Three-Colorings of Cubic Graphs and Tensor Operators

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September 30, 2010

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Abstract

Penrose's work [6] established a connection between the edge 3-colorings of cubic planar graphs and tensor algebras. We exploit this point of view in order to get algebraic representations of the category of cubic graphs with free ends.

keywords: 3-colorings of cubic planar graphs, tensor algebras, Penrose invariant, monoidal categories.

1   Introduction

Although it first appeared as a simple geometric curiosity the Four Color Problem became one of the most important fields of research in discrete mathematics linking several areas and having dozens of equivalent formulations.

The original statement said the following:

Theorem 1 *(4-CT) Every planar map can be colored using no more than four colors in such a way that no pair of adjacent regions receive the same color.*

Avoiding the rigorous definitions of map, region or adjacent regions, this result can be given an equivalent but simpler statement:

Theorem 2 *Every planar simple graph can be colored using no more than four colors in such a way that no pair of adjacent vertices receive the same color.*

We say that a graph with such a coloring is a 4-colorable graph or it has a (vertex) 4-coloring.

Since any simple planar graph can be embedded in the graph (i.e. the 1-skeleton) of a triangulation of the sphere, the 4-Color Theorem is equivalent to the following theorem:

Theorem 3 *The graph of any sphere triangulation is 4-colorable or has a loop edge.*

Now regarding a 4-coloring ϕ on the graph of a sphere triangulation $\mathcal{T}$ as a 0-cochain in the simplicial cohomology of $\mathcal{T}$ with coefficients in the field of order 4, $\mathbb{F}_4$, its coboundary $\delta\phi$ gives a 3-coloring on the edges (with colors in $\mathbb{F}_4 \setminus \{0\}$) such that, for any face f of $\mathcal{T}$, the three edges e_1, e_2 and e_3 of its boundary receive different colors (the only way to have $\delta\phi(e_1) + \delta\phi(e_2) + \delta\phi(e_3) = \delta\phi(\partial f) = 0$ with $\delta\phi(e_i) \in \mathbb{F}_4 \setminus \{0\}$). Note that, since $\mathbb{F}_4$ is a field of characteristic 2, it does not matter what order the simplices (faces, edges or vertices) of $\mathcal{T}$ have.

On the other hand if we have a 3-coloring ψ on the edges of the triangulation $\mathcal{T}$ assigning different colors to the three edges of the boundary of any face of $\mathcal{T}$, then ψ can be regarded as a closed 1-cochain. Thus ψ should be

the coboundary of some 0-cochain ϕ which would be a (vertex) 4-coloring of the triangulation $\mathcal{T}$.

This proves a result due to Tait that says that the Four Color Theorem is equivalent to the following proposition:

**Theorem 4** *Every planar bridgeless cubic graph is edge 3-colorable.*

A *cubic graph* is a graph where each vertex is adjacent to three edges. If a cubic graph is planar then it is the dual graph of a triangulation of a sphere. A graph is *bridgeless* if there is no edge that after being removed increases the number of the connected components. A planar cubic graph is bridgeless if and only if it is the dual graph of a triangulation without loops of a sphere.

Much of the research in this area focuses mainly on the Tait version of the Four Color Theorem. The references [7, 8] provied a good overview about the Four Color Theorem and its ramifications.

2   Category of cubic graphs (with free ends)

It is possible to study the edge 3-colorings of cubic graphs by introducing a category of cubic graphs with free ends.

Consider the following (monoidal) category **CG**. The objects of **CG** are the non-negative integer numbers and a morphism from m to n is a regular immersion of a cubic graph with $m + n$ free ends in the strip $\mathbb{R} \times [0,1]$ such that the free ends are placed at the points $(1,1), \ldots ,(m,1)$ and $(1,0), \ldots ,(n,0)$ (see the next figure).
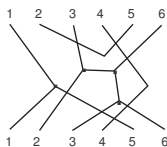

To simplify the treatment we consider piecewise linear immersions rather than smooth immersions.

The composition in this category is defined in the following way. Given two morphisms $g_1 : l \to m$ and $g_2 : m \to n$ then their composition $g_2 g_1 :$

$l \to n$ would be the immersion $g(f(g_1) \cup g_2)$ where $f(x, y) = (x, y + 1)$ and $g(x, y) = (x, y/2)$ (see the next figure)[1].
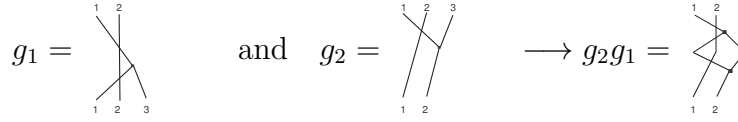
This category places the same role for cubic graphs as the category of the tangles is for links and like the latter it has a monoidal structure. Given two morphisms $g_1 : k \to l$ and $g_2 : m \to n$ we get a new morphism $g_1 \otimes g_2 : k + m \to l + n$ by putting the two graph immersions side by side (see the next figure).
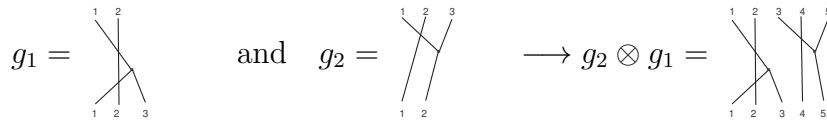
It is easy to see that with these two operations the category $\mathcal{CG}$ is generated by the following morphisms:

$$\cap : 0 \to 2 \quad , \quad \cup : 2 \to 0$$

$$\lambda : 1 \to 2 \quad , \quad y : 2 \to 1$$

$$x : 2 \to 2 \quad \text{and} \quad I := id_1 : 1 \to 1$$
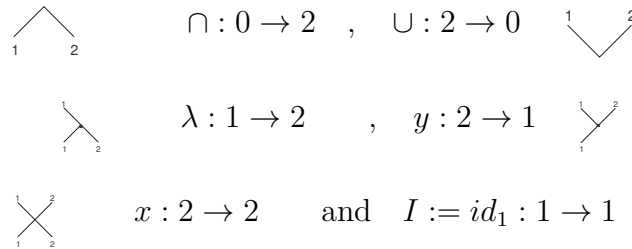
There are some relations that these generators should satisfy:

$$(\cup \otimes I)(I \otimes \cap) = I = (I \otimes \cup)(\cap \otimes I)$$

$$(x \otimes I)(I \otimes \cap) = (I \otimes x)(\cap \otimes I)$$

$$\cup x = \cup$$

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[1] In this paper, the downward direction composition is used, some authors use the opposite direction.

$$xx = I \otimes I$$

$$(I \otimes x)(x \otimes I)(I \otimes x) = (x \otimes I)(I \otimes x)(x \otimes I)$$

$$(\lambda \otimes I)\cap = I = (I \otimes \lambda)\cap$$

$$(y \otimes I)(I \otimes \cap) = \lambda = (I \otimes y)(\cap \otimes I)$$

$$(x \otimes I)(I \otimes \lambda)x = (I \otimes x)(\lambda \otimes I)$$

Besides these relations, as a strict monoidal category, **CG** should satisfy the following equality:

Given two morphisms $f : k \to l$ and $g : m \to n$

$$f \otimes g = (f \otimes id_n)(id_k \otimes g) = (id_l \otimes g)(f \otimes id_m)$$

As a consequence of this identity and the previous relations we have that this representation is invariant under ambient isotopies.

If we drop the generator x we get a subcategory **PCG** of **CG** which only contains planar cubic graphs with free ends.

Now let $\mathbb{K}$ be a field of characteristic zero and let V be a 3-dimensional $\mathbb{K}$-vector space. We fix a canonical basis $\{e_1, e_2, e_3\}$ for V and introduce the following (monoidal) functor from **CG** to the (monoidal) category $\mathbf{Vect}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$:

$$F : \mathbf{CG} \to \mathbf{Vect}_{\mathbb{K}}$$

defined on the objects by

$$F(n) = V^{\otimes n} \quad (F(0) = \mathbb{K})$$

and on the morphisms by the following definitions on the generators:

$$
\begin{aligned}
F(\cap) : \quad \mathbb{K} \quad &\longrightarrow \quad V \otimes V \\
\alpha \quad &\longmapsto \quad \alpha e_i \otimes e_i
\end{aligned}
$$

$$F(\cup): \quad V \otimes V \quad \longrightarrow \quad \mathbb{K}$$
$$\sum_{i,j=1}^{3} \alpha_{i,j} e_i \otimes e_j \quad \longmapsto \quad \sum_{i=1}^{3} \alpha_{i,i}$$

$$F(\lambda): \quad V \quad \longrightarrow \quad V \otimes V$$
$$\sum_{i=1}^{3} \alpha_i e_i \quad \longmapsto \quad \sum_{\{i,j,k\}=\{1,2,3\}} \alpha_i e_j \otimes e_k$$

$$F(y): \quad V \otimes V \quad \longrightarrow \quad V$$
$$\sum_{i,j=1}^{3} \alpha_{i,j} e_i \otimes e_j \quad \longmapsto \quad \sum_{\{i,j,k\}=\{1,2,3\}} \alpha_{i,j} e_k$$

$$F(x): \quad V \otimes V \quad \longrightarrow \quad V \otimes V$$
$$\sum_{i,j=1}^{3} \alpha_{i,j} e_i \otimes e_j \quad \longmapsto \quad \sum_{i,j=1}^{3} \alpha_{i,j} e_j \otimes e_i$$

It is easy to see that this functor is well defined under the relation (i.e. $F[(\cup \otimes I)(I \otimes \cap)] = F(I) = F[(I \otimes \cup)(\cap \otimes I)]$, $F[(x \otimes I)(I \otimes \cap)] = F[(I \otimes x)(\cap \otimes I)]$, ..., etc). In fact, we have the following theorem.

Theorem 5 *If a morphism $g : m \to n$ represents a cubic graph with $m + n$ free ends then for each element, $e_{i(1)} \otimes \cdots \otimes e_{i(m)}$, of the canonical basis of $V^{\otimes m}$*

$$F(g)(e_{i(1)} \otimes \cdots \otimes e_{i(m)}) = \sum_{j(1),...,j(n)} \chi_{j(1),...,j(n)}^{i(1),...,i(m)} e_{j(1)} \otimes \cdots \otimes e_{j(n)}$$

where $\chi_{j(1),...,j(n)}^{i(1),...,i(m)}$ is the number of edge 3-colorings of the graph such that it has the free edges on the top colored by $i(1),...,i(m)$ (in this order) and the free edges on the bottom colored by $j(1),...,j(n)$ in this order.

Example 1 $g : 2 \to 2 :$

egde 3-colorings with colors $1, 1$ on the top: $\{$, $\}$,

and
$$F(g)(e_1 \otimes e_1) = 2 e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

Proof.

It is enough to check the statement on the generators $\cap$, $\cup$, λ, y and x (which is straightforward) and to note that the composition and the monoidal operation on **CG** satisfy also the required:

$$\chi^{i(1),\ldots,i(k)}_{j(1),\ldots,j(n)}(gf) = \sum_{\alpha(1),\ldots,\alpha(m)} \chi^{\alpha(1),\ldots,\alpha(m)}_{j(1),\ldots,j(n)}(g)\chi^{i(1),\ldots,i(k)}_{\alpha(1),\ldots,\alpha(m)}(f)$$

$$\chi^{i(1),\ldots,i(k+m)}_{j(1),\ldots,j(l+n)}(g \otimes f) = \chi^{i(1),\ldots,i(k)}_{j(1),\ldots,j(l)}(g)\chi^{i(k+1),\ldots,i(k+m)}_{j(l+1),\ldots,j(l+n)}(f)$$

■

The following corollary is an immediate consequence of the theorem.

Corollary 6 *Given a cubic graph g (without free end edges) viewed as a morphism $g : 0 \longrightarrow 0$ the value $F(g)(1)$ is the number of edge 3-colorings of the graph. In particular g is edge 3-colorable if and only if $F(g)(1) \neq 0$.*

Next, we introduce another functor $\tilde{F} : \mathbf{CG} \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ which is a small modification of the functor F.

$\tilde{F}$ is equal to F on the objects and on all the generator morphisms except on the morphism x where

$$\tilde{F}(x): \quad \begin{array}{ccc} V \otimes V & \longrightarrow & V \otimes V \\ \sum_{i,j=1}^{3} \alpha_{i,j}e_i \otimes e_j & \longmapsto & \sum_{i,j=1}^{3} \tilde{\delta}_{i,j}\alpha_{i,j}e_j \otimes e_i \end{array}$$

with $\tilde{\delta}_{i,j} = -1 + 2\delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta.

We have that $\tilde{F} = F$ when restricted to the subcategory $\mathbf{PCG}$ (the planar cubic graphs).

The special feature of the functor $\tilde{F}$ is that it satisfies the Penrose formula:

$$\tilde{F}(\;\gtrless\;) = \tilde{F}(|\;\;|) - \tilde{F}(\;\times\;)$$

Thus it also satisfies the IHX identity on chinese characters (see [1])

$$\tilde{F}(\;\gtrless\;) = \tilde{F}(\;\rightarrowtail\!\!\prec\;) - \tilde{F}(\;\asymp\;)$$

We also have the formula

$$\tilde{F}(\;\Leftrightarrow\;) = -\tilde{F}(\;\wedge\;)$$

To see what the functor $\tilde{F}$ gives let us introduce the notion of the *sign* of an edge 3-coloring of a cubic graph. The sign of an edge 3-coloring of a cubic graph projected on the plane is $+1$ or -1 if the number of crossings of edges of different colors is even or odd. Then we have the following result.

Theorem 7 *If a morphism $g : m \to n$ represents a cubic graph with $m + n$ free ends then for each element, $e_{i(1)} \otimes \cdots \otimes e_{i(m)}$, of the canonical basis of $V^{\otimes m}$*

$$\tilde{F}(g)(e_{i(1)} \otimes \cdots \otimes e_{i(m)}) = \sum_{j(1),\ldots,j(n)} \tilde{\chi}_{j(1),\ldots,j(n)}^{i(1),\ldots,i(m)} e_{j(1)} \otimes \cdots \otimes e_{j(n)}$$

where $\tilde{\chi}_{j(1),\ldots,j(n)}^{i(1),\ldots,i(m)}$ is the sum of the signs of all edge 3-colorings of the graph which have the free edges on the top colored by $i(1),\ldots,i(m)$ (in this order) and the free edges on the bottom colored by $j(1),\ldots,j(n)$ in this order.

Since the functor $\tilde{F}$ satisfies the Penrose identity and, since, for a planar graph $g \in \hom(0,0)$, $\tilde{F}(g)(1)$ is equal to the number of edge 3-colorings, we have that this functor generalizes the Penrose invariant [6].

3   Binary tree and Eliahou-Kryuchkov conjecture

From now on, we will restrict ourselves to the study of the subcategory **PCG** where $F = \tilde{F}$.

Recall that a graph is Hamiltonian if there close path the passes for all the vertices of the graph. There is a well-known theorem on graph coloring theory due to Whitney [9] that states the following.

Theorem 8 *If every Hamiltonian planar graph is 4-colorable then the Four Color Theorem is true.*

Let call a morphism $g : 1 \longrightarrow n$ in **PCG** generated only by the generators λ and I a descendant binary n-tree, and call a morphism $g : n \longrightarrow 1$ in **PCG** generated only by the generators y and I an ascendant binary n-tree.

When we look at this result in its dual form we have that the Four Color Theorem is equivalent to the following.

Theorem 9 *If a morphism $g : 1 \longrightarrow 1$ is a composition of a descendant binary n-tree with an ascendant binary n-tree then $F(g)$ is non-null.*

In **CG** (or **PCG**) there is a natural involution, called the adjoint, $*$: $\mathbf{CG}^{op} \to \mathbf{CG}$ defined by $\lambda^* = y$, $\cap^* = \cup$ and $x^* = x$ (by definition of involution we have $y^* = \lambda$, $\cup^* = \cap$, $(f \circ g)^* = g^* \circ f^*$ and $(f \otimes g)^* = f^* \otimes g^*$). Geometrically this involution takes the form of a reflection of the graph in an horizontal line (see the next figure).
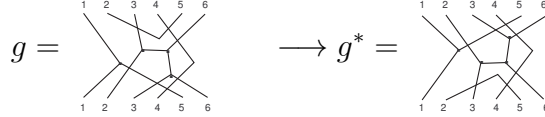
For instance a descendant tree is the adjoint of an ascendant tree.

Considering the inner products in $\{V^{\otimes n}\}_{n \in \mathbb{N}}$ defined by their canonical bases and the involution on $\mathbf{Vect}_{\mathbb{K}}$ defined by the inner products ($\langle Tx, y \rangle = \langle x, T^*y \rangle$), we have that the functors F and $\tilde{F}$ preserve the involution structure (i.e. $F(g^*) = (F(g))^*$). In particular, we have the following proposition

Proposition 10 *If $f : 1 \to n$ and $g : 1 \to n$ are two descendant binary n-trees then $F(f^*g) = \langle F(g)(e_1), F(f)(e_1) \rangle id_V$*

Proof.

We have that

$$
\begin{aligned}
F(f^*g)(e_i) &= \sum_{j=1}^{3} \langle F(f^*g)(e_i), e_j \rangle e_j \\
&= \sum_{j=1}^{3} \langle F(g)(e_i), F(f)(e_j) \rangle e_j
\end{aligned}
$$

So we have to prove that $\langle F(g)(e_i), F(f)(e_j) \rangle = \delta_{i,j} \langle F(g)(e_1), F(f)(e_1) \rangle$

If we identify e_1, e_2 and e_3 with the three non-zero elements of the field $\mathbb{F}_4$ we have that, for any morphism $g : 1 \to n$, if $\langle F(g)(e_i), e_{i(1)} \otimes \cdots \otimes e_{i(n)} \rangle \neq 0$ then $e_i = e_{i(1)} + e_{i(2)} + \cdots + e_{i(n)}$.

This proves that $\langle F(g)(e_i), F(f)(e_j) \rangle = 0$ if $i \neq j$.

On the other hand, if

$$
F(g)(e_i) = \sum \chi^i_{i(1),\ldots,i(n)} e_{i(1)} \otimes \cdots \otimes e_{i(n)}
$$

and σ is a permutation on $\{1, 2, 3\}$ then

$$
F(g)(\sigma(e_i)) = \sum \chi^i_{i(1),\ldots,i(n)} e_{\sigma(i(1))} \otimes \cdots \otimes e_{\sigma(i(n))}
$$

This proves that $\langle F(g)(e_1), F(f)(e_1) \rangle = \langle F(g)(e_2), F(f)(e_2) \rangle = \langle F(g)(e_3), F(f)(e_3) \rangle$.

■

Now we consider the following decomposition of the operator $F(\lambda)$:

$$F(\lambda) = F(\lambda^+) + F(\lambda^-) \tag{1}$$

where

$$F(\lambda^+)(e_i) = e_j \otimes e_k \quad \text{such that} \quad (i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$$

and

$$F(\lambda^-)(e_i) = e_j \otimes e_k \quad \text{such that} \quad (i, j, k) \in \{(2, 1, 3), (1, 3, 2), (3, 2, 1)\}$$

In the same way as a descendant tree is a morphism generated by λ and I, a descendant signed tree is a morphism generated by λ^+, λ^- and I. One simple observation that we can make from (1) is that for any descendant binary tree $g : 1 \to n + 1$ with n nodes, $F(g)$ is equal to the sum of the 2^n signed trees corresponding to g. Thus

$$F(g)(e_1) = \sum \chi^1_{i(1), \cdots, i(n+1)} e_{i(1)} \otimes \cdots \otimes e_{i(n+1)}$$

with $\chi^1_{i(1), \cdots, i(n+1)} = 1$ for some 2^n indices and zero for the others.

As a consequence of this we have:

Proposition 11 *For any descendant binary n-tree $g : 1 \to n + 1$ we have $F(g^*g) = 2^n id_V$.*

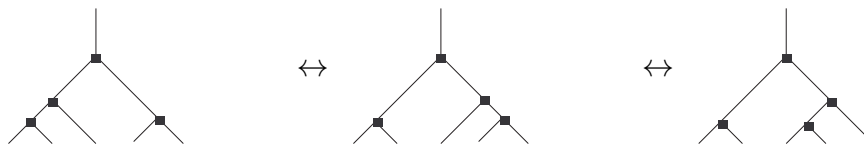Another observation that we can make is a signed reassociation identity:

$$F((I \otimes \lambda^+)\lambda^+) = F((\lambda^- \otimes I)\lambda^-) \quad \text{and} \quad F((I \otimes \lambda^-)\lambda^-) = F((\lambda^+ \otimes I)\lambda^+)$$

It is well known that any pair of binary trees with the same number of nodes is connected by a finite sequence of (non-signed) reassociation moves:
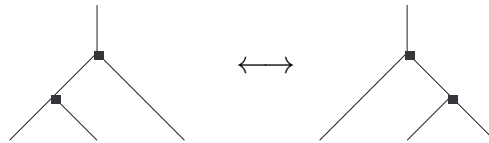
For instance:

In fact, the graph for which the vertices are the n- dimensional descendant binary trees and the edges represent reassociation moves between two trees is the 1-skeleton of the $n - 2$-dimensional *associahedron* $\mathcal{A}_{n-2}$ or Stasheff polytope (see [3] for the definition).

However when we take the analogous graph $\mathcal{A}_{n-2}^s$ for signed trees and signed reassociation moves we get a non-connected graph. Indeed, two signed trees f and g are connected if $F(f) = F(g)$.

This last graph can be projected in a natural way onto the first but it is not true that any path on the associahedron $\mathcal{A}_{n-2}$ can be lifted to a path on $\mathcal{A}_{n-2}^s$.

Eliahou [2] and Kryuchkov (cited from [5]) conjectured the following:

Conjecture 12 *(Eliahou-Kryuchkov) For any pair of vertices on $\mathcal{A}_{n-2}$ there exists a path connecting them that can be lifted to a path on the graph $\mathcal{A}_{n-2}^s$*

It is easy to see that this conjecture implies the Four Color Theorem since two signed trees connected by a sequence of signed reassociation moves give the same colors on the ends.

In the paper [4] Gravier and Payan proved that this conjecture is, in fact, equivalent to the Four Color Theorem.

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$$x \qquad y \qquad z$$

$$(xy)z$$

$$
\begin{array}{ccc}
 & y & \\
 & \overline{} & \\
x & \diagdown\ xy & z \\
 & \overline{} & \\
 & (xy)z &
\end{array}
$$

x   y   z

x(yz)

```
1          2
┌─────────┐
│         │
│         │
│         │
│         │
└─────────┘
1          2
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