Normal forms of matrices over the ring of formal series

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ABSTRACT. Matrices over the ring of formal power series are considered. Normal forms with respect to various sub-groups of the two-sided transformations are constructed. The construction is based on the special property of the action: it induces a filtration by projectors on sub-spaces of polynomial maps.

CONTENTS

1. INTRODUCTION

Let G be a group acting on a space X. Then X fibres into the disjoint union of G-orbits:

$$(1) \qquad X = \bigcup_{x \in X} Gx, \qquad Gx = \{(gx) \in X | g \in G\}$$

Recall that a subset $N \subset X$ is called a *normal form with respect to the action of G* if N intersects all the orbits. An element $z \in N \bigcap Gx$ is called a *normal form of x*. If the intersection is one point set for every x, then the normal form is called *canonical*. A canonical normal form solves the problem of classification of orbits: elements x_1 and x_2 lie in the same orbit if and only if their canonical forms coincide. The are many examples of normal and canonical forms in linear algebra [Gantmacher-book] and in analysis [AGLV-book].

Here we consider normal forms in spaces of matrices over the ring of formal power series. Various groups of invertible matrices act on these spaces. Our considerations are based on the existence of filtration preserved by these natural actions.

1.1. **Matrices of formal series.** Let $\mathbb{K}$ be a field of zero characteristic., not necessarily algebraically closed, char$\mathbb{K} = 0$. Fix some natural m, n, p, denote by

$$(2) \qquad Mat(m, n, p) = \mathrm{Mat}(m \times n, \ \mathbb{K}[[x_1 \ldots x_p]])$$

the space of all $m \times n$ *formal matrices*, i.e. matrices $A(x) = (a_{ij}(x))$ whose entries $a_{ij} \in \mathbb{K}[[x_1 \ldots x_p]]$ are formal series of p variables over $\mathbb{K}$.

Every matrix $A \in Mat(m, n, p)$ can be represented as the formal series $A(x) = \sum\limits_{|I|=0}^{\infty} A_I x^I$.

Here $I = (I_1 \ldots I_p)$ is an integer multi-index,

$$(3) \qquad I_j \geq 0, \qquad |I| = I_1 + \ldots + I_p, \qquad x^I = x_1^{I_1} \ldots x_p^{I_p},$$

and A_I is an $m \times n$ matrix over $\mathbb{K}$.

The space $Mat(m, m, p)$ is a $\mathbb{K}$-algebra. Its element $U(x) = U_0 + \sum_{|I| \geq 1} U_i x^I$ is invertible if and only if $\det U_0 \neq 0$. The subset $GL(m, \mathbb{K}[[x_1 \ldots x_p]])$ of all invertible elements is a group acting on the space $Mat(m, n, p)$ by multiplication from the left. Similarly, the group $GL(n, \mathbb{K}[[x_1 \ldots x_p]])$ acts by multiplication from the right. The direct product

$$(4) \qquad G(m, n, p) = GL(m, \mathbb{K}[[x_1 \ldots x_p]]) \times GL(n, \mathbb{K}[[x_1 \ldots x_p]])$$

acts from the two sides:

$$(5) \qquad (g.A)(x) = U(x)A(x)V^{-1}(x), \qquad g = (U, V).$$

1.2. **Types of equivalence.** Formal matrices $A(x)$ and $B(x)$ are *two-sided equivalent* if they are equivalent by the action of the group $G(m, n, p)$. Similarly, they are *left equivalent* if

$$(6) \qquad B(x) = U(x)A(x), \qquad U \in GL(m, \mathbb{K}[[x_1 \ldots x_p]])$$

and *right equivalent* if

$$(7) \qquad B(x) = A(x)V^{-1}(x), \qquad V \in GL(n, \mathbb{K}[[x_1 \ldots x_p]]).$$

If the matrices are square, $m = n$, then they are *conjugate* if

$$(8) \qquad B(x) = U(x)A(x)U^{-1}(x), \qquad U \in GL(m, \mathbb{K}[[x_1 \ldots x_p]]),$$

and are *congruent* if

$$(9) \qquad B(x) = U(x)A(x)(U(x))^T, \qquad U \in GL(m, \mathbb{K}[[x_1 \ldots x_p]]).$$

Here U^T means the transposition.

The main aim of the paper is the construction of normal forms with respect to the action of various sub-groups $G \subset G(m, n, p)$. We extend here the approach suggested in [Belitskii-1979-1], [Belitskii-1979-2] for locally analytic problems.

1.3. **One variable case.** In the simplest case of a single variable, i.e. $p = 1$, a normal form with respect to the two-sided equivalence can be stated immediately. Set

$$(10) \qquad N = \left\{ \begin{pmatrix} R(x) & 0 \\ 0 & 0 \end{pmatrix} \Big| \ R(x) = x^{k_1} \mathbb{1}_{m_1} \oplus \ldots \oplus x^{k_s} \mathbb{1}_{m_s} \right\}$$

where $\mathbb{1}_r$ is the identity $r \times r$ matrix, and

$$(11) \qquad m_1 + \ldots + m_s \leq \min(m, n), \quad k_s > k_{s-1} > \ldots > k_1.$$

Proposition 1.1. [Birkhoff-1913][Grothendieck-1957] *Every matrix of formal series in one variable is two-sided equivalent to a unique matrix in N.*

One can easily obtain similar normal form statements for other types of equivalences.

1.4. **Results.** Of course, the case $p \geq 2$ is essentially more complicated. Let us present some corollaries of our construction.

Let $\mathbb{K} \subset \mathbb{C}$. Given a polynomial $m \times n$ matrix

$$(12) \qquad A(x) = \sum_{|I| \leq k} A_I x^I, \qquad A_I \in \mathrm{Mat}(m \times n, \mathbb{K}),$$

we introduce the differential operator $A^* \left(\frac{\partial}{\partial x} \right) := \sum_{|I| \leq k} A_I^* \frac{\partial^{|I|}}{\partial x^I}$, where $A_I^* := \overline{A_I^T}$ is the conjugation. It acts on the space $Mat(m, n, p)$ by

$$(13) \qquad A^* \left(\frac{\partial}{\partial x} \right) b(x) = \sum_{|I| \leq k} A_I^* \frac{\partial^{|I|} b(x)}{\partial x^I}.$$

Theorem 1.2. *Let $A^{(k)}(x) = \sum\limits_{|I|=k} A_I x^I$, $A_I \in \mathrm{Mat}(m \times n, \mathbb{K})$ be a homogeneous polynomial matrix of degree k, and let $A(x) = A^{(k)}(x) + \big(\text{terms of orders} \geq k+1\big)$. Then the matrix A is two-sided equivalent to a matrix $B(x) = A^{(k)}(x) + b(x)$ satisfying both of the relations*

$$(14) \qquad A^{(k)*}\left(\frac{\partial}{\partial x}\right) b(x) = 0, \qquad (A^{(k)T})^*\left(\frac{\partial}{\partial x}\right) b^T(x) = 0$$

Hence, the set $N \subset Mat(m,n,p)$, consisting of matrices B satisfying equation (14), is a normal form with respect to the action of the group $G(m,n,p)$. It is not canonical: matrices $B, \tilde{B} \in N$ may be two-sided equivalent.

This theorem follows from general Theorem 5.4 proved in §5.3. It presents normal forms with respect to various sub-groups $G \subset G(m,n,p)$, including the left and the right equivalence, the conjugacy and so on. These normal forms turn out to be canonical with respect to the "unipotent parts" of the respective groups consisting of transformations with identity linear part.

Two trivial cases:
- For $k = 0$, the matrix $A^{(0)}(x) = A_0$ is a constant matrix, and we obtain:

Corollary 1.3. *Every matrix $A(x) = A_0 + (\text{terms of orders} > 0)$ is two-sided equivalent to a matrix $B(x) = A_0 + b(x)$ satisfying $A_0^* b(x) = 0$ and $b(x) A_0^* = 0$.*

In commutative algebra the analogous statement is known as the reduction to the minimal resolution [Eisenbud-book]. Similar statements for other types of equivalence are obtained in §5.4.
- For one variable case, i.e. $p = 1$, Theorem 1.2 implies Proposition 1.1

Let now $m = n$ and $A^{(k)}(x) = \mathrm{diag}\big(l_1(x) \dots l_m(x)\big)$ a diagonal matrix with homogeneous polynomials $l_i(x)$ of degree k. If $b(x) = (b_{ij}(x))$, then

$$(15)$$
$$A^{(k)*}\left(\frac{\partial}{\partial x}\right) b(x) = \left\{ l_i^*\left(\frac{\partial}{\partial x}\right) b_{ij}(x) \right\}_{i,j=1}^m \quad \text{and} \quad (A^{(k)T})^*\left(\frac{\partial}{\partial x}\right) b^T(x) = \left\{ l_j^*\left(\frac{\partial}{\partial x}\right) b_{ij}(x) \right\}_{i,j=1}^m$$

Here

$$(16) \qquad l_i^*\left(\frac{\partial}{\partial x}\right) = \sum_{|I|=k} \overline{(l_i)}_I \frac{\partial^{|I|}}{\partial x^I}.$$

Then Theorem 1.2 leads to

Corollary 1.4. *Every matrix $A(x) = A^{(k)}(x) + (\text{terms of orders} > k)$ with $A^{(k)}(x) = \mathrm{diag}(l_1(x) \dots l_m(x))$ is two-sided equivalent to a matrix $B(x) = A^{(k)}(x) + (b_{ij}(x))$ satisfying*

$$(17) \qquad l_i^*\left(\frac{\partial}{\partial x}\right) b_{ij}(x) = 0, \qquad l_j^*\left(\frac{\partial}{\partial x}\right) b_{ij}(x) = 0$$

1.5. **Matrices with formal/locally convergent/rational entries.** In many applications the matrix functions are considered as local objects, defined near the origin. (For example in singularity theory or local algebraic geometry.) Accordingly one has various notions of locality:
- the neighborhoods of Zariski topology (corresponding to the rational functions, regular at the origin),
- the neighborhoods of the classical topology (locally convergent functions)
- formal neighborhoods (formal series).

While the formal neighborhoods are better for theoretical considerations (e.g. no issues of convergence), in practice one works usually with the locally convergent series or rational

functions. Correspondingly one has various comparison questions of the three cases. We discuss this in §A.

2. Filtration in the spaces of formal matrices.

Given $A \in Mat(m, n, p)$ and $j \in \mathbb{N}$, consider the j'th jet of the matrix, $\pi_j A := \sum_{|I| \le j} A_I x^I$. Then $\{\pi_j\}$ is an increasing system of projectors, i.e. $\pi_i \pi_j = \pi_j \pi_i = \pi_i$, $i \le j$. The image $Mat_j(m, n, p)$ of the projector π_j consists of all polynomial matrices of degree $\le j$.

The *homogeneous summand* $A^{(j)}(x) := \sum_{|I|=j} A_I x^I$ can be represented in the form

$$(18) \qquad\qquad A^{(j)} = (\pi_j - \pi_{j-1})A, \pi_{-1} = 0.$$

Correspondingly, the projector to the j'th homogeneous component is $\pi^{(j)} := \pi_j - \pi_{j-1}$. Its image, $Mat^{(j)}(m, n, p)$, consists of all the homogeneous matrices of degree j. Thus

$$(19) \qquad\qquad \pi^{(j)}\pi^{(i)} = \begin{cases} \mathbb{0}, & i \ne j \\ \pi^{(i)}, & j = i \end{cases}$$

The system $\{\pi_j\}$ generates a "sequential topology": a sequence $\{A_k\}$ *converges* to A if

$$(20) \qquad \forall j \in \mathbb{N} \text{ there exists } k_0(j) \in \mathbb{N} \text{ such that } \forall k \ge k_0(j): \ \pi_j A_k = \pi_j A,$$

The "convergence Cauchy criterion" states that a sequence $\{A_k\}$ converges if and only if it stabilizes:

$$(21) \qquad\qquad \pi_j A_k = \pi_j A_{k'}, k, k' \ge k_0(j), j = 0, 1, \ldots.$$

Then the matrix A with the jets

$$(22) \qquad\qquad \pi_j A = \pi_j A_k, \ k \ge k_0(j)$$

is the limit.

A subset $S \subset Mat(m, n, p)$ is *closed* if the limit of every converging sequence $\{A_k\} \subset S$ belongs to S.

Example 2.1. • A one-point set is a closed subset in $Mat(m, n, p)$, and is not open: for every $A \in Mat(m, n, p)$ the complement $Mat(m, n, p) \setminus \{A\}$ is not closed.

• Let $P \in Mat(m, n, p)$ and let $s \ge 0$ be a fixed integer. The subset

$$(23) \qquad\qquad \{A \in Mat(m, n, p) \mid \pi_s A = \pi_s P\}$$

is simultaneously closed and open.

Consider now the direct product $Mat(m, m, p) \times Mat(n, n, p)$. Each $A \in Mat(m, m, p) \times Mat(n, n, p)$ determines the two-sided action, so we will write

$$(24) \qquad\qquad A = (A_l, A_r), \ A_l \in Mat(m, m, p), \ A_r \in Mat(n, n, p).$$

The j-jet projectors to the subspaces of polynomials, $\{\pi_j\}$, act on $Mat(m, m, p) \times Mat(n, n, p)$ as previously. In what follows we denote the projectors by the same letters: $\pi_j A = (\pi_j A_l, \pi_j A_r)$.

3. Lie groups of formal transformations.

The group $GL(m, \mathbb{K}[[x_1 \ldots x_p]])$ is a "countably dimensional Lie group", possessing an exponential map, as we explain now.

3.1. The exponential and logarithmic maps.
Let $\lambda \in Mat(m, n, p)$ be constant term free, i.e. $\pi_0 \lambda = 0$. Then the sequence

$$(25) \qquad U_k(x) = \sum_{i=0}^{k} \frac{\lambda^i(x)}{i!}$$

converges to a matrix $\exp \lambda \in GL(m, \mathbb{K}[[x_1 \ldots x_p]])$ such that $\pi_0 \exp \lambda =$id.

The image of the map exp coincides with the sub-group of all elements $U \in GL(m, \mathbb{K}[[x_1 \ldots x_p]])$ such that $\pi_0 U = \mathbb{1}$. Indeed, let $U(x) = \mathbb{1} + u(x)$ with $\pi_0 u = \mathbb{0}$. Then the sequence

$$(26) \qquad \lambda_k(x) = \sum_{i=1}^{k} \frac{(-1)^{i-1}}{i} u^i(x)$$

converges to the matrix $\log U \in Mat(m, m, p)$ with $\pi_0 \log U = \mathbb{0}$, satisfying $\exp(\log U) = U$.

3.2. Lie groups and their algebras.
Given an $m \times m$ matrix λ with $\pi_0 \lambda = \mathbb{0}$, the map

$$(27) \qquad \Phi : \mathbb{K} \to GL(m, \mathbb{K}[[x_1 \ldots x_p]])$$

defined by $\Phi(t) = \exp(t\lambda)$ is a *flow*, i.e. satisfies the relation $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$. Its image is a one-parametric sub-group. In this sense $GL(m, \mathbb{K}[[x_1 \ldots x_p]])$ is a Lie group. The Lie algebra of this group is

$$(28) \qquad \mathcal{L}ie := \{\lambda \in Mat(m, m, p) : \pi_0 \lambda = \mathbb{0}\}$$

with the usual commutator $[\lambda_1, \lambda_2] := \lambda_1 \lambda_2 - \lambda_2 \lambda_1$.

Similarly, the group $G(m, n, p) = GL(m, \mathbb{K}[[x_1 \ldots x_p]]) \times GL(n, \mathbb{K}[[x_1 \ldots x_p]])$ is Lie with the Lie algebra

$$(29) \qquad \mathcal{L}ie = \{\lambda = (\lambda_l, \lambda_r) \in Mat(m, m, p) \times Mat(n, n, p), \quad \pi_0 \lambda_l = \mathbb{0}, \ \pi_0 \lambda_r = \mathbb{0}\}$$

and the commutator $[\nu, \lambda] := ([\nu_l, \lambda_l], [\nu_r, \lambda_r])$.

More generally, let $G \subset G(m, n, p)$ be a sub-group. Define its subgroup of matrices that are "close to being idempotent":

$$(30) \qquad G^0 = \{g = (U, V) \in G | \ \pi_0 U = \mathbb{1}, \ \pi_0 V = \mathbb{1}\}.$$

Then G^0 is contained in the image of the exponential map.

Definition 3.1. *A closed sub-group $G \subset G(m, n, p)$ is called Lie if for every $g = \exp \lambda \in G^0$ the one-parametric group $\{\exp(t\lambda), \ t \in \mathbb{K}\}$ is contained in G entirely.*

In the sequential topology the sub-group G^0 is an open neighborhood of the identity in the Lie group G.

Note, if G is a Lie group then so is G^0 and their Lie algebras coincide.

Given a Lie group G we denote by $\mathcal{L}ie(G)$ its Lie algebra. If $\lambda = (\lambda_l, \lambda_r) \in \mathcal{L}ie(G)$, then $g = \exp \lambda \in G^0$ and $g.A = \exp \lambda_l A \exp(-\lambda_r)$.

Example 3.2. Here are some commonly used Lie groups and their algebras.
• We denote the "biggest" group $G(m, n, p)$ by G_{lr}. Matrices equivalent with respect to this group are called *two-sided equivalent*. Obviously,

$$(31) \qquad \mathcal{L}ie(G_{lr}) = \{\lambda \in Mat(m, m, p) \times Mat(n, n, p) : \pi_0 \lambda_l = \mathbb{0} = \pi \lambda_r\}.$$

• The group G_l of the left transformations $g = (U, \mathbb{1})$ is Lie too. We call G_l-equivalent matrices *left equivalent.* The corresponding Lie algebra is

$$(32) \qquad \mathcal{L}ie(G_l) = \{\lambda : \lambda_r = \mathbb{0}, \ \pi_0 \lambda_l = \mathbb{0}\}$$

• The group G_r of the right transformations $g = (\mathbb{1}, V)$ is Lie, and its Lie algebra is

$$(33) \qquad \mathcal{L}ie(G_r) = \{\lambda : \lambda_l = \mathbb{0}, \ \pi_0 \lambda_r = \mathbb{0}\}.$$

• Matrices equivalent with respect to the diagonal subgroup

$$(34) \qquad G_c = \{g = (U, U^{-1})\} \subset G(m, m, p) \times G(m, m, p)$$

are called *conjugate.* Obviously, $\mathcal{L}ie(G_c) = \{\nu : \lambda_l = -\lambda_r, \ \pi_0 \lambda_l = \mathbb{0}\}$.

• The group $G_T = \{g = (U, U^T)\} \subset G(m, m, p) \times G(m, m, p)$ is Lie too, and

$$(35) \qquad \mathcal{L}ie(G_T) = \{\lambda : \lambda_r = \lambda_l^T, \pi_0 \lambda_l = \mathbb{0}\}.$$

G_T-equivalent matrices are called *congruent.*

The action of any such group $G \subset G(m, n, p)$ on $Mat(m, n, p)$ is consistent with the projectors:

$$(36) \qquad \pi_j(g.A) = \pi_j g.\pi_j A, \ j = 0, 1, \dots$$

If $g \in G^0$, then, in addition, the implication

$$(37) \qquad \pi_{j-1} A = \pi_{j-1} B \ \Rightarrow \ \pi_j(g.A - g.B) = \pi_j(A - B)$$

holds. These properties allow us to adjust jets of a given formal matrix A successively.

3.3. **Linearization of the Lie groups actions.** First we study the *stabilizer* group of a given matrix, $\text{St}(A) := \{g| \ g.A = A\} \subset G^0(m, n, p)$.

Proposition 3.3. *The stabilizer is a Lie group.*

Proof. Let $g = \exp \nu \in \text{St}(A)$. Consider the function

$$(38) \qquad \begin{aligned} f : \mathbb{K} &\to Mat(m, n, p) \\ f(t) &= \exp(t\nu).A - A \end{aligned}$$

Since $\text{St}(A)$ is a sub-group, $f(n) = 0$ for $n = 0, \pm 1, \dots$. Hence, the functions

$$(39) \qquad f_j(t) = \pi_j f(t)$$

vanish on an infinite set of points. Being polynomials, they are zero identically. Hence, $\exp t\nu \in \text{St}(A)$ for all $t \in \mathbb{K}$. ∎

Similarly, the sub-groups of matrices stabilizing specific jets:

$$(40) \qquad \text{St}_j(A) := \{g \in G^0 \mid \pi_j g.A = \pi_j A\}, j = 0, 1, \dots$$

are Lie too.

Definition 3.4. *Given $\nu \in \mathcal{L}ie(G_{lr})$, define the linear map $S_A'(0)\nu := f'(0) = \frac{d}{dt}(\exp t\nu.A - A)|_{t=0}$:*

$$(41) \qquad \begin{aligned} S_A'(0) &: \text{Lie}(G_{lr}) \to Mat(m, n, p) \\ \nu = (\nu_l, \nu_r) &\mapsto \nu A = \nu_l A - A\nu_r \end{aligned}$$

It is called the linearization of the action of G_{lr}.

Lemma 3.5. *1. An element $\exp \nu$ belongs to $\text{St}_j(A)$ if and only if $\nu \in \text{Ker}\left(\pi_j S_A'(0)\right)$, i.e. $\pi_j(\nu A) = \mathbb{0}$.*
2. An element $\exp \nu$ belongs to $\text{St}(A)$ if and only if $\nu \in \text{Ker}S_A'(0)$, i.e. $\nu A = \mathbb{0}$.
3. If $\exp \nu \in \text{St}_{j-1}(A)$, then

$$(42) \qquad \pi_j \exp \nu.A = \pi_j A + \pi_j(S_A'(0)\nu)$$

Proof. 1. If $\exp \nu \in St_j(A)$ then $\pi_j(\exp t\nu.A - A) = \mathbb{0}$ for all $t \in \mathbb{K}$. Thus $\pi_j S_A'(0)\nu = \mathbb{0}$.
Conversely, let the latter equality hold. Then $\pi_j(\nu^k A) = \mathbb{0}$, i.e. $\pi_j \nu_l^k A = \pi_j A \nu_r^k$, for any k. Hence, $\pi_j \exp \nu_l.A = \pi_j A \exp \nu_r$, meaning that $\exp \nu \in \mathrm{St}_j(A)$.

2. If $\exp \nu \in St(A)$ then $\exp \nu \in \mathrm{St}_j(A)$, hence $\pi_j S_A'(0)\nu = 0$ for all $j = 0, 1, \ldots$.
Conversely, let $S_A'(0)\nu = 0$. Then $\exp \nu \in \mathrm{St}_j(A)$ for all j, implying $\exp \nu \in St(A)$.

3. Let $\exp \nu \in \mathrm{St}_{j-1}(A)$. Then $\exp t\nu \in \mathrm{St}_{j-1}(A)$ for all $t \in \mathbb{K}$. It follows from equation (37) that

(43) $$\pi_j(\exp t\nu.\exp s\nu.A - \exp t\nu.A) = \pi_j(\exp s\nu.A - A)$$

for all $t, s \in \mathbb{K}$. Hence,
(44)
$$f_j(t+s) = \pi_j(\exp(t+s)\nu.A - A) = \pi_j(\exp t\nu.\exp \nu s.A - \exp t\nu.A) + \pi_j(\exp t\nu.A - A) = f_j(t) + f_j(s),$$

meaning that f_j is an additive function. Being a polynomial (and as $char(\mathbb{K}) = 0$) it is linear:

(45) $$f_j(t) = f_j'(0)t = \pi_j S_A'(0)t\nu$$

∎

4. Determinacy by jets.

4.1. Jet-by-jet equivalence.
Let $G \subset G(m, n, p)$ be a sub-group. Formal matrices $A(x)$ and $B(x)$ are called *jet-by-jet G-equivalent* if there is a sequence $\{g_j\} \subset G$ such that

(46) $$\pi_j g_j.A = \pi_j B, \ j = 0, 1, \ldots.$$

A subgroup $G \subset G(m, n, p)$ is called *countably algebraic* over $\mathbb{K}$, if it is determined by a finite or infinite system of polynomial equations in the matrix coefficients of monomials. More precisely, this means:

$g = (U, V) \in G$, with $U(x) = \sum_I U_I x^I$ and $V(x) = \sum_I V_I x^I$, if and only if

(47) $$P_k(\{U_I\}, \{V_I\}) = 0, |I| \le N_k, \ k = 1, 2, \ldots$$

for some polynomials P_k over $\mathbb{K}$ and numbers N_k.

Obviously, the groups $G_l, G_r, G_{lr}, G_c, G_T$ are countably algebraic.

Theorem 4.1. *Let the field $\mathbb{K}$ be algebraically closed, and let G be a countably algebraic group. Then jet-by-jet G-equivalence implies G-equivalence.*

Proof. Since G is countably algebraic, the condition (46) is an infinite system of polynomial equations with respect to coefficients of the transformations g_j. By Lang's Theorem there is a common solution g satisfying all equations. ∎

Let us remind the *Lang Theorem* [Lang-1952].

Theorem 4.2. *Consider an infinite polynomial system*

(48) $$h_j(a_1 \ldots a_{m_j}) = 0, \ j = 0, 1, \ldots, \qquad m_j \to \infty$$

over a field $\mathbb{K}$. Assume that for every $k = 0, 1, 2, \ldots$ the finite sub-system

(49) $$h_j(a_1 \ldots a_{m_j}) = 0, \ j = 0, 1, \ldots, k$$

is solvable. If $\mathbb{K}$ is algebraically closed, then the total initial system has a solution $\overline{a} = (a_1, a_2, \ldots)$.

If $\mathbb{K}$ is not closed the statement fails.

Example 4.3. Let $\mathbb{K} = \mathbb{R}$. Consider the system of equations:

$$(50) \qquad a_j^2 = a_1 - j, \; j = 1, 2, \ldots.$$

Given an integer $k \geq 0$, the sequence $a_1 = k$, $a_j = \pm\sqrt{k - j}$ is a solution of the system for $j \leq k$. However, the infinite system (50) has no real solution $a_1, a_2, \ldots$. On the other hand, for every $a_1 \in \mathbb{C}$ the sequence

$$(51) \qquad a_j = \pm\sqrt{a_1 - j}, \quad j = 2, 3, \ldots$$

is a complex-valued solution of (50).

However, at least for some groups the statement of the last theorem is valid over an arbitrary field.

Proposition 4.4. *Let G be one of the groups G_{lr}, G_l, G_r, G_c. Then the jet-by-jet G-equivalence implies G-equivalence.*

Proof. Let the matrices $A(x)$ and $B(x)$ be jet-by-jet G-equivalent over $\mathbb{K}$. Let $\mathbb{K} \subset \bar{\mathbb{K}}$ be the algebraic closure. Then, by Theorem 4.1, the matrices are G-equivalent over $\bar{\mathbb{K}}$, i.e. $U(x)A(x) = B(x)V(x)$ with $U(0), V(0)$ non-degenerate.

Let $\{w_\alpha\}_\alpha$ be a Hamel basis of $\bar{\mathbb{K}}$ as a vector space over $\mathbb{K}$, i.e. a maximal set of $\mathbb{K}$-linearly independent elements, cf. [Rudin-book, pg.53]. So, any element of $\bar{\mathbb{K}}$ is presentable as $\sum a_\alpha w_\alpha$, for $a_\alpha \in \mathbb{K}$ and the sum is *finite*. Thus any series $f \in \bar{\mathbb{K}}[[x]]$ decomposes:

$$(52) \qquad f = \sum_{j=0}^{\infty} \sum_{deg(I)=j} \sum_{\alpha} a_{I,\alpha} w_\alpha x^I$$

Here $\sum_{j=0}^{\infty} \sum_{deg(I)=j} a_{I,\alpha} x^I \in \mathbb{K}[[x]]$. Note that for each fixed I the inner sum $\sum_{\alpha} a_{I,\alpha} w_\alpha x^I$ is finite, hence there is no problem of convergence.

Similarly decomposes every matrix with entries in $\bar{\mathbb{K}}[[x]]$. Let $\{U_\alpha(x)\}$ and $\{V_\alpha(x)\}$ be the projections of $U(x), V(x)$ onto the "$w_\alpha \mathbb{K}[[x]]$" subspaces, i.e. matrices with entries in $\mathbb{K}[[x]]$. Hence from $U(x)A(x) = B(x)V(x)$ one has: $U_\alpha(x)A(x) = B(x)V_\alpha(x)$ for any α.

We claim that there exists a sequence of numbers $\{\lambda_\alpha \in \mathbb{K}\}$ such that $\sum \lambda_\alpha U_\alpha(0)$ and $\sum \lambda_\alpha V_\alpha(0)$ are non-degenerate matrices. Then $\left(\sum \lambda_\alpha U_\alpha(x)\right) A(x) = B(x) \left(\sum \lambda_\alpha V_\alpha(0)\right)$ proves the statement.

Indeed, consider the polynomials $\det(\sum y_\alpha U_\alpha(0))$ and $\det(\sum y_\alpha V_\alpha(0))$ where $\{y_\alpha\}$ are independent variables. (As previously there is a finite number of variables.) As the matrices $U(0) = \sum w_\alpha U_\alpha(0)$ and $V(0) = \sum w_\alpha V_\alpha(0)$ are non-degenerate these polynomial are not identically zero. Thus they are not identically zero for some value $y_1 = \lambda_1 \in \mathbb{K}$. Fix this value then there exists $y_2 = \lambda_2 \in \mathbb{K}$ such that $\det\left(\sum y_\alpha U_\alpha(0)\right)|_{\substack{y_1=\lambda_1 \\ y_2=\lambda_2}} \neq 0$ and $\det\left(\sum y_\alpha V_\alpha(0)\right)|_{\substack{y_1=\lambda_1 \\ y_2=\lambda_2}} \neq 0$. Continue by induction to build the needed (finite) sequence $\{\lambda_\alpha \in \mathbb{K}\}$. $\blacksquare$

Obviously, G_T-equivalence of real matrices over $\mathbb{C}$ does not imply their congruence over $\mathbb{R}$. Nevertheless, the statement of Theorem 4.1 is true over an arbitrary field at least for the unipotent part G^0 of a Lie group G.

Theorem 4.5. *Let G be a Lie group over an arbitrary $\mathbb{K}$. Then the jet-by-jet G^0-equivalence implies G^0-equivalence.*

Proof. Let a matrix B be jet-by-jet G^0-equivalent to A. Then the sets

$$(53) \qquad M_k = \{\nu \in \mathcal{L}ie(G) \mid \pi_k \exp \nu.A = \pi_k B\}$$

are non-empty and decrease: $M_{k+1} \subset M_k$.

Step 1. We prove that the sequence of jets stabilizes:

$$(54) \qquad \pi_j M_{k+1} = \pi_j M_k, \ k \geq k_0(j), \ j = 0, 1, \ldots.$$

This is immediate if $\mathbb{K}$ is algebraically closed. Indeed, for any finite j the subset $\pi_j M_k \subset \pi_j Mat(n, m, p)$ is an algebraic subvariety of finite dimension. Hence the decreasing sequence $\pi_j M_{k+1} \subset \pi_j M_k$ necessarily stabilizes. Namely, for any j: $\pi_j M_{k+1} = \pi_j M_k$, for $k \geq k_0(j)$.

For an arbitrary field we prove as follows. Fix a sequence $\{\nu_k \in M_k\}_{k=0,1,\ldots}$. Then for any $\nu \in M_k$ have: $\exp(-\nu_k) \exp \nu \in St_k(A)$, i.e. there exists $\tau \in \mathrm{Ker} \pi_k S'_A(0)$ such that $\exp \nu = \exp \nu_k \exp \tau$. And conversely, if $\exp \tau \in St_k(A)$ then $\exp \nu_k \exp \tau \in M_k$. This defines the set-theoretic bijection:

$$(55) \qquad \begin{aligned} \nu : \ &\mathrm{Ker} \pi_k S'_A(0) \to M_k \\ &\tau \mapsto \exp(\nu_k) \exp(\tau) \end{aligned}$$

Similarly the map $\nu_j : \pi_j\big(\mathrm{Ker} \pi_k S'_A(0)\big) \to \pi_j(M_k)$ is bijection too.

Now we have a decreasing sequence of vector spaces:

$$(56) \qquad \pi_j(\mathrm{Lie}) \supset .. \supset \pi_j\Big(Ker \pi_k S'_A(0)\Big) \supset \pi_j\Big(Ker \pi_{k+1} S'_A(0)\Big) \supset ...$$

They are subspaces of a finite dimensional space, hence the sequence stabilizes:

$$(57) \qquad \cap_k \pi_j\Big(Ker \pi_k S'_A(0)\Big) = \pi_j\Big(Ker \pi_{k_0(j)} S'_A(0)\Big)$$

By the bijection above we get $\cap_k \pi_j M_k = \pi_j M_{k_0(j)}$.

Step 2. Denote

$$(58) \qquad S_j = \pi_j M_k, \ k \geq N(j).$$

We can assume the sequence $N(j)$ is increasing. Then for $i \leq j$:

$$(59) \qquad \pi_i S_j = \pi_i \pi_j M_k = \pi_i M_k = S_i, \ k \geq N(j).$$

i.e. the projection $S_j \to S_i$ is surjective. Hence, we can choose successively elements $\nu_0 = 0$, $\nu_1 \in S_1$, $\nu_2 \in S_2$, with $\pi_1 \nu_2 = \nu_1$, then $\nu_3 \in S_3$ with $\pi_2 \nu_3 = \nu_2$ and so on. The sequence $\{\nu_j\}$ converges. Its limit ν satisfies

$$(60) \qquad \pi_k \exp \nu . A = \pi_k B, \ k = 0, 1, \ldots.$$

Therefore, A and B are G^0-equivalent. $\blacksquare$

4.2. **Finite determinacy.** Now we discuss when a finite jet determines the G-equivalence class of the matrix.

Let us recall, that a matrix $A(x)$ is called *k-determined* with respect to a group G if every matrix B whose k-jet $\pi_k(B)$ equals $\pi_k A$ is G-equivalent to A. The minimal such k is called the *order of determinacy* with respect to the group G. A matrix is called *finitely determined* if it is k-determined with $k < \infty$. Otherwise the matrix is called *infinitely determined*.

Proposition 4.6. *1. The order of determinacy (finite or infinite, with respect to any group) is invariant with respect to G_{lr} action.*
2. If a matrix is k-determined with respect to a group G, then it is G-equivalent to a matrix whose entries are polynomials of degrees at most k.

Proof. 1. Suppose $A(x)$ is k-determined with respect to G. Let $B(x)_{\geq k}$ be a matrix with $\pi_{k-1}(B(x)_{\geq k}) = \mathbb{0}$. So the matrix $A(x) + B(x)_{\geq k}$ is G-equivalent to $A(x)$. Then for any $g \in G_{lr}$ the matrix $gA(x)$ is k-determined too:

$$(61) \qquad gA(x) + B(x)_{\geq k} = g\big(A(x) + g^{-1}B(x)_{\geq k}\big) \overset{G}{\sim} gA(x)$$

because $\pi_{k-1}\big(g^{-1}B(x)_{\geq k}\big) = \pi_{k-1}g^{-1}\pi_{k-1}\big(B(x)_{\geq k}\big) = \mathbb{0}$.
2. Immediately. ∎

Example 4.7. Many matrices are not G_{lr} equivalent to polynomial matrices. Consider a 1×1 matrix $A = \{y - xf(x)\}$, where $f(x)$ is a locally analytic but not rational function. For example $f(x) = \exp(x)$. Then the curve $\{y - xf(x) = 0\} \subset \mathbb{C}^2$ is locally analytic but not algebraic. Hence any equivalent matrix cannot be a polynomial (as it must define the same non-algebraic curve).

Theorem 4.8. *Let G be a Lie group over a field $\mathbb{K}$. For $A \in Mat(m, n, p)$ let $S'_A(0)$ be the map as defined in 3.4.*
1. Assume that $Mat^{(j)}(m, n, p) \subset \pi_j S'_A(0)(\mathcal{L}ie(G)) = \pi_j\big(\mathcal{L}ie(G)A\big)$ for $j \geq k + 1$. Then the matrix A is k-determined with respect to G^0 and, as a consequence also with respect to G.
2. Conversely, if A is k-determined with respect to G^0 then $Mat^{(j)}(m, n, p) \subset \pi_j S'_A(0)(\mathcal{L}ie(G))$ for $j \geq k + 1$.

Proof. 1. Suppose the condition holds, let $B(x) = A(x) + P(x)$ and $\pi_k P(x) = 0$. We should show that B is G^0-equivalent to A.
Let $P^{(k+1)} = \pi_{k+1}S'_A(0)\nu_1$, for $\nu_1 \in \mathcal{L}ie(G)$. Set $g_1 = \exp\nu_1$. Then

$$(62) \qquad \pi_{k+1}(g_1.A - B) = \pi_{k+1}(g_1 A - A - P^{(k+1)}) = 0.$$

Hence, $g_1^{-1}B = A + P_1$ with $\pi_{k+1}P_1 = 0$. Further, let $P_1^{(k+2)} = \pi_{k+2}S'_A(0)\nu_2$ for $\nu_2 \in \mathcal{L}ie(G)$. Then $g_2^{-1}g_1^{-1}B = A + P_2$ with $\pi_{k+2}P_2 = 0$.
In general, for every $j = k+1, k+2\ldots$, there is a transformation $h_j = g_j^{-1}g_{j-1}^{-1}\ldots g_1^{-1} \in G^0$ such that $\pi_j h_j B = \pi_j A$. This means that A and B are jet-by-jet G^0-equivalent. By Theorem 4.2 they are G^0-equivalent.

2. Conversely, let A be k-determined with respect to G^0. Then every matrix $B(x) = A(x) + P^{(j)}(x)$, for $j \geq k + 1$ is G^0-equivalent to A. Hence,

$$(63) \qquad \exp\nu.A = A + P^{(j)}(x), \ \nu \in \mathcal{L}ie(G).$$

Since $\pi_{j-1}(A + P^{(j)}) = \pi_{j-1}A$, the element $g = \exp\nu$ lies in the stabilizer $St_{j-1}(A)$. By Lemma 3.5

$$(64) \qquad P^{(j)} = \pi_j(\exp\nu.A - A) = \pi_j S'_A(0)\nu.$$

∎

Corollary 4.9. *For a given A, suppose the equation $P = S'_A(0)\nu$ has a solution $\nu \in \mathcal{L}ie(G)$ for every formal matrix P with $\pi_k P = 0$. Then A is k-determined with respect to G^0.*

Indeed, if $P \in S'_A(0)(\mathcal{L}ie(G))$ for every matrix P with $\pi_k P = 0$, then the condition of the theorem is satisfied.

Example 4.10. Let $A = $ const be a constant $m \times n$ matrix. It is 0-determined with respect to the group G_l^0 if and only if it is invertible from the left, i.e. $\text{Ker}A = \{0\}$, or $\text{rank}A = n$. Otherwise it is not finitely determined. Similarly, A is 0-determined with respect to the group G_r^0 if and only if it is invertible from the right, i.e. $\text{rank}A = m$. The matrix A is 0-determined

with respect to two-sided transformations if and only if $\text{rank}A = \min(m, n)$. This means that the linear transformation given by A is either surjective or injective. Indeed, assume that the equation

$$(65) \qquad \lambda_l A + A\lambda_r = P$$

has a solution with an $m \times m$ matrix λ_l and an $n \times n$ matrix λ_r. Assume $Ax = 0$, $x \neq 0$. Then the equation takes on the form $A\lambda_r x = Px$. Since the vector Px is arbitrary, $\text{rank}A = m$.

However, a constant matrix is not finitely determined with respect to conjugacy. Indeed, the equation

$$(66) \qquad \lambda A - A\lambda = P$$

is solvable only for P with $\text{trace}(P) = 0$.

5. Normal forms

5.1. **Construction of the normal form.** In this section we give a constructive description of the normal form. Or, in elementary terms: given $A \in Mat(m, n, p)$ and a group $G \subset G(m, n, p)$, how to reduce A modulo the orbit GA.

As $Mat(m, n, p)$ is a vector space graded by the total degree, i.e. $Mat(m, n, p) = \oplus_j Mat(m, n, p)^{(j)}$, it is natural to apply the jet-by-jet reduction. Namely, at the j'th step we adjust the j'th jet, preserving the (j-1)'st jet.

Define the stabilizer

$$(67) \qquad St_{j-1}(A) := \{g \in G | \pi_{j-1}(gA) = \pi_{j-1}(A)\} = \{\nu \in \mathcal{L}ie(G) | \pi_{j-1}(\nu A) = 0\}$$

Here the last equality is due to Lemma 3.5.

The subgroup $St_{j-1}(A) \subset G$ defines the orbit $St_{j-1}(A)A$. Consider its j'th jet:

$$(68) \qquad \pi_j\Big(St_{j-1}(A)A\Big) = \{\pi_j(gA) | \pi_{j-1}(gA) = \pi_{j-1}(A)\} = \{\pi_j(A) + \pi_j(\nu A) | \pi_{j-1}(\nu A) = 0\}$$

Again the second equality is due to Lemma 3.5.

This defines the vector space

$$(69) \qquad Mat(m, n, p)^{(j)} \supset V^{(j)}(A) := \{\pi_j(\nu A) | \nu \in \mathcal{L}ie(G), \ \pi_{j-1}(\nu A) = 0\}$$

Note that $V^{(j)}(A) = V^{(j)}(\pi_{j-1}A)$. Let $W^{(j)}(A)$ be a complementary subspace, i.e. $V^{(j)}(A) \oplus W^{(j)}(A) = Mat(m, n, p)^{(j)}$. Define

$$(70) \qquad N(G) = \{B \in Mat(m, n, p) : B^{(j)} \in W^{(j)}(B), j = 0, 1, \ldots\}$$

Theorem 5.1. *1. The set $N(G)$ is a normal form with respect to the action of G on $Mat(m, n, p)$. 2. It is a canonical form with respect to G^0.*

Proof. 1. Given a matrix $A(x)$, we construct inductively $g \in G^0$ and $B \in Mat(m, n, p)$, such that $B = gA \in N(G)$. Set $\pi_0 B = A^{(0)}$. Suppose we have built $g_{i-1} \in G^0$ such that the jet $B_{j-1} := \pi_{j-1}(B) = \pi_{j-1}(g_{j-1}A)$ is in the normal form, i.e. $B_{j-1}^{(i)} \in W^{(i)}(B_{i-1})$, for $i \leq j - 1$.

Let $\pi_j(g_{j-1}A) = B_{j-1} + v_j + w_j$, where $v_j \in V^{(j)}(g_{j-1}A)$ and $w_j \in W^{(j)}(g_{j-1}A)$. By construction $v_j = \pi_j(\nu_j A)$ for some ν_j. Hence

$$(71) \qquad \pi_j(\exp(-\nu_j)g_{j-1}A) = \pi_j((1 - \nu_j)g_{j-1}A) = B_{j-1} + w_j =: B_j$$

Thus define $g_j := \exp(-\nu_j)g_{j-1}$ and we get that $B_j = \pi_j(\exp(-\nu_j)g_{j-1}A)$ is in the normal form.

As a result we construct a sequence $\{B_j\}$ converging to $B \in N(G)$. The matrix A is jet-by-jet equivalent to B and, by Theorem 4.5 it is G^0-equivalent to the normal form.

2. To check uniqueness, let the matrices $B, \tilde{B} \in N(G)$ be G^0-equivalent, i.e. $\tilde{B} = g.B$ for $g = \exp \nu \in G^0$. Then $\pi_0 \tilde{B} = \pi_0 B$. Assume the equality $\pi_{j-1}\tilde{B} = \pi_{j-1}B$ is proved. Then $\pi_j(\tilde{B} - B) \in Mat^{(j)}(m, n, p)$ and

$$(72) \qquad \pi_{j-1}(g.B - B) = \pi_{j-1}(\tilde{B} - B) = 0.$$

Hence, $g \in \mathrm{St}_{j-1}(B)$, and $\nu \in \mathrm{Ker}\pi_{j-1}S'_B(0)$. Further, by Lemma 3.1

$$(73) \qquad \pi_j(\tilde{B} - B) = \pi_j(g.B - B) = \pi_j S'_B(0)\nu \in V_B^{(j)}.$$

Since $\tilde{B}, B$ are in the normal form, the inclusion $\pi_j(\tilde{B} - B) \in W^{(j)}(\pi_{j-1}B)$ holds. Hence $\pi_j \tilde{B} = \pi_j B$ for any j. Hence, $\tilde{B} = B$. ∎

Theorem 5.1 together with Theorem 4.8 imply:

Corollary 5.2. *If a matrix is k-determined then its normal form is a matrix of polynomials of degrees $\leq k$.*

Indeed, the complementary spaces $W^{(j)}(A)$ will be zero for $j \geq k$.

5.2. The inner product and the differential operators.
Let now $\mathbb{K} \subset \mathbb{C}$. Then we can apply the Euclidian structure to choose the complements $W^{(j)}(B)$.

Let $A(x) = \sum_{|I| \leq j} A_I x^I$ and $B(x) = \sum_{|I| \leq j} B_I x^I$ be two polynomial matrices. We introduce the inner product

$$(74) \qquad \langle A, B \rangle = \sum_{|I| \leq j}(A_I, B_I)I!, \qquad I! = I_1! \ldots I_P!$$

where $(A_I, B_I) = \mathrm{trace}B_I^* A_I$. In the space of the j'th jets of matrices, $Mat_j(m, m, p) \times Mat_j(n, n, p)$, we introduce the similar inner product

$$(75) \qquad \langle \nu, \mu \rangle = \langle \nu_l, \mu_l \rangle + \langle \nu_r, \mu_r \rangle$$

for $\nu = (\nu_l, \nu_r)$, $\mu = (\mu_l, \mu_r)$. Then the projectors π_j are self-adjoint: $\langle \pi_j \nu, \mu \rangle = \langle \nu, \pi_j \mu \rangle$.

Given a formal matrix $B(x) = \sum_I B_I x^I$, we introduce the formal differential operators

$$(76) \qquad B^* \left(\frac{\partial}{\partial x} \right) := \sum_I B_I^* \frac{\partial^{|I|}}{\partial x^I}, \qquad (B^*)^T \left(\frac{\partial}{\partial x} \right) := \sum_I (B_I^*)^T \frac{\partial^{|I|}}{\partial x^I}.$$

where $B_I^* := \overline{B_I^T}$.

The first acts on the space of all polynomial $m \times n$ matrices, while the second on the space of all polynomial $n \times m$ matrices.

The action of the differential operator D_B on a polynomial matrix $P(x)$ is defined as

$$(77) \qquad \begin{aligned} & D_B P := \nu \in Mat(m, m, p) \times Mat(n, n, p) \\ & \nu_l(x) = \left((B^*)^T \left(\frac{\partial}{\partial x} \right) P^T(x) \right)^T, \qquad \nu_r(x) = -B^* \left(\frac{\partial}{\partial x} \right) P(x). \end{aligned}$$

5.3. The normal form based on an inner product.
Let G be a Lie group and $\mathcal{L}ie(G)$ its Lie algebra. For each j there is the natural inclusion: $\mathcal{L}ie^{(j)}(G) \subset Mat^{(j)}(m, m, p) \times Mat^{(j)}(n, n, p)$. Using the inner product we can define the orthogonal complement of $\mathcal{L}ie^{(j)}(G)$ and the orthogonal projection onto $\mathcal{L}ie^{(j)}(G)$. Hence we have the collection of orthogonal projectors:

$$(78) \qquad \delta_j : Mat^{(j)}(m, m, p) \times Mat^{(j)}(n, n, p) \to \mathcal{L}ie^{(j)}(G)$$

Example 5.3. For the Lie groups introduced above the projectors δ_j have a very simple form:

(79)
$$\begin{array}{l}
\delta_j\nu = (\mathbb{1} - \pi_0)\nu \text{ for } G = G_{lr}; \\
\delta_j\nu = (\mathbb{1} - \pi_0)(\nu_l, 0) \text{ for } G = G_l; \\
\delta_j\nu = (\mathbb{1} - \pi_0)(0, \nu_r) \text{ for } G = G_r; \\
\delta_j\nu = (\mathbb{1} - \pi_0)\left(\frac{\nu_l+\nu_r}{2}, \frac{\nu_l+\nu_r}{2}\right) \text{ for } G = G_c; \\
\delta_j\nu = (\mathbb{1} - \pi_0)\left(\frac{\nu_l-\nu_r^T}{2}, \frac{\nu_r-\nu_l^T}{2}\right) \text{ for } G = G_T.
\end{array}$$

In the following we denote the projectors just by δ, assuming that they act on the corresponding subspaces.

Theorem 5.4. *Every formal matrix $A(x)$ is G^0-equivalent to a unique matrix $B(x)$ whose homogeneous summands $B^{(j)}$ satisfy the equation*

(80)
$$\delta(D_B B^{(j)})(x) = \delta(D_B \pi_{j-1} f_j)(x), \quad j = 0, 1, \ldots$$

with some formal matrices f_j.

Hence, the subset $N(G) \subset Mat(m, n, p)$ of formal matrices B satisfying the condition (80) is a normal form with respect to G and a canonical one with respect to G^0.

Proof. Recall from Theorem 5.1 that for each j, having built $\pi_{j-1}(B)$, we should fix a complement to the vector space $V^{(j)}(B) = \{\pi_j(\nu B)| \ \nu \in \mathcal{L}ie(G), \ \pi_{j-1}(\nu B) = 0\}$. Recall also that $V^{(j)}(B)$ depends on the $(j-1)$'st jet of B only, i.e. $V^{(j)}(B) = V^{(j)}(\pi_{j-1}B)$. Set

(81)
$$Mat^{(j)}(m, n, p) \supset W^{(j)}(B) := (V^{(j)}(B))^{\perp} = \left(\pi_j S_B'(0)(\mathrm{Ker}\pi_{j-1}S_B'(0))\right)^{\perp}$$

where $\perp$ means the orthogonal complement. It suffices to show that the matrix B lies in the set $N(G)$, i.e. $\forall j \in \mathbb{N}: B^{(j)} \in W^{(j)}(B)$, if and only if equation (80) holds.

Consider the linear map

(82)
$$\begin{array}{c}
Mat(m, m, p) \times Mat(n, n, p) \overset{\mathbb{B}}{\to} Mat(m, n, p) \\
\nu = (\nu_l, \nu_r) \mapsto \mathbb{B}\nu = \nu_l B - B\nu_r
\end{array}$$

Then, for $\nu \in \mathcal{L}ie(G)$ have $\pi_j S_B'(0)\nu = \pi_j \mathbb{B}\delta\nu$.

We claim that $B^{(j)} \in W^{(j)}(B)$ iff

(83)
$$(\pi_j\mathbb{B}\delta)^* B^{(j)} \in Im(\pi_{j-1}\mathbb{B}\delta)^*$$

Indeed, suppose (83) holds. Then $(\pi_j\mathbb{B}\delta)^* B^{(j)} \in (\mathrm{Ker}\pi_{j-1}\mathbb{B}\delta)^{\perp}$, implying

(84)
$$B^{(j)} \in (\pi_j\mathbb{B}\delta)(\mathrm{Ker}\pi_{j-1}\mathbb{B}\delta)^{\perp} = W^{(j)}(B)$$

Conversely, suppose (84) is valid. Then

(85)
$$< B^{(j)}, \pi_j\mathbb{B}\delta\nu >= 0, \nu \in \mathrm{Ker}\pi_{j-1}\mathbb{B}\delta.$$

Therefore

(86)
$$(\pi_j\mathbb{B}\delta)^* B^{(j)} \in (\mathrm{Ker}\pi_{j-1}\mathbb{B}\delta)^{\perp} = Im(\pi_{j-1}\mathbb{B}\delta)^*,$$

i.e. (83) holds.

In order to finish the proof it remains to compute the conjugate maps $(\pi_j\mathbb{B}\delta)^*$ and $(\pi_{j-1}\mathbb{B}\delta)^*$. Note that $\pi_j\mathbb{B}\delta\nu = \pi_j\delta(\nu_l)B - \pi_j B\delta(\nu_r)$ for $\nu = (\nu_l, \nu_r)$. We claim that

(87)
$$(\pi_j\mathbb{B}\delta)^* P = \delta\nu = \left(\delta(B^{*T}P^T)^T, \delta B^* P\right)$$

To check this we compute $\langle Q, (\pi_j \mathbb{B}\delta)^* P\rangle$ for $Q = \sum Q_I x^I$.

(88)
$$\langle Q, (\pi_j \mathbb{B}\delta)^* P\rangle = \langle \pi_j \mathbb{B}\delta Q, P\rangle = \langle \sum (Q_l)_J B_I x^{I+J} - \sum B_I (Q_r)_J x^{I+J}, \pi_j(P)\rangle =$$
$$= \langle Q_l, B^*(\tfrac{\partial}{\partial x})\pi_j P\rangle - \langle Q_r, \left(B^*(\tfrac{\partial}{\partial x})^T \pi_j P^T\right)^T\rangle = \langle Q, (\mathbb{B}\delta)^* \pi_j P\rangle$$

As a result,

(89)
$$(\pi_j \mathbb{B}\delta)^* P = \delta D_B \pi_j, \ \ (\pi_{j-1}\mathbb{B}\delta)^* P = \delta D_B \pi_{j-1},$$

proving the statement.∎

5.4. Corollaries and examples.
Let $A^{(k)}(x)$ be a homogeneous matrix of degree k and let $A(x) = A^{(k)}(x) + $ terms of orders $\geq k+1$.

Then the biggest powers in equation (80) for the above listed groups are equal to $k - j$, and arise only in the left side. If $G = G_c$ then the same is true for the matrix

(90)
$$A(x) = \lambda \mathbb{1} + A^{(k)}(x)\ldots, \ \lambda \in \mathbb{K}.$$

Taking into account the structure of the projectors δ, we arrive at

Corollary 5.5. *Let $A(x) = A^{(k)}(x) + (\text{terms of orders} \geq k+1)$. Then*
1. The matrix A is left equivalent to a matrix $A^{(k)}(x) + b(x)$ satisfying

(91)
$$(A^{(k)^*})^T \left(\frac{\partial}{\partial x}\right) b^T(x) = 0$$

2. The matrix A is right equivalent to a matrix $A^{(k)}(x) + b(x)$ satisfying

(92)
$$A^{(k)^*} \left(\frac{\partial}{\partial x}\right) b(x) = 0$$

3. The matrix A is two-sided equivalent to a matrix $A^{(k)}(x) + b(x)$ satisfying both of the relations (91) and (92).
4. If $m = n$, then the matrix A is congruent to a matrix $A^{(k)}(x) + b(x)$ satisfying

(93)
$$(A^{(k)^*})^T \left(\frac{\partial}{\partial x}\right) b^T(x) + A^{(k)^*} \left(\frac{\partial}{\partial x}\right) b(x) = 0.$$

5. If $m = n$, then every matrix $\lambda \mathbb{1} + A$, $\lambda \in \mathbb{K}$ is conjugate to a matrix $\lambda \mathbb{1} + P + b$ satisfying

(94)
$$\left((A^{(k)^*})^T \left(\frac{\partial}{\partial x}\right) b^T(x)\right)^T = A^{(k)^*} \left(\frac{\partial}{\partial x}\right) b(x).$$

The statement 3 proves Theorem 1.2 from the Introduction.

The relation similar to Corollary 1.3 for the group G_{lr} from the Introduction takes on the form

(95)
$$\begin{array}{l}\Lambda^* b(x) = 0 \text{ for the group } G_l \\ b(x)\Lambda^* = 0 \text{ for the group } G_r \\ \Lambda^* b(x) = b(x)\Lambda^* \text{ for the conjugacy} \\ (\Lambda^*)^T b^T(x) + b(x)\Lambda^* = 0 \text{ for the congruence}\end{array}$$

In addition to Corollary 1.2 from Introduction we obtain

Corollary 5.6. *Let $m = n$ and $A^{(k)}(x) = \text{diag}\big(l_1(x)\ldots l_m(x)\big)$ with homogeneous polynomials of degree k. Then*
1. Every matrix $A(x) = A^{(k)}(x) + (\text{terms of orders} \geq k+1)$ is left equivalent to a matrix $A^{(k)}(x) + b(x)$ satisfying $l_j^* \left(\frac{\partial}{\partial x}\right) b_{ij}(x) = 0$.
2. The matrix $A(x)$ is right equivalent to a matrix $A^{(k)}(x) + b(x)$ satisfying $l_i^* \left(\frac{\partial}{\partial x}\right) b_{ij}(x) = 0$.
3. If $m = n$, then the matrix $A(x)$ is congruent to a matrix $A^{(k)}(x) + b(x)$ satisfying $l_I^* \left(\frac{\partial}{\partial x}\right) b_{ij}(x) +$

$l_j^* \left(\frac{\partial}{\partial x} \right) b_{ij}(x) = 0.$

4. If $m = n$, then every matrix $\lambda \mathbb{1} + A(x)$, $\lambda \in \mathbb{K}$ is conjugate to a matrix $\lambda \mathbb{1} + A^{(k)}(x) + (b_{ij}(x))$ satisfying $\left(l_i^* \left(\frac{\partial}{\partial x} \right) - l_j^* \left(\frac{\partial}{\partial x} \right) \right) b_{ij}(x) = 0.$

Example 5.7. Let $m = n$ and suppose the polynomials l_i are linear, i.e. $k = 1$ and $l_i(x) = \sum_{s=1}^{p} \alpha_{is} x_s$, for $i = 1, \dots, m$. Then Corollary 1.2 for the group G_{lr} gives

$$(96) \qquad \sum_{s=1}^{p} \overline{\alpha}_{is} \frac{\partial b_{ij}(x)}{\partial x_s} = 0, \qquad \sum_{s=1}^{p} \overline{\alpha}_{is} \frac{\partial b_{ij}(x)}{\partial x_s} = 0.$$

In particular, if $p = 2$ and the functionals l_s are pair wise non-colinear, then $b_{ij} = 0$ for $i \neq j$. Hence, the normal form is a diagonal matrix.

Example 5.8. Let $m = n$ and $A^{(k)}(x) = l_1(x) \mathbb{1} + l_2(x) J$. Here l_1, l_2 are homogeneous polynomials of degree k, while

$$(97) \qquad J = \begin{pmatrix} 0 & 1 & . & 0 \\ . & . & . & . \\ 0 & . & 0 & 1 \\ 0 & . & . & 0 \end{pmatrix}$$

is the nilpotent Jordan block. Then the relations of Corollary 1.4 take on the form

$$(98) \qquad l_1^* \left(\frac{\partial}{\partial x} \right) b_{1j}(x) = 0, \quad l_1^* \left(\frac{\partial}{\partial x} \right) b_{ij}(x) + l_2^* \left(\frac{\partial}{\partial x} \right) b_{i-1j} = 0$$

for $j = 1, 2, \dots, m$ and $i \geq 2$. Besides,

$$(99) \qquad l_1^* \left(\frac{\partial}{\partial x} \right) b_{im}(x) = 0, \quad l_1^* \left(\frac{\partial}{\partial x} \right) b_{ij}(x) + l_2^* \left(\frac{\partial}{\partial x} \right) b_{ij+1} = 0$$

for $i = 1, 2, \dots, m$ and $j \leq m - 1$. It follows that

$$(100) \qquad l_1^* \left(\frac{\partial}{\partial x} \right) b_{ij}(x) = 0, \qquad l_2^* \left(\frac{\partial}{\partial x} \right) b_{ij}(x) = 0$$

for $j \geq i + 1$. Hence, every matrix

$$(101) \qquad A(x) = l_1(x) \mathbb{1} + l_2(x) J + \text{terms of orders } \geq k + 1$$

is two-sided equivalent to a matrix $B(x) = l_1(x) \mathbb{1} + l_2(x) J + (b_{ij}(x))$ where b_{ij} satisfy (100).

Example 5.9. In particular, consider the case of two variables, $p = 2$, and assume $l_i(x)$ are linear and linearly independent. Then equation (100) implies $b_{ij} = 0$ for $j \geq i + 1$. Hence, in this case the matrix A is two-sided equivalent to a matrix of the form

$$(102) \qquad l_1(x) \mathbb{1} + l_2(x) J + \begin{pmatrix} * & 0 & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 \\ . & . & . & \dots & . \\ * & * & * & \dots & * \end{pmatrix}$$

By the linear change of coordinates we can make $l_1(x) = x_1$ and $l_2(x) = x_2$, then the entries of the low triangular matrix satisfy:

$$(103) \qquad \partial_x b_{ij} + \partial_y b_{i-1,j} = 0, \qquad \partial_x b_{ij} + \partial_y b_{i,j+1} = 0$$

This implies $\partial_y b_{i-1,j} = \partial_y b_{i,j+1}$. Combining with the last equations it gives $\partial_x b_{i+1,j} = \partial_x b_{i,j-1}$. As b_{ij} is of order at least two we get: $b_{i,j} = b_{i+1,j+1} =: \gamma_{i-j}$. Finally we obtain that A is two-sided equivalent to

$$(104) \qquad x \mathbb{1} + y J + \begin{pmatrix} \gamma_0 & 0 & 0 & \dots & 0 \\ \gamma_1 & \gamma_0 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ \gamma_{n-1} & \gamma_{n-2} & . & \dots & \gamma_0 \end{pmatrix}, \qquad \partial_x \gamma_i + \partial_y \gamma_{i-1} = 0$$

Appendix A. Dependence on the choice of the base ring

Let the field $\mathbb{K}$ have a non-trivial valuation, so that the convergence of a series is defined. One can consider matrices whose entries are
• formal series- $\mathbb{K}[[x_1, .., x_p]]$ or
• locally converging series- $\mathbb{K}\{x_1, .., x_p\}$ or
• rational functions that are regular at the origin- $\mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}$, (i.e. fractions of polynomials, whose denominators do not vanish at the origin). Here $\mathfrak{m} \subset \mathbb{K}[x_1, .., x_p]$ is the maximal ideal.

Correspondingly we have the notions of formal/locally converging/rational-G-equivalences, formal/locally converging/rational-G-determinacy etc. Two questions occur naturally:

(injectivity) Let A_1, A_2 be matrices with rational/locally converging entries. Suppose they are formally-G-equivalent. Are they rationally/locally converging-G equivalent?

(surjectivity) Which formal matrices are G-equivalent to matrices with locally converging/rational entries?

In the discussion below many things are well known in Commutative Algebra (e.g. [Eisenbud-book], [Yoshino-book]), but they seem to be less known in other areas.

Recall that even for a formal series $f \in \mathbb{K}[[x_1, .., x_p]]$ one can speak about the corresponding hypersurface $\{f = 0\} \subset (\mathbb{K}^p, 0)$, its singularities, local irreducibility etc. Though the series might not converge off the origin.

A.1. **Injectivity.** We consider here only those subgroups $G \subset G(m, n, p)$ that are defined by polynomial equations in matrix entries. More precisely $(U, V) \in G$ iff the entries of the matrices (U, V) satisfy a finite collection of polynomial equations with constant coefficients. (For example $G_l, G_r, G_{lr}, G_c, G_T$ are such subgroups.)

Theorem A.1. *1. Let A_1, A_2 be matrices with locally converging entries, let $G \subset G(m, n, p)$. If A_1, A_2 are formally-G-equivalent, then they are locally convergent-G-equivalent.*
2. Let A_1, A_2 be matrices with rational entries. Let G be a subgroup of $G(m, n, p)$ defined by equations linear in matrix entries. For example G is one of G_l, G_r, G_{lr}, G_c. If A_1, A_2 are formally-G-equivalent, then they are rationally-G-equivalent.

Proof. 1. If A_1, A_2 are formally-G-equivalent then $UA_1 = A_2V$, where U, V are invertible at the origin, and satisfy some additional algebraic conditions (depending on G). So, if the entries of A_1, A_2 are locally converging, then by Artin approximation theorem [Artin68] the matrices U, V can be chosen with locally converging entries.

2. Suppose A_1, A_2 have rational entries and G satisfies the assumption. Then the conditions on (U, V) are:
• *linear* equations for the entries of U, V. These arise from $UA_1 = A_2V$ and from the defining conditions of the group (e.g. for G_l: $V = \mathbb{1}$, for G_c: $U = V$). The coefficients in these equations are constants or the entries of A_1, A_2, i.e. rational functions.
• the *non-degeneracy* condition: U and V are invertible at the origin.

Note that some of the linear equations above can be non-homogeneous, e.g. $V = \mathbb{1}$ for G_l. So the set of all the pairs of matrices satisfying these linear conditions is an affine space, but in general not a linear one. Hence we 'homogenize' the equations by introducing new variables. For example for G_l replace $V = \mathbb{1}$ by $V = \tilde{V}$, with $\tilde{V}$ the matrix whose entries are additional variables, at the end we will impose the additional constraint $\tilde{V} = \mathbb{1}$. Note that the equations of these additional constraints are linear, with constant coefficients.

Let $E \subset Mat\big(m \times m, \mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}\big) \oplus Mat\big(n \times n, \mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}\big) \oplus ..$ be the set of all the tuples $(U, V, additional\ variables)$, whose entries are rational functions regular at the origin, such that the tuple satisfy the homogenized linear conditions as above. So E is a vector space. In fact E is a module over $\mathbb{K}[x_1, .., x_p]_{\mathfrak{m}}$ with the action $f(U, V, ..) := (fU, fV, ..)$.

Let $E_{formal} \subset Mat\big(m \times m, \mathbb{K}[[x_1, .., x_p]]\big) \oplus Mat\big(n \times n, \mathbb{K}[[x_1, .., x_p]]\big) \oplus ..$ be the set of all the tuples $(U, V, ..)$, with formal entries, satisfying the homogenized linear conditions as above. So E_{formal} is a module over $\mathbb{K}[[x_1, .., x_p]]$. We claim that $E_{formal} = \mathbb{K}[[x_1, .., x_p]]E$, i.e. one can choose a basis of E_{formal} consisting of the elements of E.

First, observe that E is a finitely generated module over $\mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}$. For example, consider the ideal in $\mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}$ generated by $U_{1,1}$ for all $(U, V, ..) \in E$. By Hilbert basis theorem [Eisenbud-book, pg.371] this ideal has a finite basis, say $\{(U_i, V_i, ..)\}_i$. Correspondingly, the module E decomposes: $E = E' \oplus Span\big(\cup_i (U_i, V_i, ..)\big)$. Here $Span\big(\cup_i (U_i, V_i, ..)\big)$ is the submodule of E generated by $\{(U_i, V_i, ..)\}_i$, while E' is the submodule generated by those $(U, V, ..)$ that have $U_{11} = 0$. Continue in this way over all the entries of U and V, to get a finite basis for E.

Next, consider the free injective resolution of E:

$$(105) \qquad 0 \to E \to Mat\big(m \times m, \mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}\big) \oplus Mat\big(n \times n, \mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}\big) \oplus .. \xrightarrow{\phi} ...$$

Here $Mat\big(m \times m, \mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}\big) \oplus Mat\big(n \times n, \mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}\big) \oplus ..$ is considered as a free module over $\mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})}$. The map ϕ corresponds to all the linear homogenized equations imposed on $(U, V, ..)$.

Now take the completion $\mathbb{K}[x_1, .., x_p]_{(\mathfrak{m})} \to \mathbb{K}[[x_1, .., x_p]]$. As the completion functor is exact, [Eisenbud-book, pg. 198], the resolution is preserved:

$$(106) \quad 0 \to \mathbb{K}[[x_1, .., x_p]]E \to Mat\big(m \times m, \mathbb{K}[[x_1, .., x_p]]\big) \oplus Mat\big(n \times n, \mathbb{K}[[x_1, .., x_p]]\big) \oplus .. \xrightarrow{\phi} ...$$

But the last row is the resolution of E_{formal}. Hence $E_{formal} = \mathbb{K}[[x_1, .., x_p]]E$.

Now impose the additional conditions on the new variables introduced to homogenize the initial conditions (e.g. $\tilde{V} = \mathbb{1}$, for G_l). As they are all linear, with constant coefficients, we still have the property: if (U, V) is a formal solution of the initial linear equations, then $(U, V) = \sum f_i(U_i, V_i)$ for some rational solutions (U_i, V_i) and $f_i \in \mathbb{K}[[x_1, .., x_p]]$.

Finally, suppose $UA_1 = A_2V$ has a formal solution for $(U, V) \in G$, in particular (U, V) are invertible at the origin. Hence, $U' := \sum jet_0(f_i)U_i$ and $V' := \sum jet_0(f_i)V_i$ are locally invertible matrices of rational functions satisfying $U'A_1 = A_2V'$. ∎

Remark A.2. The second statement of the proposition is not true for $G = G_T$ as in this case the conditions on U, V are non-linear. For example, let $A_2 = (1 + x)A_1$ be 1×1 matrices, i.e. functions. Then for $A_1 = UA_2U^T$ one has $U^2 = 1 + x$, i.e. U cannot be rational.

Corollary A.3. *Let the matrix A have locally converging entries. Suppose A is formally-finitely-G-determined, i.e. the conditions of Theorem 4.8 or Corollary 4.9 are satisfied. Then, A is locally-converging-finitely-G-determined and A is locally-converging-G-equivalent to a matrix of polynomials. Further, by Corollary 5.2 the normal form of A is polynomial.*

A.2. **Surjectivity.** Most matrices with formal/locally convergent entries are not G_{lr}-equivalent to locally convergent/rational matrices.

Example A.4. Let $A(x, y) = y - xf(x)$ be a "1×1 matrix" of two variables, where $f(x)$ is a formal (but not locally converging) series or a locally converging series (but not a rational function).

Assume there exist a formal 1×1 matrix $U(x, y)$, invertible at the origin, such that $U(x, y)A(x, y)$ is a locally converging series/a rational function. Note that if $U(x, y)A(x, y)$ vanishes at some point then $A(x, y)$ vanishes too.

Hence if $U(x, y)A(x, y)$ is a rational function then $\{U(x, y)A(x, y) = 0\} \subset \mathbb{K}^2$ is an algebraic curve, which is defined also as $\{y - xf(x) = 0\}$. Implying that $f(x)$ is rational, contradiction.

Similarly, if $U(x,y)A(x,y)$ is a locally convergent power series, then it defines a locally analytic curve. On this curve $y = xf(x)$, i.e. $f(x)$ must be convergent at every point of this curve, contradiction.

An immediate necessary condition for a square matrix to be equivalent to a matrix of locally convergent series/rational functions is: $\det(A)$ is a locally convergent series/rational function, up to an invertible factor. Or, the ideal $\langle \det(A) \rangle \subset \mathbb{K}[[x_1, .., x_p]]$ is generated by a locally convergent series/rational function.

A stronger condition: let $I_k(A)$ be the ideal in $\mathbb{K}[[x_1, .., x_p]]$ generated by all the $k \times k$ minors of A. Note that these ideals are invariant under G_{lr} equivalence. Hence, if A is equivalent to a matrix of locally convergent series/rational functions, then all the ideals $I_k(A) \subset \mathbb{K}[[x_1, .., x_p]]$ are generated by locally convergent series/rational functions. All of these conditions are relevant, as the following example shows.

Example A.5. Consider the matrix with entries in $\mathbb{K}[[x, y, z, q, w]]$:

$$(107) \qquad A = \begin{pmatrix} z & y + x^2 f_1(x) & 0 \\ 0 & w & x + y^2 f_2(y) \\ 0 & 0 & q \end{pmatrix}$$

It has a polynomial determinant and the ideal of its entries is the maximal ideal, $I_1(A) = < x, y, z, w, q >$. In particular this ideal is polynomially generated. If $f_1(x)$, $f_2(x)$ are formal but not locally convergent/locally convergent but not rational, then A is not G_{lr} equivalent to a matrix with locally convergent/rational entries. Because $I_2(A)$ is not generated by locally convergent/rational elements.

Remark A.6. As has been proved recently, [Keller-Murfet-Van den Bergh2008, proposition 1.6], for any formal matrix $A \in Mat(m, m, p)$, with arbitrary field $\mathbb{K}$, there exists a matrix $B \in Mat(n, n, p)$ such that $A \oplus B$ is G_{lr} equivalent to a matrix of rational functions.

A.3. **The case of two variables.** Every formal matrix of one variable is G_{lr} equivalent to a polynomial matrix, e.g. see the normal form §1.3. We prove that to some extent this is true in the case of two variables.

In this section $\mathbb{K}$ is algebraically closed, A is an $m \times m$ matrix, with entries in $\mathbb{K}[[x, y]]$. We always assume $A|_0 = \mathbb{0}$ (cf. Corollary 1.3) and $\det(A) \not\equiv 0$. In addition we assume: the plane curve $C := \{det(A) = 0\}$ is *reduced*, i.e. it has no multiple components. (Note though that C can be reducible.)

First, recall the situation with functions.

Proposition A.7. *Let $f, g \in \mathbb{K}[[x, y]]$ be relatively prime. Then $g = ug' \mod (f)$, where $u \in \mathbb{K}[[x, y]]$ is invertible and $g' \in \mathbb{K}[x, y]$.*

Mote generally, let $h = (f, g)$ be the greatest common divisor, then $g = uhg' \mod (f)$ with u invertible and g' a polynomial. This property is well known, but we could not find a reference. Hence we give a proof.

Proof. Step 1. First suppose the formal curve $C = \{f = 0\} \subset (\mathbb{K}^2, 0)$ is locally irreducible. Let $\tilde{C} \xrightarrow{\nu} C$ be the normalization, i.e. a smooth curve germ and a finite morphism that is an isomorphism outside the singularity of C. (See e.g. [Eisenbud-book, pg.125-129].) This corresponds to the embedding of the local rings: $\mathbb{K}[[x, y]] \big/ (f) \xrightarrow{\nu^*} \mathbb{K}[[t]]$. By the finiteness of the morphism, the quotient $\mathbb{K}[[t]] \big/ \mathbb{K}[[x, y]]/(f)$ is a finite dimensional vector space.

The normalization induces the valuation

$$(108) \qquad val : \mathbb{K}[[x, y]] \to \mathbb{N}, \quad g \to val(g) := ord_t \nu^*(g) = ord_t(g(x(t), y(t)))$$

By the finiteness of the quotient above, there exists the conductor, i.e. the minimal number $c \in \mathbb{N}$ such that any bigger number $d > c$ is realized as the valuation of some function: $d = val(g)$, for $g = \mathbb{K}[[x,y]]$.

Step 2. Let $g \in \mathbb{K}[[x,y]]$, not a polynomial. Then can decompose g into the sum of a polynomial and some series of high valuation: $g = g_{pol} + g_{high}$, where $val(g_{high}) > c + val(g_{pol})$. By the existence of conductor, there exists $h \in \mathbb{K}[[x,y]]$ such that $val(h) = val(g_{high}) - val(g_{pol})$, i.e. $val(hg_{pol}) = val(g_{high})$. Then for some number β one has: $val(hg_{pol} + \beta g_{high}) > val(g_{high})$. Hence $(1 + h)g = g_{pol} + g'_{high}$ with $val(g'_{high}) > val(g_{high})$.

Continue this process, to get in the limit: $g = ug_{pol} + g_{high}$ such that $val(g_{high}) = \infty$ and u is invertible. But then g_{high} is divisible by f. Hence the statement.

Step 2'. If the curve $C = \{f = 0\}$ is locally reducible, $C = \cup_{i=1}^{r} C_i$, then the normalization $\coprod \tilde{C}_i \to \cup C_i$ induces the multi-valuation $\{val_i\} : \mathbb{K}[[x,y]] \to \mathbb{N}^{\oplus r}$. The quotient of the local ring is still a finite dimensional vector space, hence the conductor still exists. Continue as above, using that g, f are mutually prime. ∎

Theorem A.8. *Let $A \in Mat(m, m, 2)$ be a formal matrix in two variables. Assume $\det(A)$ is locally convergent (up to an invertible factor). Then A is G_{lr} equivalent to a matrix with locally converging entries.*

Proof. Step 1. We can consider A as a matrix with entries in $\mathbb{K}[[x,y]] \big/ (\det A)$. (Recall that $\det A \not\equiv 0$ and is reduced, and $A|_0 = \mathbb{0}$.) Namely we consider $A \mod (\det A)$. The G_{lr} equivalence over $\mathbb{K}[[x,y]]$ descends to that over $\mathbb{K}[[x,y]] \big/ (\det A)$.

Conversely, if $A \sim B$ over $\mathbb{K}[[x,y]] \big/ (\det A)$ then they are equivalent over $\mathbb{K}[[x,y]]$. Indeed, suppose $A = UBV + (\det A)Q$, where U, V are invertible and Q is some formal matrix. Recall that $\det(A) = AA^{\vee}$, where $A^{\vee}$ is the adjoint matrix. Thus we get $A(\mathbb{1} - A^{\vee}Q) = UBV$. As $A|_0 = \mathbb{0}$ we get that $(A^{\vee}Q)|_0 = \mathbb{0}$, hence the matrix $(\mathbb{1} - A^{\vee}Q)$ is invertible. Therefore: $A = (\mathbb{1} - A^{\vee}Q)^{-1}UBV$.

So, it is enough to show that A is equivalent to a locally convergent matrix modulo $\det A$. In fact, we will show this for $A^{\vee}$ and then achieve the statement for A too. Note that an equivalence transformation $A \to UAV$ results in the equivalence $A^{\vee} \to V^{-1}A^{\vee}U^{-1}$.

Step 2. From now on consider A and $A^{\vee}$ modulo $\det A$. By the previous proposition we can assume $A^{\vee}$ in the form $\{u_{ij}g_{ij}\}$, where u_{ij} are invertible and g_{ij} are locally converging. Consider the equivalence transformation

$$(109) \qquad A^{\vee} \to UA^{\vee}V, \quad U = \begin{pmatrix} u_{11}^{-1} & 0 & .. & 0 \\ 0 & u_{21}^{-1} & .. & 0 \\ .. & .. & & \\ 0 & .. & 0 & u_{m1}^{-1} \end{pmatrix}, \quad V = \begin{pmatrix} u_{11}^{-1} & 0 & .. & 0 \\ 0 & u_{12}^{-1} & .. & 0 \\ .. & .. & & \\ 0 & .. & 0 & u_{1m}^{-1} \end{pmatrix}$$

So the first row and column of $UA^{\vee}V$ have locally convergent entries. From now on we assume $A^{\vee}$ in this form.

Step 3. By definition of det, the matrix A is degenerate, when restricted to the curve $C = \{\det A = 0\}$. At a point $pt \in C$ the corank of A is not bigger than the multiplicity $mult(C, pt)$. Hence, as C is reduced, the corank of A at the smooth points of C is one.

Recall that $AA^\vee|_C = \mathbb{0}|_C$, hence at the smooth points of C the rank of $A^\vee$ is one. Namely, any two rows/columns are dependent. Thus in particular, for any entry $A_{ij}^\vee$ one has: $A_{ij}^\vee = \frac{A_{i1}^\vee A_{1j}^\vee}{A_{11}^\vee} \in \mathbb{K}[[x,y]]\big/(\det A)$. Note that the right hand side is *locally convergent*.

Hence, the above equivalence transformation results in a locally convergent matrix $A^\vee$. Thus $A \sim (A^\vee)^\vee$ is locally convergent too. ∎

Theorem A.9. *Suppose A is a formal matrix in two variables and its determinant is a non-zero, reduced polynomial, up to an invertible factor.*
* If $\det A$ is irreducible (as a formal series) then A is G_{lr}-equivalent to a polynomial matrix.
* More generally, suppose the decomposition of $\det A$ into irreducibles is the same over formal series and over rational functions. Namely, in the decomposition into irreducible formal factors $\det A = f_1 \cdots f_k$ all $\{f_i\}$ can be chosen as polynomials (up to multiplication by an invertible). Them A is G_{lr} equivalent to a polynomial matrix.

Proof. The proof goes precisely as in the locally convergent case. After *Step 2.* we have a matrix $A^\vee$ whose first row and column are polynomials. Hence in *Step 3.* we get $A_{ij}^\vee = \frac{A_{i1}^\vee A_{1j}^\vee}{A_{11}^\vee}$, where on the right we have a rational function and on the left a regular function. So $A_{ij}^\vee$ is a regular rational function, i.e. a fraction of two polynomials, with non-vanishing denominator.

Finally, multiply $A^\vee$ by all such (invertible) denominators, to get: $A^\vee$ is a polynomial matrix. From here obtain that A is polynomial too. ∎

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