THE FIRST BOUNDARY VALUE PROBLEM FOR ABREU'S EQUATION

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ABSTRACT. In this paper we prove the existence and regularity of solutions to the first boundary value problem for Abreu's equation, which is a fourth order nonlinear partial differential equation closely related to the Monge-Ampère equation. The first boundary value problem can be formulated as a variational problem for the energy functional. The existence and uniqueness of maximizers can be obtained by the concavity of the functional. The main ingredients of the paper are the a priori estimates and an approximation result, which enable us to prove that the maximizer is smooth in dimension 2.

1. INTRODUCTION

Abreu's equation was first introduced by M. Abreu [Ab] in the study of existence of extremal metrics on toric Kähler manifolds. It is a fourth order equation given by

$$\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = f \tag{1.1}$$

where u is a convex function in a bounded domain Ω in $\mathbb{R}^n$, $f \in L^\infty(\Omega)$, and (u^{ij}) is the inverse matrix of the Hessian (u_{ij}). This equation was later studied by S. Donaldon. In a series of papers [D1, D2, D3, D4], Donaldon established various a priori estimates for Abreu's equation and proved the existence of constant scalar curvature metrics on toric Kähler surfaces under the assumption of K-stability.

Abreu's equation can also be written as

$$U^{ij} w_{ij} = f, \tag{1.2}$$

where (U^{ij}) is the cofactor matrix of (u_{ij}) and

$$w = [\det D^2 u]^{-1}. \tag{1.3}$$

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The energy functional of Abreu's equation is given by

$$J_0(u) = A_0(u) - \int_\Omega f u \, dx, \tag{1.4}$$

where

$$A_0(u) = \int_\Omega \log \det D^2 u \, dx. \tag{1.5}$$

We formulate a variational problem for Abreu's equation. Let

$$S[\varphi, \Omega] = \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \mid u \text{ is convex } u|_{\partial\Omega} = \varphi(x), Du(\Omega) \subset D\varphi(\overline{\Omega})\}, \tag{1.6}$$

where φ a smooth, uniformly convex function defined in a neighborhood of $\overline{\Omega}$. The problem is to find a function u in $S[\varphi, \Omega]$ such that

$$J_0(u) = \sup\{J_0(v) \mid v \in S[\varphi, \Omega]\}. \tag{1.7}$$

The main result in this paper is as follows.

Theorem 1.1. *Suppose the domain Ω is bounded and smooth. Assume $f \in C^\infty(\Omega) \cap L^\infty(\Omega)$. If $n = 2$, there exists a unique, smooth, locally uniformly convex maximizer u of the variational problem (1.7).*

The variational problem (1.7) corresponds to the *first boundary value problem* for equation (1.1),

$$u = \varphi \quad \text{on } \partial\Omega, \tag{1.8}$$

$$Du = D\varphi \quad \text{on } \partial\Omega. \tag{1.9}$$

Indeed, if we have a classical, locally uniformly convex solution $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ to (1.1), (1.8) and (1.9), u will also solve (1.7) uniquely. The uniqueness follows from the concavity of the functional A_0.

A motivation for our investigation of the above problem is that the study of boundary value problems for elliptic equations has been a focus of attention since 1950s. The Dirichlet problem for Monge-Ampère type equations, which is somehow related to our boundary condition (1.8) above, has been studied by many people, see [CNS, GS1, Li, S, TW4, U1]. The second boundary problem for the Monge-Ampère equation, which is related to our boundary condition (1.9) above, has also been studied in [Caf2, Del, U2].

Another motivation to study the above problem is due to the increasing interest in nonlinear fourth order partial differential equations. In recent years, nonlinear fourth order equations, such as the affine mean curvature equation and Willmore surface equation, have attracted considerable attention. Abreu's equation is similar to the affine mean curvature equation, which is given by

$$U^{ij} w_{ij} = f, \tag{1.10}$$

where

$$w = [\det D^2 u]^{-(1-\theta)}, \quad \theta = \frac{1}{n+2}. \tag{1.11}$$

When $f = 0$, (1.10) is called the affine maximal surfaces equation. The energy functional of affine mean curvature equation is

$$J_\theta(u) = A_\theta(u) - \int_\Omega f u \, dx, \tag{1.12}$$

where

$$A_\theta(u) = \int_\Omega [\det D^2 u]^\theta \, dx \tag{1.13}$$

is called *affine area functional* [Cal, LR]. In [TW2, TW5], N. Trudinger and X.-J. Wang studied the first boundary value problem for the affine maximail surface equation, and the more general affine Plateau problem, which can also be reduced to a similar variational problem. In [TW2], Trudinger and Wang proved the existence and uniqueness of smooth maximizers of J_θ in $S[\varphi, \Omega]$ in dimension 2. Theorem 1.1 above is an analogue to their result. Very recently, they also obtained the regularity of maximizers to the affine Plateau problem in high dimensions [TW5].

Our proof of Theorem 1.1 is inspired by Trudinger and Wang's variational approach and their regularity argument in solving the affine Plateau problem. But due to the singularity of the function $\log d$ near $d = 0$, the approximation argument in [TW2, TW5] does not apply directly to our problem. To avoid this difficulty we introduce in Section 2 a sequence of modified functionals J_k to approximate J_0, such that the integrand in J_k is Hölder continuous at $d = 0$. We prove the existence and uniqueness of a maximizer of the functional J_k (Theorem 2.6) in the set $\overline{S}[\varphi, \Omega]$, the closure of $S[\varphi, \Omega]$ under uniform convergence.

The regularity of the maximizer is our main concern. In Section 3 we establish a uniform (in k) a priori estimates for the corresponding Euler equation of the functional J_k. Unlike the affine maximal surface equation, Abreu's equation is not invariant under linear transformation of coordinates $\mathbb{R}^{n+1}$. When we rotate the coordinates in $\mathbb{R}^{n+1}$ we get a more complicated 4th order pde (§4). In Section 4, we establish the uniform (in k) a priori estimates for the equations obtained after rotation of coordinates in $\mathbb{R}^{n+1}$.

As the maximizer may not be smooth, to apply the a priori estimates we need to prove that the maximizer can be approximated by smooth solutions. We cannot prove the approximation for the functional J_0 directly as $\log d$ is singular near $d = 0$. But for maximizers of J_k, the approximation can be proved similarly as for the affine Plateau problem [TW2, TW5]. The approximation solutions are constructed by considering the *second boundary value problem*, namely the Euler equation of J_k (see (2.6)) subject to

$$u = \varphi \quad \text{on } \partial\Omega, \tag{1.14}$$

$$w = \psi \quad \text{on } \partial\Omega. \tag{1.15}$$

We can prove the existence of locally smooth solutions to the boundary value problem (2.6), (1.14) and (1.15), in a way similar to that in [TW2, TW5]. For reader's convenience we include a proof in the Appendix.

The a priori estimates in Sections 3 and 4 rely on the strict convexity of solutions. In Sections 6 and 7 we are devoted to the proof of the strict convexity of solutions. The proof for one case is similar to that for affine mean curvature equation in [TW1, TW2] and is included in Section 6. But the proof for the other case uses the a priori estimates, the Legendre transform and in particular a strong approximation (Theorem 7.1) and is contained in Section 7. [1]

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2. A modified functional

In this section we introduce a modified functional J and prove the existence and uniqueness of a maximizer of J.

We begin with some terminologies. Let u be a convex function in a domain $\Omega \subset \mathbb{R}^n$ and $z \in \Omega$ be an interior point. The *normal mapping* of u at z, $N_u(x)$, is the set of gradients of the supporting functions of u at x, that is

$$N_u(x) = \{p \in \mathbb{R}^n \mid u(y) \geq u(x) + p \cdot (y - x)\}.$$

[1] This paper was submitted to a journal for publication in June. Recently, Chen-Li-Sheng posted a related paper [CLS]. In their paper, the boundary value problem for (1.1) with $u = \varphi$, $Du = \infty$ and $w = \infty$ was studied. They use solutions to the second boundary value problem of Abreu's equation directly as the approximating solutions. Their approach does not apply to the case considered in this paper.

For any subset $\Omega' \subset \Omega$, denote $N_u(\Omega') = \bigcup_{x \in \Omega'} N_u(x)$. If u is C^1, the normal mapping N_u is exactly the gradient mapping Du.

For a convex function u on Ω, the *Monge-Ampère measure* $\mu[u]$ is a Radon measure given by

$$\mu[u](E) = |N_u(E)|$$

for any Borel set E. By a fundamental result of Aleksandrov, $\mu[u]$ is weekly continuous with respect to the convergence of convex functions [P, TW3]. It follows that if $\{u_j\}$ converges to u in L^1_{loc}, then for any closed $E \subset \Omega$,

$$\limsup_{j \to \infty} \mu[u_j](E) \leq \mu[u](E). \tag{2.1}$$

Since the set $S[\varphi, \Omega]$ is not closed, we introduce

$$\overline{S}[\varphi, \Omega] = \{u \in C^0(\overline{\Omega}) \mid u \text{ is convex } u|_{\partial\Omega} = \varphi(x), N_u(\Omega) \subset D\varphi(\overline{\Omega})\}. \tag{2.2}$$

Note that $\overline{S}[\varphi, \Omega]$ is closed under the locally uniform convergence of convex functions. In [ZZ], we proved that A_0 is well defined and upper semi-continuous in another set of convex functions. By a similar argument, we can also prove that A_0 is well defined and upper semi-continuous in $\overline{S}[\varphi, \Omega]$, which implies the existence of a maximizer of J_0 in $\overline{S}[\varphi, \Omega]$.

To apply the a priori estimates to the maximizer, we need a sequence of smooth solutions to Abreu's equation to approximate the maximizer. Since the penalty method in [TW2] does not apply to J_0, we must have a sequence of modified smooth approximation solutions. For this purpose, we consider a functional of the form

$$J(u) = A(u) - \int_\Omega f u \, dx, \tag{2.3}$$

where

$$A(u) = \int_\Omega G(\det D^2 u) \, dx. \tag{2.4}$$

Here $G(d) = G_\delta(d)$ is a smooth concave function on $[0, \infty)$ which depends on a constant $\delta \in (0, 1)$ and satisfies the following conditions.

(a) $G(d) = \log d$ when $d \geq \delta$.

(b) $G'(d) > 0$ and there exist constants $C_1, C_2 > 0$ independent of δ such that for any $d > 0$

$$G''(d) \geq -C_1 d^{-2},$$

$$\left| \frac{dG'''(d)}{G''(d)} \right| \leq C_2.$$

(c) The function $F(t) = G(d)$, where $t = d^{\frac{1}{n}}$, is smooth in $(0, +\infty)$ and satisfies

$$F(0) > -\infty, \quad F''(t) < 0,$$

$$\lim_{t\to 0} F'(t) = \infty, \quad \lim_{t\to 0} tF'(t) \le C_3,$$

where C_3 is a positive constant.

Remark 2.1.
(i) The condition $F''(t) < 0$ in (c) implies that the functional $A(u)$ is concave.
(ii) The concavity of F, $F''(t) < 0$, is equivalent to $dG''(d) + \frac{n-1}{n}G'(d) < 0$; and $\lim_{t\to 0} F'(t) = \infty$ is equivalent to $d^{\frac{n-1}{n}}G'(d) \to \infty$ as $d \to 0$.
(iii) We point out the existence of functions G satisfying properties (a)-(c) above. A function in our mind is

$$G(d) = \begin{cases} \frac{\delta^{-\theta}}{\theta(1-\theta)} d^\theta - \frac{\theta\delta^{-1}}{1-\theta} d + \log\delta - \frac{1+\theta}{\theta}, & d < \delta, \\ \log d, & d \ge \delta, \end{cases} \tag{2.5}$$

where $\theta = \frac{1}{n+2}$. One can check that $G \in C^{2,1}(0,\infty)$ and C^3 except at $d = \delta$. It is easy to see that G satisfies (a) and (c). We can also check that G satisfies (b) except at $d = \delta$. Hence, we can always mollify G to have a sequence of smooth functions satisfying the properties (a)-(c) to approximate it.

The Euler equation of the functional J is

$$U^{ij} w_{ij} = f, \tag{2.6}$$

where

$$w = G'(\det D^2 u) \tag{2.7}$$

and (U^{ij}) is the cofactor matrix of $D^2 u$.

Remark 2.2. *Equation (2.6) is invariant under unimodular linear transformation. If we make a general non-degenerate linear transformation $T: \ y = Tx$ and let $\tilde{u}(y) = u(x)$, then $\tilde{u}(y)$ is a solution of*

$$\tilde{U}^{ij}\tilde{w}_{ij} = f, \ \tilde{w} = \tilde{G}'(\det D^2\tilde{u}),$$

where $\tilde{G}(\tilde{d}) = G(|T|^2\tilde{d})$, $\tilde{d} = \det D^2\tilde{u}$. Here $\tilde{G}$ is a smooth concave function satisfying (a), (b), (c) with $\tilde{\delta} = |T|^{-2}\delta$, $\tilde{C}_1 = C_1$, $\tilde{C}_2 = C_2$, $\tilde{C}_3 = C_3$.

Now we study the existence and uniqueness of maximizers to the functional $J(u)$. The treatment here is same as that in [TW2, ZZ], so we will only sketch the proof.

First, we extend the functional J to $\overline{S}[\varphi, \Omega]$. It is clear that the linear part in J is naturally well-defined. It suffices to extend $A(u)$ to $\overline{S}[\varphi, \Omega]$. Since u is convex, u is almost everywhere twice-differentiable, i.e., the Hessian matrix (D^2u) exists almost everywhere. Denote the Hessian matrix by $(\partial^2 u)$ at those twice-differentiable points in Ω. As a Radon measure, $\mu[u]$ can be decomposed into a regular part and a singular part as follows,

$$\mu[u] = \mu_r[u] + \mu_s[u].$$

It was proved in [TW2] that the regular part $\mu_r[u]$ can be given explicitly by

$$\mu_r[u] = \det \partial^2 u \, dx$$

and $\det \partial^2 u$ is a locally integrable function. Therefore for any $u \in \overline{S}[\varphi, \Omega]$, we can define

$$A(u) = \int_\Omega G(\det \partial^2 u) \, dx. \tag{2.8}$$

Next, we state an important property of $A(u)$. For any Lebesgue measurable set E, by the concavity of G and Jensen's inequality,

$$\begin{aligned} \int_E G(\det \partial^2 u) \, dx &\leq |E| G\left(\frac{\int_E \det \partial^2 u \, dx}{|E|}\right) \\ &\leq |E| G(|E|^{-1} \mu[u](E)). \end{aligned} \tag{2.9}$$

By the assumption (a), $d^{-1}G(d) \to 0$ as $d \to \infty$. Note that G is bounded from below. So the above integral goes to 0 as $|E| \to 0$. With this property, we have an approximation result for the functional $A(u)$. For $u \in \overline{S}[\varphi, \Omega]$, let

$$u_h(x) = h^{-n} \int_{B_1(0)} \rho(\frac{x-y}{h}) u(y) \, dy,$$

where $h > 0$ is a small constant and $\rho \in C_0^\infty(B_1(0))$ with $\int_{B_1(0)} \rho = 1$. Suppose that u is defined in a neighborhood of Ω such that u_h is well-defined for any $x \in \Omega$. A fundamental result is that $(D^2 u_h) \to (\partial^2 u)$ almost everywhere in Ω [Z]. Combining it with (2.9), we have therefore obtained as in [TW1],

Lemma 2.3. *Let $u \in \overline{S}[\varphi, \Omega]$, we have*

$$\int_\Omega G(\det \partial^2 u) \, dx = \lim_{h \to 0} \int_\Omega G(\det \partial^2 u_h) \, dx.$$

Finally, the existence of maximizers of J in $\overline{S}[\varphi, \Omega]$ follows from the following upper semi-continuity of the functional $A(u)$ with respect to uniform convergence.

Lemma 2.4. *Suppose that $u_n \in \overline{S}[\varphi, \Omega]$ converge locally uniformly to u. Then*

$$\limsup_{n \to \infty} \int_\Omega G(\det \partial^2 u_n) \, dx \leq \int_\Omega G(\det \partial^2 u) \, dx.$$

Proof. The proof is also inspired by [Lu, TW1], see also [ZZ]. Subtracting G by the constant $G(0)$, we may suppose that $G(0) = 0$. By Lemma 2.3, it suffices to prove it for $u_n \in C^2(\overline{\Omega})$ and we may assume that u_n converges uniformly to u in $\overline{\Omega}$.

Denote by S the supporting set of $\mu_s[u]$, whose Lebesgue measure is zero. By the upper semi-continuity of the Monge-Ampère measure, for any closed subset $F \subset \Omega \backslash S$,

$$\limsup_{n\to\infty} \int_F \det D^2 u_n \, dx \le \int_F \det \partial^2 u \, dx. \tag{2.10}$$

For given $\epsilon, \epsilon' > 0$, let

$$\Omega_k = \{x \in \Omega \setminus S \mid (k-1)\epsilon \le \det \partial^2 u < k\epsilon\}, \; k = 0, 1, 2, ...,$$

and $\omega_k \subset \Omega_k$ be a closed set such that

$$|\Omega_k \backslash \omega_k| < \frac{\epsilon'}{2^{|k|}}.$$

For each ω_k, by concavity of G and (2.10), we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{|\omega_k|} \int_{\omega_k} G(\det D^2 u_n) \, dx &\le \limsup_{n\to\infty} G\left(\frac{\int_{\omega_k} \det D^2 u_n \, dx}{|\omega_k|}\right) \\
&\le G\left(\frac{\int_{\omega_k} \det \partial^2 u \, dx}{|\omega_k|}\right) \\
&\le G(k\epsilon).
\end{aligned}$$

It follows

$$\begin{aligned}
\limsup_{n\to\infty} \int_{\omega_k} G(\det D^2 u_n) \, dx &\le G(k\epsilon)|\omega_k| \\
&\le G((k-1)\epsilon)|\omega_k| + G(\epsilon)|\omega_k| \\
&\le \int_{\Omega_k} G(\det \partial^2 u) \, dx + G(\epsilon)|\Omega_k|.
\end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \int_{\bigcup \omega_k} G(\det D^2 u_n) \, dx \le \int_\Omega G(\det \partial^2 u) \, dx + G(\epsilon)|\Omega|.$$

By (2.9), letting ϵ go to 0, we can replace the domain of the left hand side integral by Ω. The lemma is proved. □

For the uniqueness of maximizers, we first prove a lemma.

Lemma 2.5. *For any maximizer u of $J(\cdot)$, the Monge-Ampère measure $\mu[u]$ has no singular part.*

Proof. We use an argument from [TW2] to prove the lemma. Suppose $\mu[u]$ has non-vanishing singular part $\mu_s[u]$. Then for any $M > 0$, there must exist a ball $B_r \subset \Omega$ such that

$$\mu_s[u](B_r) \geq M(\mu_r[u](B_r) + |B_r|). \tag{2.11}$$

We consider the following Dirichlet problem for Monge-Ampère operator,

$$\begin{cases} \mu[v] = M\mu_r[u] + M \text{ in } B_r, \\ v = u \text{ on } \partial B_r. \end{cases}$$

By the Alexander theorem, the above equation has a unique convex solution v. Note

$$\det \partial^2 v = M \det \partial^2 u + M, \text{ in } B_r. \tag{2.12}$$

By comparison principle, $u \leq v$ in B_r, and the set $E = \{v > u\}$ is not empty. Define another convex function $\tilde{u}$ by

$$\begin{cases} \tilde{u} = u \text{ in } \Omega \setminus E, \\ \tilde{u} = v \text{ in } E. \end{cases}$$

Then $\tilde{u} \in \overline{S}[\varphi, \Omega]$. We claim $J(\tilde{u}) < J(u)$, so we get a contradiction to the assumption that u is a maximizer. In fact, using (2.12), we have

$$\begin{aligned} J(\tilde{u}) - J(u) &= \int_E G(\det \partial^2 v)\, dx - \int_E G(\det \partial^2 u)\, dx - \int_E f(v - u)\, dx \\ &= \int_E G(\det \partial^2 u) - G(M(1 + \det \partial^2 u))\, dx - \int_E f(v - u)\, dx. \end{aligned}$$

By the definition of G, the first integral goes to $-\infty$ as M goes to ∞. The second integral is bounded since f is bounded. The lemma is proved. □

In conclusion, we have obtained the existence and uniqueness of maximizers of J in $\overline{S}[\varphi, \Omega]$.

Theorem 2.6. *Let Ω be a bounded, Lipschitz domain in $\mathbb{R}^n$. Suppose φ is a convex Lipschitz function defined in a neighborhood of $\overline{\Omega}$ and $f \in L^\infty(\Omega)$. There exists a unique function in $\overline{S}[\varphi, \Omega]$ maximizing J.*

Proof. The existence follows from the upper semi-continuity of $A(u)$. For the uniqueness, note that by the concavity of the functional, if there exist two maximizers u and v, then $\partial^2 u = \partial^2 v$ almost everywhere. Hence by Lemma 2.5 we have $\mu[u] = \mu[v]$. By the uniqueness of generalized solutions to the Dirichlet problem of the Monge-Ampère equation, we conclude that $u = v$. □

In Theorem 2.6, we only need the Lipschitz condition on Ω and φ. But later for the regularity, we must assume the smoothness as stated in Theorem 1.1. We point out again that the above argument applies to the functional J_0, and the existence and uniqueness of maximizers also hold for J_0. But we will not study the maximizer of J_0 obtained in this way.

For our purpose of studying J_0, we choose a sequence of functions $G_k = G_{\delta_k}$ satisfying (a)-(c) with $\delta_k \to 0$ as $k \to \infty$, and consider the functionals

$$J_k(u) = A_k(u) - \int_\Omega f u \, dx, \tag{2.13}$$

where

$$A_k(u) = \int_\Omega G_k(\det D^2 u) \, dx. \tag{2.14}$$

By Theorem 2.6, there exists $u^{(k)} \in \overline{S}[\varphi, \Omega]$ maximizing the functional J_k in $\overline{S}[\varphi, \Omega]$. It is clear that $u^{(k)}$ converges to a convex function u_0 in $\overline{S}[\varphi, \Omega]$. We will prove that in dimension 2, u_0 solves the problem (1.7). The main point is to prove the smoothness of u_0. Once we have the regularity of u_0, the uniqueness follows immediately by the concavity of A_0 and the uniqueness of generalized solutions to the Dirichlet problem of the Monge-Ampère equation. Hence, In the rest of this paper, we prove that u_0 is smooth in Ω and satisfies Abreu's equation.

3. Interior estimates

In this section, we establish the interior estimates for equation (2.6).

Lemma 3.1. *Let u be a convex smooth solution to (2.6) in a convex domain Ω. Assume that $u < 0$ in Ω and $u = 0$ on $\partial\Omega$. Then there is a positive constant C depending only on n, $\sup |\nabla u|$, $\sup |u|$, $\sup |f|$ and independent of δ, such that*

$$(-u)^n \det D^2 u \le C.$$

Proof. Let

$$z = -\log d - \log (-u)^\beta - |\nabla u|^2,$$

where β is a positive number to be determined later. Then z attains its minimum at a point p in Ω. We may assume that $d(p) > \delta$ so that $w = d^{-1}$ in a small neighborhood of p. Otherwise, the estimate follows directly. Hence, at p, it holds

$$z_i = 0, \; u^{ij} z_{ij} \ge 0.$$

We can rewrite z as

$$z = \log w - \log (-u)^{\beta} - |\nabla u|^2$$

near p. By computation,

$$z_i = \frac{w_i}{w} - \frac{\beta u_i}{u} - 2u_{ki}u_k, \tag{3.1}$$

$$z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta u_{ij}}{u} + \frac{\beta u_i u_j}{u^2} - 2u_{kij}u_k - 2u_{ki}u_{kj}. \tag{3.2}$$

On the other hand, since $\det D^2 u = w^{-1}$,

$$u^{ab}u_{abi} = (-\log w)_i = -\frac{w_i}{w}.$$

Therefore we have

$$u^{ij}z_{ij} = u^{ij}\frac{w_{ij}}{w} - u^{ij}\frac{w_i w_j}{w^2} - \frac{\beta n}{u} + \beta\frac{u^{ij}u_i u_j}{u^2} + 2\frac{w_k}{w}u_k - 2\triangle u.$$

By (3.1),

$$\begin{aligned} u^{ij}\frac{w_i w_j}{w^2} &= \beta^2 u^{ij}\frac{u_i u_j}{u^2} + \frac{4\beta|Du|^2}{u} + 4u_{ij}u_i u_j, \\ \frac{w_k}{w}u_k &= \frac{\beta|Du|^2}{u} + 2u_{ij}u_i u_j. \end{aligned}$$

It follows

$$u^{ij}z_{ij} = f - \frac{\beta n}{u} - 2\triangle u - \frac{2\beta|Du|^2}{u} - (\beta^2 - \beta)u^{ij}\frac{u_i u_j}{u^2} \geq 0.$$

Choosing $\beta = n$, we have

$$(-u)[\det D^2 u]^{\frac{1}{n}} \leq (-u)\triangle u \leq C$$

at p. The lemma follows. □

For the lower bound estimate of the determinant, we consider the Legendre function u^* of u. If u is smooth, u^* is defined on $\Omega^* = Du(\Omega)$, given by

$$u^*(y) = x \cdot y - u(x),$$

where x is the point determined by $y = Du(x)$. Differentiating $y = Du(x)$, we have

$$\det D^2 u(x) = [\det D^2 u^*(y)]^{-1}.$$

The dual functional with respect to the Legendre function is given by

$$J^*(u^*) = A^*(u^*) - \int_{\Omega^*} f(Du^*)(yDu^* - u^*)\det D^2 u^*\, dy,$$

where

$$A^*(u^*) = \int_{\Omega^*} G([\det D^2 u^*]^{-1})\det D^2 u^*\, dy.$$

If u is a solution to equation (2.6) in Ω, it is a local maximizer of the functional J. Hence u^* is a critical point of J^* under local perturbation, so it satisfies the Euler equation of the dual functional J^*, namely in Ω^*

$$u^{*ij} w^*_{ij} = -f(Du^*), \tag{3.3}$$

where

$$w^* = G(d^{*-1}) - d^{*-1} G'(d^{*-1}), \ \ d^* = \det D^2 u^*. \tag{3.4}$$

Note that on the left hand side of (3.3), it is u^{*ij}, the inverse of (u^*_{ij}).

Lemma 3.2. *Let u^* be a smooth convex solution to (3.3) in Ω^* in dimension 2. Assume that $u^* < 0$ in Ω^* and $u^* = 0$ on $\partial\Omega^*$. Then there is a positive constant C depending only on $\sup|\nabla u^*|$, $\sup|u^*|$, $\inf f$ and independent of δ such that*

$$(-u^*)^2 \det D^2 u^* \le C.$$

Proof. We consider

$$z = -\log d^* - \log(-u^*)^\beta - \alpha|\nabla u^*|^2,$$

where α, β are positive numbers to be determined below. Since z tends to ∞ on $\partial\Omega^*$, it must attain its minimum at some point $p \in \Omega^*$. At p we have

$$z_i = 0, \ \ u^{*ij} z_{ij} \geqslant 0.$$

By (3.4), we compute

$$\begin{aligned} w^*_i &= G''(d^{*-1}) d^{*-3} d^*_i, && (3.5)\\ w^*_{ij} &= -G'''(d^{*-1}) d^{*-5} d^*_i d^*_j - 3G''(d^{*-1}) d^{*-4} d^*_i d^*_j + G''(d^{*-1}) d^{*-3} d^*_{ij}. && (3.6) \end{aligned}$$

On the other hand, by computation,

$$z_i = -\frac{d^*_i}{d^*} - \beta\frac{u^*_i}{u^*} - 2\alpha u^*_{ki} u^*_k, \tag{3.7}$$

$$z_{ij} = -\frac{d^*_{ij}}{d^*} + \frac{d^*_i d^*_j}{d^{*2}} - \beta\frac{u^*_{ij}}{u^*} + \beta\frac{u^*_i u^*_j}{u^{*2}} - 2\alpha u^*_{kij} u^*_k - 2\alpha u^*_{ki} u^*_{kj}. \tag{3.8}$$

It follows

$$u^{*ij} z_{ij} = -\frac{u^{*ij} d^*_{ij}}{d^*} + \frac{u^{*ij} d^*_i d^*_j}{d^{*2}} - \beta\frac{n}{u^*} + \beta\frac{u^{*ij} u^*_i u^*_j}{u^{*2}} - 2\alpha\frac{d^*_k}{d^*} u^*_k - 2\alpha \triangle u^*.$$

By (3.6) and equation (3.3), we have

$$\frac{u^{*ij} d^*_{ij}}{d^*} = -\frac{d^{*2}}{G''(d^{*-1})} f + \frac{d^{*-1} G'''(d^{*-1})}{G''(d^{*-1})} \frac{u^{*ij} d^*_i d^*_j}{d^{*2}} + 3\frac{u^{*ij} d^*_i d^*_j}{d^{*2}}.$$

We may assume that $f(p) < 0$. By condition (b) for G,

$$\frac{d^{*2}}{G''(d^{*-1})} \le -C_1^{-1}, \ \ \left|\frac{d^{*-1} G'''(d^{*-1})}{G''(d^{*-1})}\right| \le C_2.$$

Hence,

$$\frac{u^{*ij}d^*_{ij}}{d^*} \geq C_1^{-1}\inf f + (3 - C_2)\frac{u^{*ij}d^*_i d^*_j}{d^{*2}}.$$

So we have

$$u^{*ij}z_{ij} \geq -C_1^{-1}\inf f + (C_2 - 2)\frac{u^{*ij}d^*_i d^*_j}{d^{*2}} - \frac{\beta n}{u^*} + \beta\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} - 2\alpha\frac{d^*_k}{d^*}u^*_k - 2\alpha \triangle u^*.$$

By (3.7),

$$\begin{aligned} \frac{u^{*ij}d^*_i d^*_j}{d^{*2}} &= \beta^2\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} + 4\alpha\beta\frac{|\nabla u^*|^2}{u^*} + 4\alpha^2 u^*_{lk}u^*_l u^*_k, \\ \frac{d^*_k}{d^*}u^*_k &= -\beta\frac{|\nabla u^*|^2}{u^*} - 2\alpha u^*_{lk}u^*_l u^*_k. \end{aligned}$$

Therefore

$$\begin{aligned} &-C_1^{-1}\inf f + [\beta + (C_2 - 2)\beta^2]\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} - \frac{n\beta}{u^*} \\ &\quad + [4(C_2 - 2) + 2]\alpha\beta\frac{|\nabla u^*|^2}{u^*} + [4(C_2 - 2) + 4]\alpha^2 u^*_{lk}u^*_l u^*_k - 2\alpha \triangle u^* \geq 0. \end{aligned}$$

Choose α small enough depending on $\sup|\nabla u^*|$ such that

$$[4(C_2 - 2) + 4]\alpha^2 u^*_{lk}u^*_l u^*_k \leq \alpha \triangle u^*.$$

Using the fact $u^{*11} + u^{*22} = \frac{\triangle u^*}{\det D^2 u^*}$ in dimension 2, we have

$$\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} \leq \frac{|\nabla u^*|^2}{u^{*2}}\frac{\triangle u^*}{\det D^2 u^*}.$$

It follows

$$-C_1^{-1}\inf f + C'\frac{|\nabla u^*|^2}{u^{*2}}\frac{\triangle u^*}{\det D^2 u^*} - \frac{\beta n}{u^*} + C''\frac{|\nabla u^*|^2}{u^*} - \alpha \triangle u^* \geq 0,$$

where C', C'' are constants depending only on α, β, C_1 and C_2. If

$$\frac{\alpha}{2} \triangle u^* - C'\frac{|\nabla u^*|^2}{u^{*2}}\frac{\triangle u^*}{\det D^2 u^*} \leq 0,$$

we obtain

$$(-u^*)^2 \det D^2 u^* \leqslant C$$

at p. Otherwise, we have

$$-C_1^{-1}\inf f - \frac{\beta n}{u^*} + C''\frac{|\nabla u^*|^2}{u^*} - \frac{\alpha}{2} \triangle u^* \geq 0.$$

Hence, we also obtain

$$(-u^*)^2 \det D^2 u^* \leq (\triangle u^*)^2(-u^*)^2 \leqslant C$$

at p. The lemma follows by choosing $\beta = n = 2$. □

Remark 3.3.
(i) The determinant estimates above is independent of δ. This leads us to use the approximation $\{G_k\}$;
(ii) The estimate depends only on $\inf f$. This is crucial in Section 7;
(iii) In Lemma 3.2, the estimate only holds in dimension 2. Since if we do not have the relation $u^{*11} + u^{*22} = \frac{\triangle u^*}{\det D^2 u^*}$, we can not deal with the term $\frac{u^{*ij} u_i^* u_j^*}{u^{*2}}$ in the proof. This is why we can not extend Theorem 1.1 to higher dimensions.

To apply the above determinant estimates, we first introduce the *modulus of convexity* for convex functions. The modulus of convexity of u at x is defined by

$$h_{u,x}(r) = \sup\{\delta \geq 0 \mid S_{\delta,u}(x) \subset B_r(x)\}, \ r > 0 \tag{3.9}$$

and the modulus of convexity of u on Ω is defined by

$$h_{u,\Omega}(r) = \inf_{x \in \Omega} h_{u,x}(r), \tag{3.10}$$

where

$$S_{\delta,u}(x) = \{y \in \Omega \mid u(y) < \delta + a_x(y)\}$$

and a_x is a tangent plane of u at x. When no confusions arise, we will also write $S_{\delta,u}(x)$ as $S_{\delta,u}$ or S_δ, for brevity.

Lemma 3.4. *Let $u \in C^4(\Omega)$ be a locally uniformly convex solution to (2.6) in dimension 2.*

(i) Assume $f \in L^\infty(\Omega)$. Then

$$\|u\|_{W^{4,p}(\Omega')} \leq C$$

for any $p > 1$ and $\Omega' \subset\subset \Omega$, where C depends on n, p, $\sup |f|$, $dist(\Omega', \partial\Omega)$ and the modulus of convexity of u.

(ii) Assume $f \in C^\alpha(\Omega)$. Then

$$\|u\|_{C^{4,\alpha}(\Omega')} \leq C$$

for any $\alpha \in (0, 1)$ and $\Omega' \subset\subset \Omega$, where C depends on n, α, $\sup |f|$, $dist(\Omega', \partial\Omega)$ and the modulus of convexity of u.

Proof. For any $x \in \Omega$, by Lemma 3.1, we have

$$\det D^2 u(x) \leq C$$

where C is a constant depending only on f, $\delta = dist(x, \partial\Omega)$ and $h_{u,\Omega}$. Let $y = Du(x) \in \Omega^*$. By (3.9), (3.10), we have

$$S_{\delta^*,u^*}(y) \subset \Omega^*,$$

where $\delta^* = h_{u,\Omega}(\frac{\delta}{2})$. Furthermore, since $|Du^*| \le diam(\Omega)$, we also have

$$dist(y, \partial\Omega^*) \ge \frac{\delta^*}{2diam(\Omega)}.$$

Hence, by Lemma 3.2,

$$\det D^2u(x) = [\det D^2u^*(y)]^{-1} \ge C',$$

where C' is a constant depending only on f, δ and $h_{u,\Omega}$.

Once the determinant $\det D^2u$ is bounded, we also have the Holder continuity of $\det D^2u$ by Caffarelli-Gutierrez's Hölder continuity for linearized Monge-Ampère equation [CG]. Then we have the $W^{2,p}$ and $C^{2,\alpha}$ regularity for u by Caffarelli's $W^{2,p}$ and $C^{2,\alpha}$ estimates for Monge-Ampère equation [Caf1, JW], respectively. Higher regularity then follows from the standard elliptic regularity theory [GT]. □

We will estimate in Section 6 and 7 the modulus of convexity for the solution u in dimension 2. In Section 4 we consider the change of equation (2.6) under a coordinate transformation and establish the a priori estimates for the equation after the transformation.

4. Equations after rotations in $\mathbb{R}^{n+1}$

Equation (2.6) is invariant under transformations of the x-coordinates in $\mathbb{R}^n$, but it changes when taking transformations in $\mathbb{R}^{n+1}$. We note that the affine maximal surface equation is invariant under uni-modular transformations in $\mathbb{R}^{n+1}$, which plays an important part [TW1]. In order to establish the estimate of the modulus of convexity, we also need to consider the equation under rotations in $\mathbb{R}^{n+1}$. In this section we will derive the new equation under a rotation in $\mathbb{R}^{n+1}$ and establish the a priori estimates for it.

For our purpose it suffices to consider the rotation $z = Tx$, given by

$$z_1 = -x_{n+1}, \tag{4.1}$$

$$z_2 = x_2, \ ..., \ z_n = x_n, \tag{4.2}$$

$$z_{n+1} = x_1, \tag{4.3}$$

which fixes $x_2, ..., x_n$ axes. Assume that the graph of u, $\mathcal{M} = \{(x, u(x)) \in \mathbb{R}^{n+1} \mid x \in \Omega\}$, can be represented by a convex function $z_{n+1} = v(z_1, ..., z_n)$ in z-coordinates, in a domain $\hat{\Omega}$. To derive the equation for v, we compute the change of the functional A_0.

$$(4.4)\qquad \begin{aligned} A(u) &= \int_\Omega G(\det D^2u)\,dx \\ &= \int_\Omega G\left(\frac{\det D^2u}{(1+|Du|^2)^{\frac{n+2}{2}}}(1+|Du|^2)^{\frac{n+2}{2}}\right)dx \\ &= \int_{\mathcal{M}} G\left(K(1+|Du|^2)^{\frac{n+2}{2}}\right)(1+|Du|^2)^{-\frac{1}{2}}\,d\Sigma, \end{aligned}$$

where K is the Gaussian curvature of $\mathcal{M}$ and $d\Sigma$ the volume element of the hypersurface. It is easy to verify that

$$(4.5)\qquad u_1 = -\frac{1}{v_1},\ u_2 = \frac{v_2}{v_1},\ \dots,\ u_n = \frac{v_n}{v_1},$$

where $v_i = \frac{\partial v}{\partial z_i}$. So we have

$$1 + |Du|^2 = \frac{1+|Dv|^2}{v_1^2}.$$

Hence we obtain

$$(4.6)\qquad A(u) = \int_{\hat\Omega} G(v_1^{-(n+2)}\det D^2v)(v_1^2)^{\frac{1}{2}}\,dz := \hat{A}(v).$$

In addition,

$$\begin{aligned} \int_\Omega f(x)u(x)dx &= \int_{\mathcal{M}} f\cdot u\cdot(1+|Du|^2)^{-\frac{1}{2}}\,d\Sigma \\ &= \int_{\hat\Omega} f(v, z_2, \dots, z_n)\cdot(-z_1)\cdot(v_1^2)^{\frac{1}{2}}\,dz. \end{aligned}$$

Let

$$\hat{J}(v) = \hat{A}(v) - \int_{\hat\Omega} f(v, z_2, \dots, z_n)\cdot(-z_1)\cdot(v_1^2)^{\frac{1}{2}}\,dz.$$

After computing the Euler equation for the functional $\hat{J}(v)$, we have

Lemma 4.1. *Let u be a solution of (2.6). Let T and v be as above. Then v satisfies the equation*

$$(4.7)\qquad V^{ij}(d^{-1})_{ij} = g - f_1z_1v_1 + f_1z_1 + f$$

in the set $\{z \mid v_1^{-(n+2)}d > \delta\}$, where (V^{ij}) is the cofactor matrix of (v_{ij}), $d = \det D^2v$ and

$$\begin{aligned} g &= 2v^{kl}v_{kl1}\frac{1}{v_1} - (n+2)\frac{v_{11}}{v_1^2}, \\ f &= f(v, z_2, \dots, z_n), \\ f_1 &= \frac{\partial f}{\partial x_1}(v, z_2, \dots, z_n). \end{aligned}$$

Remark 4.2. *In the proof of strict convexity in Section 6, we will use the upper bound estimate for* $\det D^2 v$ *given below. Since the lower bound for* $\det D^2 v$ *will not be used, we do not need the explicit form of the equation for* v *outside the set* $\{z \mid v_1^{-(n+2)} d > \delta\}$. *Therefore in (4.7), we calculate the Euler equation only in the set* $\{z \mid v_1^{-(n+2)} d > \delta\}$.

Next we prove a determinant estimate for v. Assume v satisfies

$$\begin{array}{l} v \geq 0,\ v \geq z_1,\ v_1 \geq 0, \ \text{ and } v(0) \text{ is as small as we want such that} \\ \text{for the positive constants } \epsilon \text{ and } c \text{ in } (0, \tfrac{1}{2}),\ \hat{\Omega}_{\epsilon,c} \text{ is a nonempty open set,} \end{array} \tag{4.8}$$

where

$$\hat{v} = v - \epsilon z_1 - c \text{ and } \hat{\Omega}_{\epsilon,c} = \{z \mid \hat{v}(z) < 0\}.$$

Then $\hat{v}$ satisfies

$$\hat{V}^{ij}(\hat{d}^{-1})_{ij} = \hat{g} - \hat{f}_1 z_1(\hat{v}_1 + \epsilon) + \hat{f}_1 z_1 + \hat{f} \tag{4.9}$$

in the set $\{z \mid (\hat{v}_1 + \epsilon)^{-(n+2)} \hat{d} > \delta\} \cap \hat{\Omega}_{\epsilon,c}$, where $\hat{d} = \det D^2 \hat{v}$ and

$$\hat{g} = 2\hat{v}^{kl}\hat{v}_{kl1}\frac{1}{\hat{v}_1 + \epsilon} - (n+2)\frac{\hat{v}_{11}}{(\hat{v}_1 + \epsilon)^2}, \tag{4.10}$$

$$\hat{f} = f(\hat{v} + \epsilon z_1 + c, z_2, ..., z_n), \tag{4.11}$$

$$\hat{f}_1 = \frac{\partial f}{\partial x_1}(\hat{v} + \epsilon z_1 + c, z_2, ..., z_n). \tag{4.12}$$

Lemma 4.3. *Let* $\hat{v}$ *be as above. Then there exists* $C > 0$ *depending only on* $\sup |\hat{f}|$, $\sup |\nabla \hat{f}|$, $\sup_{\hat{\Omega}_{\epsilon,c}} |\hat{v}|$ *and* $\sup_{\hat{\Omega}_{\epsilon,c}} |D\hat{v}|$, *but independent of* δ, *such that*

$$(-\hat{v})^n \det D^2 \hat{v} \leq C.$$

Proof. Consider

$$\eta = \log w - \beta \log(-\hat{v}) - A|D\hat{v}|^2,$$

where $w = \hat{d}^{-1}$, and β, A are positive numbers to be determined below. Then η attains its minimum at a point p in $\hat{\Omega}_{\epsilon,c}$. Hence, at p, it holds

$$\eta_i = 0,\ \hat{v}^{ij}\eta_{ij} \geq 0.$$

We can suppose that $p \in \{z \mid (\hat{v}_1 + \epsilon)^{-(n+2)} \hat{d} > \delta\}$. Otherwise, we have

$$(\hat{v}_1 + \epsilon)^{-(n+2)} \hat{d} \leq \delta$$

and then the estimate follows. By computation,

$$\eta_i = \frac{w_i}{w} - \frac{\beta \hat{v}_i}{\hat{v}} - 2A\hat{v}_{ki}\hat{v}_k, \tag{4.13}$$

$$\eta_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta \hat{v}_{ij}}{\hat{v}} + \frac{\beta \hat{v}_i \hat{v}_j}{\hat{v}^2} - 2A\hat{v}_{kij}\hat{v}_k - 2A\hat{v}_{ki}\hat{v}_{kj}, \tag{4.14}$$

$$\frac{w_k}{w} = -\hat{v}^{ij}\hat{v}_{ijk}. \tag{4.15}$$

By (4.15),

$$\hat{g} = -2\frac{w_1}{w}\frac{1}{\hat{v}_1+\epsilon} - (n+2)\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2}.$$

Therefore we have

$$\begin{aligned}\hat{v}^{ij}\eta_{ij} &= -\frac{\hat{v}^{ij}w_iw_j}{w^2} - \frac{w_1}{w}\frac{2}{\hat{v}_1+\epsilon} - \frac{\beta n}{\hat{v}} - (n+2)\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} + \frac{\beta\hat{v}^{ij}\hat{v}_i\hat{v}_j}{\hat{v}^2} + 2A\frac{w_k}{w}\hat{v}_k \\ &\quad -2A\triangle\hat{v} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f}.\end{aligned}$$

By (4.13),

$$\begin{aligned}\frac{\hat{v}^{ij}w_iw_j}{w^2} &= \beta^2\hat{v}^{ij}\frac{\hat{v}_i\hat{v}_j}{\hat{v}^2} + 4A^2\hat{v}_{ij}\hat{v}_i\hat{v}_j + 4A\beta\frac{|D\hat{v}|^2}{\hat{v}}, \\ \frac{w_1}{w}\frac{2}{\hat{v}_1+\epsilon} &= \frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} + 4A\frac{\hat{v}_{1k}\hat{v}_k}{\hat{v}_1+\epsilon}, \\ \frac{w_k}{w}\hat{v}_k &= \beta\frac{|D\hat{v}|^2}{\hat{v}} + 2A\hat{v}_{ij}\hat{v}_i\hat{v}_j.\end{aligned}$$

Hence, we have

$$\begin{aligned}\hat{v}^{ij}\eta_{ij} &= -(n+2)\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} - 4A\left(\frac{\hat{v}_{11}\hat{v}_1}{\hat{v}_1+\epsilon} + \sum_{k=2}^{n}\frac{\hat{v}_{1k}\hat{v}_k}{\hat{v}_1+\epsilon}\right) - \frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} - 2A\triangle\hat{v} \\ &\quad -\frac{\beta n}{\hat{v}} - 2A\beta\frac{|D\hat{v}|^2}{\hat{v}} - (\beta^2-\beta)\hat{v}^{ij}\frac{\hat{v}_i\hat{v}_j}{\hat{v}^2} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f}. \qquad (4.16)\end{aligned}$$

We choose $\beta > 1$ such that $\beta^2 - \beta > 0$. By the positive definiteness of $\hat{v}_{ij}$, it holds $\hat{v}_{1k}^2 \le \hat{v}_{11}\hat{v}_{kk}$ for any $k = 2, ..., n$, so there is C' depending on n and $|D\hat{v}|$, such that

$$\sum_{k=2}^{n}\frac{|\hat{v}_{1k}\hat{v}_k|}{\hat{v}_1+\epsilon} \le \frac{1}{4}\sum_{k=2}^{n}\hat{v}_{kk} + C'\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} \le \frac{1}{4}\triangle\hat{v} + C'\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2}. \qquad (4.17)$$

It follows

$$\begin{aligned}&-\frac{(n+2-4AC')\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} - 4A\frac{\hat{v}_{11}\hat{v}_1}{\hat{v}_1+\epsilon} - \frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} - A\triangle\hat{v} - \frac{\beta n}{\hat{v}} \\ &-2A\beta\frac{|D\hat{v}|^2}{\hat{v}} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f} \ge 0. \qquad (4.18)\end{aligned}$$

Choosing A small enough such that $n+2-4AC' > 0$. Then by a Schwarz inequality, there exists a $C_0 > 0$ depending only on $|D\hat{v}|$ such that

$$-\frac{(n+2-4AC')\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} - 4A\frac{\hat{v}_{11}\hat{v}_1}{\hat{v}_1+\epsilon} \le C_0A^2\hat{v}_{11}. \qquad (4.19)$$

By (4.18), (4.19), we have

$$0 \le C_0A^2\hat{v}_{11} - \frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} - \frac{\beta n}{\hat{v}} - A\triangle\hat{v} - 2A\beta\frac{|D\hat{v}|^2}{\hat{v}} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f}.$$

Choosing A small enough furthermore such that $C_0A^2 \leq \frac{A}{2}$, and observing that

$$\frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} = \frac{2\beta}{\hat{v}} - \frac{2\beta\epsilon}{(\hat{v}_1+\epsilon)\hat{v}} \geq \frac{2\beta}{\hat{v}},$$

we have

$$-\frac{\beta(n+2)}{\hat{v}} - \frac{A}{2} \triangle \hat{v} - 2A\beta\frac{|D\hat{v}|^2}{\hat{v}} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f} \geq 0,$$

which implies

$$(-\hat{v})\triangle\hat{v} \leq C$$

at p. Hence, choosing $\beta = n$, the lemma follows by

$$e^{\eta(x)} \geq e^{\eta(p)} = \hat{d}^{-1}(-\hat{v})^{-n}e^{-A|D\hat{v}|^2} \geq [\frac{(-\hat{v})\triangle\hat{v}}{n}]^{-n}e^{-A|D\hat{v}|^2} \geq C.$$

□

5. Approximation

We will use a penalty method and solutions to the second boundary value problem to construct a sequence of smooth convex solutions to (2.6) to approximate the maximizer of $J(u)$. This section is similar to §6 in [TW2].

First, we consider a second boundary value problem with special non-homogenous term f. Let $B = B_R(0)$ be a ball with $\Omega \subset\subset B$ and $\varphi \in C^2(\overline{B})$ be a uniformly convex function in B vanishing on ∂B. Suppose H is a nonnegative smooth function defined in the interval $(-1,1)$ such that

$$H(t) = \begin{cases} (1-t)^{-2n}, & t \in (\frac{1}{2}, 1), \\ (1+t)^{-2n}, & t \in (-1, \frac{1}{2}). \end{cases} \tag{5.1}$$

Extend the function f to B such that

$$f(x,u) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ h(u-\varphi(x)) & \text{if } x \in B \setminus \Omega, \end{cases}$$

where $h(t) = H'(t)$.

Lemma 5.1. *Let $f(x,u)$ be as above. Suppose $\partial\Omega$ is Lipschitz continuous. Then there exists a locally uniformly convex solution to the second boundary problem*

$$\begin{aligned} U^{ij}w_{ij} &= f(x,u) \quad \text{in } B, \\ w &= G'(d), \quad \text{in } B, \\ u &= \varphi \quad \text{on } \partial B, \\ w &= 1 \quad \text{on } \partial B \end{aligned} \tag{5.2}$$

with $u \in W^{4,p}_{loc}(B) \cap C^{0,1}(\overline{B})$*, for all* $p < \infty$*, and* $w \in C^0(\overline{\Omega})$*.*

Proof. By the discussion of the second boundary problem in the Appendix, it suffices to prove that for any solution u to (5.2), $|f(x,u)| \le C$ for some constant C independent of u. Note that by our choice of H, a solution to (5.2) is bounded from below.

First, we prove an estimate of the determinant near the boundary ∂B. By the definition of H and the convexity of u, f is bounded from above near ∂B. For any boundary point $x_0 \in \partial B$, we suppose by a rotation of axes that $x_0 = (R, 0, ..., 0)$. There exists $\delta_0 > 0$ independent of x_0 such that f is bounded from above in $B \cap \{x_1 > R - \delta_0\}$. Choose a linear function $l = ax_1 + b$ such that $l(x_0) < u(x_0) = 0$ and $l > u$ on $x_1 = R - \delta_0$. Let

$$z = w + \log w - \beta \log(u - l),$$

where $\beta > 0$ is to be determined below. If z attains its minimum at a boundary point on ∂B, by the boundary condition $w = 1$, $z \ge -C$ near ∂B. If z attains its minimum at a interior point $y_0 \in \{u > l\}$, we have, at y_0,

$$0 = z_i = w_i + \frac{w_i}{w} - \beta \frac{(u-l)_i}{u-l}, \tag{5.3}$$

$$z_{ij} = w_{ij} + \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \beta \frac{(u-l)_{ij}}{u-l} + \beta \frac{(u-l)_i (u-l)_j}{(u-l)^2}. \tag{5.4}$$

By (5.3),

$$\frac{w_i}{w} = \frac{\beta}{1+w} \frac{(u-l)_i}{u-l}.$$

It follows by (5.4) and equation (5.2)

$$0 \le u^{ij} z_{ij} = \frac{f}{d} + \frac{f}{dw} - \frac{\beta n}{u-l} + \left[\beta - \frac{\beta^2}{(1+w)^2}\right] \frac{u^{ij}(u-l)_i(u-l)_j}{(u-l)^2}.$$

We may suppose that $w \le 1$. Choose β large enough such that

$$\beta - \frac{\beta^2}{(1+w)^2} \le 0.$$

So we have $w(y_0) \ge C$. Therefore, $\det D^2 u \le C$ near ∂B.

By the above determinant estimate near ∂B, it follows that $|Du|$ is bounded near ∂B. By the convexity of u,

$$\sup_B |Du| \le C.$$

Next, we prove that f is bounded from below. We note that by the Lipschitz continuity of $\partial\Omega$, there exists positive constants r, κ such that for any $p \in B \setminus \Omega$, there is a unit vector γ such that the round cone $\mathcal{C}_{p,\gamma,r,\kappa} \subset B \setminus \Omega$, where

$$\mathcal{C}_{p,\gamma,r,\kappa} := \{x \in \mathbb{R}^n \mid |x-p| < r,\ \langle x-p, \gamma\rangle > \cos\kappa\}.$$

Assume that $M = -\inf_B f$ is attained at $x_0 \in B$. If $x_0 \in \Omega$, then $M = \|f\|_{L^\infty(\Omega)}$. If $x_0 \in B \setminus \Omega$, we have

$$M = 2n[1 + u(x_0) - \varphi(x_0)]^{-2n-1},$$

that is,

$$u(x_0) - \varphi(x_0) = \left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}} - 1.$$

Let l_0 be the tangent plane of φ at x_0. Since we have the gradient estimate of u, there exists a uniform δ_0 such that

$$0 \le 1 + u(x) - \varphi(x) \le 2\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}}$$

and

$$0 \le 1 + u(x) - l_0(x) \le 2\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}}$$

in the cone $\mathcal{C}_{x_0,\gamma,\delta_0\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}},\kappa}$. Let $\omega_0 = \{x \mid u(x) < l_0(x)\}$. It is clear that when M is sufficiently large,

$$\mathcal{C}_{x_0,\gamma,\delta_0\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}},\kappa} \subset \omega_0.$$

Integrating by parts, we have

$$\begin{aligned}\int_{\omega_0} U^{ij} w_{ij}(u-l_0)\,dx &= -\int_{\omega_0} U^{ij} w_j (u-l_0)_i\,dx \\ &= -\int_{\partial\omega_0} wU^{ij}(u-l_0)_i\gamma_j\,dS + \int_{\omega_0} w \det D^2u\,dx,\end{aligned}$$

where dS is the volume element of $\partial\omega_0$. $u - l_0$ vanishes on the boundary, so $U^{ij}(u-l_0)_i\gamma_j \ge 0$. The first integral on the right-hand side is negative. Hence, we obtain

$$\int_{\omega_0} f(x,u)(u-l_0)\,dx \le \int_{\omega_0} w \det D^2u\,dx \le C. \tag{5.5}$$

Note that the last inequality follows by the condition $\lim_{t\to 0} tF'(t) \le C_3$ in the assumption (c) on G. Estimating the integral in the cone, we have

$$\int_{\omega_0} f(x,u)(u-l_0)\,dx \ge 2^{-2n-1}M \cdot \left[1 - 2\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}}\right] \cdot C \cdot \left(\frac{M}{2n}\right)^{-\frac{n}{2n+1}}. \tag{5.6}$$

Therefore $M \le C$ follows from (5.5), (5.6).

Finally, we prove that f is bounded from above. For any $\delta > 0$, let

$$\Omega_\delta = \{u < -\delta\} \subset B$$

and γ be the unit outward normal on $\partial\Omega_\delta$. We have

$$\begin{aligned}
\int_{\Omega_\delta} U^{ij} w_{ij}(u+\delta)\,dx &= -\int_{\Omega_\delta} U^{ij} w_j u_i\,dx \\
&= -\int_{\partial\Omega_\delta} wU^{ij}u_i\gamma_j\,dS + \int_{\Omega_\delta} w\det D^2u\,dx \\
&\geq -\int_{\partial\Omega_\delta} wU^{ij}u_i\gamma_j\,dS \\
&= -\int_{\partial\Omega_\delta} wU^{\gamma\gamma}u_\gamma\,dS \\
&= -\int_{\partial\Omega_\delta} wu_\gamma^n K_s\,dS \\
&\geq -C\sup_{\partial\Omega_\delta} w \sup_B |Du|^n,
\end{aligned}$$

where dS is the volume element of $\partial\Omega_\delta$ and K_s is the Gaussian curvature of $\partial\Omega_\delta$. Letting $\delta \to 0$, by $w = 1$ on ∂B and the gradient estimate,

$$\int_B f(x,u)u\,dx \geq -C.$$

By a similar argument as in the proof of lower bound, if $u-\varphi$ is sufficiently close to 1 at some point $x \in B \setminus \Omega$, $u-\varphi$ is sufficiently close to 1 nearby in $B\setminus\Omega$. This implies the integral can be arbitrary large, which is a contradiction. Hence, f is bounded and the lemma follows. □

Now we prove that the maximizer of $J(u)$ can be approximated by smooth solutions to (2.6). This approximation was proved for the affine Plateau problem in [TW2] by a penalty method. We will also use this method.

Theorem 5.2. *Let Ω and φ be as in Theorem 2.6. Suppose $\partial\Omega$ is Lipschitz continuous. Then there exist a sequence of smooth solutions to equation (2.6) converging locally uniformly to the maximizer u.*

Proof. The proof for this approximation in [TW2] is very complicated, so we use a simplified proof in [TW5].

Let $B = B_R(0)$ be a large ball such that $\Omega \subset B_R$. By assumption, φ is defined in a neighborhood of Ω, so we can extend u to B such that φ is convex in B, $\varphi \in C^{0,1}(\overline{B})$

and φ is constant on ∂B. Adding $(|x| - R + \frac{1}{2})_+^2$ to φ, where

$$(|x| - R + \frac{1}{2})_+ = \max\{|x| - R + \frac{1}{2}, 0\},$$

we assume that φ is uniformly convex in $\{x \in \mathbb{R}^n \mid R - \frac{1}{2} < |x| < R\}$. Consider the second boundary value problem (5.2) with

$$f_j(x,u) \;=\; \begin{cases} f & \text{in } \Omega, \\ H_j'(u-\varphi) & \text{in } B_R \setminus \Omega, \end{cases}$$

where $H_j(t) = H(4^j t)$ and H is defined by (5.1). By Lemma 5.1, there is a solution u_j satisfying

$$|u_j - \varphi| \le 4^{-j},\ x \in B_R \setminus \Omega. \tag{5.7}$$

By the convexity, u_j sub-converges to a convex function $\bar{u}$ in B_R as $j \to \infty$. Note that $\bar{u} = \varphi$ in $B_R \setminus \Omega$. Hence, $\bar{u} \in \overline{S}[\varphi, \Omega]$ when restricted in Ω. We claim that $\bar{u}$ is the maximizer.

Let v_j be an extension of u, given by

$$v_j = \sup\{l \mid l \in \Phi_j\},$$

where Φ_j is the set of linear functions in B_R satisfying

$$\begin{aligned} l(x) \le \varphi(x) &\qquad \text{when } |x| = R \text{ or } |x| \le R - \frac{1}{j}, \text{ and} \\ l(x) \le u_j(x) &\qquad \text{when } R - \frac{1}{j} < |x| < R. \end{aligned}$$

By our assumption, φ is uniformly convex in $B_R \setminus B_{\frac{R}{2}}$. By (5.7), $|u_j - \varphi| \le 4^{-j} = o(j^{-2})$, $x \in B_R \setminus \Omega$. So we have

$$v_j = u_j \qquad \text{in } B_R \setminus B_{R-\frac{1}{2j}}, \tag{5.8}$$

$$v_j = \varphi \qquad \text{in } B_{R-\frac{2}{j}} \setminus \Omega, \tag{5.9}$$

$$|v_j - \varphi| \le |u_j - \varphi| \qquad \text{in } B_{R-\frac{1}{2j}} \setminus B_{R-\frac{2}{j}} := D_j. \tag{5.10}$$

Now we consider the functional

$$J_j(v) = \int_{B_R} G(\det \partial^2 v)\, dx - \int_\Omega f v\, dx - \int_{B_R \setminus \Omega} H_j(v - \varphi)\, dx.$$

Subtracting G by the constant $G(0)$, we may assume that $G(0) = 0$. Note that u_j is the maximizer of J_j in $\overline{S}[u_j, B_R]$ and $v_j \in \overline{S}[u_j, B_R]$. So we have

$$J_j(v_j) \le J_j(u_j).$$

In the following, we denote by $J_j(v, E)$ the functional J_j over the domain E. By (5.8), we have

$$J_j(v_j, B_{R-\frac{1}{2j}}) \le J_j(u_j, B_{R-\frac{1}{2j}}). \tag{5.11}$$

By (5.9), (5.10), we obtain

$$-\int_{B_{R-\frac{1}{2j}}\setminus\Omega} H_j(u_j-\varphi)\,dx \le -\int_{B_{R-\frac{1}{2j}}\setminus\Omega} H_j(v_j-\varphi)\,dx. \tag{5.12}$$

For any $\epsilon > 0$, by the upper semi-continuity of the functional $A(u)$,

$$\begin{aligned}\int_{B_{R-\frac{2}{j}}\setminus\Omega} G(\det\partial^2 u_j)\,dx &\le \int_{B_{R-\frac{2}{j}}\setminus\Omega} G(\det\partial^2\varphi)\,dx+\epsilon\\ &= \int_{B_{R-\frac{2}{j}}\setminus\Omega} G(\det\partial^2 v_j)\,dx+\epsilon\end{aligned} \tag{5.13}$$

provided j is large enough. In addition, by (2.9),

$$0 \le \int_{D_j} G(\det\partial^2 v)\,dx \le |D_j| G(|D_j|^{-1}\mu[v](D_j)) \to 0 \tag{5.14}$$

as $j \to \infty$, where $v = u_j$ or v_j.

Hence, by (5.11)-(5.14) and the upper semi-continuity of the functional $A(u)$,

$$J(u) = J(v_j) \le J(u_j) + 2\epsilon \le J(\bar u) + 3\epsilon.$$

provided j is large enough. By taking $\epsilon \to 0$, this implies $\bar u$ is the maximizer. By the uniqueness of maximizers in Theorem 2.6, we obtain $\bar u = u$. □

Remark 5.3. *We remark that the above approximation does not holds for the maximizer of the functional J_0. The reason is that since $\log d$ is not bounded from below, we do not have the property*

$$\left|\int_E \log\det\partial^2 u\,dx\right| \longrightarrow 0,$$

as $|E| \to 0$. This is why we introduce the function G and consider the modified functional $J(u)$.

By Theorem 5.2, for each k, there exists a smooth solutions $u_j^{(k)}$ to

$$U^{ij} w_{ij} = f, \tag{5.15}$$

where

$$w = G_k'(\det D^2 u), \tag{5.16}$$

which converges locally uniformly to the maximizer $u^{(k)}$ of (2.13). Then we have

$$u_j^{(k)} \longrightarrow u_0, \ \ j, k \to \infty. \tag{5.17}$$

As we explained in Section 3, if u_0 is strictly convex, the interior a priori estimates of $u_j^{(k)}$ will be independent of k and j. Hence, by taking limit, we have the interior regularity of u_0 in Ω. Moreover, by the construction of G_k, u_0 will be a solution to Abreu's equation (1.1). Therefore we have

Theorem 5.4. *Let u_0 be as above. Assume that $f \in C^\infty(\Omega)$. Then if u_0 is a strictly convex function, $u_0 \in C^\infty(\Omega)$ and solves (1.7).*

In the last two sections, we will show the strict convexity of u_0.

6. Strict convexity I

We prove the strict convexity of u_0 in dimension 2. Let $\mathcal{M}_0$ be the graph of u_0. If u_0 is not strictly convex, $\mathcal{M}_0$ contains a line segment. Let $l(x)$ be a tangent function of u_0 at the segment and denote by

$$\mathcal{C} = \{x \in \Omega \mid u_0(x) = l(x)\}$$

the contact set.

We first recall the definition of extreme points. Let Ω be a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$. A boundary point $x \in \partial\Omega$ is an *extreme point* of Ω if there is a hyperplane H such that $\{x\} = H \cap \partial\Omega$, namely x is the unique point in $H \cap \partial\Omega$.

According to the distribution of extreme points of $\mathcal{C}$, we consider two cases as follows.

Case (a) $\mathcal{C}$ has an extreme point x_0 which is an interior point of Ω.

Case (b) All extreme points of $\mathcal{C}$ lie on $\partial\Omega$.

In this section, we exclude Case (a).

Proposition 6.1. *$\mathcal{C}$ contains no extreme points in the interior of Ω.*

Proof. We prove this proposition by contradiction arguments as in [TW1]. By (5.17), we can choose a sequence of smooth functions $u_k = u_{j_k}^{(k)}$ converging to u_0 such that u_k is the solution to (5.15). Let $\mathcal{M}_k$ be the graph of u_k. Then $\mathcal{M}_k$ converges in Hausdorff distance to $\mathcal{M}_0$. There is no loss of generality in assuming that $l(x) = 0$, x_0 is the origin and the segment $\{(x_1, 0) \mid 0 \leq x_1 \leq 1\} \subset \mathcal{C}$.

For any $\epsilon > 0$, we consider a linear function

$$l_\epsilon = -\epsilon x_1 + \epsilon$$

and a subdomain $\Omega_\epsilon = \{u < l_\epsilon\}$. Let T_ϵ be the coordinates transformation that normalizes Ω_ϵ. Define

$$u_\epsilon(y) = \frac{1}{\epsilon}u(x),\ u_{k,\epsilon} = \frac{1}{\epsilon}u_k(x),\quad y \in \tilde{\Omega}_\epsilon \tag{6.1}$$

where $y = T_\epsilon x$ and $\tilde{\Omega}_\epsilon = T_\epsilon(\Omega_\epsilon)$. After this transformation, we have the following observations:

(i) By Remark 2.2, $u_{k,\epsilon}$ satisfies the equation (2.6) with

$$G = G_{k,\epsilon}(d) = G_k(\epsilon|T_\epsilon|^2 d),\quad \delta = \delta_{k,\epsilon} = \frac{\delta_k}{\epsilon|T_\epsilon|^2}$$

and the right hand term ϵf. Note that $|T_\epsilon| \ge C\epsilon^{-1}$, so $\delta_{k,\epsilon} \le C\delta_k \to 0$ for a constant C independent of ϵ.

(ii) Denote by $\mathcal{M}_\epsilon$, $\mathcal{M}_{k,\epsilon}$ the graphs of u_ϵ, $u_{k,\epsilon}$, respectively. Taking $k \to \infty$, it is clear that $u_{k,\epsilon} \to u_\epsilon$ and $\mathcal{M}_{k,\epsilon}$ converges in Hausdorff distance to $\mathcal{M}_\epsilon$. Then taking $\epsilon \to 0$, we have that the domains $\tilde{\Omega}_\epsilon$ sub-converges to a normalized domain $\tilde{\Omega}$ and u_ϵ sub-converges to a convex function $\tilde{u}$ defined in $\tilde{\Omega}$. We also have $\mathcal{M}_\epsilon$ sub-converges in Hausdorff distance to a convex surface $\tilde{\mathcal{M}}_0 \in \mathbb{R}^3$.

(iii) The convex surface $\tilde{\mathcal{M}}_0$ satisfies

$$\tilde{\mathcal{M}}_0 \subset \{y_1 \ge 0\} \cap \{y_3 \ge 0\} \tag{6.2}$$

and $\tilde{\mathcal{M}}_0$ contains two segments

$$\{(0,0,y_3) \mid 0 \le y_3 \le 1\},\ \{(y_1,0,0) \mid 0 \le y_1 \le 1\}. \tag{6.3}$$

Hence, by (i), (ii), (iii), we can suppose that there is a solution $\tilde{u}_k$ to

$$U^{ij}w_{ij} = \epsilon_k f \quad \text{in } \tilde{\Omega}_k, \tag{6.4}$$

where

$$w = \tilde{G}'_{\tilde{\delta}_k}(\det D^2 u), \tag{6.5}$$

and $\tilde{\delta}_k, \epsilon_k \to 0$, such that the normalized domain $\tilde{\Omega}_k$ converges to $\tilde{\Omega}$, $\tilde{u}_k$ converges to $\tilde{u}$ and the graph of $\tilde{u}_k$, denoted by $\tilde{\mathcal{M}}_k$ converges in Hausdorff distance to $\tilde{\mathcal{M}}_0$. It is clear that in y-coordinates, $\tilde{\mathcal{M}}_0$ is not a graph of a function near the origin, so we need to rotate the $\mathbb{R}^3$ coordinates. Since the equation (2.6) is invariant under unimodular transformation, we may suppose

$$\tilde{\Omega} \subset \{y_1 \ge 0\}.$$

Adding a linear function to $\tilde{u}$, $\tilde{u}_k$, we replace (6.2), (6.3) by

$$\tilde{\mathcal{M}}_0 \subset \{y_1 \ge 0\} \cap \{y_3 \ge -y_1\} \tag{6.6}$$

and $\tilde{\mathcal{M}}_0$ contains two segments

$$\{(0,0,t) \mid 0 \leq t \leq 1\}, \ \{(t,0,-t) \mid 0 \leq t \leq 1\}. \tag{6.7}$$

Let

$$L = \{(y_1, y_2, y_3) \in \tilde{\mathcal{M}}_0 \mid y_1 = y_3 = 0\}.$$

L must be a single point (Case I) or a segment (Case II). In Case II, we may also suppose that 0 is an end point of the segment which is

$$\{(0,t,0) \mid -1 < t < 0\}.$$

Later, we will discuss the two cases separately.

Now we make the rotation

$$z_1 = -y_3, \quad z_2 = y_2, \quad z_3 = y_1$$

such that $\tilde{\mathcal{M}}_0$ can be represented by a convex v near the origin. By convexity, $\tilde{\mathcal{M}}_k$ can also be represented by $z_3 = v^{(k)}(z_1, z_2)$ near p_0, respectively. $v^{(k)}$ is a solution of the equation given in Lemma 4.1 near the origin. As we know that $\tilde{\mathcal{M}}_k$ converges in Hausdorff distance to $\tilde{\mathcal{M}}_0$, in new coordinates, $v^{(k)}$ converges locally uniformly to v. It is clear that

$$\begin{aligned} &v(0) = 0, \quad v \geq 0, \text{ when } -1 \leq z_1 \leq 0 \text{ and} \\ &v \geq z_1, \text{ when } 0 \leq z_1 \leq 1 \end{aligned}$$

and the two line segments

$$\{(t,0,0) \mid -1 \leq t \leq 0\}, \quad \{(t,0,t) \mid 0 \leq t \leq 1\}$$

lie on the graph of v.

As in (4.9), let $\hat{v}^{(k)} = v^{(k)} - \frac{1}{2}z_1$ and $\hat{v} = v - \frac{1}{2}z_1$. In the following computation we omit the hat for simplicity. Then

$$v \geq \frac{1}{2}|z_1| \quad \text{and} \quad v(z_1, 0) = \frac{1}{2}|z_1|. \tag{6.8}$$

Let

$$\tilde{\mathcal{C}} = \{z \mid v(z) = 0\}.$$

Observe that

$$L = \{(z_1, z_2, 0) \mid (z_1, z_2) \in \tilde{\mathcal{C}}\}$$

in z-coordinates.

Case I. In this case, v is strictly convex at $(0,0)$. The strict convexity implies that Dv is bounded on $S_{h,v}(0)$ for small $h > 0$. Hence, by locally uniform convergence,

$Dv^{(k)}$ are uniformly bounded on $S_{\frac{h}{2},v^{(k)}}(0)$. By Lemma 4.3, we have the determinant estimate

$$\det D^2 v^{(k)} \leq C \tag{6.9}$$

near the origin.

For $\delta \leq \frac{h}{2}$, by (6.8), $S_{\delta,v}(0) \subset \{-\frac{\delta}{2} \leq y_1 \leq \frac{\delta}{2}\}$ and $(\pm\frac{\delta}{2}, 0) \in \partial S_{\delta,v}(0)$. In the z_2 direction, we define

$$\kappa_\delta = \sup\{|z_2| \mid (z_1, z_2) \in S_{\delta,v}(0)\}.$$

By comparing the images of $S_{\delta,v}(0)$ under normal mapping of v and the cone with bottom at $\partial S_{\delta,v}(0)$ and top at the origin,

$$|N_v(S_{\delta,v}(0))| \geq C\frac{\delta}{\kappa_\delta}.$$

By the lower semi-continuity of normal mapping,

$$N_v(S_{\delta,v}(0)) \subseteq \liminf_{k\to\infty} N_{v^k}(S_{\delta,v}(0)),$$

then

$$N_v(S_{\delta,v}(0)) = N_v(S_{\delta,v}(0)) \subseteq \liminf_{k\to\infty} N_{v^{(k)}}(S_{\delta,v}(0)).$$

By (6.9),

$$\begin{aligned} |N_v(S(\delta))| &\leq \liminf_{k\to\infty} |N_{v^{(k)}}(S_{\delta,v}(0))| \\ &= \liminf_{k\to\infty} \int_{S_{\delta,v}(0)} \det D^2 v^{(k)}\, dz \\ &\leq C|S_{\delta,v}(0)| \\ &\leq C\delta\kappa_\delta. \end{aligned} \tag{6.10}$$

Hence, $\kappa_\delta \geq C > 0$, where C is independent of δ. Again by the strict convexity, $\kappa_\delta \to 0$ as $\delta \to 0$. The contradiction follows.

Case II. In this case,

$$\tilde{\mathcal{C}} = \{(0, z_2) \mid -1 < z_2 < 0\}.$$

We define the following linear function:

$$l_\epsilon(z) = \delta_\epsilon z_2 + \epsilon$$

and $\omega_\epsilon = \{z \mid v(z) \leq l_\epsilon\}$, where δ_ϵ is chosen such that

$$v(0, \frac{\epsilon}{\delta_\epsilon}) = l(0, \frac{\epsilon}{\delta_\epsilon}) = 2\epsilon,\ v(0, -\frac{\epsilon}{\delta_\epsilon}) = l(0, -\frac{\epsilon}{\delta_\epsilon}) = 0.$$

We can suppose that ϵ is small enough such that ω_ϵ is contained in a small ball near the origin. Hence, $Dv^{(k)}$ is uniformly bounded. By comparing the image of ω_ϵ under normal mapping of v and the cone with bottom at $\partial\omega_\epsilon$ and top at the origin,

$$|N_v(\omega_\epsilon)| \geq C\delta_\epsilon. \tag{6.11}$$

On the other hand, $\omega_\epsilon \subset \{-\epsilon \leq z_1 \leq \epsilon\}$ since $v \geq |z_1|$. By the convexity and the assumption above, $\omega_\epsilon \subset \{-\frac{\epsilon}{\delta_\epsilon} \leq z_2 \leq \frac{\epsilon}{\delta_\epsilon}\}$. Therefore,

$$|\omega_\epsilon| \leq C\frac{\epsilon^2}{\delta_\epsilon}.$$

Furthermore, subtracting all $v^{(k)}$ by l_ϵ, they still satisfy the same equation. By the determinant estimate in Lemma 4.3 and a similar argument as in (6.10),

$$|N_v(\omega_\epsilon \cap \{z \mid \xi_1 \geq 0\})| \leq C\frac{\epsilon^2}{\delta_\epsilon}. \tag{6.12}$$

Combining (6.11) and (6.12),

$$\frac{\epsilon^2}{\delta_\epsilon^2} \geq C.$$

However, according to our construction, $\frac{\epsilon}{\delta_\epsilon}$ goes to 0 as ϵ goes to 0. The contradiction follows. □

Remark 6.2. *The following property has been used in the above proof. Assume that u is a 2-dimensional convex function satisfying*

$$u(0) = 0, \quad u(x) > 0 \text{ for } x \neq 0 \text{ and } u(x_1, 0) \geq C|x_1|. \tag{6.13}$$

Then

$$\frac{|N_u(S_{h,u}(0))|}{|S_{h,u}(0)|} \to \infty \text{ as } h \to 0.$$

In other words, if

$$\det D^2u \leq C$$

and u vanishes on boundary, then u is C^1 in Ω. This property can be extended to high dimension if

$$u(0) = 0, \quad u(x', x_n) \geq C|x_n| \text{ and } u(x', x_n) \geq C|x'|^2, \tag{6.14}$$

where $x' = (x_1, ..., x_{n-1})$.

It is also known that a generalized solution to

$$\det D^2u \geq C$$

in a domain in $\mathbb{R}^2$ must be strictly convex. This result was first proved by Aleksandrov but a simple proof can be found in [TW3].

7. Strict convexity II

In this section, we rule out the Case (b) that all extreme points of $\mathcal{C}$ lie on the boundary $\partial\Omega$.

First, we need a stronger approximation. In the case of the affine Plateau problem, this approximation was obtained by [TW5]. Here, we extend it to our functional $J(u)$.

Theorem 7.1. *Let φ, Ω be as in Theorem 2.6 and u be the maximizer of the functional J in $\overline{S}[\varphi,\Omega]$. Assume that $\partial\Omega$ is lipschitz continuous. Then there exist a sequence of smooth solutions $u_m \in W^{4,p}(\Omega)$ to*

$$U^{ij}w_{ij} = f_m = f + \beta_m \chi_{D_m} \text{ in } \Omega \tag{7.1}$$

such that

$$u_m \longrightarrow u \quad \text{uniformly in } \Omega, \tag{7.2}$$

where $D_m = \{x \in \Omega \mid dist(x, \partial\Omega) < 2^{-m}\}$, χ is the characteristic function, and β_m is a constant. Furthermore, we can choose β_m sufficient large ($\beta_m \to \infty$ as $m \to \infty$) such that for any compact subset $K \subset N_\varphi(\Omega)$,

$$K \subset N_{u_m}(\Omega) \tag{7.3}$$

provided m is sufficient large.

Proof. By subtracting the constant $G(0)$, we assume that $G(0) = 0$ and $G \geq 0$. The proof is divided into four steps.

(i) Let $B = B_R(0)$ be a large ball such that $\Omega \subset B_R$. By assumption, φ is defined in a neighborhood of Ω, so we can extend u to B such that φ is convex in B, $\varphi \in C^{0,1}(\overline{B})$ and ϕ is constant on ∂B. Consider the second boundary value problem with

$$f_{m,j} = \begin{cases} f + \beta_m \chi_{D_m} & \text{in } \Omega, \\ H_j'(u-\varphi) & \text{in } B_R \setminus \Omega, \end{cases}$$

where $H_j(t) = H(4^j t)$ is given by (5.1). By Lemma 5.1, there is a solution $u_{m,j}$ satisfying

$$|u_{m,j} - \varphi| \leq 4^{-j}, \; x \in B_R \setminus \Omega. \tag{7.4}$$

(ii) By the convexity, $u_{m,j}$ sub-converges to a convex function u_m as $j \to \infty$ and $u_m = \varphi$ in $B_R \setminus \Omega$. Note that $u_m \in \overline{S}[\varphi,\Omega]$ when restricted in Ω. By Theorem 5.2,

u_m is the maximizer of the functional

$$J_m(v) = \int_\Omega G(\det \partial^2 v)\, dx - \int_\Omega (f + \beta_m \chi_{D_m}) v\, dx \tag{7.5}$$

in $\overline{S}[\varphi, \Omega]$.

(iii) Since $u_m \in \overline{S}[\varphi, \Omega]$, u_m converges to a convex function u_∞ in $\overline{S}[\varphi, \Omega]$ as $m \to \infty$. We claim that u_∞ is the maximizer u. The proof is as follows.

Define

$$\varphi_* = \sup\{l(x) \mid l \text{ is a tangent plane of } \varphi \text{ at some point in } B_R \setminus \overline{\Omega}\}.$$

Then $\varphi_* \in \overline{S}[\varphi, \Omega]$ and $v \geq \varphi_*$ for any $v \in \overline{S}[\varphi, \Omega]$. We consider the maximizer u. Let

$$\tilde{u}_m = \sup\{l(x) \mid l \text{ is linear, } l \leq u \text{ in } \Omega \text{ and } l \leq \varphi_* \text{ in } D_m\}.$$

Then $\tilde{u}_m \in \overline{S}[\varphi, \Omega]$ and $\tilde{u}_m = \varphi_*$ in D_m. Since u is convex, it is twice differentiable almost everywhere. By the definition of $\tilde{u}_m$, $\tilde{u}_m = u$ at any point where $D^2 u > 0$ when m is sufficiently large. Therefore, we have $\det \partial^2 \tilde{u}_m \to \det \partial^2 u$ a.e.. By the upper semi-continuity of the functional $A(u)$ and Fatou lemma,

$$\lim_{m \to \infty} \int_\Omega G(\det \partial^2 \tilde{u}_m)\, dx = \int_\Omega G(\det \partial^2 u)\, dx.$$

It follows that for a sufficiently small $\epsilon_0 > 0$,

$$J(u) \leq J(\tilde{u}_m) + \epsilon_0 \tag{7.6}$$

provided m is sufficiently large.

On the other hand, we consider the functional J_m. By (ii), u_m is the maximizer of J_m in $\overline{S}[\varphi, \Omega]$, so we have

$$J_m(\tilde{u}_m) \leq J_m(u_m). \tag{7.7}$$

Note that $u_m \geq \varphi_* = \tilde{u}_m$ in D_m. Hence, we obtain

$$\int_{D_m} \beta_m u_m\, dx \geq \int_{D_m} \beta_m \tilde{u}_m\, dx.$$

By the definition of J_m, it follows

$$J(\tilde{u}_m) \leq J(u_m) + \epsilon_0. \tag{7.8}$$

for sufficiently large m.

Finally, by (7.6), (7.8) and the upper semi-continuity of $A(u)$,

$$\begin{aligned} J(u) &\leq J(\tilde{u}_m) + \epsilon_0 \\ &\leq J(u_m) + \epsilon_0 \\ &\leq J(u_\infty) + 2\epsilon_0. \end{aligned}$$

By taking $\epsilon_0 \to 0$, this implies that u_∞ is the maximizer. By the uniqueness of maximizers, $u_\infty = u$.

(iv) It remains to prove (7.3). We claim that for any fixed m,

$$\lim_{\beta_m \to \infty} u_m(x) \leq \varphi_*(x). \tag{7.9}$$

We prove it by contradiction. Suppose that there is $x_0 \in D_m$ such that $u_m(x_0) \geq \varphi_*(x_0) + \epsilon_0$ for some $\epsilon_0 > 0$. Since u_m and φ_* are uniformly Lipschitz continuous, $u_m(x) \geq \varphi_*(x) + \frac{\epsilon_0}{2}$ in a ball $B_{C\epsilon_0}(x_0)$ for some constant C. Let

$$u_{m*} = \sup\{l(x) \mid l \text{ is linear, } l \leq u_m \text{ in } \Omega \text{ and } l \leq \varphi_* \text{ in } D_m\}.$$

Then $u_{m*} \in \overline{S}[\varphi, \Omega]$, and satisfies

$$u_{m*} \leq u_m \text{ in } \Omega,\ u_{m*} = \varphi_* \text{ in } B_{C\epsilon_0}(x_0).$$

Hence,

$$J_m(u_m) - J_m(u_{m*}) = J(u_m) - J(u_{m*}) - \beta_m \int_{D_m} u_m - u_{m*}\, dx$$

becomes negative when β_m is sufficiently large. This is a contradiction to that u_m is a maximizer of J_m. □

Remark 7.2. *If $\varphi \in C^1$, we can restate (7.3) in the theorem as*

$$|D(u_m - \varphi)| \to 0 \quad \textit{uniformly on } \partial\Omega. \tag{7.10}$$

Now we deal with Case (b). By Theorem 7.1, there exists a solution $u_m^{(k)}$ to

$$U^{ij} w_{ij} = f_m, \tag{7.11}$$

where

$$w = G_k'(\det D^2 u), \tag{7.12}$$

such that

$$u_m^{(k)} \longrightarrow u^{(k)},\ m \to \infty.$$

and for any compact set $K \subset D\varphi(\Omega)$,

$$K \subset Du_m^{(k)}(\Omega) \tag{7.13}$$

for large m. Hence, we can choose a sequence $m_k \to \infty$ such that

$$u_k := u_{m_k}^{(k)} \longrightarrow u_0. \tag{7.14}$$

Lemma 7.3. *Assume that Ω and φ are smooth. Then $\mathcal{M}_0$ contains no line segments with both endpoints on $\partial\mathcal{M}_0$.*

Proof. Suppose that L is a line segment in $\mathcal{M}_0$ with both end points on $\partial\mathcal{M}_0$. By subtracting a linear function, we suppose that $u_0 \geq 0$ and l lies in $\{x_3 = 0\}$. By a translation and a dilation of the coordinates, we may further assume that

$$L = \{(0, x_2, 0) \mid -1 \leq x_2 \leq 1\} \tag{7.15}$$

with $(0, \pm 1) \in \partial\Omega$. Note that by Remark 2.2, these transformations do not change the essential properties of equation (2.6).

Since φ is a uniformly convex function in a neighborhood of Ω and $\varphi = u_0$ at $(0, \pm 1)$, L must be transversal to $\partial\Omega$ at $(0, \pm 1)$. Hence, by $u_0 = \varphi$ on $\partial\Omega$ and the smoothness of φ and $\partial\Omega$, we have

$$u_0(x) = \varphi(x) \leq \frac{C}{2} x_1^2, \; x \in \partial\Omega.$$

By the convexity of u_0,

$$u_0(x) \leq \frac{C}{2} x_1^2, \; x \in \Omega. \tag{7.16}$$

Now we consider the Legendre function u_0^* of u_0 in $\Omega^* = D\varphi(\Omega)$, given by

$$u_0^*(y) = \sup\{x \cdot y - u_0(x), x \in \Omega\}, \; y \in \Omega^*.$$

Note that $(0, \pm 1) \in \partial\Omega$. By the uniformly convexity of φ, $0 \notin D\varphi(\partial\Omega)$. Hence, $0 \in \Omega^*$. By (7.15), (7.16) and the smoothness of φ, we have

$$u^*(0, y_2) \geq |y_2|, \tag{7.17}$$

$$u^*(y) \geq \frac{1}{2C} y_1^2. \tag{7.18}$$

On the other hand, by the approximation (7.13), (7.14), the Legendre function of u_k, denoted by u_k^*, is smooth in

$$\Omega_{\epsilon_k}^* = \{y \in \Omega^* \mid dist(y, \partial\Omega^*) > \epsilon_k\}.$$

with $\epsilon_k \to 0$ as $k \to \infty$ and satisfies the equation

$$u^{*ij} {w^*}_{ij} = -f_{m_k}(Du^*) \tag{7.19}$$

in $\Omega_{\epsilon_k}^*$, where

$$w^* = G_k(d^{*-1}) - d^{*-1} G_k'(d^{*-1}). \tag{7.20}$$

By (7.17), (7.18), u_0^* is strictly convex at 0. Then $\{y \mid u_0^* < h\} \subset \Omega^*_{\epsilon_k}$ providing m is sufficiently large. Note that u_k^* converges to u_0^*. By Lemma 3.2, we have the estimate

$$\det D^2 u_k^* \le C$$

near the origin in Ω^*. Note also that in Lemma 3.2, C depends on $\inf f$ but not on $\sup f$. In other words, the large constant β_{m_k} in (7.1) does not affect the bound C. Therefore sending $k \to \infty$, we obtain

$$\det D^2 u_0^* \le C$$

in the sense that the Monge-Ampère measure of u_0^* is an L^∞ function. This is a contradiction with (7.17), (7.18) according to Remark 6.2. □

In conclusion, we have proved that u_0 is strictly convex in Ω in dimension 2. Theorem 1.1 follows from Theorem 5.4.

8. Appendix: Second boundary value problem

In order to construct approximation solutions to the maximizer of $J(u)$, we employ the second boundary value problem for equation (2.6). This section is just a modification of the second boundary problem in [TW2]. We include it here for completeness. Throughout this section, we will denote by d the determinant $\det D^2 u$ for simplicity.

We study the existence of smooth solutions to the following problem.

$$U^{ij} w_{ij} = f(x, u), \text{ in } \Omega, \tag{8.1}$$

$$w = G'(d), \text{ in } \Omega, \tag{8.2}$$

$$w = \psi, \text{ on } \partial\Omega, \tag{8.3}$$

$$u = \varphi, \text{ on } \partial\Omega, \tag{8.4}$$

where Ω is a smooth, uniformly convex domain in $\mathbb{R}^n$, φ, ψ are smooth functions on $\partial\Omega$ with

$$0 < C_0^{-1} \le \psi \le C_0.$$

$f \in L^\infty(\Omega \times R)$ is nondecreasing in u and there is $t_0 \le 0$ such that

$$f(x,t) \le 0, \ t \le t_0.$$

We note that this condition is not needed if u is bounded from below.

By Inverse Function Theorem, $w = G'(d)$ has an inverse function $d = g(w)$. g is an decreasing function which goes to 0 as $w \to \infty$ and goes to ∞ as $w \to 0$. To solve the problem (8.1)-(8.4), we first consider the approximating problem

$$U^{ij}w_{ij} = f, \text{ in } \Omega, \tag{8.5}$$

$$\det D^2u = \eta_k g(w) + (1-\eta_k), \text{ in } \Omega, \tag{8.6}$$

where φ and ψ satisfy (8.3), (8.4) and $\eta_k \in C_0^\infty(\Omega)$ is the cut-off function satisfying $\eta_k = 1$ in $\Omega_k = \{x \in \Omega \mid dist(x, \partial\Omega) > \frac{1}{k}\}$.

Lemma 8.1. *Suppose that $f \in L^\infty$ satisfies the condition above. If (u, w) is the C^2 solution of (8.5), (8.6), there is a constant depending only on $diam(\Omega)$, f, φ, ψ and independent of k, such that*

$$C^{-1} \le w \le C, \ \textit{in } \Omega, \tag{8.7}$$

$$|w(x) - w(x_0)| \le C|x - x_0|, \ \textit{for any } x \in \Omega, x_0 \in \partial\Omega. \tag{8.8}$$

Proof. The proof of the upper bound for w is totally the same as that for affine maximal surface equation in [TW2] by considering the auxiliary function

$$z = \log w + A|x|^2,$$

where $A > 0$ is a constant to be determined later. Suppose that z attains its minimum at the point x_0. If x_0 is a boundary point, then $z(x_0) \ge C$, and hence $w \ge C$. If x_0 lies in the interior of Ω, we have, at x_0,

$$\begin{aligned} 0 = z_i &= \frac{w_i}{w} + 2Ax_i, \\ 0 \ge z_{ij} &= \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + 2A\delta_{ij}. \end{aligned}$$

Then

$$\begin{aligned} 0 &\ge u^{ij}z_{ij} \\ &= \frac{f}{dw} - 4A^2u^{ij}x_ix_j + 2Au^{ii} \\ &\ge \frac{f}{dw} + Ad^{-\frac{1}{n}}. \end{aligned}$$

Note here we choose A small. Therefore,

$$d^{\frac{n-1}{n}}w \le C.$$

combining with the definition of d, w and using the condition $F'(0) = \infty$ in (c), we obtain $w \le C$.

By $w \le C$, we have $\det D^2u \ge C$. Suppose that v is a smooth, uniformly convex function such that $D^2v \ge K > 0$ and $v = \psi$ on $\partial\Omega$. Then, if K is large,

$$U^{ij}v_{ij} \ge KU^{ii} \ge K[\det D^2v]^{\frac{n-1}{n}} \ge CK \ge f,$$

which implies $U^{ij}(v-w)_{ij} \geq 0$. By maximum principle, $v - w \leq 0$. We thus obtain

$$w(x) - w(x_0) \geq -C|x - x_0|, \text{ for any } x \in \Omega, \ x_0 \in \partial\Omega. \tag{8.9}$$

To prove the lower bound of w, let

$$z = \log w + w - \alpha h(u),$$

where $\alpha > 0$ is a constant to be determined later and h is a convex, monotone increasing function such that,

$$h(t) = t, \text{ when } t \geq -t_0 \text{ and } h \geq -t_0 - 1, \text{ when } t \leq -t_0.$$

Assume that z attains its minimum at x_0. If x_0 is near $\partial\Omega$, by (8.9), $z(x_0) \geq -C$. Otherwise, x_0 is away from the boundary. Hence, we have, at x_0,

$$\begin{aligned} 0 &= z_i = \frac{w_i}{w} + w_i - \alpha h'(u)u_i, \\ 0 &\leq z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + w_{ij} - \alpha h''(u)u_i u_j - \alpha h'(u)u_{ij}. \end{aligned}$$

By maximum principle,

$$\begin{aligned} 0 \leq u^{ij} z_{ij} &= \frac{f}{dw} - \frac{u^{ij} w_i w_j}{w^2} + \frac{f}{d} - \alpha h''(u) u^{ij} u_i u_j - \alpha h'(u) n \\ &\leq \frac{f}{dw} + \frac{f}{d} - \alpha h'(u) n. \end{aligned}$$

If $u(x_0) \leq t_0$, $f \leq 0$, which immediately induces a contradiction. Hence, $u(x_0) \geq t_0$, and $h'(u(x_0)) \geq h'(t_0)$. Then choosing α large enough, we obtain $d \leq C$ at x_0 by the assumption (a). Using the relation between w and d, we have $w(x_0) \geq C$. By definition,

$$z = \log w + w - \alpha h(u) \geq z(x_0) \geq -C.$$

This implies $w \geq C$.

Similarly, with the upper bound of the determinant, we can construct a barrier function v from above for w and prove

$$w(x) - w(x_0) \leq C|x - x_0|.$$

In conclusion, the lemma has been proved. □

Proposition 8.2. *There is a solution $u \in C^{2,\alpha}(\overline{\Omega}) \cap W^{4,p}(\Omega)$ to the approximation problem (8.5), (8.6). If furthermore $f \in C^{\alpha}(\overline{\Omega})$, then $u \in C^{4,\alpha}(\overline{\Omega})$.*

Proof. By (8.7), using Caffarelli-Gutierrez's Hölder continuity for linearized Monge-Ampère equation [CG] we have the interior C^{α} estimate for $\det D^2 u$, for some $\alpha \in (0,1)$. Then by Caffarelli's $W^{2,p}$ and $C^{2,\alpha}$ estimates for Monge-Ampère equation

[**?**, JW], we have interior $W^{2,p}$ estimate for u for some $p > 1$ and $C^{2,\alpha}$ estimate when $f \in C^\alpha(\overline{\Omega})$. Then the interior $W^{4,p}$ and $C^{4,\alpha}$ estimates follow from the standard elliptic regularity theory. Note that $\det D^2 u$ is constant near the boundary of Ω, we also have the boundary $W^{4,p}$ and $C^{4,\alpha}$ estimates by [CNS, GT, K]. In conclusion, we have

$$\|u\|_{W^{4,p}(\Omega)} \leq C, \tag{8.10}$$

where C depends on n, p, φ, ψ and f. and

$$\|u\|_{C^{4,\alpha}(\overline{\Omega})} \leq C \tag{8.11}$$

when $f \in C^\alpha(\overline{\Omega})$, where C depends on n, α, φ, ψ and f.

Now we use the degree theory to prove the existence of solutions to the approximating problem (8.5), (8.6).

For any positive $w \in C^{0,1}(\overline{\Omega})$, let $u = u_w$ be the solution of (8.6) with $u = \varphi$ on $\partial\Omega$. Next, let w_t, $t \in [0,1]$, be the solution of

$$U^{ij} w_{ij} = tf \text{ in } \Omega, \;\; w_t = t\psi + (1-t) \text{ on } \partial\Omega. \tag{8.12}$$

Therefore, we have a compact mapping

$$T_t : w \in C^{0,1}(\overline{\Omega}) \longrightarrow w_t \in C^{0,1}(\overline{\Omega}).$$

By estimate (8.10), the degree $deg(T_t, B_R, 0)$ is well defined, where B_R is the set of all functions satisfying $\|w\|_{C^{0,1}(\overline{\Omega})} \leq R$. When $t = 0$, T_0 has a unique fixed point $w = 1$ by (8.12). Hence, $deg(T_0, B_R, 0) = 1$. By degree theory, we have $deg(T_1, B_R, 0) = 1$. Namely, there is a unique solution when $t = 1$. The proposition follows. □

Finally, taking $k \to \infty$, we obtain

Theorem 8.3. *The second boundary problem (8.1)-(8.4) admits a solution $u \in W^{2,p}_{loc} \cap C^{0,1}(\overline{\Omega})(p > 1)$ with $\det D^2 u \in C^0(\overline{\Omega})$. Moreover, if $f \in C^\alpha(\overline{\Omega} \times \mathbb{R})$ $(0 < \alpha < 1)$, then $u \in C^{4,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$.*

Remark 8.4. *The second boundary problem we consider here is for the equation (2.6). By checking the proof, it is easy to see that Theorem 8.3 also holds for Abreu's equation.*

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