Radon partitions in convexity spaces*

Boris Bukh†

Abstract

Tverberg's theorem asserts that every $(k-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into k parts, so that the convex hulls of the parts have a common intersection. Calder and Eckhoff asked whether there is a purely combinatorial deduction of Tverberg's theorem from the special case $k=2$. We dash the hopes of a purely combinatorial deduction, but show that the case $k=2$ does imply that every set of $O(k^2 \log^2 k)$ points admits a Tverberg partition into k parts.

Introduction

Radon's lemma [Rad21] states that every set P of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two classes $P = P_1 \cup P_2$ so that the convex hulls of P_1 and P_2 intersect. Birch [Bir59] (for $d=2$) and Tverberg [Tve66] (for general d) extended Radon's theorem to the analogous statement for partitions of a set into more than two parts: For a set $P \subset \mathbb{R}^d$ of $|P| \geq (k-1)(d+1)+1$ points there is a partition $P = P_1 \cup \dots \cup P_k$ into k parts, such that the intersection of the convex hulls $\operatorname{conv} P_1 \cap \dots \cap \operatorname{conv} P_k$ is non-empty. The bound of $(k-1)(d+1)+1$ is sharp, as witnessed by any set of points in sufficiently general position.

Calder [Cal71] conjectured and Eckhoff [Eck79] speculated that Tverberg's theorem is a consequence of Radon's theorem in the context of abstract convexity spaces. The conjecture, which we now present, is commonly referred as "Eckhoff's conjecture", and we will maintain this tradition to avoid additional confusion. If true, the conjecture would have provided a purely combinatorial proof of Tverberg's theorem. However, we will show that the conjecture is false.

A *convexity space* on the ground set X is a family $\mathcal{F} \subset 2^X$ of subsets of X, called *convex sets*, such as both $\emptyset$ and X are convex, and intersection of any collection of convex sets is convex. For example, the familiar convex sets in $\mathbb{R}^d$ form a convexity space on $\mathbb{R}^d$. Among the other examples are axis-parallel boxes in $\mathbb{R}^d$, finite subsets on any ground set, closed sets in any topological space (see the book [vdV93] for a through overview of convexity spaces). If the ground set X in the convexity space $(X, \mathcal{F})$ is clear from the

†B.Bukh@dpmms.cam.ac.uk. Centre for Mathematical Sciences, Cambridge CB3 0WB, England and Churchill College, Cambridge CB3 0DS, England.

context, we will speak simply of a convexity space $\mathcal{F}$. The *convex hull* of a set $P \subset X$, denoted $\operatorname{conv} P$, is the intersection of all the convex sets containing P. We write $\operatorname{conv}_{\mathcal{F}} P$ if the convexity space is not clear from the context. The k-th *Radon number* of $(X, \mathcal{F})$ is the minimum natural number r_k, if it exists, so that every set $P \subset X$ of at least r_k points admits a partition $P = P_1 \cup \cdots \cup P_k$ into k parts whose convex hulls have an element in common. It is not hard to show[1] that if r_2 is finite, then so is r_k. Eckhoff's conjecture states that $r_k \le (k-1)(r_2-1)+1$ in every convexity space. The conjecture has been proved for $r_2 = 3$ by Jamison [JW81], and for convexity space with at most $2r_2$ points by Sierksma and Boland [SB83]. In section 4 we reproduce a version of Jamison's proof.

The best bounds on r_k are

$$r_{k_1k_2} \le r_{k_1} r_{k_2} \qquad \text{(due to Jamison [JW81]),}$$
$$r_{2k+1} \le (r_2-1)(r_{k+1}-1) + r_k + 1 \qquad \text{(due to Eckhoff [Eck00]).}$$

In particular,

$$r_k \le k^{\lceil \log_2 r_2 \rceil}. \tag{1}$$

The following result improves on (1).

Theorem 1. *Let $(X, \mathcal{F})$ be a convexity space, and assume that r_2 is finite. Then*

$$r_k \le c(r_2) k^2 \log^2 k,$$

where $c(r_2)$ is a constant that depends only on r_2.

Though this bound is not far from Eckhoff's conjecture, the conjecture itself is false.

Theorem 2. *For each $k \ge 3$ there is a convexity space $(X, \mathcal{F})$ such that $r_2 = 4$, but $r_k \ge 3(k-1)+2$.*

Despite the failure of Eckhoff's conjecture, we have been unable to rule out that the convexity spaces with finite r_2 might behave similarly to Euclidean spaces. It is conceivable that r_k is bounded by a linear function of k for each r_2. Moreover, it is possible that other results from combinatorial convexity extend to such spaces. For instance, Radon proved the lemma now bearing his name to give an alternative proof of Helly's theorem that if in some family of convex sets in $\mathbb{R}^d$ every $d+1$ sets intersect, then all of them do. One of the easy but startling consequences of Helly's theorem is the centrepoint theorem. The centrepoint theorem asserts that for every finite set $P \subset \mathbb{R}^d$ there is a point $p \in \mathbb{R}^d$ (the "centrepoint") such that every convex set containing more than $\frac{d}{d+1}|P|$ points of P also contains p. Both the deduction of Helly's theorem from Radon's theorem, and the deduction of centrepoint theorem from Helly's theorem remain valid in the context of the convexity spaces with finite r_2. This prompts the following question:

[1] According to [Eck00] it was first shown by R.E.Jamison (1976). The first published proofs appear to be in [DRS81] and [JW81].

Question 3 (Weak epsilon-nets)**.** *Suppose $(X, \mathcal{F})$ is a convexity space with finite r_2. Let $\varepsilon > 0$ be given. Let $P \subset X$ be a set of points in the space. Is there a set N of $|N| \le f(\epsilon, r_2)$ points such that every convex set S containing more than $\varepsilon|P|$ points of P also contains at least one point of N?*

The set N as in the question above is called a *weak ε-net* (with respect to convex sets) for P. In $\mathbb{R}^d$ it is known that there are weak ε-nets of size only $(1/\varepsilon)^d \log^{c_d}(1/\varepsilon)$. The discussion above shows that the answer to the question is positive if $\epsilon > 1 - 1/(r_2 - 1)$. It is unclear whether the weak ε-nets of size $f(\varepsilon, r_2)$ exist for any $\epsilon < 1 - 1/(r_2 - 1)$.

Bárány [Bár82] showed that if P is an n-point set in $\mathbb{R}^d$, then there is a point p in $c_d\binom{n}{d+1}$ of all the $\binom{n}{d+1}$ simplices spanned by P, where c_d is a positive constant that depends only on d. In $\mathbb{R}^1$, it is immediate that $c_1 = 1/2$ is admissible, and is best possible. The situation for convexity spaces with bounded r_2 is again unclear, except if $r_2 = 3$:

Proposition 4 (Selection theorem)**.** *Let $(X, \mathcal{F})$ be a space with $r_2 = 3$. Let $P \subset X$ be point set. Then there is a point $p \in X$ that is contained in at least $\frac{1}{3}\binom{n}{2} + O(n)$ of all the sets* $\operatorname{conv}\{x, y\}$.

Question 5. *Does the preceding proposition hold with $1/2$ in place of $1/3$?*

The standard greedy argument of Alon, Bárány, Füredi, Kleitman [ABFK92, Section 8] shows that the selection theorem implies an affirmative answer to Question 3. In particular, it gives $f(\epsilon, 3) \le O\big((1/\epsilon)^2\big)$, which is probably not sharp.

The rest of the paper is organized as follows. In section 1 we introduce our only technical tool, the nerves of convex sets. In lemma 7 we will show that the nerves encode all the information about the convexity space that we need. In section 2 we present a counterexample to Eckhoff's conjecture. It is then followed in section 3 by the proof of Theorem 1. We conclude the paper with a short discussion of convexity spaces with $r_2 = 3$.

1 Nerves

Let P be a set of points in a some convexity space. We associate to P a collection $\boldsymbol{\mathcal{N}}(P)$ of subsets of 2^P. A family $\mathcal{F} \subset 2^P$ belongs to $\boldsymbol{\mathcal{N}}(P)$ if and only if the intersection $\bigcap_{S \in \mathcal{F}} \operatorname{conv} S$ is non-empty. In the conventional terminology one would say that the collection $\boldsymbol{\mathcal{N}}(P)$ is the nerve of the family of convex sets $\{\operatorname{conv} S : S \subset P\}$. Since we will not use the nerves of any other families of sets, in this paper we abuse the language and say that $\boldsymbol{\mathcal{N}}(P)$ is the nerve of P.

Proposition 6. *If $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{N}}(P)$, then $\boldsymbol{\mathcal{N}}$ satisfies the following properties:*

(N1) $\boldsymbol{\mathcal{N}}$ is a downset: if $\mathcal{F} \in \boldsymbol{\mathcal{N}}$ and $\mathcal{F}' \subset \mathcal{F}$, then $\mathcal{F}' \in \boldsymbol{\mathcal{N}}$.

(N2) If $\mathcal{F}$ is in $\boldsymbol{\mathcal{N}}$, then so is $\hat{\mathcal{F}} \stackrel{def}{=} \{S' : S' \supset S \in \mathcal{F}\}$.

(N3) For every $p \in P$ the family $\mathcal{F}(p) \stackrel{def}{=} \{S : p \in S\}$ is in $\boldsymbol{\mathcal{N}}$.

(N4) The set P can be partitioned into k parts $P = P_1 \cup \cdots \cup P_k$ so that $(\operatorname{conv} P_1) \cap \cdots \cap (\operatorname{conv} P_k) \neq \emptyset$ if and only if there is a family $\{P_1, \ldots, P_k\} \in \boldsymbol{\mathcal{N}}$ consisting of k disjoint sets.

(N5) If r_t exists, then for every set of r_t families $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_1, \ldots, \mathcal{F}_{r_t}\} \subset \boldsymbol{\mathcal{N}}$ there is a partition $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_1 \cup \cdots \cup \boldsymbol{\mathcal{F}}_t$ of $\boldsymbol{\mathcal{F}}$ into t parts so that $(\bigcap \boldsymbol{\mathcal{F}}_1) \cup \cdots \cup (\bigcap \boldsymbol{\mathcal{F}}_t) \in \boldsymbol{\mathcal{N}}$.

Proof. The first properties four properties are immediate from the definition of $\boldsymbol{\mathcal{N}}(P)$.

The final property is easy too: Suppose $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_1, \ldots, \mathcal{F}_{r_t}\}$ is given. Let q_i be any point in $\bigcap_{S \in \mathcal{F}_i} \operatorname{conv} S$. The set $Q = \{q_1, \ldots, q_{r_t}\}$ of r_t points can be partition into t parts $Q = Q_1 \cup \cdots \cup Q_t$ so that $(\operatorname{conv} Q_1) \cap \cdots \cap (\operatorname{conv} Q_t)$ is non-empty, thus containing some point p. The partition $Q = Q_1 \cup \cdots \cup Q_t$ naturally induces the partition $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_1 \cup \cdots \cup \boldsymbol{\mathcal{F}}_t$. It is easy to see that the point q belongs to $\bigcap_{S \in \cap \boldsymbol{\mathcal{F}}_i} \operatorname{conv} S$ for each $i = 1, \ldots, t$. □

Thanks to the following lemma we can avoid the convexity spaces in the rest of the paper, and work exclusively with nerves.

Lemma 7. *Let P be a set, and let $\boldsymbol{\mathcal{N}}$ be a collection of subsets of 2^P that satisfies the first three properties in the Proposition 6. Then there are a ground set $X \supset P$ and a convexity space on X so that $\boldsymbol{\mathcal{N}}(P) = \boldsymbol{\mathcal{N}}$.*

Proof. For an arbitrary family $\mathcal{F}$ let $C(\mathcal{F}) = \{\mathcal{F}' \in \boldsymbol{\mathcal{N}} : \mathcal{F} \subset \mathcal{F}'\}$, and denote by $\mathcal{C}$ the family of all the sets of the form $C(\mathcal{F})$. Put $X = \boldsymbol{\mathcal{N}}$. We claim that $\mathcal{C}$ forms a desired convexity space on X. It is clear that $\emptyset, X \in \mathcal{C}$. Since $C(\mathcal{F}_1) \cap C(\mathcal{F}_2) = C(\mathcal{F}_1 \cup \mathcal{F}_2)$, and similarly for intersections of more than two sets, the collection $\mathcal{C}$ indeed forms a convexity space on X. Define $\phi \colon P \to X$ by $\phi(p) = \mathcal{F}(p)$. The map ϕ is well-defined by property (N3), and provides the embedding of P into X. We need to check that $\boldsymbol{\mathcal{N}}(\phi(P)) = \phi(\boldsymbol{\mathcal{N}})$

Since $\mathcal{F}(p) \in C(\mathcal{F})$ if and only if $\mathcal{F} \subset \mathcal{F}(p)$, it follows that $\{\mathcal{F}(p_1), \ldots, \mathcal{F}(p_t)\} \subset C(\mathcal{F})$ precisely when $\mathcal{F} \subset \bigcap \mathcal{F}(p_i)$. Hence, if $P' \subset P$, then

$$\operatorname{conv}_{\mathcal{C}}(\phi(P')) = \bigcap_{\phi(P') \subset C(\mathcal{F})} C(\mathcal{F}) = \bigcap_{\mathcal{F} \subset \bigcap_{p \in P'} \mathcal{F}(p)} C(\mathcal{F}) = C\left(\bigcap_{p \in P'} \mathcal{F}(p)\right).$$

Hence, $\mathcal{F} \in \operatorname{conv}_{\mathcal{C}}(\phi(P'))$ if and only if $\{P : P' \subset P\} \subset \mathcal{F}$. The intersection $\bigcap_{P \in \mathcal{F}'} \operatorname{conv}_{\mathcal{C}}(\phi(P))$ is non-empty if and only if there is an $\mathcal{F} \in \boldsymbol{\mathcal{N}}$ so that $\hat{\mathcal{F}}' \subset \mathcal{F}$. Thus by the properties (N1) and (N2)

$$\bigcap_{P \in \mathcal{F}'} \operatorname{conv}_{\mathcal{C}}(\phi(P)) \neq \emptyset \iff F' \in \boldsymbol{\mathcal{N}}.$$

Therefore $\boldsymbol{\mathcal{N}}_{\mathcal{C}}(\phi(P)) = \phi(\boldsymbol{\mathcal{N}})$ as claimed. □

2 Counterexample to Eckhoff's conjecture

Proof of Theorem 2. We shall use the Lemma 7 to construct the requisite convexity space. Let $P = [3(k-1)+1]$. Consider the three kinds of families:

$$\begin{aligned}
A[x] &= \bigl\{\{x\}\bigr\} \cup \binom{P}{4}, \\
B[xy:zw] &= \bigl\{\{x,y\},\{z,w\}\bigr\} \cup \bigl\{S \in \binom{P}{3} : \{x,y,z,w\} \cap S \neq \emptyset\bigr\} \cup \binom{P}{4}, \quad \text{distinct } x,y,z,w \\
C[xy] &= \bigl\{\{x,y\}\bigr\} \cup \binom{P}{3}, \qquad x,y \text{ are distinct.}
\end{aligned}$$

Here x, y, z, w are elements of $P = [3(k-1)+1]$. Let $\hat{A[x]}, \hat{B}[xy : zw]$ and $\hat{C}[xy]$ be as in Proposition 6 property (N2). Let $\boldsymbol{\mathcal{N}}$ consist of all the families, $\hat{A[x]}, \hat{B}[xy : zw]$ and $\hat{C}[xy]$ and all their subfamilies. Let $\boldsymbol{\mathcal{N}}_0$ consist only of families $\hat{A}[x]$, $\hat{B}[xy : zw]$ and $\hat{C}[xy]$. As $\boldsymbol{\mathcal{N}}$ automatically satisfies properties (N1) and (N2) in Proposition 6 and $\mathcal{F}(p) \subset \hat{A}[p]$, by Lemma 7 it is a nerve of some convexity space. As $k \geq 3$, no family of the form $A[x], B[xy : zw]$ or $C[xy]$ contains t disjoint sets. From that it follows that none of $\hat{A}[x], \hat{B}[xy : zw]$ or $\hat{C}[xy]$ contain k disjoint sets either, and same holds for every family in $\boldsymbol{\mathcal{N}}$. Therefore, to establish the theorem it remains to verify the property (N5) with $r_2 = 4$.

As $\hat{A}$-, $\hat{B}$- and $\hat{C}$-families are the maximal families in $\boldsymbol{\mathcal{N}}$, it suffices to show that whenever $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_1, \ldots, \mathcal{F}_4\}$ is a collections of four families in $\boldsymbol{\mathcal{N}}_0$, then there is a partition $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_1 \cup \boldsymbol{\mathcal{F}}_2$ so that $(\bigcap \boldsymbol{\mathcal{F}}_1) \cup (\bigcap \boldsymbol{\mathcal{F}}_2)$ is contained in some $\mathcal{F} \in \boldsymbol{\mathcal{N}}_0$.

To every family $\mathcal{F}$ we associate a subset

$$e(\mathcal{F}) = \mathcal{F} \cap \binom{P}{2}.$$

That is

$$\begin{aligned}
e(\hat{A}[x]) &= \bigl\{\{x,y\} : y \in P \setminus \{x\}\bigr\} \\
e(\hat{B}[xy:zw]) &= \bigl\{\{x,y\},\{z,w\}\bigr\}, \\
e(\hat{C}[xy]) &= \bigl\{\{x,y\}\bigr\}.
\end{aligned}$$

Note that $e(\mathcal{F}_1 \cap \mathcal{F}_2) = e(\mathcal{F}_1) \cap e(\mathcal{F}_2)$. It is convenient think of $e(\mathcal{F})$ as an edge of a hypergraph on the ground set $\binom{P}{2}$.

Note that if $\mathcal{F}_1, \mathcal{F}_2 \in \boldsymbol{\mathcal{N}}_0$ are two distinct families, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is contained in a $\hat{C}$-set. Moreover, if $e(\mathcal{F}_1) \cap e(\mathcal{F}_2) = \emptyset$, then $\mathcal{F}_1 \cap \mathcal{F}_2$ in contained in $\binom{P}{3}$.

Suppose $\mathcal{F}_1, \ldots, \mathcal{F}_4$ are four families in $\boldsymbol{\mathcal{N}}_0$. If $e(\mathcal{F}_1) \cap e(\mathcal{F}_2) = \emptyset$, then $\mathcal{F}_1 \cap \mathcal{F}_2 \subset \binom{P}{3}$ and $\mathcal{F}_3 \cap \mathcal{F}_4 \subset \hat{C}[xy]$ for some x, y. Hence $(\mathcal{F}_1 \cap \mathcal{F}_2) \cup (\mathcal{F}_3 \cap \mathcal{F}_4) \subset \binom{P}{3} \cup \hat{C}[xy] = \hat{C}[xy]$. We may thus assume that $e(F_1) \cap e(F_2)$ is non-empty, and similarly for other pairs of sets $\mathcal{F}_1, \ldots, \mathcal{F}_4$.

There are five cases according to the number of $\hat{A}$-families among the four families $\mathcal{F}_1, \dots \mathcal{F}_4$.

There no $\hat{A}$-families: Since every two families meet, and $e(\mathcal{F}_1), \dots, e(\mathcal{F}_4)$ contain 1 or 2 vertices each, it follows that $e(\mathcal{F}_1), \dots, e(\mathcal{F}_4)$ must have a common vertex, say $\{1,2\} \in \binom{P}{2}$. Then $(\mathcal{F}_1 \cap \mathcal{F}_2) \cup (\mathcal{F}_3 \cap \mathcal{F}_4) \subset \hat{C}[12]$.

There is a single $\hat{A}$-family $\mathcal{F}_1$: As $e(\mathcal{F}_2)$, $e(\mathcal{F}_3)$ and $e(\mathcal{F}_4)$ pairwise meet, they either have a vertex in common, or $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ are $\hat{B}$-families, and $e(\mathcal{F}_2), e(\mathcal{F}_3), e(\mathcal{F}_4)$ form a triangle. However, they cannot form the triangle because $e(\mathcal{F}_1)$ would not meet each of $e(\mathcal{F}_2)$, $e(\mathcal{F}_3)$ and $e(\mathcal{F}_4)$. Thus, $e(\mathcal{F}_1) \cap \dots \cap e(\mathcal{F}_4)$ is non-empty, and equals to say $\{1,2\} \in \binom{P}{2}$. Then $(\mathcal{F}_1 \cap \mathcal{F}_2) \cup (\mathcal{F}_3 \cap \mathcal{F}_4) \subset \hat{C}[12]$.

There are two $\hat{A}$-families $\mathcal{F}_1$ and $\mathcal{F}_2$: The intersection $e(\mathcal{F}_3) \cap e(\mathcal{F}_4)$ contains just one element, say $\{x,y\}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are just $\hat{A}[x]$ and $\hat{A}[y]$, then $(\mathcal{F}_1 \cap \mathcal{F}_2) \cup (\mathcal{F}_3 \cap \mathcal{F}_4) \subset \hat{C}[xy]$. If $\mathcal{F}_1 = \hat{A}[z]$ and $z \notin \{x,y\}$, then it necessarily follows that $e(\mathcal{F}_3) = \big\{\{x,y\},\{z,w_3\}\big\}$ and $e(\mathcal{F}_4) = \big\{\{x,y\},\{z,w_4\}\big\}$ for some w_3 and w_4. Thus $\mathcal{F}_2$ is either $\hat{A}[x]$ or $\hat{A}[y]$. In either case $(\mathcal{F}_1 \cap F_3) \cup (\mathcal{F}_2 \cap \mathcal{F}_4) \subset \hat{B}[xy : zw_3]$.

There are three $\hat{A}$-families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$: As $e(\mathcal{F}_4)$ has to meet all of $e(\mathcal{F}_1)$, $e(F_2)$, $e(F_3)$, it must be that $\mathcal{F}_4$ is a $\hat{B}$-family, implying that $(\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3) \cup \mathcal{F}_4 = \mathcal{F}_4$.

All four families are $\hat{A}$-families: Say, $\mathcal{F}_1 = \hat{A}[x]$, $\mathcal{F}_2 = \hat{A}[y]$, $\mathcal{F}_3 = \hat{A}[z]$ and $\mathcal{F}_4 = \hat{A}[w]$. In that case $(\mathcal{F}_1 \cap \mathcal{F}_2) \cup (F_3 \cap \mathcal{F}_4) \subset \hat{B}[xy : zw]$. $\square$

3 Upper bound on Radon numbers

The main ingredient in the proof of theorem 1 is a version of Kruskal–Katona theorem from [Buk10]. A *d-dimensional r-uniform family* is a d-tuple of r-element sets. In other words, if we denote by $\binom{X}{r}$ the family of all r-element subsets of X, then d-dimensional r-uniform family is a subset of $\binom{X}{r}^d$. A *shadow* of such a family $\mathcal{F} \subset \binom{X}{r}^d$ is defined to be

$$\partial\mathcal{F} \stackrel{\text{def}}{=} \big\{(S_1 \setminus \{x_i\}, \dots, S_d \setminus \{x_d\}) : (S_1, \dots, S_d) \in \mathcal{F}, \text{and } x_i \in S_i \text{ for } i = 1, \dots, d\big\}.$$

Lemma 8 (Theorem 1 of [Buk10])**.** *Suppose $\mathcal{F} \subset \binom{X}{r}^d$ is a d-dimensional r-uniform family of size*

$$|\mathcal{F}| = \binom{x}{r}^d,$$

where $x \geq r$ is a real number. Then

$$|\partial\mathcal{F}| \geq \binom{x}{r-1}^d.$$

In addition to multidimensional Kruskal–Katona theorem, we shall need four lemmas. The first two lemmas are a bound on Turán numbers of hypergraphs and a bound on the independence numbers of graphs in which every subgraph have a large independence number.

Lemma 9 ([dC83]). *If H is an s-uniform hypergraph on n vertices with fewer than $\binom{l-1}{s-1}^{-1}\frac{n-l+1}{n-s+1}\binom{|H|}{s}$ edges, then H contains an independent set on l vertices.*

Lemma 10 (Special case of Theorem 2.1 from [AS07]). *Let $t < s \le 2s-3$, and let G be a graph on n vertices. Suppose that every set of size s contains an independent set of size t. Then G contains an independent set of size $n-s+1$.*

Our third lemma is purely computational. We say that a tuple $(S_1,\ldots,S_d) \in \binom{P}{a}^d$ is *r-good* if there are r pairwise disjoint sets $S_{i_1},\ldots,S_{i_r}$ among S's.

Lemma 11. *Let P be a finite set. There are at most*

$$C(d)(a^2/|P|)^{d-r+1}\binom{|P|}{a}^d$$

r-bad tuples in $\binom{P}{a}^d$, where $C(d)$ is a constant that depends only on d.

Proof. For $S=(S_1,\ldots,S_d) \in \binom{P}{a}^d$ let $G[S]$ be a graph on $\{1,\ldots,d\}$ with ij forming an edge if $S_i \cap S_j \neq \emptyset$. A tuple S is r-bad if and only if the independence number of $G[S]$ is less than r. Suppose that the largest forest in $G[S]$ has m edges, then by contracting these edges we obtain an independent set of size $d-m$. Thus if a tuple S is r-bad, then $G[S]$ contains a forest F with $d-r+1$ edges. We say that the forest F *witnesses* that S is r-bad.

Fix a forest F. We shall bound the number of r-bad tuples S for which F is a witness that S is r-bad. Let $v_1,\ldots,v_d$ be a relabelling of $\{1,\ldots,d\}$ so that in F the vertex v_i is adjacent to at most one vertex v_j with $j<i$. Pick $S_1,\ldots,S_d$ uniformly at random from $\binom{P}{a}$. If v_i is adjacent to some v_j with $j<i$ let E_i be the event that $S_i \cap S_j \neq \emptyset$. If v_j is adjacent to none v_j with $j<i$ let E_i be the event that holds with probability 1. Then

$$\Pr[F \text{ is a witness that } S \text{ is } r\text{-bad}] = \prod_{i=1}^{d} \Pr[E_i | E_1,\ldots,E_{i-1}] = \prod_{i=1}^{d}\Pr[E_i] \le (a^2/|P|)^{d-r+1}.$$

As the number of forests on d vertices depends only on d, the lemma follows by the union bound. ☐

Finally, the third lemma that we need is a restatement of Jamison's upper bound $r_{2^t} \le r_2^t$ in terms of nerves. We include the proof for completeness.

Lemma 12. *Suppose P a set in a convexity space, and $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{N}}(P)$ is its nerve. Then for every set $P' \subset P$ of size $|P'| = r_2^t$ there is a family $\mathcal{F} \in \boldsymbol{\mathcal{N}}$ containing 2^t disjoint subsets of P'.*

Proof. The proof is by induction on t. The base case $t=0$ is trivial. Suppose $t \ge 1$. Let $P' = P'_1 \cup \cdots \cup P'_{r_2}$ be a partition of P' into sets of size r_2^{t-1}. By the induction hypothesis, there are families $\mathcal{F}_1, \dots, \mathcal{F}_{r_2}$ such that each $\mathcal{F}_i$ contains 2^{t-1} disjoint subsets of P'_i. Let these subsets be $R_{i,1}, \dots, R_{i,2^{t-1}}$. By property (N5) of the proposition 6, there is a a set $I \subset [r_k]$ so that

$$\bigcap_{i \in I} \mathcal{F}_i \cup \bigcap_{i \notin I} \mathcal{F}_i \in \boldsymbol{\mathcal{N}}.$$

By property (N2) the intersection $R_j = \bigcap_{i \in I} R_{i,j}$ is in $\bigcap_{i \in I} \hat{\mathcal{F}}_i$ for each $j = 1, \dots, 2^{t-1}$. The sets R_j are 2^{t-1} disjoint subsets of $\bigcup_{i \in I} P'_i$. Similarly one obtains 2^{t-1} disjoint subsets of $\bigcup_{i \notin I} P'_i$, for the total of 2^t disjoint subsets of P'. □

Proof of theorem 1. It suffices to show that for every nerve $\boldsymbol{\mathcal{N}}$ on $|P| = k^2 \log^2 k$ points there are k disjoint sets $S_1, \dots, S_k \subset P$ and a family $\mathcal{F}$ that contains all of these sets.

For brevity we shall write $r = r_2$ and $t = 1 + \lceil \log_2 r \rceil$. Define a $(2r-3)$-dimensional family $\mathcal{T} \subset \binom{P}{r^t}^{2r-3}$ as follows: A tuple $(S_1, \dots, S_{2r-3}) \in \binom{P}{r^t}^{2r-3}$ is in $\mathcal{T}$ if there is a family $\mathcal{F} \in \boldsymbol{\mathcal{N}}$ such that $\{S_1, \dots, S_{2r-3}\} \subset \mathcal{F}$. Let $P_0 \subset P$ be any $(2r-3)r^t$-element subset of P. Let $P' \subset P_0$ be an arbitrary r^t-element subset of P_0. By the preceding lemma there is a family $\mathcal{F}$ that contains 2^t disjoint subsets of P'. Since $2r - 3 \le 2^t$, by property (N2) it follows that $\mathcal{F}$ contains $2r-3$ disjoint subsets of size r^t each that partition P_0. In other words, P_0 gives rise to at least one tuple in $\mathcal{T}$. Since P_0 is an arbitrary $(2r-3)r^t$-element subset of P, we conclude that

$$|\mathcal{T}| \ge \binom{|P|}{(2r-3)r^t} \ge c_1(r)|P|^{(2r-3)r^t} \ge \binom{|P|}{r^t}^{2r-3} - \binom{(1-c_2(r))|P|}{r^t}^{2r-3}$$

for some positive constants $c_1(r), c_2(r)$ that depend only on r.

Let $m = \lceil \log k / c_2(r) \rceil$. Define a $(2r-3)$-dimensional family $\mathcal{T}' \subset \binom{P}{mr^t}^{2r-3}$ in the same way as $\mathcal{T}$ was defined: namely, $S \in \mathcal{T}'$ if there is an $\mathcal{F} \in \boldsymbol{\mathcal{N}}$ such that $S \subset \mathcal{F}$. Note that the property (N2) implies that if $S \in \binom{P}{mr^t}^{2r-3}$ is not in $\mathcal{T}'$, then neither is any family obtained from S by removing some elements from each set in S. Lemma 8 applied to the complement of $\mathcal{T}'$ yields

$$|\mathcal{T}'| \ge \binom{|P|}{mr^{t+1}}^{2r-3} - \binom{(1-c_2(r))|P|}{mr^t}^{2r-3}.$$

Let $H \subset \binom{\binom{P}{mr^t}}{2r-3}$ be a $(2r-3)$-uniform hypergraph on $\binom{P}{mr^t}$ with edges

$$\{S_1, \dots, S_{2r-3}\} \in H \iff (S_1, \dots, S_{2r-3}) \in \mathcal{T}' \text{ and } (S_1, \dots, S_{2r-3}) \text{ is } r\text{-good}.$$

By Lemma 11, it follows that

$$\begin{aligned} |H| &\ge \frac{1}{(2r-3)!} \left(|\mathcal{T}'| - c_3(r)(m^2 r^{2t}/|P|)^{r-2} \binom{|P|}{mr^t}^{2r-3} \right) \\ &\ge \binom{\binom{|P|}{mr^t}}{2r-3} \left(1 - (1-c_2(r))^{(2r-3)mr^t} - c_4(r)(m^2/|P|)^{r-2} \right) \end{aligned}$$

Since $m > \log k/c_2(r)$, and $|P| \geq (9c_4(r))^{1/(r-2)} m^2 k^2$ it follows that the density of H is

$$|H| / \binom{\binom{|P|}{mr^t}}{2r-3} \geq 1 - k^{(2r-3)r^t} - (3k)^{-(2r-4)} \geq 1 - (2k)^{-(2r-4)}$$

for k large enough.

By Lemma 9 the hypergraph H contains a clique on $2k$ vertices. Let $S_1, \ldots, S_{2k} \in \binom{P}{mr^t}$ be the vertices of this clique. Since edges of H are r-good among every $2r-3$ of these $2k$ sets there are r that are pairwise disjoint. Thus, by Lemma 10 there are k of them, say $S_1, \ldots, S_k$, that are pairwise disjoint.

We claim that for every $I \subset [k]$ there is a family $\mathcal{F}_I \in \boldsymbol{\mathcal{N}}$ that contains S_i for every $i \in I$. The proof is by induction on $|I|$ starting with $|I| = 2r-3$. If $|I| = 2r-3$, then the claim holds because $\{S_i : i \in I\}$ is an edge in H. Suppose $|I| > 2r-3$. Pick any r distinct $|I|-1$-element subsets $I_1, \ldots, I_r$ of I. Then by by property (N5) applied to families $\mathcal{F}_{I_1}, \ldots, \mathcal{F}_{I_r}$ it follows that there is a $J \subset [r]$ so that $\mathcal{F} = \left(\bigcap_{j\in J} \mathcal{F}_{I_j}\right) \cup \left(\bigcap_{j\notin J} \mathcal{F}_{I_j}\right) \in \boldsymbol{\mathcal{N}}$. Since the family $\mathcal{F}$ contain $\mathcal{F}_i$ for every $i \in I$, we may put $\mathcal{F}_I = \mathcal{F}$.

Finally, the family $\mathcal{F}_{[k]}$ contains k disjoint sets $S_1, \ldots, S_k$, as required. □

4 Convexity spaces with $r_2 = 3$

The space with $r_2 = 3$ are especially nice because of the following lemma, which is implicit in [JW81].

Lemma 13. *Let P be a set in a convexity space with $r_2 = 3$, and let $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{N}}(P)$ be its nerve. Then there is a family $\mathcal{F}_p \in \boldsymbol{\mathcal{N}}$ for each $p \in P$, and these families satisfy*

(J1) $\{p\} \in \mathcal{F}_p$.

(J2) If p, q, r are any three points of P, then either $\{p, q\} \in \mathcal{F}_r$ or $\{p, r\} \in F_q$ or $\{q, r\} \in \mathcal{F}_p$.

(J3) If $\{q, r\} \in \mathcal{F}_p$ and $\{r, s\} \in \mathcal{F}_q$, then $\{r, s\} \in \mathcal{F}_p$.

Proof. Let $\mathcal{F}_p$ be a maximal family containing $\{p\}$. Then the other conditions follow from the property (N5) applied to the triple of families $\mathcal{F}_p, \mathcal{F}_q, \mathcal{F}_r$. □

Proof of Proposition 4. Let $I = \{(p, q, r) : p \in \operatorname{conv}\{q, r\}\}$. Since there are $\binom{n}{3}$ triples $\{p, q, r\}$, each of which contributes at least at least one element I, the proposition follows by the pigeonhole principle. □

Since Jamison's proof of Eckhoff's conjecture is especially short in the language of nerves, we include it:

Theorem 14. *If $r_2 = 3$, then $r_k \leq 2(k-1) + 1$.*

Proof. Suppose $|P| = 2(k-1)+1$. We shall show that one of $\mathcal{F}_p$ contains k pairwise disjoint sets. We claim that there is a pair of elements $p, q \in P$ so that $\{p, q\} \in \mathcal{F}_r$ for every $r \neq p, q$. Indeed, it is true if $|P| \leq 3$. If $|P| \geq 4$, and s is any element of $\mathcal{F}_p$, then by induction there is a $p, q \in P \setminus \{s\}$ so that $\{p, q\} \in \mathcal{F}_r$ for every $r \neq p, q, s$. If in addition $\{p, q\} \in \mathcal{F}_s$, then we are done. Otherwise by property (J2) either $\{p, s\} \in \mathcal{F}_q$ or $\{q, s\} \in \mathcal{F}_p$. Say $\{p, s\} \in \mathcal{F}_q$. Then by property (J3) applied to $\{p, q\} \in \mathcal{F}_r$ and either $\{p, s\} \in \mathcal{F}_q$ we conclude that $\{p, s\}$ is in every $\mathcal{F}_r$, $r \neq p, s$. The claim is proved.

Let p, q be a pair of element so that $\{p, q\} \in \mathcal{F}_r$ for $r \neq p, q$. By the induction hypothesis applied to $P \setminus \{p, q\}$ there is $r \in \{p, q\}$ so that $\mathcal{F}_r$ contains $k-1$ disjoint sets that are also disjoint from $\{p, q\}$. Together with $\{p, q\}$ these form a desired family of disjoint sets. $\square$

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