{n,m}$, $(\mathscr{M}^\alpha f)(v)$ extends as a meromorphic function of α with the polar set $\mathbb{N} = \{1,2,3,\dots\}$.*
(b) *For every complex $\alpha \notin \mathbb{N}$, $(\mathscr{M}^\alpha f)(v)$ is an infinitely differentiable $O(m)$-invariant function on $V_{n,m}$.*
(c) *If $\alpha \notin \mathbb{N} \cup \tilde{\mathbb{N}}$, $\quad \tilde{\mathbb{N}} = \{2m-n-1, 2m-n-2, \dots\}$, then*

$$(\mathscr{M}^{2m-\alpha-n} \mathscr{M}^\alpha f)(v) = f(v) \tag{6.19}$$

for every $v \in V_{n,m}$.
(d) *The Funk transform*

$$(F_m f)(u) = \int_{\{v \in V_{n,m} :\, u'v=0\}} f(v)\, d_u v, \qquad u \in V_{n,m}, \tag{6.20}$$

can be inverted by the formula

$$(\mathscr{M}^{2m-n} F_m f)(v) = \varkappa_m f(v), \tag{6.21}$$

which holds for every $v \in V_{n,m}$.

In the case $m = 1$ these statements are known and can be obtained using decomposition in spherical harmonics; see, e.g., [Ru98, Ru08]. We expect that Conjecture 6.8 can be proved using harmonic analysis developed in [Ge, TT] together with results from [Su].

7. The method of Riesz potentials

We already know that the Funk transform can be formally inverted as $F_m^{-1} = \varkappa_m^{-1} \underset{\alpha=2m-n}{a.c.} \mathscr{M}^\alpha F_m f$. There are many ways to realize this analytic continuation. One of them amounts to Blaschke, Radon, Fuglede, and Helgason [Rad, Fug, Hel10] and consists of two steps: (a) We apply a certain back-projection operator and reduce the problem to inversion of the corresponding potential; (b) We invert that potential operator. In the case of the Funk transform on the sphere, the potential operator is realized as the sine transform, explicit inversion of which is pretty sophisticated; see, e.g., [Hel10, Ru02]. In the higher-rank case the situation is much more complicated; cf. [Gri, Theorem 3.4], [Ka, Theorem 10.4]. Semyanistyi [Se63] suggested to express the spherical sine transform through the spatial Riesz potential. In this section we generalize his idea for the higher-rank case.

There is a substantial difference between the cases, when $n-k-m$ is an even number and an odd number. This phenomenon, which is well-known in the case $m = 1$, is related to the so-called local and non-local inversion formulas.

7.1. **Local inversion formulas for the Funk transform.**

It is instructive to start with the particular case $k = m$, when evenness of $n-k-m$ is equivalent to the evenness of n.

Theorem 7.1. *Let f be a right $O(m)$-invariant function in $L^1(V_{n,m})$, $2m \le n$. If n is even and*

$$c_m = \frac{(-1)^{m(n/2-m)}\, 2^{m(2m-n)}\, \Gamma_m^2(m/2)}{\Gamma_m^2((n-m)/2)},$$

then the Funk transform $\varphi = F_m f$ can be inverted by the formula

$$E_{m-n} f = c_m\, \Delta^{n/2-m}\, E_{-m} F_m \varphi \qquad \text{(in the } S'\text{-sense)}. \tag{7.1}$$

If the right-hand side of (7.1) is locally integrable in the neighborhood of the Stiefel manifold $V_{n,m}$, then (7.1) holds pointwise a.e. on $V_{n,m}$.

Proof. By (5.26) for $\phi \in S(\mathfrak{M}_{n,m})$ we have

$$(-1)^{m\ell}(\Delta^\ell E_{-m} F_m \varphi, \hat{\phi}) = \tilde{c}_0\, (E_{m-n}\varphi, |x|_m^{2\ell}\phi) = \tilde{c}_0\, (E_{m-n+2\ell} F_m f, \phi).$$

We can choose $2\ell = n - 2m$ to get $\tilde{c}_0\, (E_{-m} F_m f, \phi)$. By (5.26) this coincides with $\tilde{c}_0^2\, (2\pi)^{-nm}\, (E_{m-n} f, \hat{\phi})$, and the result follows. □

The next statement is more general.

Theorem 7.2. *Let f be a right $O(m)$-invariant function in $L^1(V_{n,m})$, $1 \le m \le k \le n-m$. If $n-k-m$ is an even number and*

$$c_{m,k} = \frac{(-1)^{m\ell}\, 4^{-m\ell}\, \Gamma_m(m/2)\, \Gamma_m(k/2)}{\Gamma_m((n-m)/2)\, \Gamma_m((n-k)/2)}, \quad \ell = (n-k-m)/2,$$

then the Funk transform $\varphi = F_{m,k} f$ can be inverted by the formula

$$E_{m-n} f = c_{m,k}\, \Delta^\ell\, E_{-k} \overset{*}{F}_{m,k} \varphi \qquad \text{(in the } S'\text{-sense).} \tag{7.2}$$

If the right-hand side of (7.2) is locally integrable in the neighborhood of $V_{n,m}$, then (7.2) holds pointwise a.e. on $V_{n,m}$.

Proof. By (5.25) and (4.13),

$$(-1)^{m\ell}(\Delta^\ell\, E_{-k} \overset{*}{F}_{m,k}\varphi, \hat{\phi}) = d_0 \underset{\alpha=0}{a.c.} \left(\frac{E_{\alpha-k}\, \overset{*}{\mathcal{C}}{}^{\alpha}_{m,k}\varphi}{\Gamma_m(\alpha/2)}, (-1)^{m\ell}\Delta^\ell \hat{\phi} \right)$$

$$= d_1 \underset{\alpha=0}{a.c.} \left(\frac{E_{\alpha-k} M^{\alpha+m-k} F_m f}{\Gamma_m((\alpha+m-k)/2)}, (-1)^{m\ell}\Delta^\ell \hat{\phi} \right),$$

$$d_1 = \frac{\Gamma_m((n-k)/2)\, \Gamma_m(m/2)}{\Gamma_m(n/2)}.$$

Owing to (5.20), for $2\ell = n-k-m$, this expression can be written as

$$d_2 \underset{\alpha=0}{a.c.} (E_{k-n-\alpha} F_m f, |x|^{2\ell}_m \phi) = d_2\, (E_{-m} F_m f, \phi),$$

$$d_2 = 2^{m(n-k)}\, \pi^{nm/2}\, \Gamma_m((n-k)/2)/\Gamma_m(k/2).$$

Finally, by (5.26),

$$(-1)^{m\ell}(\Delta^\ell\, E_{-k} \overset{*}{F}_{m,k}\varphi, \hat{\phi}) = \tilde{c}_0\, d_2\, (E_{m-n} f, \check{\phi}) = \frac{\tilde{c}_0\, d_2}{(2\pi)^{nm}}\, (E_{m-n} f, \hat{\phi}).$$

This gives the result. □

7.2. **Sine transforms and non-local inversion formulas.** Theorems 7.1 and 7.2 show that when $n-k-m$ is odd, we have to deal with fractional powers of the operator Δ. The latter are realized by the Riesz potential I^α; see Section 2.4. For the following we need some more preparation.

7.2.1. *The Semyanistyi-Lizorkin-Samko spaces* Φ_V. From the Fourier transform formula (2.33) it is evident that the Schwartz class $\mathcal{S}$ is not well-adapted for Riesz potentials, because $\mathcal{S}$ is not invariant under multiplication by $|y|_m^{-\alpha}$. To get around this difficulty, we choose another space of test functions. Let $\Psi = \Psi(\mathfrak{M}_{n,m})$ be the collection of all functions $\psi(y) \in \mathcal{S}(\mathfrak{M}_{n,m})$ vanishing on the manifold

$$V = \{y \in \mathfrak{M}_{n,m} : \text{rank}(y) < m\} = \{y \in \mathfrak{M}_{n,m} : |y'y| = 0\} \tag{7.3}$$

with all derivatives. The manifold V is a cone in $\mathbb{R}^{nm}$ with vertex 0. Let $\Phi = \Phi(\mathfrak{M}_{n,m})$ be the Fourier image of Ψ. Since the Fourier transform is an automorphism of $\mathcal{S}$, then Φ is a closed linear subspace of $\mathcal{S}$, which is isomorphic to Ψ.

The spaces Φ and Ψ were introduced by V.I. Semyanistyi [Se] in the case $m = 1$. They have proved to be very useful in integral geometry and real analysis. Further generalizations and applications are mainly due to Lizorkin and Samko; see [Li, Ru96, Sa1, SKM] on this subject.

The following characterization of the space Φ is a consequence of a more general result by Samko [Sa1].

Theorem 7.3. *The Schwartz function $\phi(x)$ on $\mathfrak{M}_{n,m}$ belongs to the space Φ if and only if it is orthogonal to all polynomials $p(x)$ on any hyperplane τ in $\mathbb{R}^{nm}$ having the form $\tau = \{x : \text{tr}(a'x) = c\}$, $a \in V$:*

$$\int_\tau p(x)\phi(x)d\mu(x) = 0, \tag{7.4}$$

$d\mu(x)$ being the induced Lebesgue measure on τ.

When $m = 1$, the space Φ consists of Schwartz functions which are orthogonal to all polynomials on $\mathbb{R}^n$.

We denote by Φ' the space of all semilinear continuous functionals on Φ. Two $\mathcal{S}'$-distributions that coincide in the Φ'-sense, differ from each other by an arbitrary $\mathcal{S}'$-distribution with the Fourier transform supported by V. Since for any complex α, multiplication by $|y|_m^{-\alpha}$ is an automorphism of Ψ, then, according to the general theory [GSh2], I^α is an automorphism of Φ, and we have

$$\mathcal{F}[I^\alpha \phi](y) = |y|_m^{-\alpha} \mathcal{F}[\phi](y), \qquad \phi \in \Phi. \tag{7.5}$$

This gives

$$\mathcal{F}[I^\alpha f](y) = |y|_m^{-\alpha} \mathcal{F}[f](y) \tag{7.6}$$

for any Φ'-distribution f. For k even, the Riesz potential $I^k f$ can be inverted by repeated application of the Cayley-Laplace operator Δ in the sense of Φ'-distributions.

7.2.2. *Inversion of the sine transform.*

Lemma 7.4. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $2m \le n$. Then for any $\alpha \in \mathbb{C}$,*

$$\text{a.c.}\, E_{\alpha+m-n}\mathscr{Q}^\alpha f = 2^{\alpha m}\, \text{a.c.}\, I^\alpha E_{m-n} f \tag{7.7}$$

in the Φ'-sense.

Proof. Let $\phi \in \Phi,\ \psi = \check{\phi} \in \Psi$. We first suppose that

$$Re\,\alpha > n-m-1; \qquad \alpha \neq n-m, n-m+1, n-m+2, \dots. \tag{7.8}$$

Then, owing to (6.6), (4.17), and (5.20),

$$\begin{aligned}(E_{\alpha+m-n}\mathscr{Q}^\alpha f, \phi) &= d_{n,m}(\alpha)\, d_\alpha (E_{\alpha+m-n} M^{\alpha+2m-n} F_m f, \hat{\psi}) \\ &= \frac{2^{(\alpha+m)m}\, \pi^{nm/2}\, \Gamma_m(m/2)}{\Gamma_m((n-m)/2)} (E_{-m} F_m f, |x|_m^{-\alpha}\psi).\end{aligned} \tag{7.9}$$

Since $|x|_m^{-\alpha}\psi(x) = (2\pi)^{-nm}|x|_m^{-\alpha}\hat{\phi}(-x) = (2\pi)^{-nm}(I^\alpha\phi)^\wedge(-x)$, then (5.26) yields

$$(E_{\alpha+m-n}\mathscr{Q}^\alpha f, \phi) = 2^{\alpha m}\,(E_{m-n}f, I^\alpha\phi) = 2^{\alpha m}\,(I^\alpha E_{m-n} f, \phi).$$

An expression in (7.9) is an entire function of α. Hence, the result follows by analytic continuation. □

Lemma 7.4 enables us to reconstruct f from $Q^\alpha f$ in the Φ'-sense.

Corollary 7.5. *Let $g = Q^\alpha f$, where f is an integrable right $O(m)$-invariant function on $V_{n,m}$, $2m \le n$,*

$$Re\,\alpha > m-1; \qquad \alpha \neq n-2m+1, n-2m+2, \dots.$$

We set $\alpha = 2\ell - \gamma$, where ℓ is a positive integer. The following inversion formula holds in the Φ'-sense:

$$E_{m-n} f = 2^{-\alpha m}\,(-1)^{\ell m}\,\Delta^\ell I^\gamma E_{\alpha+m-n} Q^\alpha f, \tag{7.10}$$

where Δ is the Cayley-Laplace operator (2.32).

Remark 7.6. An analogue of (7.7) for $m = 1$ is contained in Lemma 3.1 from [Ru02] and invokes spherical convolutions with hypergeometric kernel. It is an interesting open problem to extend that Lemma to the higher-rank case.

7.2.3. *Non-local inversion of cosine and Funk transforms.*

Theorem 7.7. *Let f be an integrable right $O(m)$-invariant function on $V_{n,m}$, $1 \le m \le k \le n-m$, and let ℓ be a positive integer.*
(i) *If $g = \mathscr{C}^{\alpha}_{m,k} f$,*

$$Re\,\alpha > m-1, \qquad \alpha + m - k \neq 1, 2, \dots,$$

and $\alpha + n - k - m = 2\ell - \gamma$, then the following inversion formula holds in the Φ'-sense:

$$E_{m-n} f = c\,\Delta^{\ell} I^{\gamma} E_{\alpha-k} \overset{*}{F}_{m,k} \mathscr{C}^{\alpha}_{m,k} f, \tag{7.11}$$

$$c = \frac{2^{m(k+m-n-\alpha)}(-1)^{m\ell}\,\Gamma_m(k/2)}{\Gamma_m((n-m)/2)}.$$

(ii) *If $\varphi = F_{m,k} f$ and $n-k-m = 2\ell - 1$, then, similarly,*

$$E_{m-n} f = c_*\,\Delta^{\ell} I^{1} E_{-k} \overset{*}{F}_{m,k} F_{m,k} f, \tag{7.12}$$

$$c_* = \frac{2^{m(k+m-n)}(-1)^{m\ell}\,\Gamma_m(k/2)\,\Gamma_m(m/2)}{\Gamma_m((n-m)/2)\,\Gamma_m((n-k)/2)}.$$

Proof. (i) By (4.12), after normalization, we obtain

$$\overset{*}{F}_{m,k} \mathscr{C}^{\alpha}_{m,k} f = \tilde{c}_{\alpha}\, \mathscr{Q}^{\alpha+n-k-m} f, \qquad \tilde{c}_{\alpha} = \frac{c_{\alpha}\,\delta_{n,m,k}(\alpha)}{d_{n,m}(\alpha+n-k-m)}.$$

Then we apply Corollary 7.5 with α replaced by $\alpha+n-k-m$. This gives the result.

(ii) By (5.19) and (5.23), for any $h \in \mathcal{S}(\mathfrak{M}_{n,m})$ we have

$$\underset{\alpha=0}{a.c.} \left(\frac{E_{\alpha-k} \overset{*}{\mathcal{C}}{}^{\alpha}_{m,k} \varphi}{\Gamma_m(\alpha/2)}, h \right) = \lambda_1\,(E_{-k} \overset{*}{F}_{m,k} \varphi, h), \tag{7.13}$$

$$\lambda_1 = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2)\,\Gamma_m((n-k)/2)}.$$

Owing to (4.12), the left-hand side of (7.13) can be written as

$$\underset{\alpha=0}{a.c.} \frac{c_{\alpha}}{\Gamma_m(\alpha/2)}\,(E_{\alpha-k} Q^{\alpha+n-k-m} f, h).$$

Now we replace h by the Riesz potential $I^{1-2\ell}\phi = (-1)^{m\ell}\Delta^{\ell} I^{1}\phi$, where $\phi \in \Phi$, $2\ell - 1 = n-k-m$, and apply (7.7). This gives

$$\lambda_1\,(-1)^{m\ell}\,(\Delta^{\ell} I^{1} E_{-k} \overset{*}{F}_{m,k} \varphi, \phi) = \lambda_2 \underset{\alpha=0}{a.c.} (I^{\alpha} E_{m-n} f, \phi) = \lambda_2\,(E_{m-n} f, \phi),$$

$$\lambda_2 = \frac{2^{m(n-m-k)}\Gamma_m(n/2)\,\Gamma_m((n-m)/2)}{\Gamma^2_m(k/2)\,\Gamma_m(m/2)},$$

and the result follows. □

8. Appendix

8.1. **On the paper by Gelfand, Graev, and Rosu.** Sections 2 and 3 of [GGR] deal with analytic continuation of a certain auxiliary operator R_p^λ; see, e.g., formula (3.5) in [GGR]. Ignoring the normalizing factor, and changing notation $p = m,\ \lambda = \alpha + n - m$, one can write it as a cosine transform

$$(M^\alpha f)(u) = \int_{V_{n,m}} f(v)\,|u'v|^{\alpha-m}\,d_*v, \qquad u \in V_{n,m}, \tag{8.1}$$

where $2m \le n$, $Re\,\alpha > m-1$, and f is a right $O(m)$-invariant function. It was stated on p. 368, that elementary computation (which is skipped) leads to the formula of R_p^λ in coordinates; see formula (3.6) in the same paper. Below we perform computation and arrive at a different expression, which does not fall (at least, directly) into the scope of references [P67, Rai], as stated in [GGR, Proposition 1, p. 368].[2]

Let $\tilde{\mathfrak{M}}_{n-m,m}$ be the subset of $\mathfrak{M}_{n-m,m}$, which consists of matrices of rank m. Every matrix $y \in \tilde{\mathfrak{M}}_{n-m,m}$ is uniquely represented in polar coordinates as $y = \omega s^{1/2}$, $\omega \in V_{n-m,m}$, $s \in \Omega$. We set

$$v = \begin{bmatrix} \omega((I_m+s)^{-1}s)^{1/2} \\ (I_m+s)^{-1/2} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{8.2}$$

Since

$$\begin{aligned} v'v &= ((I_m+s)^{-1}s)^{1/2}\omega'\omega((I_m+s)^{-1}s)^{1/2} + (I_m+s)^{-1} \\ &= (I_m+s)^{-1}s + (I_m+s)^{-1} = I_m, \end{aligned}$$

then $v \in V_{n,m}$. Thus (8.2) defines a map $\mu : \tilde{\mathfrak{M}}_{n-m,m} \to V_{n,m}$. Conversely, given $v \in V_{n,m}$, the corresponding matrix $y = \omega s^{1/2} \in \tilde{\mathfrak{M}}_{n-m,m}$ can be reconstructed from obvious relations

$$(I_m+s)^{-1} = v'v_0v_0'v, \qquad v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n,m}, \tag{8.3}$$

$$\omega((I_m+s)^{-1}s)^{1/2} = \check{v}_0'v, \qquad \check{v}_0 = \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix} \in V_{n,n-m}.$$

Lemma 8.1. *Let $2m \le n$. If f is an integrable right $O(m)$-invariant function on $V_{n,m}$, $\tilde f(y) = (f \circ \mu)(y)$, then*

$$\int_{V_{n,m}} f(v)\,dv = \sigma_{m,m} \int_{\mathfrak{M}_{n-m,m}} \frac{\tilde f(y)}{|I_m + y'y|^{n/2}}\,dy. \tag{8.4}$$

[2]The reasoning from [P67] is reproduced in Lemma 5.3 for the more general situation.

If F is an integrable function on $\mathfrak{M}_{n-m,m}$, $\check{F}(v) = (F \circ \mu^{-1})(v)$, then

$$(8.5) \qquad \int_{\mathfrak{M}_{n-m,m}} F(y)\, dy = \frac{1}{\sigma_{m,m}} \int_{V_{n,m}} \frac{\check{F}(v)}{|v_0' v|^n}\, dv.$$

Proof. We denote by I the left-hand side of (8.4). By (2.12) with $k = n - m$,

$$I = \int_0^{I_m} d\nu(r) \int_{V_{n-m,m}} dw \int_{V_{m,m}} f\left(\begin{bmatrix} wr^{1/2} \\ u(I_m - r)^{1/2} \end{bmatrix}\right) du,$$

$$d\nu(r) = 2^{-m} |r|^{(n-2m-1)/2} |I_m - r|^{-1/2}\, dr.$$

Changing variable $r = I_m - (I_m + s)^{-1}$, we get

$$I = 2^{-m} \int_{\mathcal{P}_m} d\tilde{\nu}(s) \int_{V_{n-m,m}} dw \int_{V_{m,m}} f\left(\begin{bmatrix} \omega((I_m + s)^{-1} s)^{1/2} \\ u(I_m + s)^{-1/2} \end{bmatrix}\right) du,$$

$$d\tilde{\nu}(s) = 2^{-m} |s|^{(n-2m-1)/2} |I_m + s|^{-n/2}\, ds.$$

Since f is right $O(m)$-invariant, then integration over $V_{m,m} = O(m)$ can be suppressed and Lemma 2.1 yields (8.4). The second equality is a consequence of the first one, owing to (8.3). □

Now we return back to (8.1). In the statement below $v_0, f, \tilde{f}, F$, and $\check{F}$ have the same meaning as in Lemma 8.1; $u, v \in V_{n,m}$; $x, y \in \mathfrak{M}_{n-m,m}$; $u = \mu(x)$. We set

$$\tilde{K}_\alpha(x, y) = \frac{|I_m + x'x|^{(m-\alpha)/2}}{|I_m + y'y|^{(n+\alpha-m)/2}}, \qquad \check{K}_\alpha(u, v) = \frac{|v_0' u|^{m-\alpha}}{|v_0' v|^{n+\alpha-m}}.$$

Lemma 8.2. *The following relations hold provided that integrals in either side are absolutely convergent:*

$$(8.6) \qquad \int_{V_{n,m}} f(v)\, |u'v|^{\alpha-m}\, d_*v = \frac{\sigma_{m,m}}{\sigma_{n,m}} \int_{\mathfrak{M}_{n-m,m}} \tilde{f}(y)\, |I_m + x'y|^{\alpha-m} \tilde{K}_\alpha(x, y)\, dy,$$

$$(8.7) \qquad \int_{\mathfrak{M}_{n-m,m}} F(y)\, |I_m + x'y|^{\alpha-m}\, dy = \frac{\sigma_{n,m}}{\sigma_{m,m}} \int_{V_{n,m}} f(v)\, |u'v|^{\alpha-m} \check{K}_\alpha(u, v)\, d_*v.$$

Proof. As above, we set $x = \theta r^{1/2}$, $y = \omega s^{1/2}$, where $\theta, \omega \in V_{n-m,m}$ and $r, s \in \Omega$. Then we define $v \in V_{n,m}$ by (8.2) and, similarly,

$$(8.8) \qquad u = \begin{bmatrix} \theta((I_m + r)^{-1} r)^{1/2} \\ (I_m + r)^{-1/2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Keeping in mind that

$$r(I_m+r)^{-1}=(I_m+r)^{-1}r, \qquad ((I_m+r)^{-1}r)^{1/2}=(I_m+r)^{-1/2}r^{1/2},$$

we easily have

$$|u'v|=|u_1'v_1+u_2'v_2|=|I_m+r|^{-1/2}|I_m+r^{1/2}\theta'\omega s^{1/2}||I_m+s|^{-1/2}.$$

Hence, by (8.3),

$$|u'v|=\frac{|I_m+x'y|}{|I_m+x'x|^{1/2}\,|I_m+y'y|^{1/2}}, \qquad |I_m+x'y|=\frac{|u'v|}{|v_0'u|\,|v_0'v|}.$$

It remains to plug these expressions in the corresponding integrals and apply Lemma 8.1. □

Remark 8.3. An important factor $\tilde{K}_\alpha(x,y)$ in (8.6), that depends on α and affects the behavior at infinity, is skipped in [GGR, formula (3.6)]. Moreover, Lemma 8.1 reveals that, in general, $\tilde{f}(y)$ is not rapidly decreasing. It follows that the reasoning from [P67, Rai], related to analytic continuation of the distribution $|x|^\lambda/\Gamma_p((\lambda+p)/2)$ cannot be directly applied to (8.6). To understand the essence of the matter, the reader is encouraged to perform analytic continuation in detail for $m=1$. Note also that, the integral operator on the right-hand side of (8.7) is not $O(n)$-invariant, unlike the left-hand side of (8.6).

In the preceding work [GGŠ70], the result was deduced from [GGŠ67], using transition to the Radon transform on the space of matrices. This transition requires careful inspection because it does not lead to rapidly decreasing functions, which are assumed in [GGŠ67].

8.2. **A useful integral.**

Lemma 8.4. *Let* $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_m)\in\mathbb{C}^m$, $u\in V_{n,k}$, $v\in V_{n,m}$, $1\le m\le k\le n$. *Then*

$$\int\limits_{V_{n,m}}(v'uu'v)^{\boldsymbol{\lambda}}\,d_*v=\int\limits_{V_{n,k}}(v'uu'v)^{\boldsymbol{\lambda}}\,d_*u=\frac{\Gamma_m(m/2)\,\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{k}_0)}{\Gamma_m(k/2)\,\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)}. \tag{8.9}$$

This integral converges absolutely if and only if $Re\,\lambda_j>j-k-1$ *for each* $j=1,2,\dots,m$.

Proof. The first equality follows from Lemma 2.4. Both integrals are, in fact, constants. Thus, it suffices to evaluate

$$I=\int\limits_{V_{n,m}}(v'u_0u_0'v)^{\boldsymbol{\lambda}}\,d_*v, \qquad u_0=\begin{bmatrix} I_k \\ 0 \end{bmatrix}\in V_{n,k}. \tag{8.10}$$

Consider an auxiliary integral

$$A = \int\limits_{\mathfrak{M}_{n,m}} (x' u_0 u_0' x)^{\boldsymbol{\lambda}} \, e^{-\mathrm{tr}(x'x)} \, dx. \tag{8.11}$$

We compute it in two different ways. Let first

$$x = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad a \in \mathfrak{M}_{k,m}, \quad b \in \mathfrak{M}_{n-k,m}.$$

Then $u_0' x = a$, $x'x = a'a + b'b$, and we have

$$A = A_1 A_2, \quad A_1 = \int\limits_{\mathfrak{M}_{k,m}} (a'a)^{\boldsymbol{\lambda}} \, e^{-\mathrm{tr}(a'a)} \, da, \quad A_2 = \int\limits_{\mathfrak{M}_{n-k,m}} e^{-\mathrm{tr}(b'b)} \, db. \tag{8.12}$$

Passing to polar coordinates, owing to (2.21) and (2.23), we obtain

$$A_1 = 2^{-m} \sigma_{k,m} \int\limits_{\Omega} r^{\boldsymbol{\lambda} + \mathbf{k}_0} e^{-\mathrm{tr}(r)} d_* r = 2^{-m} \sigma_{k,m} \, \Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{k}_0), \tag{8.13}$$

provided $Re\, \lambda_j > j - k - 1, \ \forall j = 1, 2, \ldots, m$. The last condition is sharp. It gives the "only if" part of the lemma. For A_2 we have

$$A_2 = \left(\int\limits_{-\infty}^{\infty} e^{-s^2} \, ds \right)^{m(n-k)} = \pi^{m(n-k)/2}.$$

Thus

$$A = \frac{\pi^{nm/2} \, \Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{k}_0)}{\Gamma_m(k/2)}, \qquad Re\, \lambda_j > j - k - 1. \tag{8.14}$$

On the other hand, by setting $x = vt$, $v \in V_{n,m}$, $t \in T_m$, owing to Lemma 2.2, we obtain

$$A = \int\limits_{T_m} e^{-\mathrm{tr}(t't)} \, d\mu(t) \int\limits_{V_{n,m}} (t' v' u_0 u_0' v t)^{\boldsymbol{\lambda}} \, dv,$$

$$d\mu(t) = \prod_{j=1}^{m} t_{j,j}^{n-j} \, dt_{j,j} \, dt_*, \qquad dt_* = \prod_{i<j} dt_{i,j}.$$

By (2.18), one can write

$$A = BI, \tag{8.15}$$

where I is our integral (8.10) and

$$\begin{aligned} B &= \sigma_{n,m} \int\limits_{T_m} (t't)^{\boldsymbol{\lambda}} e^{-\mathrm{tr}(t't)}\, d\mu(t) = \\ &= \sigma_{n,m} \prod_{j=1}^{m} \int\limits_0^\infty t_{j,j}^{\lambda_j+n-j}\, e^{-t_{j,j}^2}\, dt_{j,j} \times \prod_{i<j} \int\limits_{-\infty}^{\infty} e^{-t_{i,j}^2}\, dt_{i,j} \\ &= 2^{-m}\, \pi^{m(m-1)/4}\, \sigma_{n,m} \prod_{j=1}^{m} \Gamma\Big(\frac{\lambda_j+n-j+1}{2}\Big) \\ &= 2^{-m}\, \sigma_{n,m}\, \Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0), \qquad Re\, \lambda_j > j-n-1. \end{aligned}$$

Combining this with (8.14) and (8.15), we obtain

$$I = \frac{A_1 A_2}{B} = \frac{\Gamma_m(m/2)\, \Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{k}_0)}{\Gamma_m(k/2)\, \Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)}.$$

□

In the case $\lambda_1 = \ldots = \lambda_m = \lambda$ we have the following.

Corollary 8.5. *Let* $1 \le m \le k \le n$, $Re\,\lambda > m-k-1$. *Then*

$$\int\limits_{V_{n,m}} |v'uu'v|^{\lambda/2}\, dv = \int\limits_{V_{n,k}} |v'uu'v|^{\lambda/2}\, du = \frac{\Gamma_m(n/2)\, \Gamma_m((\lambda+k)/2)}{\Gamma_m(k/2)\, \Gamma_m((\lambda+n)/2)}. \tag{8.16}$$

References

[Al] A. D. Aleksandrov, On the theory of mixed volumes of convex bodies.II New inequalities between mixed volumes and their applications, Mat. Sbornik N.S. 2 (1937), 12051238, (in Russian).

[A] S. Alesker, *The α-cosine transform and intertwining integrals*, Preprint, 2003.

[AB] S. Alesker, and J. Bernstein, *Range characterization of the cosine transform on higher Grassmannians*, Advances in Math., **184** (2004), 367–379.

[Bla] W. Blaschke, Kreis und Kugel, Chelsea, New York, 1949.

[Bli1] B. Blind, *Distributions homogènes sur une algèbre de Jordan*, Bull. Soc. Math. France, **125** (1997), no. 4, 493–528.

[Bli2] ——— *Distributions vectorielles homogènes sur une algèbre de Jordan* J. Funct. Anal., **208** (2004), no. 2, 482–507.

[C] F. Chatelin, Eigenvalues of matrices, John Wiley & Sons. Inc., New York, (1993).

[FK] J. Faraut, and A. Korányi, *Analysis on symmetric cones*, Clarendon Press, Oxford, 1994.

[Fug] B. Fuglede, *An integral formula*, Math. Scand., **6** (1958), 207-212.

[F11] P. G. Funk, Über Flächen mit lauter geschlossenen geodätischen Linien”, Thesis, Georg-August-Universitt Göttingen, 1911.

[F13] ——— *”Uber Flächen mit lauter geschlossenen geodätischen Linien”*, Mathematische Annalen, Band 74, 1913, 278-300.

[Ga] R.J. Gardner, *Geometric tomography*, Cambridge University Press, New York, 1995.

[Ge] S.S. Gelbart, Fourier analysis on matrix space, Memoirs of the Amer. Math. Soc., No. 108, AMS, Providence, RI, 1971.

[GGG] I. M. Gel'fand, S. G. Gindikin, and M. I. Graev, Selected topics in integral geometry, Translations of Mathematical Monographs, AMS, Providence, Rhode Island, 2003.

[GGR] I. M. Gelfand, M.I. Graev, and R. Rosu, *The problem of integral geometry and intertwining operators for a pair of real Grassmannian manifolds*, J. Operator Theory, **12** (1984), 339–383.

[GGŠ67] I.M. Gelfand, M.I. Graev, and Z.Ja. Šapiro, *Integral geometry on k-dimensional planes*, Funkcional. Anal. i Priloen 1 1967 15–31 (Russian).

[GGŠ70] ———, *A problem of integral geometry connected with a pair of Grassmann manifolds*, Dokl. Akad. Nauk SSSR, **193**, No. 2, (1970), 892-896.

[GSha] I.M. Gelfand, and Z. Ya. Shapiro, Homogeneous functions and their extensions. (Russian) Uspehi Mat. Nauk (N.S.) 10 (1955), no. 3(65), 3–70. Translated in Amer. Math. Soc. Transl. (2) 8 1958 21–85.

[GSh2] I.M. Gelfand, and G.E. Shilov, Generalized functions, Vol. 2. Spaces of fundamental and generalized functions, Academic Press, New York, 1968.

[Gi] S.G. Gindikin, *Analysis on homogeneous domains*, Russian Math. Surveys, **19** (1964), No. 4, 1–89.

[GW] H. Gluck, and F.W. Warner, *Great circle fibrations of the three-sphere*, Duke Math. J., **50** (1983), 107–132.

[Go] F. Gonzalez, *Radon transform on Grassmann manifolds*, Journal of Func. Anal., **71** (1987), 339-362.

[GK1] F. Gonzalez, and T. Kakehi, *Pfaffian systems and Radon transforms on affine Grassmann manifolds*, Math. Ann. 326 (2003), no. 2, 237–273.

[GK2] ———, Moment conditions and support theorems for Radon transforms on affine Grassmann manifolds. Adv. Math. 201 (2006), no. 2, 516–548.

[GK3] ———, *Invariant differential operators and the range of the matrix Radon transform*, Journal of Functional Analysis, **241** (2006), 232–267.

[Goo] P. Goodey, *Applications of representation theory to convex bodies*, II International Conference in "Stochastic Geometry, Convex Bodies and Empirical Measures" (Agrigento, 1996). Rend. Circ. Mat. Palermo (2) Suppl. No. 50 (1997), 179–187.

[GH1] P. Goodey, and R. Howard, *Processes of flats induced by higher-dimensional processes*, Adv. in Math., **80** (1) (1990), 92–109.

[GZ] P. Goodey, and G. Zhang, *Inequalities between projection functions of convex bodies*, Amer. J. Math. **120** (1998), 345–367.

[Gra] M.I. Graev, Integral geometry on the space L^n, where L is a matrix ring. Funkt. Anal. Prilozhen. 30(4), 7174 (1997) (Russian); translation in Funct. Anal. Appl. 30(4), 277280.

[Gri] E. Grinberg, *Radon transforms on higher rank Grassmannians*, J. Differential Geometry **24** (1986), 53-68.

[GR] E. Grinberg, and B. Rubin, *Radon inversion on Grassmannians via Gårding-Gindikin fractional integrals*, Annals of Math. **159** (2004), 809–843.

[Hel90] S. Helgason, *The totally geodesic Radon transform on constant curvature spaces*, Contemp. Math., **113** (1990), 141–149.

[Hel00] ______, Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions, Corrected reprint of the 1984 original. Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2000.

[Hel10] ______, Integral geometry and Radon transforms, Springer-Verlag, to appear.

[Herz] C. Herz, *Bessel functions of matrix argument*, Ann. of Math., **61** (1955), 474–523.

[J] A.T. James, Normal multivariate analysis and the orthogonal group, Ann. Math. Statist., **25** (1954), 40–75.

[Ka] T. Kakehi, *Integral geometry on Grassmann manifolds and calculus of invariant differential operators*, J. Funct. Anal., **168** (1999), 1-45.

[Ke] F. Keinert, *Inversion of k-plane transforms and applications in computer tomography*, SIAM Review, **31** (1989), 273–289.

[Kh1] S.P. Khekalo, *Riesz potentials in the space of rectangular matrices and iso-Huygens deformations of the Cayley-Laplace operator*, Doklady Mathematics, **63** (2001), No. 1, 35–37.

[Kh1a] ______, *The Igusa zeta function associated with a complex power function on the space of rectangular matrices*, Math. Notes **78** (2005), no. 5-6, 719–734.

[Kh2] ______, *The Cayley-Laplace differential operator on the space of rectangular matrices*, Izvestya: Mathematics **61**:1, (2005),191–219.

[K97] A. Koldobsky, Inverse formula for the Blaschke-Levy representation, Houston J. Math., **23** (1997), 95-107.

[K05] ______, Fourier analysis in convex geometry, Mathematical Surveys and Monographs, **116**, AMS, 2005.

[Lev] P. Levy, Théory de l'addition de variable aléatoires, Gauthier-Villars, Paris, 1937.

[Li] P.I. Lizorkin, *Generalized Liouville differentiation and functional spaces $L^r_p(E_n)$. Imbedding theorems*, Matem. sbornik, **60**(120) (1963), 325–353 (Russian).

[Lu] E. Lutwak, Centroid bodies and dual mixed volumes. Proc. London Math. Soc. (3) 60 (1990), 365–391.

[Mat] G. Matheron, *Un théorème d'unicité pour les hyperplans poissoniens*, J. Appl. Probability, **11** (1974), 184–189.

[Min] H. Minkowski, On bodies of constant width (in Russian), Mat. Sbornik. **25** (1904), 505–508.

[Mu] R.J. Muirhead, *Aspects of multivariate statistical theory*, John Wiley & Sons. Inc., New York, 1982.

[Na] F. Natterer, *The mathematics of computerized tomography*, Wiley, New York, 1986.

[OlR] G. Ólafsson, and B. Rubin, *Invariant Functions on Grassmannians*, Cont. math.

[OR04] E. Ournycheva, and B. Rubin, *Radon transform of functions of matrix argument*, Preprint, 2004 (math.FA/0406573).

[OR05a] ______, *Composite cosine transforms*, Mathematika, **52** (2005), 53–68.

[OR05b] ______, *An analogue of the Fuglede formula in integral geometry on matrix spaces*, Contemporary Math., **382** (2005), 305–320.

[OR06] ______, *The Composite Cosine Transform on the Stiefel Manifold and Generalized Zeta Integrals*, Contemp. Math., **405** (2006), 111–133.

[OR08a] ______, *Method of Mean Value Operators for Radon Transforms in the Space of Matrices*, Intern. J. Math., 19 (2008), 245-283.

[OR08b] ______, *Semyanistyi's integrals and Radon transforms on matrix spaces*, The Journal of Fourier Anal. and Appl., 14 (2008), 60–88.

[P67] E.E. Petrov, *The Radon transform in spaces of matrices and in Grassmann manifolds*, Dokl. Akad. Nauk SSSR, **177**, No. 4 (1967), 1504-1507.

[P70] ______, *The Radon transform in spaces of matrices*, Trudy seminara po vektornomu i tenzornomu analizu, M.G.U., Moscow, **15** (1970), 279–315 (Russian).

[Rad] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. - Nat. Kl., **69** (1917), 262–277.

[Rai] M. Raïs, *Distributions homogènes sur des espaces de matrices*, Bull. Soc. math. France, Mem., **30**, (1972), 3–109.

[RS] F. Ricci, and E. Stein, *Homogeneous distributions on spaces of Hermitean matrices*, J. Reine Angew. Math., textbf368 (1986), 142–164.

[Ru96] B. Rubin, Fractional integrals and potentials, Pitman Monographs and Surveys in Pure and Applied Mathematics, **82**, Longman, Harlow, 1996.

[Ru98] ______, *Inversion of fractional integrals related to the spherical Radon transform*, Journal of Funct. Analysis, **157** (1998), 470–487.

[Ru02] ______, *Inversion formulas for the spherical Radon transform and the generalized cosine transform*, Advances in Applied Math., **29** (2002), 471–497.

[Ru03] ______, *Notes on Radon transforms in integral geometry*, Fract. Calc. Appl. Anal. **6** (2003), No. 1, 25–72.

[Ru04] ______, Radon transforms on affine Grassmannians, Trans. Amer. Math. Soc., 356 (2004), 5045-5070.

[Ru06] ______, Riesz potentials and integral geometry in the space of rectangular matrices, Advances in Math., **205** (2006), 549–598.

[Ru08] ______, *Intersection bodies and generalized cosine transforms*, Advances in Math., **218** (2008), 696–727.

[Ru10a] ______, *Comparison of volumes of convex bodies in real, complex, and quaternionic spaces*, Advances in Math., **225** (2010), 1461-1498.

[RZ] B. Rubin, and G. Zhang, *Generalizations of the Busemann-Petty problem for sections of convex bodies*, J. Funct. Anal., 213 (2004), no. 2, 473–501.

[Sa1] S.G. Samko, *Test functions vanishing on a given set, and division by a function*, Mat. Zametki, **21** (1977), No. 5, 677–689 (Russian).

[Sa2] ______, Hypersingular Integrals and Their Applications, Taylor & Francis, Series: Analytical Methods and Special Functions, Volume 5, 2002.

[SKM] S.G. Samko, A.A. Kilbas, and O.I. Marichev, Fractional integrals and derivatives. Theory and applications, Gordon and Breach Sc. Publ., New York, 1993.

[Se61] V.I. Semyanistyi, *Homogeneous functions and some problems of integral geomery in spaces of constant cuvature*, Sov. Math. Dokl., **2** (1961), 59–61.

[Se63] ______, *Some integral transformations and integral geometry in an elliptic space*, Trudy Sem. Vektor. Tenzor. Anal., *12* (1963), 397–441 (in Russian).

[Sha] B. V. Shabat, *Introduction to Complex Analysis: Functions of Several Variables: Part II*, Translations of Mathematical Monographs, Vol. 110, American Mathematical Society, 1992.

[Sh1] L.P. Shibasov, *Integral problems in a matrix space that are connected with the functional $X^{\lambda}_{n,m}$*. Izv. Vysš. Učebn. Zaved. Matematika (1973), No. 8 (135), 101–112 (Russian).

[Sh2] ———, *Integral geometry on planes of a matrix space*. (Russian) Harmonic analysis on groups. Moskov. Gos. Zaočn. Ped. Inst. Sb. Naučn. Trudov Vyp. **39** (1974), 68–76.

[Spo] E. Spodarev, *On the rose of intersections of stationary flat processes*, Advances in Appl. Probability, *33* (2001), 584–599.

[Shi] M.A. Shifrin, Non-local inversion formula for the problem of integraal geometry for a pair of real Grassmannian manifolds (affine variant), Preprint I.P.M, Moscow, no. 91 (1981).

[St1] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, NJ, 1970.

[St2] ———, *Analysis in matrix spaces and some new representations of* SL(N, C). Ann. of Math. **(2) 86** (1967), 461–490.

[Str70] R.S. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc., **148** (1970), 461–471.

[Str75] ———, *The explicit Fourier decomposition of* $L^2(SO(n)/SO(n-m))$, Can. J. Math., **27** (1975), 294–310.

[Str81] ———, *L^p-estimates for Radon transforms in Euclidean and non-euclidean spaces*, Duke Math. J., **48** (1981), 699–727.

[Str86] ———, *Harmonic analysis on Grassmannian bundles*, Trans. of the Amer. Math. Soc., **296** (1986), 387-409.

[Su] M. Sugiura, *Fourier series of smooth functions on compact Lie groups*, Osaka J. Math., **8** (1971), 33–47.

[T] A. Terras, Harmonic analysis on symmetric spaces and applications, Vol. II, Springer, Berlin, 1988.

[TT] T. Ton-That, *Lie group representations and harmonic polynomials of a matrix variable*, Trans. Amer. Math. Soc. **216** (1976), 1–46.

[Tr] J. F. Treves, *Lectures on linear partial differential equations with constant coefficients*, Notas de Matemática, No. 27 Instituto de Matemática Pura e Aplicada do Conselho Nacional de Pesquisas, Rio de Janeiro, 1961.

[Zh1] G. Zhang, *Radon transform on real, complex and quaternionic Grassmanians*, Duke Math. J. **138** (2007), 137–160.

[Zh2] ———, *Radon, cosine and sine transforms on Grassmannian manifolds*, International Mathematics Research Notices, 2009, 1743–1772.

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803, USA

E-mail address: borisr@math.lsu.edu