Quasisymmetrically minimal homogeneous perfect sets*

Yingqing Xiao †

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Abstract: In [6], the notion of homogenous perfect set as a generalization of Cantor type sets is introduced. Their Hausdorff, lower box-counting, upper box-counting and packing dimensions are studied in [6] and [8]. In this paper, we show that the homogenous perfect set be minimal for 1-dimensional quasisymmetric maps, which generalize the conclusion in [3] about the uniform Cantor set to the homogenous perfect set.

Key words: Homogenous perfect set; Quasisymmetric map; Quasisymmetrically minimal set

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1 Introduction

Given $M \geq 1$, a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is said to be $M-$quasisymmetric if and only if

$$M^{-1} \leq \frac{|f(I)|}{|f(J)|} \leq M$$

for all pairs of adjacent intervals I, J of equal length, here and in sequel $|\cdot|$ stands for the 1-dimensional Lebesgue measure. A map is quasisymmetric if it is $M-$quasisymmetric for some $M \geq 1$. More generally a homeomorphism

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†College of Mathematics and Economics, Hunan University, Changsha 410082, China
E-mail address: ouxyq@yahoo.cn (Yingqing Xiao)

between metric spaces (X, d_X) and (Y, d_Y). If there is a homeomorphism $\eta : [0, +\infty) \to [0, +\infty)$ such that

$$\frac{d_X(a,x)}{d_X(b,x)} \le t \Rightarrow \frac{d_Y(f(a), f(x))}{d_Y(f(b), f(x))} \le \eta(t) \tag{1}$$

for all triples a, b, x of distinct points in X and $t \in [0, +\infty)$, then we call f is a quasisymmetric map. When $X = Y = \mathbb{R}^n$, we also say that f is an n-dimensional quasisymmetric map.

Let $QS(X)$ denote the collection of all quasisymmetric maps defined on X. Conformal dimension of a metric space, a concept introduced by Pansu in [5], is the infimal Hausdorff dimension of quasisymmetric images of X,

$$\mathcal{C}\dim X = \inf_{f \in QS(X)} \dim_H f(X).$$

We say X is minimal for conformal dimension or just minimal if $\mathcal{C}\dim X = \dim_H X$. Euclidean spaces with standard metric are the simplest examples of minimal spaces. Basic analytic definitions and results about the conformal dimension and the quasisymmetric map are contained in [4].

Now, we introduce the notion of the homogeneous perfect set. The general references on the homogeneous perfect set are [6, 8]. In these paper, the authors obtained the Hausdorff, lower box-counting, upper box-counting and packing dimensions of the homogeneous perfect set.

Homogeneous perfect sets. Let $J_0 = [0, 1] \subset R$ be the fixed closed interval which we call the initial interval. Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers and $\{c_k\}$ a sequence of positive real numbers such that for any $k \ge 1, n_k \ge 2$ and $0 < c_k < 1$. For any $k \ge 1$, let $D_k = \{(i_1, i_2, \cdots, i_k) : 1 \le i_j \le n_j, 1 \le j \le k\}, D = \bigcup_{k \ge 0} D_k$, where $D_0 = \{0\}$. We assume if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k) \in D_k, 1 \le j \le n_{k+1}$, then $\sigma * j = (\sigma_1, \sigma_2, \cdots, \sigma_k, j) \in D_{k+1}$.

Suppose that J_0 is the initial interval and $\mathcal{J} = \{J_\sigma : \sigma \in D\}$ is a collection of closed subintervals of J_0. We say that the collection $\mathcal{J}$ fulfills the homogenous perfect structure provided:

1. For any $k \ge 0, \sigma \in D_k, J_{\sigma*1}, J_{\sigma*2}, \cdots, J_{\sigma*n_{k+1}}$ are subintervals of J_σ. Furthermore, $\max\{x : x \in J_{\sigma*i}\} \le \min\{x : x \in J_{\sigma*(i+1)}\}, 1 \le i \le n_{k+1} - 1$, that is the interval $J_{\sigma*i}$ is located at the left of $J_{\sigma*(i+1)}$ and the interiors of the intervals $J_{\sigma*i}$ and $J_{\sigma*(i+1)}$ are disjoint.

2. For any $k \ge 1, \sigma \in D_{k-1}, 1 \le j \le n_k$, we have

$$\frac{|J_{\sigma*i}|}{|J_\sigma|} = c_k.$$

3. There exists a sequence of nonnegative real numbers $\{\eta_{k,j}, k \geq 1, 0 \leq j \leq n_k\}$ such that for any $k \geq 0, \sigma \in D_k$, we have $\min(J_{\sigma*1}) - \min(J_\sigma) = \eta_{k+1,0}, \max(J_\sigma) - \max(J_{\sigma*n_{k+1}}) = \eta_{k+1,n_{k+1}}$, and $\min(J_{\sigma*(i+1)}) - \max(J_{\sigma*i}) = \eta_{k+1,i}(1 \leq i \leq n_{k+1} - 1)$.

Suppose that the collection of intervals $\mathcal{J} = \{J_\sigma : \sigma \in D\}$ satisfies the homogeneous perfect structure.

Let

$$E_k = \bigcup_{\sigma \in D_k} J_\sigma$$

for every $k \geq 1$. The set

$$E := E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\}) = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma = \bigcap_{k \geq 0} E_k$$

is called a homogeneous perfect set and the intervals $J_\sigma, \sigma \in D_k$, the fundamental intervals of order k.

For any $k \geq 1$, if $\eta_{k,0} = \eta_{k,n_k} = 0$ and $\eta_{k,l} = e_k|J_\sigma|$ for all $1 \leq l \leq n_k - 1, \sigma \in D_{k-1}$. Then E is called a uniform Cantor set. This case has been considered by M.D. Hu and S.Y.Wen in [3]. They obtained

Theorem 1 ([3])**.** *Let E be a uniform Cantor set. If the sequence $\{n_k\}$ is bounded and if $\dim_H E = 1$. Then $\dim_H f(E) = 1$ for all 1-dimensional quasisymmetric maps f.*

In this paper, we generalize Theorem 1 to the homogeneous perfect set and show how the techniques of [3] can be applied to the homogeneous perfect set and obtain the following theorem.

Theorem 2. *Let $E := E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\})$ be a homogeneous perfect set. If the sequence $\{n_k\}$ is bounded and if $\dim_H E = 1$, then $\dim_H f(E) = 1$ for all 1-dimensional quasisymmetric map f.*

This paper is organized as following. In section 2 we introduce the basic general definitions and results in fractal geometry. The proof of Theorem 2 appears in section 3.

2 Preliminary

In order to obtain our result, we need the following lemma from [9], the lemma can also be found in [2] or [3].

Lemma 1 ([9]). *Let f be an M-quasisymmetric map. Then*

$$(1+M)^{-2}(\frac{|J|}{|I|})^q \leq \frac{|f(J)|}{|f(I)|} \leq 4(\frac{|J|}{|I|})^p \tag{2}$$

for all pairs J, I of intervals with $J \subset I$, where

$$0 < p = \log_2(1+M^{-1}) \leq 1 \leq q = \log_2(1+M). \tag{3}$$

Hausdorff dimension. In this subsection, we recall the definition of Hausdorff dimension. For more details we refer to [1, 7].

Let $K \subset \mathbb{R}^d$. For any $s \geq 0$, the $s-$dimensional Hausdorff measure of K is given in the usual way by

$$\mathbf{H}^s(K) = \liminf_{\delta \to 0} \{\sum_i |U_i|^s : K \subset \bigcup_i U_i, 0 < |U_i| < \delta\}.$$

This leads to the definition of the Hausdorff dimension of K:

$$\dim_H K = \inf\{s : \mathbf{H}^s(K) < \infty\} = \sup\{s : \mathbf{H}^s(K) > 0\}.$$

The Hausdorff dimension of the homogeneous perfect set E, which depends on $\{n_k\}, \{c_k\}$ and $\{\eta_{k,j}\}$ have been obtained in [6] as follows

Theorem 3 ([6]). *Let $E = E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\})$ be a homogeneous perfect set. Suppose $n_k \leq D$ for all k, where D is a constant, then*

$$\dim_H E = \liminf_{k \to \infty} \frac{\log(n_1 n_2 \cdots n_k)}{-\log(\sum_{l=1}^{n_{k+1}-1} \eta_{k+1,l} + n_{k+1} c_1 c_2 \cdots c_{k+1})}. \tag{4}$$

Denote by N_k the number of component intervals of E_k and by δ_k their common length. Let $e_{k,l} = \eta_{k,l}/\delta_{k-1} \geq \eta_{k,l}$ for all $k \geq 1$ and $0 \leq l \leq n_k$. From the definition we obtain

$$n_k c_k \leq 1, \quad N_k = n_k n_{k-1} \cdots n_1 \quad \text{and} \quad \delta_k = c_k c_{k-1} \cdots c_1$$

for all $k \geq 1$. So we have the total length of E_k is

$$N_k \delta_k = \prod_{i=1}^{k} n_i c_i,$$

and

$$\delta_k = \Sigma_{l=0}^{n_{k+1}} \eta_{k+1,l} + n_{k+1}\delta_{k+1} = \Sigma_{l=0}^{n_{k+1}} e_{k+1,l}\delta_k + n_{k+1}\delta_{k+1}. \tag{5}$$

Lemma 2. *Let $E = E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\})$ be a homogeneous perfect set. Suppose the sequence $\{n_k\}$ is bounded and $\dim_H E = 1$ then:*

(1) $\lim_{k\to\infty} (N_k\delta_k)^{1/k} = 1$.

(2) $\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} e_i^p = 0$ *for any* $0 < p \leq 1$*, where* $e_i = \max_{0\leq l\leq n_i} e_{i,l}$.

(3) $\lim_{k\to\infty} \sharp\{i : 0 \leq i \leq k, e_i \geq \epsilon\}/k = 0$ *for any* $\epsilon \in (0,1)$*, where* $\sharp$ *denotes the cardinality.*

Proof. (1) Since

$$N_k(\delta_k - \eta_{k,0} - \eta_{k,n_{k+1}}) \leq N_k\delta_k \leq 1,$$

Thus, we have

$$\frac{\log N_k}{-\log(\delta_k - \eta_{k,0} - \eta_{k,n_{k+1}})} \leq \frac{\log N_k}{-\log \delta_k} \leq 1.$$

As $\dim_H E = 1$, we get from Theorem 3

$$\begin{aligned} 1 = \dim_H E \ &= \liminf_{k\to\infty} \frac{\log N_k}{-\log(\delta_k - \eta_{k,0} - \eta_{k,n_{k+1}})} \\ &\leq \lim_{k\to\infty} \frac{\log N_k}{-\log \delta_k} \leq 1. \end{aligned} \tag{6}$$

Thus we obtain

$$\lim_{k\to\infty} \frac{\log N_k}{-\log \delta_k} = \lim_{k\to\infty} \frac{\log N_k}{\log N_k - \log N_k\delta_k} = 1,$$

and

$$\lim_{k\to\infty} \frac{\log N_k\delta_k}{\log N_k} = 0.$$

Let $N = 1 + \sup_k n_k < \infty$. We obtain $N_k \leq N^k$, so

$$\lim_{k\to\infty} \frac{\log N_k\delta_k}{k \log N} = 0,$$

that gives the the conclusion (1) of the lemma.

(2) Since

$$(N_k\delta_k)^{1/k} = (\prod_{i=1}^{k} n_i c_i)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} n_i c_i \leq 1.$$

Thus, we have

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}n_ic_i=1. \tag{7}$$

From the equation (5), we have

$$\delta_k=\Sigma_{l=0}^{n_{k+1}}e_{k+1,l}\delta_k+n_{k+1}c_{k+1}\delta_k. \tag{8}$$

Thus

$$e_{k+1}\le 1-n_{k+1}c_{k+1},$$

so

$$\frac{1}{k}\sum_{i}^{k}e_i\le\frac{1}{k}\sum_{i}^{k}(1-n_ic_i).$$

Since the equation (7), we obtain

$$\lim_{i}\frac{1}{k}\sum_{i}^{k}e_i=0,$$

which together with Jensen's inequality yields

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}e_i^p\le\lim_{k\to\infty}(\frac{1}{k}\sum_{i=1}^{k}e_i)^p=0$$

for any $0<p\le 1$. This proves the conclusion (2).

(3) Fixed $\epsilon\in(0,1)$, we obtain from the conclusion (2)

$$\frac{1}{k}\sharp\{i:0\le i\le k,e_i\ge\epsilon\}=\frac{1}{k}\sum_{i:1\le i\le k,e_i\ge\epsilon}1\le\frac{1}{k\epsilon}\sum_{i=1}^{k}e_i\to 0$$

as k tends to ∞. This proves the conclusion (3).

3 The proof of Theorem 2

In order to obtain our result, we need the following mass distribution principle to estate the lower bound.

Lemma 3 ([1]). *Let μ be a mass distribution supported on E. Suppose that for some t there are numbers $c > 0$ and $\eta > 0$ such that for all sets U with $|U| \le \eta$ we have $\mu(U) \le c|U|^t$. Then $\dim_H E \ge t$.*

The proof of Theorem 2: Let $E = \bigcap_{k=0}^{\infty} E_k$ be a homogeneous perfect set satisfying the conditions of Theorem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be an M-quasisymmetric map and q is the number defined as in (3). Without loss of generality assume that $f([0,1]) = [0,1]$. Then $f(E) = \bigcap_{k=1}^{\infty} f(E_k)$. The images of component intervals of E_k are component intervals of $f(E_k)$.

We define a mass distribution μ on $f(E)$ as follows: Let $\mu([0,1]) = 1$. For every $k \ge 1$ and for every component interval J of $f(E_{k-1})$, let $J_{k1}, J_{k2}, \cdots, J_{kn_k}$ denote the n_k component intervals of $f(E_k)$ lying in J. Define

$$\mu(J_{ki}) = \frac{|J_{ki}|^d}{||J||_d}\mu(J), \quad i = 1, 2, \cdots, n_k,$$

where

$$||J||_d = \sum_{i=1}^{n_k} |J_{ki}|^d$$

and

$$d \in \begin{cases} (0,1) & \text{when } q = 1, \\ (1/q, 1) & \text{when } q > 1. \end{cases} \tag{9}$$

we are going to prove that the measure μ satisfy

$$\mu(J) \le C|J|^d \tag{10}$$

for any interval $J \subset [0,1]$, where C is a positive constant independent of J. We do this as following two steps.

Step 1. Suppose that J is a component interval of $f(E_k)$, For every $i, 0 \le i \le k$, let J_i be the component interval of $f(E_i)$ such that

$$J = J_k \subset J_{k-1} \subset \cdots J_1 \subset J_0 = [0,1] \tag{11}$$

By the definition of μ, we have

$$\frac{\mu(J)}{|J|^d} = \frac{1}{||J_{k-1}||_d}\frac{|J_{k-1}|^d}{||J_{k-2}||_d}\cdots\frac{|J_1|^d}{||J_0||_d} = \frac{|J_{k-1}|^d}{||J_{k-1}||_d}\cdots\frac{|J_1|^d}{||J_1||_d}\frac{|J_0|^d}{||J_0||_d}.$$

Let

$$r_i = \frac{||J_i||_d}{|J_i|^d}, \quad i = 0, 1, 2, \cdots, k-1. \tag{12}$$

So the above equality can be rewritten as

$$\frac{\mu(J)}{|J|^d} = (\prod_{i=1}^{k} r_{i-1})^{-1}. \tag{13}$$

In order to prove (10), it suffices to show

$$\lim_{k\to\infty} \prod_{i=1}^{k} r_{i-1} = \infty. \tag{14}$$

Given an i, $1 \le i \le k$, we are going to estimate r_{i-1}. Let J_{i-1} be the component interval of $f(E_{i-1})$ in the sequence (11). Let $J_{i1}, J_{i2}, \cdots, J_{in_i}$ be the n_i component intervals of $f(E_i)$ lying in J_{i-1}. Recall that $J_i \subset J_{i-1}$ is a component interval of $f(E_i)$. So there must exist $1 \le i_0 \le n_i$ such that $J_i = J_{ii_0}$. Let $G_{i0}, G_{i1}, \cdots, G_{in_i}$ be the $n_i + 1$ gaps in the J_{i-1}. Put

$$I_{i-1} = f^{-1}(J_{i-1}), \quad I_i = f^{-1}(J_i) = f^{-1}(J_{ii_0}) \quad \text{and} \quad I_{ij} = f^{-1}(J_{ij}),$$

for $j = 1, 2, \cdots, n_i$. Then $I_{i1}, \cdots, I_{in_i}$ are component intervals of E_i lying in the component interval I_{i-1} of E_{i-1}. Since f is M-quasisymmetric, it follows Lemma 1 and the construction of E that

$$\frac{|G_{ij}|}{|J_{i-1}|} \le 4(\frac{|f^{-1}(G_{ij})|}{|f^{-1}(J_{i-1})|})^p \le 4e_i^p, \quad j = 0, 1, 2, \cdots, n_i, \tag{15}$$

where $e_i = \max_{0\le l\le n_i} e_{i,l}$ and that

$$\frac{|J_{ij}|}{|J_{i-1}|} \ge (1+M)^{-2}(\frac{|I_{ij}|}{|I_{i-1}|})^q = (1+M)^{-2}c_i^q. \tag{16}$$

Here p, q are numbers defined in Lemma 1. The inequality (15) yields

$$\frac{|J_{i1}| + \cdots + |J_{in_i}|}{|J_{i-1}|} = \frac{|J_{i-1}| - |G_{i0}| - \cdots - |G_{in_i}|}{|J_{i-1}|} \ge 1 - 4(n_i+1)e_i^p. \tag{17}$$

From inequality (16), we have

$$\begin{aligned} r_{i-1} &= \frac{|J_{i1}|^d + \cdots + |J_{in_i}|^d}{|J_{i-1}|^d} \\ &\ge n_i(\frac{|J_{ij}|}{|J_{i-1}|})^d \\ &\ge \frac{n_i}{(1+M)^{2d}} c_i^{dq}. \end{aligned} \tag{18}$$

Let

$$S(k,p) = \{i : 1 \le i \le k, e_i^p \le \min\{a, |I_i|^p\}$$

where $a = 1 - \sqrt[4N+4]{\frac{4N+4}{4N+5}}$, where $N = 1 + \sup_l n_l$. Since $\eta_{i,l} \le e_{i,l}$. Thus, If $i \in S(k,p)$ we have

$$\begin{aligned} c_i = \frac{|I_{ij}|}{|I_{i-1}|} &= \frac{|I_{ij}|}{n_i|I_{ij}| + \sum_{l=0}^{n_i} \eta_{i,l}} \\ &\ge \frac{|I_{ij}|}{n_i|I_{ij}| + (n_i+1)\eta_i} \\ &\ge \frac{1}{2n_i+1} \\ &\ge \frac{1}{2N} \end{aligned} \tag{19}$$

for $j = 1, \cdots, n_i$, where $\eta_i = \max_{0 \le l \le n_i} \eta_{i,l}$.

From the conclusion (3) of Lemma 2, we obtain

$$\lim_{k\to\infty} \frac{\sharp S(k,p)}{k} = 1. \tag{20}$$

Then follows from the left hand inequality of (2) that

$$1 \ge \frac{|J_{ij}|}{|J_i|} = \frac{|f(I_{ij})|}{|f(I_i)|} \ge (1+M)^{-2}(\frac{|I_{ij}|}{|I_{i-1}|})^q \ge A$$

for $j = 1, 2, \cdots, n_i$, where $A = \frac{(1+M)^{-2}}{(2N)^q}$. Therefore,

$$\begin{aligned} \frac{|J_i|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d}{(|J_i| + |J_{i1}| + \cdots + |J_{in_i}|)^d} &= \frac{1 + x_1^d + \cdots + x_{n_i}^d}{(1 + x_1 + \cdots + x_{n_i})^d} \\ &\ge (1+A)^{1-d}, \end{aligned} \tag{21}$$

where $x_j = \frac{|J_{ij}|}{|J_i|} \in [A, 1]$.

Note that the equality (17) and (21), for any $i \in S(k,p)$ we obtain

$$\begin{aligned} r_{i-1} &= \frac{|J_i|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d}{|J_{i-1}|^d} \\ &= \frac{|J_i|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d}{(|J_i| + |J_{i1}| + \cdots + |J_{in_i}|)^d} \frac{(|J_i| + |J_{i1}| + \cdots + |J_{in_i}|)^d}{|J_{i-1}|^d} \\ &\ge \alpha_2(1 - 4(n_i+1)e_i^p)^d, \end{aligned} \tag{22}$$

where $\alpha_2 = (1+A)^{1-d} > 1$.

Since

$$1 - mx \geq (1-x)^{m+1}$$

for all $x \in (0, 1 - \sqrt[m]{\frac{m}{m+1}})$, so we have

$$1 - 4mx \geq (1-x)^{4m+1}$$

for all $x \in (0,a)$where $a = 1 - \sqrt[4N+4]{\frac{4N+4}{4N+5}}$ and all positive inters $m \leq N$.

Note that $n_i < N$ and $e_i^p \in (0,a)$ for all $i \in S(k,p)$, thus we obtain

$$r_{i-1} \geq \alpha_2 (1 - e_i^p)^{(4n_i+4)d} \tag{23}$$

Using the estimate (18) and (23), we obtain

$$\begin{aligned}
\prod_{i=1}^{k} r_{i-1} &\geq \prod_{i \notin S(k,p)} \frac{n_i c_i^{dq}}{(1+M)^{2d}} \prod_{i \in S(k,p)} \alpha_2 (1 - 4(n_i+1)e_i^p)^d \\
&\geq \prod_{i \notin S(k,p)} \frac{n_i c_i^{dq}}{(1+M)^{2d}} \prod_{i \in S(k,p)} \alpha_2 (1 - e_i^p)^{(4n_i+4)d} \\
&= \alpha_2^{\sharp S(k,p)} [(1+M)^{-2d}]^{k - \sharp S(k,p)} \prod_{i \notin S(k,p)} n_i c_i^{dq} \prod_{i \in S(k,p)} (1 - e_i^p)^{(4n_i+4)d}.
\end{aligned} \tag{24}$$

If $q = 1$, since $n_i c_i \leq 1$ then we have

$$\prod_{i \notin S(k,p)} n_i c_i^{dq} = \prod_{i \notin S(k,p)} n_i c_i^d \geq \prod_{i \notin S(k,p)} n_i c_i \geq \prod_{i=1}^{k} n_i c_i = N_k \delta_k.$$

If $q > 1$, we have

$$\begin{aligned}
\prod_{i \notin S(k,p)} n_i c_i^{dq} &= \prod_{i \notin S(k,p)}^{k} (n_i c_i)^{dq} n_i^{1-dq} \geq \prod_{i=1}^{k} (n_i c_i)^{dq} \prod_{i \notin S(k,p)} n_i^{1-dq} \\
&= \prod_{i=1}^{k} (n_i c_i)^{dq} \prod_{i \notin S(k,p)} n_i^{1-dq} \geq (N_k \delta_k)^{dq} \prod_{i \notin S(k,p)} N^{1-dq} \\
&= (N_k \delta_k)^{dq} (N^{1-dq})^{k - \sharp S(k,p)}
\end{aligned} \tag{25}$$

for $d \in (1/q, 1)$.

Let

$$\xi_k = \alpha_2^{\sharp S(k,p)}((1+M)^{-2d})^{k-\sharp S(k,p)}(N_k\delta_k)^{dq}(N^{1-dq})^{k-\sharp S(k,p)} \tag{26}$$

and

$$\zeta_k = \prod_{i\in S(k,p)} (1-e_i^p)^{(4n_i+4)d}.$$

Thus, we have

$$\prod_{i=1}^{k} r_{i-1} \geq \xi_k\zeta_k. \tag{27}$$

It is obvious that

$$\lim_{k\to\infty} \xi_k^{1/k} = \alpha_2 > 1. \tag{28}$$

due to the conclusion (1) of Lemma 2 and the equality (20) . On the other hand, since $\log(1-x) \geq -2x$ when $0 < x < 1$, the conclusion (2) of Lemma 2, we obtain

$$\begin{aligned}
\frac{1}{k}\log\zeta_k &= \frac{1}{k}\log\prod_{i\in S(k,p)}(1-e_i^p)^{(4n_i+4)d} \\
&= \frac{1}{k}\sum_{i\in S(k,p)}\log(1-e_i^p)^{(4n_i+4)d} \\
&= \frac{1}{k}\sum_{i\in S(k,p)}(4n_i+4)d\log(1-e_i^p) \\
&\geq \frac{(4N+4)d}{k}\sum_{i\in S(k,p)}\log(1-e_i^p) \\
&\geq -2\frac{(4N+4)d}{k}\sum_{i\in S(k,p)}e_i^p \\
&\geq -2\frac{(4N+4)d}{k}\sum_{i=1}^{k}e_i^p \to 0.
\end{aligned} \tag{29}$$

as $k \to \infty$. This show that

$$\lim_{k\to\infty} \zeta_k^{1/k} = 1. \tag{30}$$

From (27), (28), (30), we obtain

$$\liminf_{k\to\infty}(\prod_{i=1}^{k} r_{i-1})^{1/k} \geq \alpha_2 > 1.$$

This implies

$$\lim_{k\to\infty}(\prod_{i=1}^{k} r_{i-1}) = \infty.$$

Step 2. Let $J \subset [0,1]$ be any interval. For such J, let k be the unique positive inter such that

$$\delta_k \leq |f^{-1}(J)| \leq \delta_{k-1},$$

where δ_k denotes the lengthen of component intervals of E_k. Then the set $f^{-1}(J)$ meets at most two component intervals of E_{k-1} and hence at most $2n_{k+1}$ component intervals of E_k. Thus, the set J meets at most $2n_{k+1}$ component intervals of $f(E_k)$.

Let $J_1, J_2, \cdots, J_l, l \leq 2n_{k+1}$, be those component intervals of $f(E_k)$ meeting J. Using the conclusion of step 1. we obtain

$$\mu(J) \leq \sum_{i=1}^{l} \mu(J_i) \leq C \sum_{i=1}^{l} |J_i|^d. \tag{31}$$

Since $\delta_k \leq |f^{-1}(J)|$, we obtain

$$f^{-1}(J_i) \subset 3f^{-1}(J), \quad i = 1, 2, 3 \cdots l,$$

where $3f^{-1}(J)$ denote the interval of lengthen $3|f^{-1}(J)|$ concentric with $f^{-1}(J)$. Thus we obtain

$$|J_i| \leq f(3f^{-1}(J)) \leq K|J|, \quad i = 1, 2, 3 \cdots l,$$

where K is a positive constant depending on M only. This together with gives

$$\mu(J) \leq ClK^d|J|^d \leq 2NCK^d|J|^d.$$

This show that (10).

By Lemma (3), it follows from that $\dim_H f(E) \geq d$ for d. As d could be chosen as closed to 1 as one would. Hence $\dim_H f(E) = 1$. This completes the proof of Theorem 2. □

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