Multidimensional Kruskal–Katona theorem*

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Abstract

We present a generalization of a version of the Kruskal–Katona theorem due to Lovász. A shadow of a d-tuple $(S_1, \ldots, S_d) \in \binom{X}{r}^d$ consists of d-tuples $(S'_1, \ldots, S'_d) \in \binom{X}{r-1}^d$ obtained by removing one element from each of S_i. We show that if a family $\mathcal{F} \subset \binom{X}{r}^d$ has size $|F| = \binom{x}{r}^d$ for a real number $x \geq r$, then the shadow of $\mathcal{F}$ has size at least $\binom{x}{r-1}^d$.

Introduction

An r-uniform set family $\mathcal{F}$ is simply a collection of r-element sets. The shadow of $\mathcal{F}$, denoted $\partial\mathcal{F}$, consists of all $(r-1)$-element sets that can be obtained by removing an element from a set in $\mathcal{F}$. If $(X, <)$ is an ordered sets, then $A \subset X$ is colexicographically smaller than $B \subset X$ if the largest element of $(A \cup B) \setminus (A \cap B)$ lies in B.

Kruskal–Katona theorem [Kru63, Kat68] is a classic result in combinatorics that states that $|\partial\mathcal{F}| \geq |\partial F_0|$, where $\mathcal{F}_0$ is the initial segment of length $|\mathcal{F}|$ in colexicographic order on r-tuples of some ordered set. Moreover the equality is achieved only if $\mathcal{F}$ is an initial segment of such a colexicographic order. As the quantitative form of Kruskal–Katona theorem is unwieldy, in applications one usually uses the weaker form due Lovász [Lov79, Ex. 13.31(b)]: if $|\mathcal{F}| = \binom{x}{r}$ for some real number[1] $x \geq r$, then $|\partial F| \geq \binom{x}{r-1}$.

In this paper we present a generalization of Lovász's theorem to multidimensional r-uniform families. A *d-dimensional r-uniform family* is a collection of d-tuples of r-element sets. In other words, if we denote by $\binom{X}{r}$ the family of all r-element subsets of X, then d-dimensional r-uniform family is a subset of $\binom{X}{r}^d$. A *shadow* of such a family $\mathcal{F} \subset \binom{X}{r}^d$ is defined to be

$$\partial\mathcal{F} \stackrel{\text{def}}{=} \{(S_1 \setminus \{x_i\}, \ldots, S_d \setminus \{x_d\}) : (S_1, \ldots, S_d) \in \mathcal{F}, \text{and } x_i \in S_i \text{ for } i = 1, \ldots, d\}.$$

The special case $d = 1$ of the following theorem is Lovász's result.

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[1]For real x and integer r the binomial coefficient $\binom{x}{r}$ is defined by $x(x-1)\cdots(x-r+1)/r!$.

Theorem 1. *Suppose $\mathcal{F} \subset \binom{X}{r}^d$ is a d-dimensional r-uniform family of size*

$$|\mathcal{F}| = \binom{x}{r}^d,$$

where $x \geq r$ is a real number. Then

$$|\partial\mathcal{F}| \geq \binom{x}{r-1}^d.$$

Moreover, the equality holds only if $\mathcal{F}$ is of the form $\binom{Y_1}{r} \times \cdots \times \binom{Y_d}{r}$ for some sets $Y_1, \ldots, Y_d \subset X$.

The rest of the paper contains the proof of this result.

Proof

For simplicity of notation we shall assume that the ground set is $[n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$, with the ordering on it being the standard ordering of the integers. This incurs no loss of generality.

A k-dimensional *section* of a d-dimensional family $\mathcal{F} \subset \binom{X}{r}^d$ is the subfamily of $\mathcal{F}$ obtaining by fixing $d-k$ coordinates. For example, for any $(d-k)$-tuple $S = (S_1, \ldots, S_{d-k}) \in \binom{X}{r}^{d-k}$ the family

$$\mathcal{F}_S \stackrel{\text{def}}{=} \{(S_{d-k+1}, \ldots, S_d) \in \binom{X}{r}^k : (S_1, \ldots, S_d) \in \mathcal{F}\}$$

is a k-section of $\mathcal{F}$. In general, any $d-k$ coordinates might be fixed, not necessarily the first $d-k$.

We say that a family is *monotone* if every 1-dimensional section is an initial segment in the colexicographic order.

Lemma 2 (Proof deferred to p. 5)**.** *For every family $\mathcal{F} \subset \binom{[n]}{r}^d$ there is a monotone family $\mathcal{F}_0 \subset \binom{[n]}{r}^d$ of the same size as $\mathcal{F}$, and such that $|\partial\mathcal{F}_0| \leq |\partial\mathcal{F}|$.*

By the Lemma 2 it suffices to restrict the attention to monotone families. The shadows of monotone families are most easily described using the colexicographic ordering. That will permits us to establish a correspondence between the d-dimensional monotone families and subsets of $\mathbb{N}^d$. Let $\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, \ldots, \}$ be the set of positive integers, and partially order $\mathbb{N}^d$ by

$$(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d) \text{ whenever } x_i \leq y_i \text{ for every } i = 1, \ldots, d. \tag{1}$$

A set $L \subset \mathbb{N}^d$ is said to be *monotone* if whenever $x = (x_1, \ldots, x_d) \in L$, then L contains all the elements smaller than x.

If $S \in \binom{[n]}{r}$ is the i'th in the colexicographic ordering on $\binom{[n]}{r}$, then we put $\operatorname{ind}_r(S) = i$. A tuple $S = (S_1, \dots, S_d) \in \binom{[n]}{r}^d$ is mapped to $\operatorname{ind}_r(S) \stackrel{\text{def}}{=} (\operatorname{ind}_r(S_1), \dots, \operatorname{ind}_r(S_d))$. In this manner every $\mathcal{F} \subset \binom{[n]}{r}^d$ is associate to its image $\operatorname{ind}_r(\mathcal{F}) \subset \mathbb{N}^d$. An *extreme point* of a monotone set $L \subset \mathbb{N}^d$ is a point $x \in L$ such that no point in L is larger than x. The set of extreme points of L will be denoted $\operatorname{extr} L$. *Monotone closure* of a set $L \subset \mathbb{N}^d$ is $\operatorname{mclos}(L) = \{x \in \mathbb{N}^d : x \leq y \text{ for some } y \in L\}$. It is clear that $L = \operatorname{mclos} \operatorname{extr} L$.

For an integer $m \geq 1$ let $KK_r(m)$ be the size of a shadow of the initial segment of length m in colexicographic order of $\binom{[n]}{r}$. The Kruskal–Katona theorem states that if $\mathcal{F} \subset \binom{[n]}{r}$, then $|\partial \mathcal{F}| \geq KK_r(|\mathcal{F}|)$.

Lemma 3 (Proof deferred to p. 6)**.** *Let $\mathcal{F} \subset \binom{[n]}{r}^d$ be a monotone family. Then its shadow $\partial \mathcal{F}$ is also a monotone family, and*

$$\operatorname{extr} \operatorname{ind}_{r-1}(\partial \mathcal{F}) = KK_r(\operatorname{extr} \operatorname{ind}_r(\mathcal{F})).$$

The preceding lemma permits us to forget about shadows of set families, and instead think about images of monotone sets under KK_r. However, as KK_r is quite an erratic function, our next step is to replace it by a smoother function. For an integer $r \geq 2$ put

$$LL_r\left(\binom{x}{r}\right) = \binom{x}{r-1} \qquad \text{if } x \geq r. \tag{2}$$

Since $\binom{x}{r}$ is an increasing function of x for $x \geq r-1$, the function LL_r is well-defined on $[1, \infty)$. We would like to extend LL_r to $[0, 1)$ while maintaining the inequality $LL_r \leq KK_r$. Furthermore, as it will become clear below, it will be essential for LL_r to be increasing, concave and to satisfy

$$x \frac{f'(x)}{f(x)} < y \frac{f'(y)}{f(y)} \qquad \text{when } x > y. \tag{3}$$

Any extension of LL_r to $[0, \infty)$ satisfying these conditions is equally good for us. For example, one permissible extension is

$$LL_r(x) = r\left(x + \frac{1}{\sum_{i=1}^{r} 1/i}(x - x^2)\right) \qquad \text{if } 0 \leq x \leq 1. \tag{4}$$

Lemma 4 (Proof deferred to p. 6)**.** *The function LL_r defined by* (2) *and* (4) *is a continuously differentiable function that is strictly increasing, concave, and satisfies* (3)*.*

Put $\mathbb{R}_+ = [0, \infty)$. Partially order $\mathbb{R}_+^d$ according to (1), and extend the definitions of the terms "monotone" and "extreme point" in the obvious way. We associate to every monotone set $L \subset \mathbb{N}^d$ the set $M \subset \mathbb{N}^d$ given by $M = L + [-1, 0]^d$. Geometrically, M is set obtaining by filling in the square lattice boxes indexed by L. The volume of M is equal to the number of points in L. The set M so obtained is monotone. Since $LL_r(0) = 0$ and $LL_r \leq KK_r$, Lemma 3 implies that if $|\partial \mathcal{F}| \leq X$ for some family $\mathcal{F} \subset \binom{[n]}{r}^d$, then there is a closed monotone set $M \subset \mathbb{R}_+^d$ for which $\operatorname{vol}(LL_r(M)) \leq X$. The Theorem 1 thus follows from the following claim.

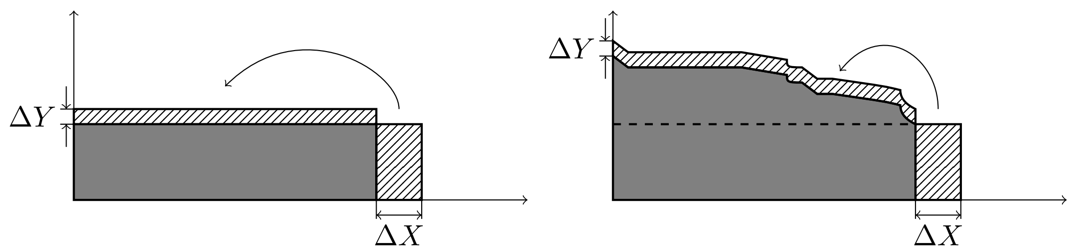

Figure 1: The area-reducing transformation for an elongated rectangle (left), and for a general monotone set (right).

Claim 5. *Suppose $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable, strictly increasing, concave function satisfying* (3) *and $f(0) = 0$, and define $f\colon \mathbb{R}_+^d \to \mathbb{R}_+^d$ by $f(x_1, \ldots, x_d) = (f(x_1), \ldots, f(x_d))$. Then for every closed monotone set $M \subset \mathbb{R}_+^d$ we have*

$$\operatorname{vol}(f(M)) \geq \operatorname{vol}(f(M_0))$$

where $M_0 = [0, \sqrt[d]{\operatorname{vol}(M)}]^d$ is the cube of the same volume as M, and one of whose vertices is at the origin. Furthermore equality holds only if $M = M_0$.

To prove the claim we shall first establish it in the dimension $d = 2$, and use that to deduce the general case. Indeed, assume that the two-dimensional case is known, $d \geq 3$, and M is not a cube. Pick any 2-dimensional coordinate plane P. On each 2-dimensional section of M by a plane parallel to P, replace the section of M by a square of the same area as the area of that section. The operation yields a monotone set, and by the case $d = 2$ of the claim, it reduces the volume of $f(M)$ unless every section of M is a square. Therefore, the only minimizer of $\operatorname{vol}(f(M))$ is the cube $[0, \sqrt[d]{\operatorname{vol}(M)}]^d$.

So assume $d = 2$. To see where the condition (3) comes from consider the case where M is a rectangle, i.e. a set of the form $M = [0, X] \times [0, Y]$, with say $X > Y$. In that case, if we are to move a small amount of mass from the shorter side to the longer one, to obtain a less elongated rectangle $M^* = [0, X - \Delta X] \times [0, Y + \Delta Y]$, then (3) is exactly what is necessary to conclude that $\operatorname{area}(f(M^*)) < \operatorname{area}(f(M))$.

The situation when M is not a rectangle is to our advantage because f is concave and we place the mass farther from the origin than in the case when M is a rectangle. The only complication is that we need to introduce continuous time to avoid technicalities arising from discrete time increments.

Since M is monotone there is a decreasing function $g_\infty\colon \mathbb{R}_+ \to \mathbb{R}_+$ so that $M = \{(x, y) \in \mathbb{R}_+^2 : y \leq g_\infty(x)\}$. Since M is closed, g_∞ is left-continuous. Define $g_t\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$g_t(x) = \begin{cases} g_\infty(x) + \frac{1}{t} \int_{[t,\infty)} g_\infty(y)\, dy & \text{if } x \leq t, \\ 0 & \text{if } x > t. \end{cases}$$

Let $M_t = \{(x, y) \in \mathbb{R}_+^2 : y \le g_t(x)\}$. Then $\text{area}(M_t) = \text{area}(M)$. Differentiating

$$\text{area}(f(M_t)) = \int_{[0,t]} f(g_t(x)) f'(x)\, dx,$$

we obtain

$$\begin{aligned}
\frac{\partial\, \text{area}(f(M_t))}{\partial t} &= f(g_t(t))f'(t) + \int_{[0,t]} f'(g_t(x)) \frac{\partial g_t}{\partial t}(x) f'(x)\, dx \\
&\ge f(g_t(t))f'(t) + f'(g_t(t)) \int_{[0,t]} \frac{\partial g_t}{\partial t}(x) f'(x)\, dx \\
&= f(g_t(t))f'(t) + f'(g_t(t)) \left(-\frac{\partial g_t}{\partial t}(t) f(t) - \int_{[0,t]} f(x) \frac{\partial^2 g_t(x)}{\partial x \partial t}\, dx \right) \\
&= f(g_t(t))f'(t) - f'(g_t(t)) \frac{\partial g_t}{\partial t}(t) f(t),
\end{aligned}$$

where the inequality holds since f is concave, and $(\partial g_t/\partial t)f'$ is negative (see Figure 1 for a geometric illustration of the inequality). Since $\partial g_t/\partial t = -g_t(t)/t$, from (3) it follows that $\text{area}(f(M_t))$ is an increasing function of t as long as $g_t(t) < t$.

Let $T = \sqrt{\text{area}(M)}$. Since $\text{area}(M_t) \ge t g_t(t)$, it follows that $g_t < t$ for every $t > T$. Thus $\text{area}(f(M_T)) \le \text{area}(f(M))$, with equality only if $M \subset [0, T] \times \mathbb{R}_+$. Since $g_T(x) \le g_\infty + \text{area}(M)/T$ it follows that if $M \subset \mathbb{R}_+ \times [0, Y]$, then $M_T \subset [0, T] \times [0, Y + \text{area}(M)/T] = [0, T] \times [0, Y + T]$. Reversing the roles of x and y axes, and applying the argument to M_T, it follows that for every closed monotone set $M \subset \mathbb{R}_+^2$ there is a compact monotone set $M' \subset [0, 2T] \times [0, T]$ for which $\text{area}(f(M')) \le \text{area}(f(M))$ with equality holding only for $M = [0, T]^2$. Since the space of compact monotone subset of $[0, 2T] \times [0, T]$ endowed with Hausdorff distance is a compact space, and $\text{area}(f(\cdot))$ is a continuous function on the space, it follows that $[0, T]^2$ is a unique set minimizing this function. This completes the proof of the Claim 5 in the case $d = 2$.

Deferred lemmas

Proof of Lemma 2. For the duration of this proof define the *weight* of $\mathcal{F} \subset \binom{[n]}{r}^d$ to be $\sum_{S \in \mathcal{F}} \|\text{ind}_r(\mathcal{F})\|_1$, where $\|(m_1, \ldots, m_d)\|_1 = m_1 + \cdots + m_d$. We may assume that $\mathcal{F}$ has smallest weight among families of size $|\mathcal{F}|$ and whose shadow does not exceed $|\partial \mathcal{F}|$.

Suppose some 1-dimensional section of $\mathcal{F}$ is not an initial segment of the colexicographic order. Without loss of generality we may assume that the section is of the form $\mathcal{F}_S$ for some S. Define a compression operator $\Delta \colon 2^{\binom{[n]}{r}} \to 2^{\binom{[n]}{r}}$ which takes $\mathcal{F} \subset \binom{[n]}{r}$ to the initial segment of $\binom{[n]}{r}$ in the colexicographic order. One can write $\mathcal{F}$ as a disjoint union of its 1-dimensional sections as

$$\mathcal{F} = \bigcup_{S \in \binom{[n]}{r}^{d-1}} \{S\} \times F_S.$$

Define

$$\mathcal{F}' = \bigcup_{S \in \binom{[n]}{r}^{d-1}} \{S\} \times \Delta F_S.$$

We claim that $|\partial\mathcal{F}'| \le |\partial\mathcal{F}|$. Indeed, let $S' \in \binom{[n]}{r-1}^{d-1}$ be arbitrary, and consider the section $(\partial F')_{S'}$. The section has at least t elements if and only if there is a $S \in \binom{[n]}{r}^{d-1}$ such that $S' \in \partial S$ and $KK_r(|\mathcal{F}_S|) \ge t$. Hence, if $|(\partial F')_{S'}| \ge t$, then by the classical Kruskal–Katona inequality $|(\partial F)_{S'}| \ge t$. Since the inequality holds for every S', it follows that

$$|\partial\mathcal{F}| = \sum_{S' \in \binom{[n]}{r-1}^{d-1}} (\partial\mathcal{F})_{S'} \ge \sum_{S' \in \binom{[n]}{r-1}^{d-1}} (\partial\mathcal{F}')_{S'} = |\partial\mathcal{F}'|$$

Since the weight of $\mathcal{F}'$ is less than that of $\mathcal{F}$, this contradicts the choice of $\mathcal{F}$. □

Proof of Lemma 3. First we establish that $\partial\mathcal{F}$ is monotone. Suppose $S = (S_1, S_2, \ldots, S_d) \in \partial\mathcal{F}$ and S_1' precedes S_1 in colexicographic order. There is an $\bar{S} = (\bar{S}_1, \ldots, \bar{S}_d) \in \mathcal{F}$ so that $S \in \partial\bar{S}$. Since shadow of an initial segment of colexicographic order is an initial segment of colexicographic order, there is an $\bar{S}_1' \in \binom{[n]}{r}$ so that $\bar{S}_1'$ precedes $\bar{S}_1$ in the order, and $S_1' \in \partial\bar{S}_1'$. Thus $(S_1', S_2, \ldots, S_d) \in \partial(\bar{S}_1', S_2, \ldots, S_d) \subset \partial\mathcal{F}$. This shows that the 1-dimensional section $(\partial\mathcal{F})_{S_2,\ldots,S_d}$ of $\partial\mathcal{F}$ is monotone. Since ordering of coordinates is arbitrary, it follows every 1-dimensional section of $\mathcal{F}$ is monotone, i.e. $\mathcal{F}$ is monotone.

From the definition of KK_r it follows that $\max \operatorname{ind}_{r-1}(\partial\mathcal{F}_0) = KK_r(|F|_0)$ whenever $\mathcal{F}_0$ is the initial segment of $\binom{[n]}{r}$ in the colexicographic order. The second claim of the Lemma is then again a consequence of the fact that an image of an initial segment of colexicographical order on $\binom{[n]}{r}$ is an initial segment on $\binom{[n]}{r-1}$. □

Proof of Lemma 4. It is clear that the function defined by (2) is a continuous monotone increasing function. The concavity of LL_r on $(1, \infty)$ follows from a simple derivative calculation: Indeed, for $x \ge r$

$$\frac{d}{dx} LL_r\Big(\binom{x}{r}\Big) = \frac{d}{dx}\binom{x}{r-1}$$

$$LL_r'\Big(\binom{x}{r}\Big)\binom{x}{r}\left(\frac{1}{x} + \cdots + \frac{1}{x-r+1}\right) = \binom{x}{r-1}\left(\frac{1}{x} + \cdots + \frac{1}{x-r+2}\right)$$

$$1/LL_r'\Big(\binom{x}{r}\Big) = \frac{x-r+1}{r}\,\frac{\frac{1}{x} + \cdots + \frac{1}{x-r+1}}{\frac{1}{x} + \cdots + \frac{1}{x-r+2}}$$

$$1/LL_r'\Big(\binom{x}{r}\Big) = \frac{1}{r}\left(x - r + 1 + \frac{1}{\frac{1}{x} + \cdots + \frac{1}{x-r+2}}\right),$$

from which it is clear that LL_r' is decreasing on $(1, \infty)$. Moreover this expression for LL_r' and

$$LL\Big(\binom{x}{r}\Big)\Big/\binom{x}{r} = \binom{x}{r-1}\Big/\binom{x}{r} = r/(x-r+1)$$

imply that

$$\frac{LL_r\left(\binom{x}{r}\right)}{\binom{x}{r}LL_r'\left(\binom{x}{r}\right)} = 1 + \frac{1}{(x-r+1)\left(\frac{1}{x}+\cdots+\frac{1}{x-r+2}\right)}.$$

Since $(x-r+1)/(x-t)$ is a decreasing function of x for every $t < r-1$, it follows that $\frac{LL_r\left(\binom{x}{r}\right)}{\binom{x}{r}LL_r'\left(\binom{x}{r}\right)}$ is increasing, i.e. LL_r satisfies (3) on $(1,\infty)$.

Since $x - x^2$ is concave, the function given by (4) is concave on $[0,1)$. For brevity of notation put $\epsilon \overset{\text{def}}{=} \sum_{i=1}^{r} 1/i$. Monotonicity of LL_r on $[0,1)$ follows from $\epsilon < 1$. Furthermore, for $x \in [0,1)$ we have

$$x\frac{LL_r'(x)}{LL_r(x)} = x\frac{1+\epsilon(1-2x)}{x+\epsilon(x-x^2)} = 2 - \frac{1+\epsilon}{1+\epsilon(1-x)},$$

from which we see that LL_r satisfies (3) on $[0,1)$. Finally, it is easy to check that at $x=1$ the function $LL_r(x)$ is continuous and the left and right derivatives agree. □

Concluding remarks

For us the original motivation for the study of shadows of d-dimensional families was in their application to convexity spaces, and Eckhoff's conjecture[Buk10]. For that application the Theorem 1 sufficed. However, it would be interesting to find the sharp multidimensional generalization of Kruskal–Katona theorem.

It is worth noting that the argument given in this paper is largely insensitive to the poset structure of 2^X. The only input it uses is the one-dimensional Kruskal–Katona theorem. First, Lemma 2 is a direct consequence of the fact in the Kruskal–Katona theorem the equality is attained only for an initial segment of a certain linear order. Secondly, a weaker quantitative form of the Kruskal–Katona theorem is used to construct in Lemma 4 a continuous function to which Claim 5 applies.

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