Average characteristic polynomials in the two-matrix model

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Abstract

The two-matrix model is defined on pairs of Hermitian matrices (M_1, M_2) of size $n \times n$ by the probability measure

$$\frac{1}{Z_n} \exp\left(\text{Tr}(-V(M_1) - W(M_2) + \tau M_1 M_2)\right) \ dM_1 \ dM_2,$$

where V and W are given potential functions and $\tau \in \mathbb{R}$. We study averages of products and ratios of characteristic polynomials in the two-matrix model, where both matrices M_1 and M_2 may appear in a combined way in both numerator and denominator. We obtain determinantal expressions for such averages. The determinants are constructed from several building blocks: the biorthogonal polynomials $p_n(x)$ and $q_n(y)$ associated to the two-matrix model; certain transformed functions $\widetilde{P}_n(w)$ and $\widetilde{Q}_n(v)$; and finally Cauchy-type transforms of the four Eynard-Mehta kernels $K_{1,1}$, $K_{1,2}$, $K_{2,1}$ and $K_{2,2}$. In this way we generalize known results for the 1-matrix model. Our results also imply a new proof of the Eynard-Mehta theorem for correlation functions in the two-matrix model, and they lead to a generating function for averages of products of traces.

Keywords: Two-matrix model, average characteristic polynomial, Eynard-Mehta theorem, biorthogonal polynomial, multiple orthogonal polynomial, Riemann-Hilbert problem, determinantal point process.

1 Introduction and statement of results

1.1 The two-matrix model

Let $\mathcal{H}_n$ be the space of Hermitian n by n matrices. The *two-matrix model* is defined on $\mathcal{H}_n \times \mathcal{H}_n$ by the probability measure

$$\frac{1}{Z_n} \exp\left(\text{Tr}(-V(M_1) - W(M_2) + \tau M_1 M_2)\right) \ dM_1 \ dM_2, \qquad (M_1, M_2) \in \mathcal{H}_n \times \mathcal{H}_n, \qquad (1.1)$$

where V and W are given real-valued *potential functions*, where $\tau \in \mathbb{R}$ is the *coupling constant*, Z_n is a normalization constant, Tr denotes the trace and $dM_1 dM_2$ is the Lebesgue measure on $\mathcal{H}_n \times \mathcal{H}_n$, see [29]. The matrix model (1.1) is probably the most popular instance of a two-matrix model in the literature, see e.g. [1, 5, 6, 7, 9, 17, 18, 19, 20, 21, 24, 28, 29, 30] among many others. We may mention however that other kinds of two-matrix models exist as well, see e.g. [8] and the references therein.

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The normalization constant Z_n in (1.1) is called the *partition function*. It is defined as the total integral

$$Z_n = \int_{\mathcal{H}_n} \int_{\mathcal{H}_n} \exp\left(\mathrm{Tr}(-V(M_1) - W(M_2) + \tau M_1 M_2)\right) \, dM_1 \, dM_2. \tag{1.2}$$

We assume that the potential functions V and W have sufficient increase at infinity in order that the integral (1.2) is convergent. Throughout this paper, we will assume that V and W are polynomials of even degree with positive leading coefficients.

Due to the *Harish-Chandra formula* (or *Itzykson-Zuber formula*) [29], integrals over the two-matrix model (1.1) can often be rewritten in terms of the eigenvalues of M_1 and M_2. We need the following version of this result. Assume that the functions $f, g : \mathcal{H}_n \to \mathbb{C}$ depend only on the eigenvalues $(\lambda_i)_{i=1}^n$ of M_1 and $(\mu_i)_{i=1}^n$ of M_2, respectively, (taking into account eigenvalue multiplicities), in the sense that

$$f(M_1) = \tilde{f}(\lambda_1, \ldots, \lambda_n), \qquad g(M_2) = \tilde{g}(\mu_1, \ldots, \mu_n),$$

where $\tilde{f}, \tilde{g} : \mathbb{R}^n \to \mathbb{C}$ are symmetric functions of n variables. Then

$$\frac{1}{Z_n} \int_{\mathcal{H}_n} \int_{\mathcal{H}_n} f(M_1) g(M_2) e^{\mathrm{Tr}(-V(M_1) - W(M_2) + \tau M_1 M_2)} \, dM_1 \, dM_2$$

$$= \frac{1}{\tilde{Z}_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \tilde{f}(\lambda_1, \ldots, \lambda_n) \tilde{g}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \left(e^{-V(\lambda_i)} e^{-W(\mu_i)} \right)$$

$$\times \Delta(\lambda) \Delta(\mu) \det(e^{\tau \lambda_i \mu_j})_{i,j=1}^n \prod_{i=1}^{n} (d\lambda_i \, d\mu_i), \quad (1.3)$$

for a new normalization constant $\tilde{Z}_n$, see [29]. The factors $\Delta(\lambda)$ and $\Delta(\mu)$ in (1.3) denote the Vandermonde determinants $\prod_{1 \leq i < j \leq n}(\lambda_j - \lambda_i)$ and $\prod_{1 \leq i < j \leq n}(\mu_j - \mu_i)$ respectively. In (1.3) we also assume that $\tilde{f}$ and $\tilde{g}$ are such that the integrals converge.

1.2 Biorthogonal polynomials and kernels

It is known that the two-matrix model (1.1) can be analyzed by means of biorthogonal polynomials and kernel functions that we describe below [19, 20, 29]. We will also introduce certain (non-standard) Cauchy-type transforms of these objects, which will be needed in the statement of our main theorems.

The monic *biorthogonal polynomials* $\{p_i(x)\}_{i=0}^\infty$ and $\{q_j(y)\}_{j=0}^\infty$ are defined as the monic polynomials

$$p_i(x) = x^i + O(x^{i-1}), \qquad q_j(y) = y^j + O(y^{j-1}),$$

that satisfy the orthogonality relations

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_i(x) q_j(y) e^{-V(x) - W(y) + \tau xy} \, dx \, dy = 0, \qquad i \neq j. \tag{1.4}$$

These polynomials exist and are unique, see [19].

We define the *transformed functions* P_i and Q_j by

$$P_i(y) = \int_{-\infty}^{\infty} p_i(x) e^{-V(x) - W(y) + \tau xy} \, dx, \tag{1.5}$$

$$Q_j(x) = \int_{-\infty}^{\infty} q_j(y) e^{-V(x) - W(y) + \tau xy} \, dy, \tag{1.6}$$

and we let h_i^2 be defined by

$$h_i^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_i(x)q_i(y)e^{-V(x)-W(y)+\tau xy} \ dx \ dy. \tag{1.7}$$

The integral (1.7) is positive, see [19].

The *Eynard-Mehta kernels* $K_{1,1}$, $K_{1,2}$, $K_{2,1}$ and $K_{2,2}$ are defined as [20, 11]

$$K_{1,1}(x_1, x_2) = \sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x_1)Q_i(x_2), \tag{1.8}$$

$$K_{1,2}(x, y) = \sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x)q_i(y), \tag{1.9}$$

$$K_{2,1}(y, x) = \left(\sum_{i=0}^{n-1} \frac{1}{h_i^2} P_i(y)Q_i(x)\right) - e^{-V(x)-W(y)+\tau xy}, \tag{1.10}$$

$$K_{2,2}(y_1, y_2) = \sum_{i=0}^{n-1} \frac{1}{h_i^2} P_i(y_1)q_i(y_2). \tag{1.11}$$

It is known that the correlation functions for the two-matrix model (1.1) can be written as determinants of matrices built out of the four Eynard-Mehta kernels; see [20] and see also Section 3.2 below.

In what follows, we will need certain Cauchy-type transforms of the above objects. First we define the Cauchy transforms $\widetilde{P}_i(w)$ and $\widetilde{Q}_j(v)$ of the functions P_i and Q_j in (1.5)–(1.6):

$$\widetilde{P}_i(w) = \int_{-\infty}^{\infty} \frac{P_i(\eta)}{w - \eta} \ d\eta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{p_i(\xi)e^{-V(\xi)-W(\eta)+\tau\xi\eta}}{w - \eta} \ d\xi \ d\eta, \tag{1.12}$$

$$\widetilde{Q}_j(v) = \int_{-\infty}^{\infty} \frac{Q_j(\xi)}{v - \xi} \ d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{q_j(\eta)e^{-V(\xi)-W(\eta)+\tau\xi\eta}}{v - \xi} \ d\xi \ d\eta. \tag{1.13}$$

We will also need the following Cauchy-type transforms of the Eynard-Mehta kernels:

$$\widetilde{K}_{1,1}(x, v) = \frac{1}{x - v} \int_{-\infty}^{\infty} \frac{x - \xi}{v - \xi} K_{1,1}(x, \xi) \ d\xi, \tag{1.14}$$

$$\widetilde{K}_{1,2}(x, y) = K_{1,2}(x, y), \tag{1.15}$$

$$\widetilde{K}_{2,1}(w, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{K_{2,1}(\eta, \xi)}{(w - \eta)(v - \xi)} \ d\xi \ d\eta, \tag{1.16}$$

$$\widetilde{K}_{2,2}(w, y) = \frac{1}{y - w} \int_{-\infty}^{\infty} \frac{y - \eta}{w - \eta} K_{2,2}(\eta, y) \ d\eta. \tag{1.17}$$

See Section 2.4 for several properties of the kernels $\widetilde{K}_{i,j}$, $i, j = 1, 2$. Note that in the above formulas, we consistently use 'ξ' and 'η' to denote integration variables which play the role of an 'x-variable' or a 'y-variable' respectively.

The following lemma gives some alternative expressions for the kernels $\widetilde{K}_{i,j}$, $i, j = 1, 2$.

Lemma 1.1. *(Summation formulas for the kernels $\widetilde{K}_{i,j}$:) We have the following equivalent*

expressions for the kernels $\widetilde{K}_{i,j}$ in (1.14)–(1.17):

$$\widetilde{K}_{1,1}(x,v) = \left(\sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x)\widetilde{Q}_i(v) \right) - \frac{1}{v-x}, \tag{1.18}$$

$$\widetilde{K}_{1,2}(x,y) = \sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x)q_i(y), \tag{1.19}$$

$$\widetilde{K}_{2,1}(w,v) = \left(\sum_{i=0}^{n-1} \frac{1}{h_i^2} \widetilde{P}_i(w)\widetilde{Q}_i(v) \right) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-V(\xi)-W(\eta)+\tau\xi\eta}}{(w-\eta)(v-\xi)}\, d\xi\, d\eta, \tag{1.20}$$

$$\widetilde{K}_{2,2}(w,y) = \left(\sum_{i=0}^{n-1} \frac{1}{h_i^2} \widetilde{P}_i(w)q_i(y) \right) - \frac{1}{w-y}, \tag{1.21}$$

where we use the notations in (1.12)–(1.13).

Lemma 1.1 is proved in Section 2.1.

1.3 Average characteristic polynomials

Let the matrices $M_1, M_2 \in \mathcal{H}_n$ be distributed according to the two-matrix model (1.1). The *average characteristic polynomial* of the matrix M_1 is defined by (the notation in the left hand side is explained below)

$$P^{[1,0,0,0]}(x) := \frac{1}{Z_n} \int_{\mathcal{H}_n}\int_{\mathcal{H}_n} \det(xI_n - M_1)\, e^{\mathrm{Tr}(-V(M_1)-W(M_2)+\tau M_1 M_2)}\, dM_1\, dM_2, \tag{1.22}$$

where I_n denotes the identity matrix of size n. Similarly, the average characteristic polynomial of the matrix M_2 is defined by

$$P^{[0,1,0,0]}(y) := \frac{1}{Z_n} \int_{\mathcal{H}_n}\int_{\mathcal{H}_n} \det(yI_n - M_2)\, e^{\mathrm{Tr}(-V(M_1)-W(M_2)+\tau M_1 M_2)}\, dM_1\, dM_2. \tag{1.23}$$

In this paper, we consider more general versions of (1.22)–(1.23). We are interested in averages of products and ratios of characteristic polynomials, where both matrices M_1 and M_2 may appear in a combined way in both numerator and denominator. Our object of study is of the form

$$P_n^{[I,J,K,L]}(x_1,\ldots,x_I; y_1,\ldots,y_J; v_1,\ldots,v_K; w_1,\ldots,w_L) :=$$
$$\frac{1}{Z_n} \int_{\mathcal{H}_n}\int_{\mathcal{H}_n} \frac{\prod_{i=1}^{I}\det(x_iI_n - M_1)\prod_{j=1}^{J}\det(y_jI_n - M_2)}{\prod_{k=1}^{K}\det(v_kI_n - M_1)\prod_{l=1}^{L}\det(w_lI_n - M_2)}\, e^{\mathrm{Tr}(-V(M_1)-W(M_2)+\tau M_1 M_2)}\, dM_1\, dM_2. \tag{1.24}$$

Here $I, J, K, L \in \mathbb{N}\cup\{0\}$ are nonnegative integers and we assume that $x_1,\ldots,x_I, y_1,\ldots,y_J \in \mathbb{C}$, that $v_1,\ldots,v_K, w_1,\ldots,w_L \in \mathbb{C}\setminus\mathbb{R}$, that the numbers in the set $(x_1,\ldots,x_I,\ v_1,\ldots,v_K)$ are pairwise distinct, and similarly the numbers in the set $(y_1,\ldots,y_J,\ w_1,\ldots,w_L)$ are pairwise distinct. The numbers x_i, y_j, v_k, w_l are sometimes called *external sources* [4].

Due to (1.3), the expression (1.24) can be written alternatively as

$$
P_n^{[I,J,K,L]}(x_1, \ldots, x_I; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_L)
$$
$$
= \frac{1}{\tilde{Z}_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{m=1}^{n} \left(\frac{\prod_{i=1}^{I}(x_i - \lambda_m) \prod_{j=1}^{J}(y_j - \mu_m)}{\prod_{k=1}^{K}(v_k - \lambda_m) \prod_{l=1}^{L}(w_l - \mu_m)} \right)
$$
$$
\times \prod_{i=1}^{n} \left(e^{-V(\lambda_i)} e^{-W(\mu_i)} \right) \Delta(\lambda) \Delta(\mu) \det(e^{\tau \lambda_i \mu_j})_{i,j=1}^{n} \prod_{i=1}^{n} (d\lambda_i \, d\mu_i). \quad (1.25)
$$

In principle, an expression for (1.24) can be obtained as follows: if we define the *perturbed* potentials

$$
\widehat{V}(x) = V(x) - \log \left(\frac{\prod_{i=1}^{I}(x_i - x)}{\prod_{k=1}^{K}(v_j - x)} \right), \qquad \widehat{W}(y) = W(y) - \log \left(\frac{\prod_{j=1}^{J}(y_j - y)}{\prod_{l=1}^{L}(w_l - y)} \right),
$$

then (1.24) is just the scaled partition function corresponding to these perturbed potentials:

$$
P_n^{[I,J,K,L]}(x_1, \ldots, x_I; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_L) = \frac{\widehat{Z}_n}{Z_n}, \qquad (1.26)
$$

with

$$
\widehat{Z}_n := \int_{\mathcal{H}_n} \int_{\mathcal{H}_n} \exp \left(\mathrm{Tr}(-\widehat{V}(M_1) - \widehat{W}(M_2) + \tau M_1 M_2) \right) \, dM_1 \, dM_2.
$$

Our interest, however, is to express (1.24) in terms of the *unperturbed* potentials V and W. In this way we will obtain analogues of known results for the 1-matrix model, which go back to Brézin-Hikami [13], Fyodorov-Strahov [22, 32] and others [3, 4, 12, 25, 26].

To start with, we consider the case where only one external source is present.

Theorem 1.2. *(Average characteristic polynomials of M_1 and M_2:) For the average characteristic polynomials in (1.22)–(1.23), we have*

$$
P_n^{[1,0,0,0]}(x) = p_n(x),
$$
$$
P_n^{[0,1,0,0]}(y) = q_n(y),
$$

where p_n and q_n denote the biorthogonal polynomials in (1.4).

Theorem 1.3. *(Average inverse characteristic polynomials of M_1 and M_2:) With (I, J, K, L) equal to $(0, 0, 1, 0)$ or $(0, 0, 0, 1)$ in (1.24), we have*

$$
P_n^{[0,0,1,0]}(v) = \frac{\widetilde{Q}_{n-1}(v)}{h_{n-1}^2},
$$

$$
P_n^{[0,0,0,1]}(w) = \frac{\widetilde{P}_{n-1}(w)}{h_{n-1}^2},
$$

where we use the notations in (1.7) and (1.12)–(1.13).

Theorems 1.2 and 1.3 will be proved in Section 2.2. We note that the proofs will be quite different from the proofs of the corresponding statements for the Hermitian 1-matrix model, see [3, 12, 13, 22, 32]. The latter proofs typically use a Heine-type representation of the orthogonal

polynomials on the real line in terms of a moment (Hankel) determinant [33]. Such formulas also exists in the biorthogonal case [19], but it seems that a straightforward generalization of the arguments for the 1-matrix case is not available. Instead, our proofs of Theorems 1.2 and 1.3 will rely on the *multiple* orthogonality properties of p_n and q_n due to Kuijlaars-McLaughlin [28].

Next we consider the case where there are exactly two external sources in (1.24).

Theorem 1.4. *(Two external sources:) With (I, J, K, L) equal to $(1, 0, 1, 0)$, $(1, 1, 0, 0)$, $(0, 0, 1, 1)$ or $(0, 1, 0, 1)$ in (1.24), we have*

$$
\begin{aligned}
P_n^{[1,0,1,0]}(x, v) &= (x - v)\widetilde{K}_{1,1}(x, v), \\
P_n^{[1,1,0,0]}(x, y) &= h_n^2 \widetilde{K}_{1,2}(x, y), \\
P_n^{[0,0,1,1]}(v, w) &= -h_{n-1}^{-2}\widetilde{K}_{2,1}(w, v), \\
P_n^{[0,1,0,1]}(y, w) &= (y - w)\widetilde{K}_{2,2}(w, y),
\end{aligned}
$$

where we use the notations in (1.7) and (1.18)–(1.21), but with the index n in (1.19) and (1.20) replaced by $n + 1$ and $n - 1$ respectively.

Theorem 1.4 is a consequence of the more general Theorem 1.6 below.

Finally we consider the general case in (1.24). We first state the result when all external sources are in the numerator.

Theorem 1.5. *(External sources in numerator only:) Assume that $K = L = 0$ in (1.24), i.e., (1.24) has external sources in the numerator only. Also suppose that $I \geq J$. Then*

$$
P_n^{[I,J,0,0]}(x_1, \ldots, x_I; y_1, \ldots, y_J) = \frac{\prod_{i=0}^{J-1} h_{n+i}^2}{\prod_{1 \leq i < j \leq I}(x_j - x_i)\prod_{1 \leq i < j \leq J}(y_j - y_i)}
$$
$$
\times \det \begin{pmatrix}
\widetilde{K}_{1,2}(x_1, y_1) & \ldots & \widetilde{K}_{1,2}(x_1, y_J) & p_{n+J}(x_1) & \ldots & p_{n+I-1}(x_1) \\
\vdots & & \vdots & \vdots & & \vdots \\
\widetilde{K}_{1,2}(x_I, y_1) & \ldots & \widetilde{K}_{1,2}(x_I, y_J) & p_{n+J}(x_I) & \ldots & p_{n+I-1}(x_I)
\end{pmatrix}. \quad (1.27)
$$

Here $\widetilde{K}_{1,2}$ is as defined in (1.19) but with n replaced by $n + p$ where $p \in \{J, \ldots, I\}$ may be any fixed number.

Theorem 1.5 will be proved in Section 2.3. The proof will be based on formulas of Christoffel type [3, 33].

In the statement of Theorem 1.5 we assume that the kernel $\widetilde{K}_{1,2}$ is defined in (1.19) but with n replaced by $n + p$ for an arbitrary but fixed number $p \in \{J, \ldots, I\}$. The fact that all values of $p \in \{J, \ldots, I\}$ give the same determinant can be seen by applying elementary column operations to (1.27). More precisely, to the jth column of (1.27), $j \in \{1, \ldots, J\}$, we may add a suitable linear combination of the last $I - J$ columns.

Now we turn to the general case.

Theorem 1.6. *(The general case:) Suppose that $I - K \geq J - L \geq -n$. Then*

$$P_n^{[I,J,K,L]}(x_1, \ldots, x_I; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_L)$$

$$= C \frac{\prod_{i,k}(x_i - v_k)}{\prod_{1 \leq i < j \leq I}(x_j - x_i) \prod_{1 \leq k < l \leq K}(v_k - v_l)} \frac{\prod_{j,l}(y_j - w_l)}{\prod_{1 \leq i < j \leq J}(y_j - y_i) \prod_{1 \leq k < l \leq L}(w_k - w_l)}$$

$$\times \det \begin{pmatrix} \widetilde{K}_{1,1}(x_1, v_1) & \ldots & \widetilde{K}_{1,1}(x_I, v_1) & \widetilde{K}_{2,1}(w_1, v_1) & \ldots & \widetilde{K}_{2,1}(w_L, v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,1}(x_1, v_K) & \ldots & \widetilde{K}_{1,1}(x_I, v_K) & \widetilde{K}_{2,1}(w_1, v_K) & \ldots & \widetilde{K}_{2,1}(w_L, v_K) \\ \widetilde{K}_{1,2}(x_1, y_1) & \ldots & \widetilde{K}_{1,2}(x_I, y_1) & \widetilde{K}_{2,2}(w_1, y_1) & \ldots & \widetilde{K}_{2,2}(w_L, y_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,2}(x_1, y_J) & \ldots & \widetilde{K}_{1,2}(x_I, y_J) & \widetilde{K}_{2,2}(w_1, y_J) & \ldots & \widetilde{K}_{2,2}(w_L, y_J) \\ p_{n+J-L}(x_1) & \ldots & p_{n+J-L}(x_I) & \widetilde{P}_{n+J-L}(w_1) & \ldots & \widetilde{P}_{n+J-L}(w_L) \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{n+I-K-1}(x_1) & \ldots & p_{n+I-K-1}(x_I) & \widetilde{P}_{n+I-K-1}(w_1) & \ldots & \widetilde{P}_{n+I-K-1}(w_L) \end{pmatrix}, \quad (1.28)$$

with

$$C = \begin{cases} (-1)^{(I+K)L} \prod_{i=0}^{J-L-1} h_{n+i}^2, & \text{if } J - L \geq 0, \\ (-1)^{(I+K)L} \prod_{i=J-L}^{-1} h_{n+i}^{-2}, & \text{if } J - L < 0. \end{cases}$$

Here the kernels $\widetilde{K}_{i,j}$ are as defined in (1.18)–(1.21) but with n replaced by $n + p$ where $p \in \{J - L, \ldots, I - K\}$ may be any fixed number.

  Similarly, if $J - L \geq I - K \geq -n$ then

$$P_n^{[I,J,K,L]}(x_1, \ldots, x_I; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_L)$$

$$= C \frac{\prod_{i,k}(x_i - v_k)}{\prod_{1 \leq i < j \leq I}(x_j - x_i) \prod_{1 \leq k < l \leq K}(v_k - v_l)} \frac{\prod_{j,l}(y_j - w_l)}{\prod_{1 \leq i < j \leq J}(y_j - y_i) \prod_{1 \leq k < l \leq L}(w_k - w_l)}$$

$$\times \det \begin{pmatrix} \widetilde{K}_{1,1}(x_1, v_1) & \ldots & \widetilde{K}_{1,1}(x_1, v_K) & \widetilde{K}_{1,2}(x_1, y_1) & \ldots & \widetilde{K}_{1,2}(x_1, y_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,1}(x_I, v_1) & \ldots & \widetilde{K}_{1,1}(x_I, v_K) & \widetilde{K}_{1,2}(x_I, y_1) & \ldots & \widetilde{K}_{1,2}(x_I, y_J) \\ \widetilde{K}_{2,1}(w_1, v_1) & \ldots & \widetilde{K}_{2,1}(w_1, v_K) & \widetilde{K}_{2,2}(w_1, y_1) & \ldots & \widetilde{K}_{2,2}(w_1, y_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{2,1}(w_L, v_1) & \ldots & \widetilde{K}_{2,1}(w_L, v_K) & \widetilde{K}_{2,2}(w_L, y_1) & \ldots & \widetilde{K}_{2,2}(w_L, y_J) \\ \widetilde{Q}_{n+I-K}(v_1) & \ldots & \widetilde{Q}_{n+I-K}(v_K) & q_{n+I-K}(y_1) & \ldots & q_{n+I-K}(y_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{Q}_{n+J-L-1}(v_1) & \ldots & \widetilde{Q}_{n+J-L-1}(v_K) & q_{n+J-L-1}(y_1) & \ldots & q_{n+J-L-1}(y_J) \end{pmatrix}, \quad (1.29)$$

with

$$C = \begin{cases} (-1)^{(I+K)L} \prod_{i=0}^{I-K-1} h_{n+i}^2, & \text{if } I - K \geq 0, \\ (-1)^{(I+K)L} \prod_{i=I-K}^{-1} h_{n+i}^{-2}, & \text{if } I - K < 0. \end{cases}$$

Here $\widetilde{K}_{i,j}$ are as in (1.18)–(1.21) but with n replaced by $n + p$ for any arbitrary but fixed number $p \in \{I - K, \ldots, J - L\}$.

Note that it would be more natural to write the matrix in (1.28) in its transposed form; we have written it in its present form because of typographical reasons.

Theorem 1.6 will be proved in Section 2.5. The proof will use Theorem 1.5 together with a mechanism to transform an external source in the matrix M_1 (or M_2) in the *numerator*, into an external source in the matrix M_2 (or M_1 respectively) in the *denominator*.

In (1.28) we assume that the kernels $\widetilde{K}_{i,j}$ are defined in (1.18)–(1.21) but with n replaced by $n + p$ for an arbitrary but fixed number $p \in \{J - L, \ldots, I - K\}$. The fact that all values of $p \in \{J - L, \ldots, I - K\}$ give the same determinant can be seen by applying elementary row operations to (1.28). More precisely, to the jth row of (1.28), $j \in \{1, \ldots, J + K\}$, we may add a suitable linear combination of rows $J + K + 1, \ldots, I + L$, taking into account (1.18)–(1.21). A similar remark applies to (1.29).

In the special case where $J = L = 0$ in (1.24), i.e., when there are external sources in the matrix M_1 only, then Theorem 1.6 could be obtained from the Riemann-Hilbert characterization in Section 2.3 together with the results in [15]; see also Remark 2.5 in Section 2.3. A similar remark holds of course if $I = K = 0$, i.e., when there are external sources in the matrix M_2 only. We will not use this approach however in the proofs.

In Theorem 1.6 we assume that

$$\min(I - K, J - L) \geq -n. \tag{1.30}$$

We believe that this assumption is sufficiently mild to cover most of the applications. We also expect that similar determinantal formulae may exist if the assumption (1.30) fails. See [4] for the statement of such formulae in a similar context, requiring a lot of case distinctions to be made. We will not address this issue in this paper.

We wish to point out that the above formulas have a lot of similarity with those obtained in the physical paper [4] by Bergère in the context of the *normal 1-matrix model*. Recall that the normal 1-matrix model is defined by the probability distribution

$$\frac{1}{Z_n} e^{-\mathrm{Tr}(V(M,M^*))} \, dM, \tag{1.31}$$

on the space $\mathcal{N}_n$ of all normal $n \times n$ matrices M. Here the superscript * denotes the Hermitian conjugate, and the potential function V in (1.31) is such that its trace is *real-valued* and depends only on the eigenvalues of M. In this context, one has the following analogue of (1.3):

$$\frac{1}{Z_n} \int_{\mathcal{N}_n} f(M)g(M^*)e^{-\mathrm{Tr}(V(M,M^*))} \, dM$$
$$= \frac{1}{\tilde{Z}_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \tilde{f}(\lambda_1, \ldots, \lambda_n)\tilde{g}(\overline{\lambda_1}, \ldots, \overline{\lambda_n}) \prod_{i=1}^{n} \left(e^{-V(\lambda_i, \overline{\lambda_i})} \right) \Delta(\lambda)\Delta(\overline{\lambda}) \prod_{i=1}^{n} d\lambda_i, \quad (1.32)$$

for functions f and g which depend only on the eigenvalues and which are such that the integrals converge. Here the bar denotes complex conjugation.

In the physical paper [4], Bergére obtains determinantal formulae for averages of products and ratios of characteristic polynomials over the normal matrix model (1.31). His formulas are written as determinants constructed from (bi)orthogonal polynomials and kernels in a similar way as in our results. We should stress however that the two-matrix model (1.1) and the normal 1-matrix model (1.31) are truly different models, exhibiting very different properties. Viewed from this perspective, the similarity of our results with those of Bergère [4] may appear rather remarkable. It would be interesting to obtain a better understanding on this issue.

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1.4 Organization of this paper

The rest of this paper is organized as follows. In Section 2 we prove the main theorems. In Section 3 we give two applications: a proof of the Eynard-Mehta theorem for correlation functions, and a formula for averages of products of traces. Finally, Section 4 contains some concluding remarks on the more general model of matrices coupled in a chain.

2 Proofs

In this section we prove the main theorems. This section is organized as follows. In Section 2.1 we prove Lemma 1.1. Section 2.2 contains the proof of Theorems 1.2 and 1.3. In Section 2.3 we prove Theorem 1.5. In Section 2.4 we discuss several properties of the kernels $\widetilde{K}_{i,j}$. In Section 2.5 we use these properties to prove the general result in Theorem 1.6.

2.1 Proof of Lemma 1.1

Equations (1.19) and (1.20) are immediate by the definitions. Now let us prove (1.18). Using the decomposition $x - \xi = (x - v) + (v - \xi)$ in the integrand in (1.14), we obtain

$$
\begin{aligned}
\widetilde{K}_{1,1}(x,v) &= \int_{-\infty}^{\infty} \frac{1}{v-\xi} K_{1,1}(x,\xi) \, d\xi - \frac{1}{v-x} \int_{-\infty}^{\infty} K_{1,1}(x,\xi) \, d\xi \\
&= \int_{-\infty}^{\infty} \frac{1}{v-\xi} K_{1,1}(x,\xi) \, d\xi - \frac{1}{v-x},
\end{aligned} \tag{2.1}
$$

where we used that

$$
\begin{aligned}
\int_{-\infty}^{\infty} K_{1,1}(x,\xi) \, d\xi &:= \int_{-\infty}^{\infty} \sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x) Q_i(\xi) \, d\xi \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x) q_i(\eta) e^{-V(\xi)-W(\eta)+\tau\xi\eta} \, d\xi \, d\eta \\
&= \sum_{i=0}^{n-1} p_i(x) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{h_i^2} p_0(\xi) q_i(\eta) e^{-V(\xi)-W(\eta)+\tau\xi\eta} \, d\xi \, d\eta \right) \\
&= 1,
\end{aligned} \tag{2.2}
$$

due to the biorthogonality relations and the fact that $p_0(\xi) \equiv 1$. Now by inserting (1.8) in (2.1), we find that

$$
\widetilde{K}_{1,1}(x,v) = \sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x) \left(\int_{-\infty}^{\infty} \frac{1}{v-\xi} Q_i(\xi) \, d\xi \right) - \frac{1}{v-x},
$$

which is (1.18). The proof of (1.21) is similar, this time using the following analogue of (2.1):

$$
\widetilde{K}_{2,2}(w,y) = \int_{-\infty}^{\infty} \frac{1}{w-\eta} K_{2,2}(\eta,y) \, d\eta - \frac{1}{w-y}. \tag{2.3}
$$

$\square$

Remark 2.1. (Reproducing properties of $K_{1,1}$ and $K_{2,2}$:) The argument used to obtain (2.2) in the above proof shows more generally that

$$\int_{-\infty}^{\infty} p(\xi)K_{1,1}(x,\xi) \, d\xi = p(x) \tag{2.4}$$

and similarly

$$\int_{-\infty}^{\infty} q(\eta)K_{2,2}(\eta,y) \, d\eta = q(y) \tag{2.5}$$

whenever p (or q) is equal to one of the biorthogonal polynomials p_i (or q_i respectively) with $i = 0, \ldots, n-1$. By linearity, (2.4)–(2.5) then remain true for any polynomials p and q of degree at most $n - 1$.

We also remark that in addition to (2.4)–(2.5), there are many other formulas of reproducing type for the kernels $K_{i,j}$, $i, j = 1, 2$. We prefer to state them not all separately but will sometimes encounter them in the proofs below.

2.2 Proofs of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3. The proofs will use the Riemann-Hilbert characterization of Kuijlaars-McLaughlin [28].

We recall our assumption that the potential W is a polynomial of even degree:

$$W(y) = c_q y^q + O(y^{q-2}), \qquad c_q > 0, \qquad q := \deg W \in 2\mathbb{N}. \tag{2.6}$$

Following [28], we define the weight functions

$$w_j(x) := \int_{-\infty}^{\infty} y^j e^{-V(x)-W(y)+\tau xy} \, dy, \qquad j = 0, \ldots, q-2. \tag{2.7}$$

We stack these weight functions together in the row vector

$$\mathbf{w}(x) := \begin{pmatrix} w_0(x) & \ldots & w_{q-2}(x) \end{pmatrix}. \tag{2.8}$$

The Riemann-Hilbert problem (RH problem) of Kuijlaars-McLaughlin [28] is as follows. See also [7, 9, 19, 24] for alternative Riemann-Hilbert problems for the two-matrix model.

RH problem 2.2. *We look for a matrix-valued function $Y(z)$ of size q by q such that*

(1) $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;

(2) For $x \in \mathbb{R}$, it holds that

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \mathbf{w}(x) \\ 0 & I_{q-1} \end{pmatrix}, \tag{2.9}$$

where I_{q-1} denotes the identity matrix of size $q - 1$, where $\mathbf{w}(x)$ is defined in (2.8), and where the notation $Y_+(x), Y_-(x)$ denotes the limit of $Y(z)$ with z approaching $x \in \mathbb{R}$ from the upper or lower half plane in $\mathbb{C}$, respectively;

(3) As $z \to \infty$, we have that

$$Y(z) = (I_q + O(1/z)) \operatorname{diag}(z^n, z^{-n_0}, \ldots, z^{-n_{q-2}}), \tag{2.10}$$

where $n_k := \lfloor \frac{n+q-2-k}{q-1} \rfloor$, $k = 0, \ldots, q-2$.

Similarly, we may consider the following dual RH problem.

RH problem 2.3. *We look for a matrix-valued function $X(z)$ of size q by q such that*

(1) $X(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;

(2) For $x \in \mathbb{R}$, it holds that

$$X_+(x) = X_-(x) \begin{pmatrix} I_{q-1} & \mathbf{w}^T(x) \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

where the superscript ^{T} denotes the transpose;

(3) As $z \to \infty$, we have that

$$X(z) = (I_q + O(1/z)) \operatorname{diag}(z^{n_0}, \dots, z^{n_{q-2}}, z^{-n}). \tag{2.12}$$

Partition the matrix $Y = Y(z)$ as

$$Y(z) = \begin{matrix} & 1 & q-1 \\ \begin{matrix} 1 \\ q-1 \end{matrix} & \begin{pmatrix} Y_{1,1}(z) & Y_{1,2}(z) \\ Y_{2,1}(z) & Y_{2,2}(z) \end{pmatrix} \end{matrix}, \tag{2.13}$$

where the partition is such that $Y_{1,1}$ has size 1×1. Similarly, partition the matrix $X = X(z)$ as

$$X(z) = \begin{matrix} & q-1 & 1 \\ \begin{matrix} q-1 \\ 1 \end{matrix} & \begin{pmatrix} X_{1,1}(z) & X_{1,2}(z) \\ X_{2,1}(z) & X_{2,2}(z) \end{pmatrix} \end{matrix}, \tag{2.14}$$

where the partition is such that $X_{1,1}$ has size $(q-1) \times (q-1)$.

The solutions to the RH problems 2.2 and 2.3 are related through the formula

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} = \begin{pmatrix} Y_{2,2} & -Y_{2,1} \\ -Y_{1,2} & Y_{1,1} \end{pmatrix}^{-T}, \tag{2.15}$$

where the superscript $^{-T}$ denotes the inverse transpose. This was first shown by Van Assche et al. in [36], see also [2, 14].

Lemma 2.4. *With the partitions (2.13) and (2.14) as described above, we have that*

$$Y_{1,1}(z) = p_n(z), \tag{2.16}$$

which is the biorthogonal polynomial in (1.4), and

$$X_{2,2}(z) = \frac{1}{h_{n-1}^2} \widetilde{Q}_{n-1}(z), \tag{2.17}$$

where h_{n-1} and $\widetilde{Q}_{n-1}$ are as defined in (1.7) and (1.13) respectively.

Proof. The statement (2.16) was established by Kuijlaars-McLaughlin [28].

Next we prove (2.17). To this end we recall some of the identities of Van Assche et al. [36]: it is observed in the latter paper that

$$X_{2,1}(z) = \begin{pmatrix} A_0(z) & \dots & A_{q-2}(z) \end{pmatrix} \tag{2.18}$$

for certain polynomials $A_j(z)$ of degree at most $n_j - 1$, $j = 0, \ldots, q - 2$. These polynomials are such that the following 'type 1' orthogonality relations hold:

$$\int_{-\infty}^{\infty} x^j X_{2,1}(x) \mathbf{w}^T(x) \, dx = -2\pi i \delta_{j,n-1}, \tag{2.19}$$

for $j = 0, 1, \ldots, n - 1$, where $\delta_{j,n-1}$ denotes the Kronecker delta. Finally, [36] also shows that

$$X_{2,2}(z) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{z - x} X_{2,1}(x) \mathbf{w}^T(x) \, dx. \tag{2.20}$$

Now we are going to use the special form of the weight functions $w_j(x)$ in (2.7)–(2.8). Using (2.18) and an integration by parts argument as in [28] it is easy to see that

$$\begin{aligned} X_{2,1}(x) \mathbf{w}^T(x) &:= A_0(x) w_0(x) + \ldots + A_{q-2}(x) w_{q-2}(x) \\ &= \int_{-\infty}^{\infty} B_{n-1}(y) e^{-V(x) - W(y) + \tau x y} \, dy, \end{aligned} \tag{2.21}$$

for a certain polynomial $B_{n-1}(y)$ of degree at most $n - 1$. Inserting this in (2.19)–(2.20), these relations are transformed into

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j B_{n-1}(y) e^{-V(x) - W(y) + \tau x y} \, dx \, dy = -2\pi i \delta_{j,n-1}, \tag{2.22}$$

for $j = 0, 1, \ldots, n - 1$, and

$$X_{2,2}(z) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{z - x} B_{n-1}(y) e^{-V(x) - W(y) + \tau x y} \, dx \, dy, \tag{2.23}$$

respectively.

Now (2.22) shows that B_{n-1} is precisely the biorthogonal polynomial q_{n-1} in (1.4), up to a constant, i.e.,

$$B_{n-1}(y) = C_{n-1} q_{n-1}(y),$$

for certain $C_{n-1} \in \mathbb{C}$. Then (2.23) implies in turn that

$$\begin{aligned} X_{2,2}(z) &= -\frac{C_{n-1}}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{z - x} q_{n-1}(y) e^{-V(x) - W(y) + \tau x y} \, dx \, dy \tag{2.24} \\ &= -\frac{C_{n-1}}{2\pi i} \widetilde{Q}_{n-1}(z), \tag{2.25} \end{aligned}$$

where the second equality follows by virtue of (1.13).

Finally, we want to identify the normalization constant C_{n-1} in (2.25). To this end, we substitute the expansion $\frac{1}{z-x} = \frac{1}{z} + \frac{x}{z^2} + \frac{x^2}{z^3} + \ldots$ and the orthogonality relations (1.4) in (2.24) to obtain that

$$X_{2,2}(z) = -\frac{C_{n-1}}{2\pi i} h_{n-1}^2 z^{-n} + O(z^{-n-1}), \qquad z \to \infty.$$

Comparing this with the asymptotics in (2.12), we see that

$$-\frac{C_{n-1}}{2\pi i} h_{n-1}^2 = 1.$$

Inserting this expression for C_{n-1} in (2.25), we obtain the desired formula (2.17). This ends the proof of the lemma. $\qquad \square$

Proof of Theorems 1.2 and 1.3. Taking into account Lemma 2.4, Theorems 1.2 and 1.3 now follow directly from the results in [10] and [16] respectively; see also [15, 27]. □

Remark 2.5. The kernels $K_{1,1}$ in (1.8) and $\widetilde{K}_{1,1}$ in (2.1) can be expressed in terms of the RH problem for $Y(z)$ as follows:

$$K_{1,1}(x_1, x_2) = \frac{1}{2\pi i(x_1 - x_2)} \begin{pmatrix} 0 & \mathbf{w}(x_2) \end{pmatrix} Y^{-1}(x_2)Y(x_1) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad (2.26)$$

and

$$\widetilde{K}_{1,1}(x, v) = \frac{1}{x - v} \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} Y^{-1}(v)Y(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (2.27)$$

These formulas can be obtained from the results in [14] and [15] respectively. Note that if $x_1, x_2 \in \mathbb{R}$ (or $x \in \mathbb{R}$) then we should replace Y in (2.26) (or (2.27) respectively) by one of its boundary values Y_+ or Y_-; both boundary values lead to the same formulas. Also note that, by means of (2.27) (or (2.1)) and the results in [15], it is possible to obtain determinantal formulas for $P_n^{[I,0,K,0]}$ for any I and K; however we will not follow this route here.

2.3 Proof of Theorem 1.5

In this section we prove Theorem 1.5. To this end we will establish two formulas of Christoffel type, see [3, 33].

Proposition 2.6. *Define the weight function*

$$\tilde{w}(x, y) := \left(\prod_{i=1}^{I}(x - x_i) \right) e^{-V(x)-W(y)+\tau xy}, \qquad (2.28)$$

with $I \geq 0$. The monic biorthogonal polynomial $A_n(x)$ of degree n with respect to this weight function, which is defined by the biorthogonality relations

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_n(x)q(y)\tilde{w}(x, y) \, dx \, dy = 0 \qquad (2.29)$$

for any polynomial $q(y)$ of degree at most $n - 1$, is given by

$$A_n(x) := \frac{1}{\prod_{i=1}^{I}(x - x_i)} \frac{\det \begin{pmatrix} p_n(x_1) & \ldots & p_{n+I}(x_1) \\ \vdots & & \vdots \\ p_n(x_I) & \ldots & p_{n+I}(x_I) \\ p_n(x) & \ldots & p_{n+I}(x) \end{pmatrix}}{\det \begin{pmatrix} p_n(x_1) & \ldots & p_{n+I-1}(x_1) \\ \vdots & & \vdots \\ p_n(x_I) & \ldots & p_{n+I-1}(x_I) \end{pmatrix}}. \qquad (2.30)$$

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Proof. Clearly $A_n(x)$ is a monic polynomial in x of degree n. So it suffices to check the biorthogonality relations (2.29). Observe that in the integrand $A_n(x)q(y)\tilde{w}(x,y)$, the factor $\prod_{i=1}^{I}(x-x_i)$ of $\tilde{w}(x,y)$ in (2.28) cancels with the prefactor of $A_n(x)$ in (2.30). By the linearity of determinants, the double integral in (2.29) can be performed entrywise inside the last row of the matrix in the numerator of (2.30). Then it suffices to prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{n+i}(x)q(y)w(x,y)\ dx\ dy = 0, \qquad i = 0,\ldots,I,$$

which is a direct consequence of the biorthogonality relations. $\square$

Proposition 2.7. *Define the weight function*

$$\tilde{w}(x,y) := \left(\prod_{i=1}^{I}(x-x_i)\prod_{j=1}^{J}(y-y_j)\right)e^{-V(x)-W(y)+\tau xy}, \tag{2.31}$$

where $I > J \geq 0$. The monic biorthogonal polynomial $B_n(y)$ of degree n with respect to this weight function, which is defined by the biorthogonality relations

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)B_n(y)\tilde{w}(x,y)\ dx\ dy = 0 \tag{2.32}$$

for any polynomial p of degree at most $n-1$, is given by

$$B_n(y) := \frac{h_{n+J}^2}{\prod_{j=1}^{J}(y-y_j)}$$

$$\times \frac{\det\begin{pmatrix} \widetilde{K}_{1,2}(x_1,y_1) & \ldots & \widetilde{K}_{1,2}(x_1,y_J) & \widetilde{K}_{1,2}(x_1,y) & p_{n+J+1}(x_1) & \ldots & p_{n+I-1}(x_1) \\ \vdots & & \vdots & \vdots & & & \vdots \\ \widetilde{K}_{1,2}(x_I,y_1) & \ldots & \widetilde{K}_{1,2}(x_I,y_J) & \widetilde{K}_{1,2}(x_I,y) & p_{n+J+1}(x_I) & \ldots & p_{n+I-1}(x_I) \end{pmatrix}}{\det\begin{pmatrix} \widetilde{K}_{1,2}(x_1,y_1) & \ldots & \widetilde{K}_{1,2}(x_1,y_J) & p_{n+J}(x_1) & p_{n+J}(x_1) & \ldots & p_{n+I-1}(x_1) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,2}(x_I,y_1) & \ldots & \widetilde{K}_{1,2}(x_I,y_J) & p_{n+J}(x_1) & p_{n+J}(x_1) & \ldots & p_{n+I-1}(x_I) \end{pmatrix}}. \tag{2.33}$$

Here we define $\widetilde{K}_{1,2}(x,y)$ as in (1.19) but with n replaced by $n+p$ for an arbitrary but fixed number $p \in \{J+1,\ldots,I\}$.

Proof. Clearly $B_n(y)$ is a monic polynomial in y of degree n. Next we check the biorthogonality relations (2.32). To this end we will take the index p in the statement of the lemma equal to I. Observe that in the integrand $p(x)B_n(y)\tilde{w}(x,y)$, the factor $\prod_{j=1}^{J}(y-y_j)$ of $\tilde{w}(x,y)$ in (2.31) cancels with the prefactor of $B_n(y)$ in (2.33). By linearity we can then take the double integral in (2.32) entrywise inside the $(J+1)$th column of the matrix in the numerator of (2.33). Hence the integral (2.32) can be written as a linear combination of terms of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)\prod_{i=1}^{I}(x-x_i)\widetilde{K}_{1,2}(x_k,y)e^{-V(x)-W(y)+\tau xy}\ dx\ dy$$

$$= \int_{-\infty}^{\infty} p(x)\prod_{i=1}^{I}(x-x_i)K_{1,1}(x_k,x)\ dx, \tag{2.34}$$

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for $k = 1, \ldots, I$, where we used (1.8) and (1.19). But the integral (2.34) is zero for any $k = 1, \ldots, I$, because of the reproducing property (2.4) for $K_{1,1}$. (Recall that we are taking the index p in the statement of the lemma equal to I.) $\qquad\square$

Theorem 1.5 now follows from Theorem 1.2 and Propositions 2.6 and 2.7 by an easy induction argument, see e.g. [3]. $\qquad\square$

2.4 Properties of the kernels $\widetilde{K}_{i,j}$

Our next goal is to prove Theorem 1.6. In the present section we first collect some preliminary results on the kernels $\widetilde{K}_{i,j}$ which will be needed in the proof.

First we discuss the kernel $\widetilde{K}_{1,2}(x, y) = K_{1,2}(x, y)$. Observe that by the definition (1.19), the kernel $\widetilde{K}_{1,2}(x, y)$ is a bivariate polynomial in x and y, of the form

$$\widetilde{K}_{1,2}(x, y) = \sum_{i,j=0}^{n-1} c_{i,j} x^i y^j, \tag{2.35}$$

for suitable coefficients $c_{i,j} \in \mathbb{C}$.

Lemma 2.8. *(Reproducing property of $\widetilde{K}_{1,2}$:) The kernel $\widetilde{K}_{1,2}$ is reproducing in the sense that*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{1,2}(x, y) p(\xi) e^{-V(\xi)-W(y)+\tau\xi y} \, d\xi \, dy = p(x), \tag{2.36}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{1,2}(x, y) q(\eta) e^{-V(x)-W(\eta)+\tau x\eta} \, dx \, d\eta = q(y), \tag{2.37}$$

for any polynomials p and q of degree at most $n-1$. Moreover, the reproducing property (2.36) (or (2.37)) uniquely characterizes $\widetilde{K}_{1,2}(x, y)$ over all bivariate polynomials of the form (2.35).

Proof. Taking into account (1.8), (1.11) and (1.19), the reproducing properties (2.36)–(2.37) are just a restatement of (2.4) and (2.5) respectively. The claim about uniqueness is also easily shown, see also [15]. $\qquad\square$

Next we discuss the kernels $\widetilde{K}_{1,1}$ and $\widetilde{K}_{2,2}$. We will need the following result.

Lemma 2.9. *(Vanishing properties of $\widetilde{K}_{1,1}$ and $\widetilde{K}_{2,2}$:) The kernel $\widetilde{K}_{1,1}$ satisfies the 'vanishing property'*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{1,1}(\xi, v) q(\eta) e^{-V(\xi)-W(\eta)+\tau\xi\eta} \, d\xi \, d\eta = 0 \tag{2.38}$$

for any $v \in \mathbb{C} \setminus \mathbb{R}$ and all polynomials q of degree at most $n-1$. Similarly, the kernel $\widetilde{K}_{2,2}$ satisfies the vanishing property

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{2,2}(w, \eta) p(\xi) e^{-V(\xi)-W(\eta)+\tau\xi\eta} \, d\xi \, d\eta = 0, \tag{2.39}$$

for any $w \in \mathbb{C} \setminus \mathbb{R}$ and all polynomials p of degree at most $n-1$.

Proof. This lemma can be obtained from (2.27) and the vanishing property in [15]. For completeness, we include a direct proof. Using (2.1) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{K}_{1,1}(\xi,v)q(\eta)e^{-V(\xi)-W(\eta)+\tau\xi\eta}\,d\xi\,d\eta = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{q(\eta)e^{-V(\xi)-W(\eta)+\tau\xi\eta}}{v-\xi}\,d\xi\,d\eta$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{K_{1,1}(\xi,x)q(\eta)e^{-V(\xi)-W(\eta)+\tau\xi\eta}}{v-x}\,d\xi\,d\eta\,dx. \quad (2.40)$$

Now we use the following formula of reproducing type:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{1,1}(\xi,x)q(\eta)e^{-V(\xi)-W(\eta)+\tau\xi\eta}\,d\xi\,d\eta = \int_{-\infty}^{\infty} q(\eta)e^{-V(x)-W(\eta)+\tau x\eta}\,d\eta, \quad (2.41)$$

for any polynomial q of degree at most $n-1$. (As usual, by linearity it suffices to check (2.41) when q equals one of the biorthogonal polynomials q_i for $i=0,\dots,n-1$, in which case it follows easily by the biorthogonality relations.) Using (2.41) in the term with the triple integral in (2.40), we see that both terms in the right hand side of (2.40) cancel each other and so (2.38) follows. The proof of (2.39) is similar. $\square$

Lemma 2.10. *(Asymptotic behavior of biorthogonal polynomials and kernels:) We have*

$$p_n(x) = x^n + O(x^{n-1}), \qquad x \to \infty, \quad (2.42)$$

$$q_n(x) = y^n + O(y^{n-1}), \qquad y \to \infty, \quad (2.43)$$

$$\widetilde{P}_n(w) = h_n^2 w^{-n-1} + O(w^{-n-2}), \qquad w \to \infty, \quad (2.44)$$

$$\widetilde{Q}_n(v) = h_n^2 v^{-n-1} + O(v^{-n-2}), \qquad v \to \infty, \quad (2.45)$$

and

$$\widetilde{K}_{1,1}(x,v) = h_{n-1}^{-2}\widetilde{Q}_{n-1}(v)x^{n-1} + O(x^{n-2}), \qquad x \to \infty, \quad (2.46)$$

$$\widetilde{K}_{1,1}(x,v) = -p_n(x)v^{-n-1} + O\left(v^{-n-2}\right), \qquad v \to \infty, \quad (2.47)$$

$$\widetilde{K}_{1,2}(x,y) = h_{n-1}^{-2}q_{n-1}(y)x^{n-1} + O(x^{n-2}), \qquad x \to \infty, \quad (2.48)$$

$$\widetilde{K}_{1,2}(x,y) = h_{n-1}^{-2}p_{n-1}(x)y^{n-1} + O(y^{n-2}), \qquad y \to \infty, \quad (2.49)$$

and

$$\widetilde{K}_{2,1}(w,v) = -\widetilde{Q}_n(v)w^{-n-1} + O\left(w^{-n-2}\right), \qquad w \to \infty, \quad (2.50)$$

$$\widetilde{K}_{2,1}(w,v) = -\widetilde{P}_n(w)v^{-n-1} + O\left(v^{-n-2}\right), \qquad v \to \infty, \quad (2.51)$$

$$\widetilde{K}_{2,2}(w,y) = -q_n(x)w^{-n-1} + O\left(w^{-n-2}\right), \qquad w \to \infty, \quad (2.52)$$

$$\widetilde{K}_{2,2}(w,y) = h_{n-1}^{-2}\widetilde{P}_{n-1}(w)y^{n-1} + O(y^{n-2}), \qquad y \to \infty. \quad (2.53)$$

Proof. Equations (2.42)–(2.43) are obvious by definition. Let us check (2.44). By substituting the series $\frac{1}{w-\eta} = \sum_{i=1}^{\infty}\frac{\eta^{i-1}}{w^i}$ in (1.12) we find that

$$\widetilde{P}_n(w) = \sum_{i=1}^{\infty}\frac{1}{w^i}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(\xi)\eta^{i-1}e^{-V(\xi)-W(\eta)+\tau\xi\eta}\,d\xi\,d\eta\right).$$

Now by the biorthogonality relations we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(\xi)\eta^{i-1}e^{-V(\xi)-W(\eta)+\tau\xi\eta}\,d\xi\,d\eta = h_n^2\delta_{i,n+1},$$

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for any $i = 1, \ldots, n + 1$. So we obtain (2.44). In a similar way one checks (2.45).

Equation (2.46) follows by virtue of (1.18) and (2.42). Next we check (2.47). By substituting the expansion $\frac{1}{v-\xi} = \sum_{i=1}^{\infty} \frac{\xi^{i-1}}{v^i}$ in (1.14) we obtain

$$\widetilde{K}_{1,1}(x, v) = \frac{1}{x - v} \sum_{i=1}^{\infty} \frac{1}{v^i} \left(\int_{-\infty}^{\infty} (x - \xi) \xi^{i-1} K_{1,1}(x, \xi) \, d\xi \right). \tag{2.54}$$

Now

$$\int_{-\infty}^{\infty} (x - \xi) \xi^{i-1} K_{1,1}(x, \xi) \, d\xi = \left[(x - \xi) \xi^{i-1} \right]_{\xi=x} = 0, \qquad \text{if } i = 1, \ldots, n - 1,$$

because of the reproducing property (2.4). For $i = n$ we have

$$\int_{-\infty}^{\infty} (x - \xi) \xi^{n-1} K_{1,1}(x, \xi) \, d\xi = \int_{-\infty}^{\infty} (x - \xi) \xi^{n-1} \left(K_{1,1}(x, \xi) + \frac{1}{h_n^2} p_n(x) Q_n(\xi) \right) \, d\xi$$
$$- \int_{-\infty}^{\infty} (x - \xi) \xi^{n-1} \frac{1}{h_n^2} p_n(x) Q_n(\xi) \, d\xi. \tag{2.55}$$

The expression between brackets in the first term in the right hand side of (2.55) is nothing but the kernel $K_{1,1}(x, \xi)$ with n replaced by $n+1$ in (1.8), so this integral is zero as before. We then have

$$
\begin{aligned}
\int_{-\infty}^{\infty} (x - \xi) \xi^{n-1} K_{1,1}(x, \xi) \, d\xi &= -\frac{1}{h_n^2} p_n(x) \int_{-\infty}^{\infty} (x - \xi) \xi^{n-1} Q_n(\xi) \, d\xi \\
&= -\frac{1}{h_n^2} p_n(x) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \xi) \xi^{n-1} q_n(\eta) e^{-V(\xi) - W(\eta) + \tau \xi \eta} \, d\xi \, d\eta \\
&= p_n(x),
\end{aligned}
$$

because of biorthogonality. Using this relation in (2.54), we obtain (2.47).

Equations (2.48)–(2.49) are immediate by the definition (1.19). Equations (2.52)–(2.53) can be obtained similarly as before.

Finally, let us check (2.50). By substituting the expansion $\frac{1}{w-\eta} = \sum_{i=1}^{\infty} \frac{\eta^{i-1}}{w^i}$ in (1.16) we obtain

$$\widetilde{K}_{2,1}(w, v) = \sum_{i=1}^{\infty} \frac{1}{w^i} \int_{-\infty}^{\infty} \frac{1}{v - \xi} \left(\int_{-\infty}^{\infty} \eta^{i-1} K_{2,1}(\eta, \xi) \, d\eta \right) d\xi. \tag{2.56}$$

By (1.10) we see that

$$\int_{-\infty}^{\infty} \eta^{i-1} K_{2,1}(\eta, \xi) \, d\eta = -\int_{-\infty}^{\infty} \eta^{i-1} e^{-V(\xi) - W(\eta) + \tau \xi \eta} \, d\eta + \int_{-\infty}^{\infty} \eta^{i-1} \sum_{j=0}^{n-1} \frac{1}{h_j^2} P_j(\eta) Q_j(\xi) \, d\eta. \tag{2.57}$$

Now we claim that

$$\int_{-\infty}^{\infty} q(\eta) \sum_{j=0}^{n-1} \frac{1}{h_j^2} P_j(\eta) Q_j(\xi) \, d\eta = \int_{-\infty}^{\infty} q(\eta) e^{-V(\xi) - W(\eta) + \tau \xi \eta} \, d\eta \tag{2.58}$$

for any polynomial q of degree at most $n - 1$. (Once again, by linearity it suffices to check (2.58) when q equals one of the biorthogonal polynomials q_i for $i = 0, \ldots, n - 1$, in which case it follows directly from the biorthogonality relations.) Using (2.58) with $q(\eta) = \eta^{i-1}$, we see that

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if $i = 1, \ldots, n$ then both terms in the right hand side of (2.57) cancel each other. For $i = n + 1$ we get

$$\int_{-\infty}^{\infty} \eta^n \sum_{j=0}^{n-1} \frac{1}{h_j^2} P_j(\eta) Q_j(\xi) \, d\eta$$

$$= \left(\int_{-\infty}^{\infty} \eta^n \sum_{j=0}^{n} \frac{1}{h_j^2} P_j(\eta) Q_j(\xi) \, d\eta \right) - \int_{-\infty}^{\infty} \eta^n \frac{1}{h_n^2} P_n(\eta) Q_n(\xi) \, d\eta$$

$$= \left(\int_{-\infty}^{\infty} \eta^n e^{-V(\xi) - W(\eta) + \tau \xi \eta} \, d\eta \right) - \frac{Q_n(\xi)}{h_n^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta^n p_n(x) e^{-V(x) - W(\eta) + \tau x \eta} \, dx \, d\eta$$

$$= \left(\int_{-\infty}^{\infty} \eta^n e^{-V(\xi) - W(\eta) + \tau \xi \eta} \, d\eta \right) - Q_n(\xi),$$

where we used (2.58) (with n replaced by $n+1$) and the biorthogonality relations. Substituting this in (2.57) we find that

$$\int_{-\infty}^{\infty} \eta^{i-1} K_{2,1}(\eta, \xi) \, d\eta = -\delta_{i,n+1} Q_n(\xi),$$

for any $i = 1, \ldots, n + 1$. Using this in (2.56) we obtain

$$\widetilde{K}_{2,1}(w, v) = -w^{-n-1} \int_{-\infty}^{\infty} \frac{Q_n(\xi)}{v - \xi} \, d\xi + O(w^{-n-2}) = -w^{-n-1} \widetilde{Q}_n(v) + O(w^{-n-2}),$$

which is (2.50). Equation (2.51) can be obtained similarly. □

Finally, we will also need the following additional properties of the kernels $\widetilde{K}_{i,j}$.

Lemma 2.11. *(Integral formula relations between the kernels $\widetilde{K}_{i,j}$:)* *We have*

$$\widetilde{P}_n(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(\xi) e^{-V(\xi) - W(\eta) + \tau \xi \eta} \frac{1}{w - \eta} \, d\xi \, d\eta, \tag{2.59}$$

$$\widetilde{K}_{2,2}(w, y) = \frac{1}{y - w} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{1,2}(\xi, y) e^{-V(\xi) - W(\eta) + \tau \xi \eta} \frac{y - \eta}{w - \eta} \, d\xi \, d\eta, \tag{2.60}$$

and

$$\widetilde{K}_{2,1}(w, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{1,1}(\xi, v) e^{-V(\xi) - W(\eta) + \tau \xi \eta} \frac{1}{w - \eta} \, d\xi \, d\eta. \tag{2.61}$$

Proof. Equation (2.59) is just (1.12). Equation (2.60) follows from the definitions in (1.17), (1.11) and (1.19). Finally, equation (2.61) follows by (1.20), (1.18) and (1.12). □

2.5 Proof of Theorem 1.6

In this section we prove Theorem 1.6. To this end we use Theorem 1.5 together with a mechanism to transform an external source in the matrix M_1 (or M_2) in the *numerator*, into an external source in the matrix M_2 (or M_1 respectively) in the *denominator*. That is, we will apply transformations of the form

$$(I, J, K, L) \to (I - 1, J, K, L + 1), \tag{2.62}$$

or

$$(I, J, K, L) \to (I, J - 1, K + 1, L). \tag{2.63}$$

It will then suffice to show that each of the objects in (1.12)–(1.17) transforms in an appropriate way under these transformations; this will be achieved by virtue of Lemmas 2.8–2.11.

As in [3, 15], we will also need an appropriate use of partial fraction decomposition. For given complex numbers $y_1, \ldots, y_J, w_1, \ldots, w_{L+1} \in \mathbb{C}$, the partial fraction decomposition which is of interest to us is of the form

$$\frac{\prod_{j=1}^{J}(\eta - y_j)}{(\eta - w_{L+1})\prod_{l=1}^{L}(\eta - w_l)} = \frac{c_{L+1}}{\eta - w_{L+1}} + \sum_{l=1}^{L} \frac{c_l}{\eta - w_l} + P(\eta). \tag{2.64}$$

Here $c_1, \ldots, c_{L+1} \in \mathbb{C}$ and P is a polynomial of degree $J - L - 1$. (We put $P \equiv 0$ when $J - L - 1 < 0$.)

Note that (2.64) also implies the more complicated partial fraction decomposition

$$\frac{\prod_{j=1}^{J}(\eta - y_j)}{(\eta - w_{L+1})\prod_{l=1}^{L}(\eta - w_l)} = \frac{c_{L+1}(\eta - y_j)}{(w_{L+1} - y_j)(\eta - w_{L+1})} + \sum_{l=1}^{L} \frac{c_l(\eta - y_j)}{(w_l - y_j)(\eta - w_l)} + P(\eta) - P(y_j), \tag{2.65}$$

for any fixed $j \in \{1, \ldots, J\}$. We leave it to the reader to obtain (2.65) from (2.64).

Now we are ready for the proof of Theorem 1.6.

Proof of Theorem 1.6. By symmetry we can assume without loss of generality that $I - K \geq J - L$. We will show that the formula in (1.28) is compatible with any transformation of the form (2.62).

Consider the weight function

$$\tilde{w}(x, y) = \frac{\prod_{i=1}^{I-1}(x - x_i) \prod_{j=1}^{J}(y - y_j)}{\prod_{k=1}^{K}(x - v_k) \prod_{l=1}^{L}(y - w_l)} e^{-V(x) - W(y) + \tau xy}. \tag{2.66}$$

(Note that there are only $I - 1$ factors x_i.) Denote by $A_n(x)$ the monic biorthogonal polynomial of degree n with respect to this weight function, defined by the orthogonality relations

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_n(x) q(y) \tilde{w}(x, y) \, dx \, dy = 0 \tag{2.67}$$

for all polynomials q of degree at most $n - 1$. Theorem 1.2 implies that

$$P_n^{[I,J,K,L]}(x_1, \ldots, x_I; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_L) = A_n(x_I),$$

while from Theorem 1.3 and (1.12) it follows that

$$P_n^{[I-1,J,K,L+1]}(x_1, \ldots, x_{I-1}; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_{L+1})$$
$$= \frac{1}{\tilde{h}_{n-1}^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{n-1}(x_I) \frac{\tilde{w}(x_I, \eta)}{w_{L+1} - \eta} \, dx_I \, d\eta,$$

where $\tilde{h}_{n-1}^2$ is a constant which depends on each of the numbers x_i, y_j, v_k, w_l, but *not* on w_{L+1}. By combining these two formulas, one gets

$$P_n^{[I-1,J,K,L+1]}(x_1, \ldots, x_{I-1}; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_{L+1})$$
$$= \frac{1}{\tilde{h}_{n-1}^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_{n-1}^{[I,J,K,L]}(x_1, \ldots, x_I; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_L) \frac{\tilde{w}(x_I, \eta)}{w_{L+1} - \eta} \, dx_I \, d\eta. \tag{2.68}$$

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Now assume, by induction, that (1.28) holds for $P_n^{[I,J,K,L]}$. By substituting this expression (with $n-1$ instead of n) for $P_{n-1}^{[I,J,K,L]}$ in the right hand side of (2.68), we get

$$P_n^{[I-1,J,K,L+1]}(x_1,\ldots,x_{I-1};y_1,\ldots,y_J;v_1,\ldots,v_K;w_1,\ldots,w_{L+1})$$

$$= C_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\prod_{k=1}^{K}(x_I - v_k)}{\prod_{i=1}^{I-1}(x_I - x_i)} \frac{\tilde{w}(x_I,\eta)}{w_{L+1} - \eta}$$

$$\times \det \begin{pmatrix} \widetilde{K}_{1,1}(x_1,v_1) & \ldots & \widetilde{K}_{1,1}(x_I,v_1) & \widetilde{K}_{2,1}(w_1,v_1) & \ldots & \widetilde{K}_{2,1}(w_L,v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,1}(x_1,v_K) & \ldots & \widetilde{K}_{1,1}(x_I,v_K) & \widetilde{K}_{2,1}(w_1,v_K) & \ldots & \widetilde{K}_{2,1}(w_L,v_K) \\ \widetilde{K}_{1,2}(x_1,y_1) & \ldots & \widetilde{K}_{1,2}(x_I,y_1) & \widetilde{K}_{2,2}(w_1,y_1) & \ldots & \widetilde{K}_{2,2}(w_L,y_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,2}(x_1,y_J) & \ldots & \widetilde{K}_{1,2}(x_I,y_J) & \widetilde{K}_{2,2}(w_1,y_J) & \ldots & \widetilde{K}_{2,2}(w_L,y_J) \\ p_{n-1+J-L}(x_1) & \ldots & p_{n-1+J-L}(x_I) & \widetilde{P}_{n-1+J-L}(w_1) & \ldots & \widetilde{P}_{n-1+J-L}(w_L) \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{n-2+I-K}(x_1) & \ldots & p_{n-2+I-K}(x_I) & \widetilde{P}_{n-2+I-K}(w_1) & \ldots & \widetilde{P}_{n-2+I-K}(w_L) \end{pmatrix} dx_I \, d\eta,$$

$$(2.69)$$

where C_0 is a new constant which depends on each of the numbers x_i, y_j, v_k, w_l, but *not* on w_{L+1}. We will work this out. Note that the factor $\frac{\prod_{i=1}^{I-1}(x_I - x_i)}{\prod_{k=1}^{K}(x_I - v_k)}$ in the definition of $\tilde{w}(x_I,\eta)$ in (2.66) cancels with the prefactor in the integrand of (2.69). By linearity, we can then apply the double integration entrywise inside the Ith column of the matrix in (2.69). Then the entries in the Ith column of (2.69) transform into expressions of the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{1,1}(x_I,v_k) \frac{\prod_{j=1}^{J}(\eta - y_j)}{\prod_{l=1}^{L}(\eta - w_l)} \frac{1}{w_{L+1} - \eta} e^{-V(x_I)-W(\eta)+\tau x_I \eta} \, dx_I \, d\eta, \qquad (2.70)$$

for $k = 1,\ldots,K$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}_{1,2}(x_I,y_j) \frac{\prod_{j=1}^{J}(\eta - y_j)}{\prod_{l=1}^{L}(\eta - w_l)} \frac{1}{w_{L+1} - \eta} e^{-V(x_I)-W(\eta)+\tau x_I \eta} \, dx_I \, d\eta, \qquad (2.71)$$

for $j = 1,\ldots,J$, and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{n-1+p}(x_I) \frac{\prod_{j=1}^{J}(\eta - y_j)}{\prod_{l=1}^{L}(\eta - w_l)} \frac{1}{w_{L+1} - \eta} e^{-V(x_I)-W(\eta)+\tau x_I \eta} \, dx_I \, d\eta, \qquad (2.72)$$

for $p = J - L,\ldots,I - K - 1$.

Next, we substitute the partial fraction decomposition (2.64) in (2.70) and (2.72), and we substitute (2.65) in (2.71). Doing this for each of the entries in the Ith column of (2.69), the determinant can be split in a sum of three terms,

$$D_1 + D_2 + D_3, \qquad (2.73)$$

corresponding to the three terms on the right hand sides of (2.64)–(2.65).

For the first term D_1 (which is obtained by selecting the terms $\frac{c_{L+1}}{\eta - w_{L+1}}$ and $\frac{c_{L+1}(\eta - y_j)}{(w_{L+1} - y_j)(\eta - w_{L+1})}$ in (2.64) and (2.65) respectively), the expressions (2.70)–(2.72) transform into

$$c_{L+1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{K}_{1,1}(x_I, v_k)\frac{1}{w_{L+1} - \eta}e^{-V(x_I)-W(\eta)+\tau x_I \eta}\, dx_I\, d\eta, \qquad (2.74)$$

for $k = 1, \ldots, K$,

$$c_{L+1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{K}_{1,2}(x_I, y_j)\frac{\eta - y_j}{(w_{L+1} - y_j)(w_{L+1} - \eta)}e^{-V(x_I)-W(\eta)+\tau x_I \eta}\, dx_I\, d\eta, \qquad (2.75)$$

for $j = 1, \ldots, J$, and

$$c_{L+1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{n-1+p}(x_I)\frac{1}{w_{L+1} - \eta}e^{-V(x_I)-W(\eta)+\tau x_I \eta}\, dx_I\, d\eta, \qquad (2.76)$$

for $p = J - L, \ldots, I - K - 1$. Comparing this with Lemma 2.11, we see that (2.74)–(2.76) are nothing but c_{L+1} times $\widetilde{K}_{2,1}(w_{L+1}, v_k)$, $\widetilde{K}_{2,2}(w_{L+1}, y_j)$ and $\widetilde{P}_{n-1+p}(w_{L+1})$ respectively. The factor c_{L+1} can be taken out of the Ith column and be put in front of the determinant. Note that

$$c_{L+1} = \frac{\prod_{j=1}^{J}(w_{L+1} - y_j)}{\prod_{l=1}^{L}(w_{L+1} - w_l)},$$

a fact which is easily checked from (2.64).

For the second term D_2 in (2.73) (which is obtained by selecting the terms $\sum_{l=1}^{L}\frac{c_l}{\eta - w_l}$ and $\sum_{l=1}^{L}\frac{c_l(\eta - y_j)}{(w_l - y_j)(\eta - w_l)}$ in (2.64)–(2.65) respectively), we obtain similarly that (2.70)–(2.72) transform into a linear combination of $\widetilde{K}_{2,1}(w_l, v_k)$, $\widetilde{K}_{2,2}(w_l, y_j)$ and $\widetilde{P}_{n-1+p}(w_l)$, $l = 1, \ldots, L$. But then the Ith column in (2.69) is a linear combination of columns $I + 1, \ldots, I + L$ and so the determinant D_2 vanishes.

Finally, the third determinant D_3 (corresponding to the polynomial parts $P(\eta)$ and $P(\eta) - P(y_j)$ in (2.64)–(2.65) respectively) vanishes as well, by virtue of Lemmas 2.8 and 2.9.

Summarizing, we showed that (2.69) equals

$$P_n^{[I-1,J,K,L+1]}(x_1, \ldots, x_{I-1}; y_1, \ldots, y_J; v_1, \ldots, v_K; w_1, \ldots, w_{L+1}) = C_1\frac{\prod_{j=1}^{J}(w_{L+1} - y_j)}{\prod_{l=1}^{L}(w_{L+1} - w_l)}$$

$$\times \det \begin{pmatrix} \widetilde{K}_{1,1}(x_1, v_1) & \ldots & \widetilde{K}_{1,1}(x_{I-1}, v_1) & \widetilde{K}_{2,1}(w_1, v_1) & \ldots & \widetilde{K}_{2,1}(w_{L+1}, v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,1}(x_1, v_K) & \ldots & \widetilde{K}_{1,1}(x_{I-1}, v_K) & \widetilde{K}_{2,1}(w_1, v_K) & \ldots & \widetilde{K}_{2,1}(w_{L+1}, v_K) \\ \widetilde{K}_{1,2}(x_1, y_1) & \ldots & \widetilde{K}_{2,1}(x_{I-1}, y_1) & \widetilde{K}_{2,2}(w_1, y_1) & \ldots & \widetilde{K}_{2,2}(w_{L+1}, y_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,2}(x_1, y_J) & \ldots & \widetilde{K}_{2,1}(x_{I-1}, y_J) & \widetilde{K}_{2,2}(w_1, y_J) & \ldots & \widetilde{K}_{2,2}(w_{L+1}, y_J) \\ p_{n-1+J-L}(x_1) & \ldots & p_{n-1+J-L}(x_{I-1}) & \widetilde{P}_{n-1+J-L}(w_1) & \ldots & \widetilde{P}_{n-1+J-L}(w_{L+1}) \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{n-2+I-K}(x_1) & \ldots & p_{n-2+I-K}(x_{I-1}) & \widetilde{P}_{n-2+I-K}(w_1) & \ldots & \widetilde{P}_{n-2+I-K}(w_{L+1}) \end{pmatrix},$$

$$(2.77)$$

where $C_1 = \pm C_0$ depends on each of the numbers x_i, y_j, v_k, w_l, but *not* on w_{L+1}.

To find C_1, we compute the leading order behavior of (2.77) in w_{L+1}; this can be done by replacing each of the entries in the last column of (2.77) by their dominant term in w_{L+1} as in Lemma 2.10. (More precisely, if $I - K > J - L$ then only the entry $\widetilde{P}_{n-1+J-L}(w_{L+1})$ in (2.77) contributes to the dominant term, while if $I - K = J - L$ then there is a contribution from *all* entries in the last column of (2.77).) Taking into account that

$$P_n^{[I-1,J,K,L+1]}(x_1,\ldots,x_{I-1};y_1,\ldots,y_J;v_1,\ldots,v_K;w_1,\ldots,w_{L+1})$$
$$= w_{L+1}^{-n} P_n^{[I-1,J,K,L]}(x_1,\ldots,x_{I-1};y_1,\ldots,y_J;v_1,\ldots,v_K;w_1,\ldots,w_L) + O\left(w_{L+1}^{-n-1}\right),$$

as $w_{L+1} \to \infty$, a fact which trivially follows from (1.24), we then obtain an expression for C_1.

In a similar way we can compute the leading order behavior of (1.28) in x_I, i.e., we can write $P_n^{[I,J,K,L]}$ in the form

$$C_2 \frac{\prod_{k=1}^{K}(x_I - v_k)}{\prod_{i=1}^{I-1}(x_I - x_i)}$$

$$\times \det \begin{pmatrix} \widetilde{K}_{1,1}(x_1,v_1) & \ldots & \widetilde{K}_{1,1}(x_I,v_1) & \widetilde{K}_{2,1}(w_1,v_1) & \ldots & \widetilde{K}_{2,1}(w_L,v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,1}(x_1,v_K) & \ldots & \widetilde{K}_{1,1}(x_I,v_K) & \widetilde{K}_{2,1}(w_1,v_K) & \ldots & \widetilde{K}_{2,1}(w_L,v_K) \\ \widetilde{K}_{1,2}(x_1,y_1) & \ldots & \widetilde{K}_{2,1}(x_I,y_1) & \widetilde{K}_{2,2}(w_1,y_1) & \ldots & \widetilde{K}_{2,2}(w_L,y_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,2}(x_1,y_J) & \ldots & \widetilde{K}_{2,1}(x_I,y_J) & \widetilde{K}_{2,2}(w_1,y_J) & \ldots & \widetilde{K}_{2,2}(w_L,y_J) \\ p_{n+J-L}(x_1) & \ldots & p_{n+J-L}(x_I) & \widetilde{P}_{n+J-L}(w_1) & \ldots & \widetilde{P}_{n+J-L}(w_L) \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{n+I-K-1}(x_1) & \ldots & p_{n+I-K-1}(x_I) & \widetilde{P}_{n+I-K-1}(w_1) & \ldots & \widetilde{P}_{n+I-K-1}(w_L) \end{pmatrix}, \quad (2.78)$$

and then we can replace each of the entries in the Ith column of (2.78) by their dominant term in x_I as in Lemma 2.10. (More precisely, if $I - K > J - L$ then only the entry $p_{n+I-K-1}(x_I)$ in (2.78) contributes to the dominant term, while if $I - K = J - L$ then there is a contribution from *all* entries in the Ith column of (2.78).) Taking into account that

$$P_n^{[I,J,K,L]}(x_1,\ldots,x_I;y_1,\ldots,y_J;v_1,\ldots,v_K;w_1,\ldots,w_L)$$
$$= x_I^n P_n^{[I-1,J,K,L]}(x_1,\ldots,x_{I-1};y_1,\ldots,y_J;v_1,\ldots,v_K;w_1,\ldots,w_L) + O\left(x_I^{n-1}\right),$$

as $x_I \to \infty$, we then obtain an expression for C_2 in (2.78). A straightforward calculation now shows that the so obtained expressions for C_1 and C_2 in (2.77) and (2.78) are equal to each other, up to a shift $n \mapsto n - 1$ of the index, and a factor $(-1)^{I+J+K+1}/h_{n+J-L-1}^2$. So the constants C_1 and C_2 are related precisely in the way that is required for applying the transformation (2.62) to the prefactor in (1.28).

Summarizing, we have shown that (1.28) is compatible with any transformation of the form (2.62). In a similar way one shows this for transformations of the form (2.63). Then applying the transformations (2.62)–(2.63) repeatedly, and using Theorem 1.5 as induction basis, we obtain (1.28) in its full generality. $\qquad \square$

Remark 2.12. An alternative proof of Theorem 1.6 can be obtained by establishing formulas of Christoffel-Uvarov type [3, 33, 35]. To this end one can use similar ideas as in the proof above. We preferred the given proof since it is probably shorter.

3 Applications

In this section we give two applications of our results. Here we follow ideas for the 1-matrix model.

3.1 Generating function for averages of products of traces

In this section we show how the results in this paper allow to obtain expectation values of products of traces of the form

$$\frac{1}{Z_n} \int \int \left(\prod_{i=1}^{I} \mathrm{Tr}(M_1^{m_i}) \prod_{j=1}^{J} \mathrm{Tr}(M_2^{n_j}) \right) e^{\mathrm{Tr}(-V(M_1) - W(M_2) + \tau M_1 M_2)} \, dM_1 \, dM_2,$$

or equivalently

$$\frac{1}{\tilde{Z}_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{I} \left(\sum_{k=1}^{n} \lambda_k^{m_i} \right) \prod_{j=1}^{J} \left(\sum_{k=1}^{n} \mu_k^{n_j} \right)$$
$$\times \prod_{i=1}^{n} \left(e^{-V(\lambda_i)} e^{-W(\mu_i)} \right) \Delta(\lambda) \Delta(\mu) \det(e^{\tau \lambda_i \mu_j})_{i,j=1}^{n} \prod_{i=1}^{n} (d\lambda_i \, d\mu_i), \quad (3.1)$$

with exponents $m_1, \ldots, m_I; n_1, \ldots, n_J \in \mathbb{N} \cup \{0\}$, see (1.3). Here we follow Bergère [4]. We start from (1.25) with $I = K$ and $J = L$:

$$\frac{1}{\tilde{Z}_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{I} \left(\frac{\prod_{k=1}^{n}(x_i - \lambda_k)}{\prod_{k=1}^{n}(v_i - \lambda_k)} \right) \prod_{j=1}^{J} \left(\frac{\prod_{k=1}^{n}(y_j - \lambda_k)}{\prod_{k=1}^{n}(w_j - \lambda_k)} \right)$$
$$\times \prod_{i=1}^{n} \left(e^{-V(\lambda_i)} e^{-W(\mu_i)} \right) \Delta(\lambda) \Delta(\mu) \det(e^{\tau \lambda_i \mu_j})_{i,j=1}^{n} \prod_{i=1}^{n} (d\lambda_i \, d\mu_i). \quad (3.2)$$

Applying the operator

$$\prod_{i=1}^{I} \left(\frac{\partial}{\partial x_i} \right)_{v_i = x_i} \prod_{j=1}^{J} \left(\frac{\partial}{\partial y_i} \right)_{w_j = y_j} \quad (3.3)$$

to (3.2) leads to the (multi-variate) Cauchy transform

$$\frac{1}{\tilde{Z}_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{I} \left(\sum_{k=1}^{n} \frac{1}{x_i - \lambda_k} \right) \prod_{j=1}^{J} \left(\sum_{k=1}^{n} \frac{1}{y_j - \mu_k} \right)$$
$$\times \prod_{i=1}^{n} \left(e^{-V(\lambda_i)} e^{-W(\mu_i)} \right) \Delta(\lambda) \Delta(\mu) \det(e^{\tau \lambda_i \mu_j})_{i,j=1}^{n} \prod_{i=1}^{n} (d\lambda_i \, d\mu_i), \quad (3.4)$$

where we assume that all $x_i, y_j \in \mathbb{C} \setminus \mathbb{R}$. Note that the large x_i, y_j expansion of (3.4) is a formal power series whose coefficients are the expectation values of products of traces (3.1).

On the other hand, Theorem 1.6 with $I = K$ and $J = L$ shows that (3.2) equals

$$= \frac{\prod_{i,k=1}^{I}(x_i - v_k) \prod_{j,l=1}^{J}(y_j - w_l)}{\prod_{1 \le i < j \le I}(x_j - x_i)(v_i - v_j) \prod_{1 \le i < j \le J}(y_j - y_i)(w_i - w_j)}$$

$$\times \det \begin{pmatrix} \widetilde{K}_{1,1}(x_1, v_1) & \ldots & \widetilde{K}_{1,1}(x_1, v_I) & \widetilde{K}_{1,2}(x_1, y_1) & \ldots & \widetilde{K}_{1,2}(x_1, y_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,1}(x_I, v_1) & \ldots & \widetilde{K}_{1,1}(x_I, v_I) & \widetilde{K}_{1,2}(x_I, y_1) & \ldots & \widetilde{K}_{1,2}(x_I, y_J) \\ \widetilde{K}_{2,1}(w_1, v_1) & \ldots & \widetilde{K}_{2,1}(w_1, v_I) & \widetilde{K}_{2,2}(w_1, y_1) & \ldots & \widetilde{K}_{2,2}(w_1, y_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{2,1}(w_J, v_1) & \ldots & \widetilde{K}_{2,1}(w_J, v_I) & \widetilde{K}_{2,2}(w_J, y_1) & \ldots & \widetilde{K}_{2,2}(w_J, y_J) \end{pmatrix} . \quad (3.5)$$

Applying the operator (3.3) to this determinant, and making a small calculation, one gets

$$\det \begin{pmatrix} \widehat{K}_{1,1}(x_1, x_1) & \ldots & \widetilde{K}_{1,1}(x_1, x_I) & \widetilde{K}_{1,2}(x_1, y_1) & \ldots & \widetilde{K}_{1,2}(x_1, y_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{1,1}(x_I, x_1) & \ldots & \widehat{K}_{1,1}(x_I, x_I) & \widetilde{K}_{1,2}(x_I, y_1) & \ldots & \widetilde{K}_{1,2}(x_I, y_J) \\ \widetilde{K}_{2,1}(y_1, x_1) & \ldots & \widetilde{K}_{2,1}(y_1, x_I) & \widehat{K}_{2,2}(y_1, y_1) & \ldots & \widetilde{K}_{2,2}(y_1, y_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ \widetilde{K}_{2,1}(y_J, x_1) & \ldots & \widetilde{K}_{2,1}(y_J, x_I) & \widetilde{K}_{2,2}(y_J, y_1) & \ldots & \widehat{K}_{2,2}(y_J, y_J) \end{pmatrix}, \quad (3.6)$$

where the entries on the main diagonal are defined by

$$\widehat{K}_{1,1}(x, x) = \sum_{i=0}^{n-1} \frac{1}{h_i^2} p_i(x) \widetilde{Q}_i(x),$$

and

$$\widehat{K}_{2,2}(y, y) = \sum_{i=0}^{n-1} \frac{1}{h_i^2} \widetilde{P}_i(y) q_i(y),$$

respectively; compare with (1.18) and (1.21). So (3.6) yields a determinantal formula for the multi-variate Cauchy transform (3.4). By the above discussion, this is also the generating function for averages of products of traces (3.1).

3.2 Eynard-Mehta theorem for correlation functions

In this section we show how Theorem 1.6 can be used to give an alternative proof of the Eynard-Mehta theorem for correlation functions in the two-matrix model. Alternative proofs of this formula, which are applicable for the more general model of random matrices coupled in a chain, can be found in [11, 20, 23, 31, 34].

The *correlation function* $R_{I,J}$ is defined from (1.3) by

$$R_{I,J}(\lambda_1, \ldots, \lambda_I; \mu_1, \ldots, \mu_J) = \frac{n!}{(n-I)!} \frac{n!}{(n-J)!}$$

$$\times \frac{1}{\widetilde{Z}_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(e^{-V(\lambda_i)} e^{-W(\mu_i)} \right) \Delta(\lambda) \Delta(\mu) \det(e^{\tau \lambda_i \mu_j})_{i,j=1}^{n} \prod_{i=I+1}^{n} d\lambda_i \prod_{j=J+1}^{n} d\mu_j. \quad (3.7)$$

The *Eynard-Mehta theorem* [20] then asserts that

$$R_{I,J}(\lambda_1, \ldots, \lambda_I; \mu_1, \ldots, \mu_J)$$
$$= \det \begin{pmatrix} K_{1,1}(\lambda_1, \lambda_1) & \ldots & K_{1,1}(\lambda_1, \lambda_I) & K_{1,2}(\lambda_1, \mu_1) & \ldots & K_{1,2}(\lambda_1, \mu_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ K_{1,1}(\lambda_I, \lambda_1) & \ldots & K_{1,1}(\lambda_I, \lambda_I) & K_{1,2}(\lambda_I, \mu_1) & \ldots & K_{1,2}(\lambda_I, \mu_J) \\ K_{2,1}(\mu_1, \lambda_1) & \ldots & K_{2,1}(\mu_1, \lambda_I) & K_{2,2}(\mu_1, \mu_1) & \ldots & K_{2,2}(\mu_1, \mu_J) \\ \vdots & & \vdots & \vdots & & \vdots \\ K_{2,1}(\mu_J, \lambda_1) & \ldots & K_{2,1}(\mu_J, \lambda_I) & K_{2,2}(\mu_J, \mu_1) & \ldots & K_{2,2}(\mu_J, \mu_J) \end{pmatrix}. \quad (3.8)$$

We now establish this formula by using the formulas in Section 3.1. Here we follow [12]. We start from the multi-variate Cauchy transform in (3.4). Applying the operators

$$f \mapsto \frac{1}{2\pi i} \lim_{\epsilon \to 0+} \left(f|_{x_i = \lambda_i - \epsilon i} - f|_{x_i = \lambda_i + \epsilon i} \right)$$

subsequently for $i = 1, \ldots, I$ and then

$$f \mapsto \frac{1}{2\pi i} \lim_{\epsilon \to 0+} \left(f|_{y_j = \mu_j - \epsilon i} - f|_{y_i = \mu_i + \epsilon i} \right)$$

for $j = 1, \ldots, J$ to (3.4), we obtain by the Stieltjes-Perron inversion principle [33] precisely the correlation function (3.7). On the other hand, applying these same operations to (3.6) and using again the Stieltjes-Perron inversion principle leads to the right hand side of (3.8). This establishes (3.8).

4 Concluding remarks

The two-matrix model is a particular instance of a more general model, sometimes referred to as *random matrices coupled in a chain*, see e.g. [2, 11, 20, 23, 29, 31, 34]. The Eynard-Mehta theorem for correlation functions can be formulated for this more general model.

In view of this observation, it is natural to ask whether the results in this paper can be extended to the more general model of random matrices coupled in a chain. That is, one may ask whether the averages of products and ratios of characteristic polynomials in this model can still be written as determinants built out of (transformed) Eynard-Mehta kernels and biorthogonal polynomials. A little thought reveals that such a result, if it exists, should be a non-trivial extension of Theorem 1.6, except maybe for special configurations of the external sources. This is an open problem.

Another question of interest is whether the results in this paper have an analogue for the Cauchy two-matrix model in [8].

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