

INJECTIVITY OF A CERTAIN CYCLE MAP FOR FINITE DIMENSIONAL W-ALGEBRAS

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ABSTRACT. We study a certain cycle map defined on finite dimensional modules for the W -algebra with regular integral central character. Via comparison with the theory in positive characteristic, we show that this map injects into the top Borel-Moore homology group of a Springer fibre. This is the first result in a larger program to completely describe the finite dimensional modules for the W algebras.

1. INTRODUCTION

Recently, the subject of the finite W -algebras has come to the attention of many researchers. Although originally introduced in the physics literature, they were first defined in a mathematical context by Premet [P1], who related them to the study of (non-restricted) modular representations of semisimple lie algebras. The fundamental paper of Gan and Ginzburg [GG] reproved some of Premet's results, and recast them in the light of non-commutative algebraic geometry. Since then, many authors have made contributions to their study, c.f., e.g., [P2], [L1], [L2], [BGK],[DK], and the survey articles [L3] and [W] have appeared. In particular, the results of [BGK] and [L2] are concerned with the finite dimensional representations of W -algebras. Despite the significant progress made there, some fundamental questions remain open. For instance, given a finite W -algebra $U(\mathfrak{g}, e)$ and an integral central character λ , it is still not known how to parametrize the simple finite dimensional $U(\mathfrak{g}, e)$ modules with character λ . The goal of this paper is to provide the first step to answering this question. In fact, we will provide some detailed information on the K -group $K_{\mathbb{Q}}(\text{mod}^{f.d.}(U^{\lambda}(\mathfrak{g}, e)))$.

Our main tool will be the use of a certain characteristic cycle map which takes $K_{\mathbb{Q}}(U^{\lambda}(\mathfrak{g}, e))$ to the the homology group $H_{top}(\mathcal{B}_e, \mathbb{Q})$ - this is the top Borel-Moore homology of the Springer fibre associated to the nilpotent element e (definitions will be recalled below). The latter group has a natural basis (as a \mathbb{Q} - vector space) indexed by irreducible components of the variety \mathcal{B}_e . In addition, it has the structure of a module over the Weyl group associated to \mathfrak{g} , W . This is the classical construction of Springer, which finds all of the simple W -modules in such homology groups.

The group $K_{\mathbb{Q}}(U^{\lambda}(\mathfrak{g}, e))$ also has a natural structure of a W -module, via the action of reflection functors on the category $U^{\lambda}(\mathfrak{g}, e) - \text{mod}$. The theory of these functors, which is parallel to the classical theory of reflection functors for $U(\mathfrak{g})$ developed by Jantzen, has been worked out by Losev. We shall recall their basic properties below.

With all of this in hand, we can state the basic theorem of this paper.

Theorem 1. *The cycle map*

$$cc : K_{\mathbb{Q}}(\text{mod}^{f.d.}(U^{\lambda}(\mathfrak{g}, e))) \rightarrow H_{top}(\mathcal{B}_e, \mathbb{Q})$$

is injective and W -equivariant, with respect to the actions of W discussed above.

Let us note right away that the paper [ES] provides a weaker version of this result; namely, they prove the numerical bound

$$\dim(K_{\mathbb{Q}}(\text{mod}^{f.d.}(U^\lambda(\mathfrak{g}, e)))) \leq \dim(H_{top}(\mathcal{B}_e, \mathbb{Q}))$$

Their proof uses D -module theory in characteristic zero and the theory of Springer's representations.

The main tool of this paper will be the use of reduction mod p and a comparison with the theory of W -algebras over algebraically closed fields of positive characteristic. Following the reasoning of [BMR], we shall construct a localization theory for such algebras, and compare the resulting geometry with the geometry in characteristic zero. In positive characteristic, we have extra tools such as the Azumaya splitting and the action of the Frobenius morphism. We shall use these to deduce the result for sufficiently large positive characteristic, and then we shall transfer to characteristic zero.

We should also point out that I. Losev and V. Ostrik have a conjecture about the image of the map cc , which has been partially proved by them. To state it, let us recall that to the nilpotent element e , we can also associate a (possibly trivial) cell c in the Weyl group W (c.f. [Lu] for a complete introduction to cells and representations of Weyl groups and Hecke algebras). To this cell we can then associate

$$H_{top}(\mathcal{B}_e, \mathbb{Q})_c$$

a sub- W -representation of $H_{top}(\mathcal{B}_e, \mathbb{Q})$. They conjecture that this is the image of the map cc . Therefore, combined with the result in this paper, the proof of this conjecture would yield a complete description of $K_{\mathbb{Q}}(\text{mod}^{f.d.}(U^\lambda(\mathfrak{g}, e)))$.

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2. FINITE W-ALGEBRAS

There are a great many references which explain the basic construction of the finite W -algebras. The papers of Premet [P1], Gan-Ginzburg [GG], and Brundan-Goodwin-Kleschev [BGK] all have very complete introductions. For now we shall just recall the very basic outline of what we need.

We let \mathfrak{g} be a complex semisimple Lie algebra, and let $e \in \mathfrak{g}$ be a nonzero nilpotent element. By the Jacobson-Morozov theorem, there exist $f, h \in \mathfrak{g}$ such that $\{e, f, h\}$ form an \mathfrak{sl}_2 -triple, and we fix such a triple. We define the Slodowy slice $S \subseteq \mathfrak{g}^*$ to be the affine subspace which corresponds, via the Killing isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, to the affine space $e + \ker(\text{ad}(f))$. Slodowy's book [Slo] contains a wealth of information on these spaces and their uses in Lie theory. Three facts about these spaces are crucial for us.

The first, recorded in [G-G] section 3, is that this affine space has a natural Poisson structure inherited from the Poisson structure on \mathfrak{g}^* . The second, to be found in [G-G], section 2, is that the space S admits a natural \mathbb{C}^* action defined as follows: our chosen \mathfrak{sl}_2 -triple gives a homomorphism $\tilde{\gamma}: SL_2(\mathbb{C}) \rightarrow G$, and we define $\gamma(t) = \tilde{\gamma} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so that $\text{Ad}(\gamma(t))e = t^2e$; so we define $\bar{\rho}(t) = t^{-2}\text{Ad}^*(\gamma(t))$, a \mathbb{C}^* -action on \mathfrak{g} which stabilizes S_e and fixes χ (the element of \mathfrak{g}^* corresponding to e under the Killing isomorphism). In fact, this action contracts S to χ . So, we get a grading on $O(S)$ and it is easy to see that the Poisson multiplication respects this grading.

Finally, we wish to recall that the space S can be realized as a "Hamiltonian reduction" of the space \mathfrak{g}^* . To explain this, we let $\chi \in \mathfrak{g}^*$ be the element associated to e under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ given by the Killing form. We define a skew-symmetric bilinear form on

$\mathfrak{g}(-1)$ via $\langle x, y \rangle = \chi([x, y])$, which is easily seen to be nondegenerate. Thus, $(\mathfrak{g}(-1), \langle \cdot, \cdot \rangle)$ is a symplectic vector space, and we choose $l \subset \mathfrak{g}(-1)$ a Lagrangian subspace. We define $\mathfrak{m}_l = l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$, a nilpotent lie algebra such that $\chi|_{\mathfrak{m}_l}$ is a character of \mathfrak{m}_l . We let M_l be the unipotent connected algebraic subgroup of G such that $\text{Lie}(M_l) = \mathfrak{m}_l$. We let I denote the ideal of $\text{Sym}(\mathfrak{g}) = O(\mathfrak{g}^*)$ generated by $\{m - \chi(m) | m \in \mathfrak{m}_l\}$. Then we have an isomorphism of algebras

$$O(S) \cong (O(\mathfrak{g}^*)/I)^{M_l}$$

where M_l acts via the adjoint action (c.f. [G-G], lemma 2.1).

Given this, we can recall that the finite W-algebra associated to $e \in \mathfrak{g}$, denoted $U(\mathfrak{g}, e)$, is a filtered associative algebra whose associated graded Poisson algebra is isomorphic to $O(S)$. In fact, the algebra $U(\mathfrak{g}, e)$ can be defined as the Hamiltonian reduction of the enveloping algebra $U(\mathfrak{g})$ in a manner exactly parallel to the formula above. There is a natural map $Z(U(\mathfrak{g})) \rightarrow Z(U(\mathfrak{g}, e))$ which is an isomorphism. So we have the usual description of central characters indexed by elements of the affine space \mathfrak{h}/W (where \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} , and W is the Weyl group). Given a $\lambda \in \mathfrak{h}$, we thus get an ideal of $Z(U(\mathfrak{g}, e))$ and then an ideal J_λ of $U(\mathfrak{g}, e)$, and we define $U^\lambda(\mathfrak{g}, e) := U(\mathfrak{g}, e)/J_\lambda$.

3. THE CYCLE MAP

In the paper [DK], the author and Kobi Kremnizer gave a geometric interpretation of certain categories of modules over finite W-algebras. Inspired by the classical Beilinson-Bernstein localization theorem, we considered the singular Poisson variety $S \cap \mathcal{N} := S_{\mathcal{N}}$, where \mathcal{N} denotes the nilpotent cone of \mathfrak{g}^* . The variety \mathcal{N} has a resolution of singularities, denoted $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$, called the springer resolution (see [CG], chapter 3, for a very complete treatment). It turns out that the restriction of this map $\mu : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ is also a resolution of singularities, and the scheme theoretic preimage $\mu^{-1}(S_{\mathcal{N}})$ is denoted $\tilde{S}_{\mathcal{N}}$. Further, this variety has a natural symplectic structure which extends the Poisson structure on the base $S_{\mathcal{N}}$.

Now, the \mathbb{C}^* action constructed above lifts naturally to $\tilde{S}_{\mathcal{N}}$, and it contracts the smooth variety $\tilde{S}_{\mathcal{N}}$ to the singular variety $\mathcal{B}_\chi := \mu^{-1}(\chi)$, the springer fibre of χ . This provides the perfect setting to do geometry.

In particular, given an anti dominant regular weight λ , we constructed a sheaf of $\mathbb{C}[[h]]$ -algebras $D_h(\lambda, \chi)(0)$ on $\tilde{S}_{\mathcal{N}}$, which is a quantization in the sense of [BK], i.e., it is flat over $\mathbb{C}[[h]]$ and satisfies $D_h(\lambda, \chi)(0)/hD_h(\lambda, \chi)(0) \cong O(\tilde{S}_{\mathcal{N}})$. This algebra is related to the finite W-algebra in the following way:

The algebra $U^\lambda(\mathfrak{g}, e)$ is naturally filtered, as recalled above. Thus we can consider the Rees algebra associated to this filtered algebra (c.f. [BFG], section 2.4) which is naturally an algebra over $\mathbb{C}[[h]]$. If we then complete with respect to h , we obtain an algebra which we call $U_h^\lambda(\mathfrak{g}, e)(0)$ (One can then formally invert h to obtain a $\mathbb{C}((h))$ -algebra $U_h^\lambda(\mathfrak{g}, e)$, which is one of the main players in [DK], although it won't be used here). We then have

$$(3.1) \quad \Gamma(D_h(\lambda, \chi)(0)) = U_h^\lambda(\mathfrak{g}, e)(0)$$

With these ingredients in hand, we can explain our construction of the cycle of a finite dimensional module M over $U^\lambda(\mathfrak{g}, e)$. Given such, we choose any good filtration F on M (c.f., [HTT], appendix D). Then we have the module $\text{Rees}(M; F)$ and, after completion, the $\mathbb{C}[[h]]$ -module $\widehat{\text{Rees}(M; F)}$. We can then define a localization functor

$$\text{Loc}(M; F)(0) = D_h(\lambda, \chi)(0) \otimes_{U_h^\lambda(\mathfrak{g}, e)(0)} \widehat{\text{Rees}(M; F)}$$

which makes sense because of 3.1. Because $D_h(\lambda, \chi)(0)$ is a quantization of $\tilde{S}_{\mathcal{N}}$, we then get a coherent sheaf on $\tilde{S}_{\mathcal{N}}$ by letting

$$CS(M; F) := Loc(M; F)(0)/hLoc(M; F)(0)$$

Now we define our main object of study, the cycle map, by

$$cc(M) := CC(CS(M; F)) \in H_{top}(\mathcal{B}_{\chi}, \mathbb{Q})$$

where CC stands for the characteristic cycle of a coherent sheaf. A very complete treatment of characteristic cycles is provided by the book [F], and some details are recalled in section 8 below. For now, we just recall that there is a chern character map

$$K(X) \rightarrow H_*(X)$$

from the K theory to the total Borel-Moore homology of a projective scheme X . The projection of this map to the top graded piece of $H_*(X)$ yields the map CC .

The fact that, in our case, $CS(M; F)$ is actually supported on \mathcal{B}_{χ} will also be addressed in section 6 below.

The fact that the construction of cc does not depend on the filtration chosen (while $CS(M; F)$ does) is a standard argument (c.f. [HTT], appendix D).

The main objective of the next few sections will be to relate this construction to the positive characteristic machinery of [BMR], where the relation between K groups of representations and homology of springer fibres is very strong indeed.

4. LOCALIZATION MOD P

In this section, we'll review the main results of the localization theory for enveloping algebras in characteristic p , which can be found in [BMR I,II] and [BM]. We recall that the lie algebra \mathfrak{g} has an integral form $\mathfrak{g}_{\mathbb{Z}}$ (c.f. [H]), which then has a base extension to any field k , called \mathfrak{g}_k . Throughout the rest of the paper, we will use k to denote an algebraically closed field of positive characteristic. When $char(k) > h$ (where h is the Coxeter number of \mathfrak{g}), Bezrukavnikov-Mirkovic-Rumynin have developed a localization theory for the enveloping algebra $U(\mathfrak{g}_k)$. Since this theory is extremely important for us, we shall recall their basic notations and results in some detail.

4.1. Quantized Twisted Differential Operators. We start with the quantized sheaf of twisted differential operators on $T^*\mathcal{B}_{\mathbb{C}}$. We first recall that the original sheaf of twisted differential operators can be defined using the following two steps (c.f. [Mil], Chapter C1 for details):

First, one defines the sheaf of algebras $U^0 = \mathcal{O}_{\mathcal{B}} \otimes_{\mathbb{C}} U(\mathfrak{g})$, where the multiplication is twisted by the action of an element of \mathfrak{g} , considered as a vector field, on a local section of $\mathcal{O}_{\mathcal{B}}$. Inside U^0 , we have the sub-ideal-sheaf \mathfrak{n}^0 , which is generated at each point $x \in \mathcal{B}$ by the subspace $\mathfrak{n}_x \in \mathfrak{g}$ (thinking of \mathcal{B} as the variety of Borel subalgebras in \mathfrak{g} , each point gets a Borel \mathfrak{b}_x and a corresponding maximal nilpotent subalgebra; this is \mathfrak{n}_x). We also, therefore, have an ideal sheaf \mathfrak{b}^0 inside U^0 , and containing \mathfrak{n}^0 .

From here, we define the sheaf of algebras $D_{\mathfrak{h}} = U^0/\mathfrak{n}^0$. Thus there is a natural map from the sheaf of lie algebras $\mathfrak{h}^0 = \mathfrak{b}^0/\mathfrak{n}^0$ to the sheaf $D_{\mathfrak{h}}$, which then induces a map $\phi : U(\mathfrak{h}) \rightarrow \Gamma(\mathcal{B}, U^0/\mathfrak{n}^0)$. For any element $\lambda \in \mathfrak{h}^*$, we have an ideal $I_{\lambda} \subseteq U(\mathfrak{h})$, which, due to normalization reasons, is the ideal chosen to correspond the character $\lambda + \rho$ (where ρ is the half sum of the positive roots, as usual). Thus we have an ideal $I_{\lambda} D_{\mathfrak{h}}$, and we can finally put $D^{\lambda} = D_{\mathfrak{h}}/I_{\lambda} D_{\mathfrak{h}}$.

Now, we can quantize each step of this construction in a natural way. We start by defining, for an affine cover of \mathcal{B} , $\{U_i\}$ the sheaves $U_h^0 = \mathcal{O}_{U_i} \otimes_{\mathbb{C}[[h]]} U_h(\mathfrak{g})$, where $U_h(\mathfrak{g})$ is simply $\text{Rees}(U(\mathfrak{g}))$ (with respect to the usual PBW filtration). These are not merely sheaves on the varieties U_i , but, by using ore localization, we can view them as quantizations¹ of the varieties $U_i \times \mathfrak{g}$ (we get the structure algebras of these varieties by setting $h = 0$). This construction glues naturally, and so we get a sheaf U_h^0 on $\mathcal{B} \times \mathfrak{g}$.

Given this, we still have subsheaves \mathfrak{b}^0 and \mathfrak{n}^0 generated by the same elements, and so we can consider the quotient $D_{h,\mathfrak{h}} = U_h^0 / \mathfrak{n}^0$, with its associated map $U_h(\mathfrak{h}) \rightarrow \Gamma(D_{h,\mathfrak{h}})$. Then the element $\lambda \in \mathfrak{h}^*$ still defines an ideal of $U_h(\mathfrak{h})$ (defined as the ideal generated by $\{v - h(\lambda + \rho)(v) \mid v \in \mathfrak{h}\}$) again called I_λ , and we can now define $D_h^\lambda = D_{h,\mathfrak{h}} / I_\lambda D_{h,\mathfrak{h}}$, which are now sheaves on the space $T^*\mathcal{B}$. We also note that the sheaf $D_{h,\mathfrak{h}}$ can be considered a sheaf on the space $\tilde{\mathfrak{g}}^*$ - the full Grothendieck alteration (c.f. [CG], chapter 3)

4.2. Differential Operators in Positive Characteristic. Now suppose, in addition to the assumptions of the above section, that the element $\lambda \in \mathfrak{h}^*$ is integral. Then every object that we have used in the above construction; the enveloping algebra $U(\mathfrak{g})$, the group G , its Borel and nilpotent subalgebras, and flag variety, exists over \mathbb{Z} . Therefore, it makes perfect sense to construct the algebra $D_h^\lambda(\mathbb{Z})$ as a quantization of $T^*\mathcal{B}(\mathbb{Z})$, and further, to base change to an algebraically closed field of positive characteristic, and thus obtain an object $D_h^\lambda(k)$ (from now on in this section we shall drop the k , understanding that we are working over a closed field of positive characteristic). Upon taking the quotient $D_h^\lambda / (h - 1)$, we obtain the sheaf of crystalline differential operators as featured in [BMR], which we will simply denote D^λ . This sheaf has the feature that there is a “frobenius morphism”

$$F : \mathcal{O}(T^*\mathcal{B}^{(1)}) \rightarrow D^\lambda$$

obtained by using p^h iterates of vector fields, c.f. [BMR], section 2.

In the case of the sheaf D_h^λ , we can lift F to a morphism $F : \mathcal{O}(T^*\mathcal{B}^{(1)} \times \mathbb{A}^1) \rightarrow D_h^\lambda$ (simply by sending the extra variable to h). This means that D_h^λ (technically, its h -completion) is a “frobenius constant quantization” in the terminology of [BK]. Let us recall the definition there:

Definition 2. Let O_h be a quantization of the Poisson scheme X (defined over k). Then O_h is a frobenius constant quantization if the frobenius morphism $F : \mathcal{O}_X^p \rightarrow \mathcal{O}_X$ lifts to a morphism $F : O_h \rightarrow Z(O_h)$ (where Z is the algebra center).

In this case, the sheaf O_h can be regarded as a locally free coherent sheaf of algebras on the scheme $X^{(1)} \times \text{Spec}(k[[h]])$.

In the case of the sheaf $D_{h,\mathfrak{h}}$, we can even say a bit more. By the same reasoning, there is a morphism $F : \tilde{\mathfrak{g}}^{*(1)} \rightarrow D_{h,\mathfrak{h}}$. But in fact there is also a morphism in any characteristic (even over \mathbb{Z}) $\mathcal{O}(\mathfrak{h}^*) \rightarrow D_{h,\mathfrak{h}}$, simply by noting that $U(\mathfrak{h}) \rightarrow D_{h,\mathfrak{h}}$ by construction.

Now, in positive characteristic, both of the schemes $\tilde{\mathfrak{g}}^{*(1)}$ and \mathfrak{h}^* live over the scheme $\mathfrak{h}^{*,(1)}$. The morphism $\tilde{\mathfrak{g}}^{*(1)} \rightarrow \mathfrak{h}^{*,(1)}$ is Grothendieck’s invariants map (c.f. [CG] chapter 3), and the map $\mathfrak{h}^* \rightarrow \mathfrak{h}^{*,(1)}$ is the Artin-schreier map for p -lie algebras, which on algebras of functions is the morphism $\text{Sym}(\mathfrak{h}^{(1)}) \rightarrow \text{Sym}(\mathfrak{h})$ which takes $h \rightarrow h^p - h^{[p]}$ (c.f. [BMR], section 2.3).

¹In fact, these are not quite quantizations as we have defined them above, because these algebras are not complete with respect to h . However, they are h -free, and their h -completions, considered below, would be quantizations in the strict sense.

By comparing images, we see that we actually arrive at a morphism $\tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^* \rightarrow D_{h,\mathfrak{h}}$, which will play the role of the Frobenius morphism for this sheaf.

4.3. Localization For Lie Algebras. As in the classical case, localization involves comparing modules over a sheaf of differential operators to modules over the global sections. Similarly to the classical case, for $\lambda \in \mathfrak{h}_k$, we have that

$$\Gamma(D^\lambda) \cong U^\lambda(\mathfrak{g}_k)$$

where $U^\lambda(\mathfrak{g}_k)$ is the quotient of the algebra $U(\mathfrak{g}_k)$ by the ideal J_λ obtained as follows: the algebra $S(\mathfrak{h}_k)^{(W,\cdot)}$ occurs as a subalgebra of $Z(U(\mathfrak{g}_k))$. As in characteristic zero, this subalgebra is equal to $U(\mathfrak{g}_k)^{G_k}$, but unlike in characteristic zero, this is not the whole center. Still, any point in \mathfrak{h}_k defines an ideal of $S(\mathfrak{h}_k)^{(W,\cdot)}$, which can then be extended to an ideal of $U(\mathfrak{g}_k)$.

Then, if $\lambda \in \mathfrak{h}_k$ is regular with respect to the dot-action of W , we have [BMR, 3.2]:

Theorem 3. *There is an equivalence of categories*

$$R\Gamma : D^b(\text{mod}^c(D^\lambda)) \rightarrow D^b(\text{mod}^{f\cdot g}(U^\lambda))$$

where $\text{mod}^c(D^\lambda)$ denotes the category of coherent modules over the Azumaya algebra D^λ on $T^*\mathfrak{B}_k^{(1)}$.

The ‘‘underived’’ version of this; i.e., simply the functor Γ , does not produce an equivalence. This is an essential difference with the characteristic zero case, where the localization theorem is underived.

The next step is to exploit the full center of the algebra $U(\mathfrak{g}_k)$. As mentioned above, the subalgebra $U(\mathfrak{g}_k)^{G_k}$ does not equal the full $Z(U(\mathfrak{g}_k))$. The reason for this is the existence of the so-called p^{th} iterate map $\mathfrak{g} \rightarrow \mathfrak{g}^{(1)}$, denoted $x \rightarrow x^{[p]}$ (where (1) again denotes Frobenius twist). In the case of $\mathfrak{gl}(n)$, this map is simply the p^{th} power of a matrix; and one can define it in general (for a reductive Lie algebra) by using a suitable embedding $\mathfrak{g} \rightarrow \mathfrak{gl}(n)$.

This map allows us to construct new central elements as follows: for any $x \in \mathfrak{g}$, the element $x^p - x^{[p]} \in U(\mathfrak{g}_k)$ is now central. In fact, the algebra generated by these elements, called the p -center of $U(\mathfrak{g}_k)$, denoted Z_p , is precisely $S(\mathfrak{g}^{(1)})$, because the scalar multiplication on $x^p - x^{[p]}$ is twisted by the p^{th} power. Then, one can describe the full center of the enveloping algebra in positive characteristic (c.f. [BMR], chapter 3) as

$$Z \cong O(\mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}^*/W)$$

where W acts via the dot action, and the map from $\mathfrak{h}^* \rightarrow \mathfrak{h}^{*(1)}$ is the Artin-Schreier map, a version of the p^{th} power map, defined, e.g., in [BMR], section 2.3, and [BFG], section 3.2.

Thus, given $\chi \in \mathfrak{g}^*$, we can associate to it an element $\chi^{(1)} \in \mathfrak{g}^{(1)*}$ (c.f. [BFG], section 3.2), and from this we get a maximal ideal of $S(\mathfrak{g}^{(1)})$, and a central ideal of $U(\mathfrak{g}_k)$. Any irreducible module over $U(\mathfrak{g}_k)$ will have a well defined central character, which will restrict to characters of both Z_p and $U(\mathfrak{g}_k)^{G_k}$ (the latter is called the Harish-Chandra center of $U(\mathfrak{g}_k)$). If we suppose that the Harish-Chandra character is integral, then this implies that the Z_p -character is a nilpotent element of \mathfrak{g}^* ([BMR], section 6). We will assume that we are in this situation from now on.

Then the above localization theorem can be modified as follows: Let λ be a regular, integral central character, and let $\chi^{(1)} \in \mathfrak{g}^{*(1)}$ be a nilpotent element. Then we have $\mu^{-1}(\chi)^{(1)} = (\mu^{(1)})^{-1}(\chi^{(1)})$, the Frobenius twist of the Springer fibre of χ , which we shall usually denote \mathcal{B}_χ . We define the category $\text{mod}_\chi^c(D^\lambda)$ to be the category of coherent D^λ

modules which are set-theoretically supported on the variety $\mathcal{B}_\chi^{(1)}$. Similarly, we define $\text{mod}_\chi^{f.g.}(U^\lambda)$ to be the category of U^λ -modules which have generalized p -character χ - i.e., they are killed by some power of the maximal ideal in $S(\mathfrak{g}_k)$ associated to χ . Then we have:

Theorem 4. . *The the above localization theorem specializes to an equivalence of categories*

$$R\Gamma : D^b(\text{mod}_\chi^c(D^\lambda)) \rightarrow D^b(\text{mod}_\chi^{f.g.}(U^\lambda))$$

Finally, we can say a little more about the structure of the category $\text{mod}_\chi^c(D^\lambda)$. Since D^λ is an Azumaya algebra, it is etale locally a matrix algebra over $O_{T^*\mathcal{B}^{(1)}}$. We recall that an Azumaya algebra \mathfrak{A} on a scheme X is said to be *split* if it is in fact isomorphic to a matrix algebra over O_X . If this happens, then there is the standard Morita equivalence of categories $\text{mod}^c(\mathfrak{A}) \xrightarrow{\sim} \text{mod}^c(O_X)$. Although D^λ is not split, we have the following:

Theorem 5. *Under the same assumptions as above, we have that the restriction of D^λ to the formal neighborhood of $\mathcal{B}_\chi^{(1)}$ in $T^*\mathcal{B}^{(1)}$ is a split Azumaya algebra. This implies an equivalence*

$$\text{mod}_\chi^c(D^\lambda) \xrightarrow{\sim} \text{mod}_{\mathcal{B}_\chi^{(1)}}^c(O_{T^*\mathcal{B}^{(1)}})$$

And therefore an equivalence

$$D^b(\text{mod}_{\mathcal{B}_\chi^{(1)}}^c(O_{T^*\mathcal{B}^{(1)}})) \xrightarrow{\sim} D^b(\text{mod}_\chi^{f.g.}(U^\lambda))$$

Thus we observe a tight relation between representation theory and coherent sheaves in characteristic p . In the next several sections we relate this theory to the theory of W-algebras.

5. LOCALIZATION FOR MODULAR W-ALGEBRAS

In this section, we would like to construct the “local” object which corresponds to the version of the W -algebra in positive characteristic (we shall give a complete definition of this object in section six below). This object will, in particular, be a sheaf on the scheme $\tilde{\mathcal{S}}_{\mathcal{N}}^{(1)}$ (the Frobenius twist of the resolution of the Springer fibre). It will also be necessary to introduce a version over the extended scheme $\tilde{\mathcal{S}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$, which will be completely parallel to the first version. We’ll begin by recalling, briefly, the construction of the main object of the paper [DK], which is a quantization of the scheme $\tilde{\mathcal{S}}_{\mathcal{N},\mathbb{C}}$. We shall attempt to keep this paper as self contained as possible.

5.1. Hamiltonian Reduction. As indicated above, the W -algebra is related to the Lie algebra via the procedure of Hamiltonian reduction. Since modules over Lie algebras can be localized to D -modules on the flag variety \mathcal{B} , it stands to reason that modules over W -algebras should localize to modules over a some sort of Hamiltonian reduction of $D_{\mathcal{B}}$. One version of this (over \mathbb{C}) was worked out in [DK]. Here, we wish to construct a modular version of this localization, based on the ideas of [BMR] and the related paper [BFG].

In particular, we shall take the Hamiltonian reduction of differential operators $D_{\mathfrak{h}}^\lambda$ with respect to the action of Premet’s subgroup $M_{\mathfrak{l}}$, and character χ . In particular, we work under the assumption that $\text{char}(k)$ is large enough so that the Lie algebra \mathfrak{m}_l (c.f. section 2 above) exists as over k , and satisfies $\mathfrak{m}_l^{[p]} = 0$ (where $[p]$ is the p^{th} iterate map induced from \mathfrak{g}) (note that this is possible because \mathfrak{m}_l is nilpotent).

We recall that the action of G on the flag variety \mathcal{B} induces a “quantum” moment map $\mu : U_h(\mathfrak{g}) \rightarrow D_h^\lambda$, which has a restriction, which we shall also call $\mu : U_h(\mathfrak{m}_l) \rightarrow D_h^\lambda$. In the positive characteristic case, this map also satisfies some additional structure, as in:

Definition 6. (c.f. [BK] section 5)

Suppose that X is a symplectic variety over k , and let O_h be a Frobenius-constant quantization of X^2 . Suppose further that there is an algebraic group action $H \times X \rightarrow X$, and a morphism $\mu : U_h(\mathfrak{h}) \rightarrow O_h$ (where $\mathfrak{h} = \text{Lie}(H)$), satisfying $\mu(h) = h$.

1) This quantization is said to be “Frobenius H-constant” if H acts on the sheaf of algebras O_h in a way that preserves the subalgebra $O(X)^{(1)}[[\hbar]]$, such that H fixes the parameter h , and such that the induced action of H on $X^{(1)}$ agrees with the base change of the action of H on X .

For $\xi \in \mathfrak{h}$, and $s \in O_h$ a local section, let $\xi \cdot s$ denote the action (obtained by differentiating the action of H).

2) Given the set-up of 1), the map μ is said to be a quantum moment map if the restriction of μ to $U_h(\mathfrak{h})^{(1)} = U(\mathfrak{h})^{(1)}[[\hbar]]$ lands in $O(X)^{(1)}[[\hbar]]$, and we have the “action” relation: for all $\xi \in \mathfrak{h}$, and all local sections $s \in O_h$, $\mu(\xi)s - s\mu(\xi) = h\xi \cdot s$.

The last part of definition 2 is the standard definition of a quantum moment map in any characteristic. The fact that (the completion of) our map μ is a quantum moment map follows immediately from the definition of μ - the algebra D_h^λ was constructed out of the enveloping algebra. Further, the p^{th} powers on both sides clearly coincide.

Given this, we can state the version of Hamiltonian reduction which we will use, again following [BK]. In the rest of this section, we shall work with the h -completed algebras $U_h(\mathfrak{m}_l)(0)$ and $D_h(\lambda)(0)$ (we use this notation to match our characteristic zero notation from [DK]).

First, we consider the ideal $I_\chi \subseteq U_h(\mathfrak{m}_l)(0)$ defined as the ideal generated by $\{m - \chi(m) | m \in \mathfrak{m}_l\}$. Note that this ideal is *not* homogeneous with respect to the usual grading on $\text{Rees}(U_h) \cong U_h(0)$.

Next, we define the sheaf of algebras $D_h(\lambda)(0) / \langle \mu(I_\chi) \rangle$. This is naturally a sheaf over the scheme $\mu^{-1}(\chi)^{(1)} \times \text{Spec}(k[[\hbar]])$. To see why, note that the central subalgebra $\text{Sym}(\mathfrak{m}_l^{(1)}[[\hbar]])$ has an ideal generated by the point $\chi^{(1)} \in \mathfrak{m}_l^*$. Let $I_\chi^{(1)}$ be the ideal in $U_h(\mathfrak{m}_l)(0)$ generated by this central ideal. From the definition of moment map above, we see that $\mu(I_\chi^{(1)})$ defines the subscheme $\mu^{-1}(\chi)^{(1)}$ inside $T^*\mathcal{B}^{(1)}$. Further, $I_\chi^{(1)} = I_\chi \cap \text{Sym}(\mathfrak{m}_l^{(1)}[[\hbar]])$ (recall that we are assuming that $m^{[p]} = 0$ for all $m \in \mathfrak{m}_l$).

Finally, we look at the pushforward $p_*(D_h(\lambda)(0) / \langle \mu(I_\chi) \rangle)^{M_l}$. Then this is naturally a sheaf on the scheme $\tilde{\mathcal{S}}_\chi^{(1)} \times \text{Spec}(k[[\hbar]])$. This is our Hamiltonian reduction $D_h(\lambda, \chi)(0)$.

A slight modification gives the case of the sheaf $D_{h,\mathfrak{h}}$: we note that the moment map works the same way, and the group M_l only acts $\tilde{\mathfrak{g}}^{*(1)}$. Then after Hamiltonian reduction, we end up with a sheaf $D_{h,\mathfrak{h}}(\chi)(0)$ on the scheme $\tilde{\mathcal{S}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^* \times \text{Spec}(k[[\hbar]])$ (where $\tilde{\mathcal{S}}^{(1)} \rightarrow \mathfrak{h}^{*(1)}$ is the restriction of the map $\tilde{\mathfrak{g}}^{*(1)} \rightarrow \mathfrak{h}^{*(1)}$).

Remark 7. Below, we will see how to get rid of the parameter h and work with a sheaf simply defined on the scheme $\tilde{\mathcal{S}}_e^{(1)}$. It would have been possible to do things in the opposite order, i.e., get rid of the h on $T^*\mathcal{B}$ by working with the crystalline differential operators D^λ ,

²As noted before, our objects defined above actually satisfy the property that their h -completions are quantizations. We shall work with such completions below

and then take Hamiltonian reduction of this sheaf to get a sheaf on $\tilde{S}_{\mathcal{N}}^{(1)}$. This is the approach used for Hilbert schemes in [BFG], and it is probably the most natural approach in positive characteristic. However, this approach doesn't really have an analogue in characteristic zero, because without including the parameter h , the sheaf D^λ isn't local on $T^*\mathcal{B}$, and of course we can't pull back to the Frobenius twist because it doesn't exist. Since the main theorem of this paper involves comparing a construction in characteristic zero and positive characteristic, we have to include the “unnatural” construction in positive characteristic as well.

5.2. Azumaya Property. In this section, we use the results of [BK] (whose set-up we borrowed in the previous section) to show that the algebra we have obtained by Hamiltonian reduction is a reasonable object. This will be the key point in showing that the “localization theorem” holds in our context.

So, let us quote the results from [BK] in the form that we need:

Proposition 8. (c.f. [BK], proposition 5.8) *The sheaf $D_h(\lambda, \chi)(0)$ is a Frobenius constant quantization of the variety $\tilde{S}_{\mathcal{N}}$. Thus it can also be thought of as a coherent sheaf on the scheme $\tilde{S}_{\mathcal{N}}^{(1)} \times \text{Spec}(k[[h]])$. The same is true of $D_{h, \mathfrak{h}}(\chi)(0)$ on $\tilde{S}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^* \times \text{Spec}(k[[h]])$.*

and

Proposition 9. (c.f. [BK], proposition 3.8) *Let O_h be a Frobenius constant quantization of a variety X , and let x be a closed point of $X^{(1)}$. Then, regarding $O_h(h^{-1})$ as a coherent sheaf of algebras on $X^{(1)} \times \text{Spec}(k((h)))$, the local algebra $O_h(h^{-1})_x$ is Azumaya over $k((h))$.*

From these results we derive immediately an Azumaya property for the sheaf $D_h(\lambda, \chi)(0)$. A similar argument shows the Azumaya property of the sheaf $D_{h, \mathfrak{h}}(\chi)(0)$ (c.f. [BMR] section 2.3). However, we wish to obtain an even stronger property by getting rid of the parameter h . To do that, we shall invoke some facts about \mathbb{G}_m -equivariant quantizations.

Before doing so, let us note that all of the varieties in the above section carry the Gan-Ginzburg \mathbb{G}_m action on $T^*\mathcal{B}$, which we recall is given by

$$t(g, v) = (\gamma(t)g, \bar{\rho}(t)v)$$

where $\gamma: \mathbb{C}^* \rightarrow G$ was the natural embedding described above in section 2, and where we've identified the cotangent space at the point g with $(\mathfrak{g}/\mathfrak{b}_0)^* \cong \mathfrak{n}_0$, where \mathfrak{b}_0 is our standard Borel subalgebra, (which we choose to contain the “positive part” of our \mathfrak{sl}_2 -triple, e and h), and \mathfrak{n}_0 is its nilradical, and $\bar{\rho}(t) = t^{-2}ad(\gamma(t))$ as above. We also let \mathbb{G}_m act on \mathfrak{h}^* as $t \cdot h = t^2h$.³

We can see also that all of the sheaves considered above are equivariant with respect to this action: note that the action extends by definition to \mathcal{B} , and then to $T^*\mathcal{B}$ and D_h^λ by the usual extension of an action to differential operators (as usual, we demand $t \cdot h = t^2h$ to make the relations of D_h^λ homogeneous). In addition, the action preserves the varieties $S_{\mathcal{N}}$ and $\tilde{S}_{\mathcal{N}}$, and is respected by the moment map, by its definition. The ideal of Hamiltonian reduction, which was inhomogeneous with respect to the usual grading, is homogeneous with respect to this one. Thus we see that $D_h(\lambda, \chi)(0)$ carries this action as well, and the same for $D_{h, \mathfrak{h}^*}(\chi)(0)$.

³This action exists as long as $\text{char}k$ is sufficiently large, c.f [J]. We shall assume that $\text{char}k$ is large enough in the rest of the paper.

Although this action is poorly behaved on $T^*\mathcal{B}$, it is positive weight and contracting on $\tilde{\mathcal{S}}_{\mathcal{N}}$. So at this point we can invoke a very general lemma, which is similar to [BK], lemma 3.4, and [L1], proposition 2.1.5:

Lemma 10. 1) *Let k be any algebraically closed field, and let X be smooth a variety over k , with a quantization \mathcal{O}_h . Suppose that X is equipped with a positive weight $\mathbb{G}_m(k)$ -action, which extends to an action of \mathcal{O}_h via $t \cdot h = t^n h$ for some $n > 0$. Then the sheaf \mathcal{O}_h on $X \times \text{Spec}(k[[h]])$ is the restriction of a sheaf on the variety $X \times \text{Spec}(k[h])$.*

2) *Now let $\text{char}(k) > 0$ and suppose that \mathcal{O}_h is Frobenius constant, and, in addition, that $Fr : \mathcal{O}(X^{(1)}) \rightarrow \mathcal{O}_h$ is \mathbb{G}_m -equivariant (where we use the induced action on the Frobenius twist of a variety). Then we can say in addition that the coherent sheaf \mathcal{O}_h on $X^{(1)} \times \text{Spec}(k[[h]])$ is the restriction of a coherent sheaf on $X^{(1)} \times \text{Spec}(k[h])$.*

Proof. To prove 1), we first consider the case that X is affine, following ([L1], prop. 2.1.5). In this case, let $X = \text{Spec}(A)$. As a $k[[h]]$ -module, we have that $\mathcal{O}_h \cong A[[h]]$. We first claim that $A[[h]] = A[[h]]_{\mathbb{G}_m\text{-fin}}$ (where $(V)_{\mathbb{G}_m\text{-fin}}$ denotes the sum of the finite dimensional modules of the \mathbb{G}_m -module V). This simply follows from the obvious fact that every eigenvalue for the \mathbb{G}_m -action on $A[[h]]$ is a finite sum of terms of the form $h^k a$ where a is an eigenvalue for the \mathbb{G}_m -action on A (here we use that the action is positive weight, so that the number of h 's must be bounded).

Now, the claim implies, since \mathbb{G}_m acts on $A[[h]]$ by algebra automorphisms, that $A[[h]]$ is a subalgebra of $A[[h]]$. So this is 1) for affine X . In general, we note that taking \mathbb{G}_m -finite vectors clearly commutes with localization by a \mathbb{G}_m -stable element of A , and that any smooth variety with a \mathbb{G}_m -action has an affine \mathbb{G}_m -equivariant cover (c.f., GIT, section). So we can take the sheaf of local sections of \mathbb{G}_m -finite vectors, and this suffices for 1).

To get 2), we note that the image of $\mathcal{O}(X^{(1)})$ clearly lies in $\mathcal{O}_{h, \mathbb{G}_m\text{-fin}}$ by the assumption. Therefore the extension to $Fr : \mathcal{O}(X^{(1)})[[h]] \rightarrow \mathcal{O}_h$ obtained by sending h to h has image in $\mathcal{O}_{h, \mathbb{G}_m\text{-fin}}$ as well. But this is exactly 2). \square

With this in hand, we see right away that in fact $D_h(\lambda, \chi)(0)$ is the restriction of a sheaf, called $D_h^{\lambda, \chi}$, on $\tilde{\mathcal{S}}_{\mathcal{N}}^{(1)} \times \mathbb{A}^1$, and by the same reasoning, that $D_{h, \mathfrak{h}}(\chi)(0)$ is the restriction of a sheaf on $\tilde{\mathcal{S}}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^* \times \mathbb{A}^1$. By [BK] lemma 3.4, these sheaves are even the unique ones with this property. Further, we are now free to take the quotient $D_h^{\lambda, \chi}/(h-1)$, (respectively $D_{h, \mathfrak{h}}(\chi)(0)/(h-1)$) and obtain coherent sheaves on the variety $\tilde{\mathcal{S}}_{\mathcal{N}}^{(1)}$ (respectively $\tilde{\mathcal{S}}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*$), which we will call $D^{\lambda, \chi}$ (respectively $\tilde{D}(\chi)$, following [BMR]). We can now state the main result about these objects

Proposition 11. *$D^{\lambda, \chi}$, respectively $\tilde{D}(\chi)$, is an Azumaya algebra on the variety $\tilde{\mathcal{S}}_{\mathcal{N}}^{(1)}$, respectively $\tilde{\mathcal{S}}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*$.*

Proof. This will follow from proposition 9. To see how, let us note (for the first statement, the second is similar) that it suffices to show that, for any point $x \in \tilde{\mathcal{S}}_e^{(1)}$ with associated ideal m_x in $\mathcal{O}(\tilde{\mathcal{S}}_e^{(1)})$, $(D^{\lambda, \chi})_x/m_x(D^{\lambda, \chi})_x$ is a central simple algebra (c.f. [Milne, chapter 4]; we already know that these are locally free sheaves because they are Frobenius constant quantizations).

If \bar{I} is a nontrivial ideal of this algebra, then we can lift it to a nontrivial ideal I of $(D^{\lambda, \chi})_x/m_x(D^{\lambda, \chi})_x$. Since $h-1$ is an element of \bar{I} (by definition of $D^{\lambda, \chi}$), we see that no power of h can be an element of \bar{I} ; if it were, then some h^{p^k} would be in \bar{I} ($p = \text{char}(k)$), but since $(h-1)^{p^k} = h^{p^k} - 1$ is in \bar{I} , this contradicts non-triviality.

Now, this means that the ideal I can be extended to a nontrivial ideal of

$$[D_h(\lambda, \chi)(0)(h^{-1})]_x/m_x$$

by first completing and then inverting h . But now this is a $k((h))$ -central simple algebra by proposition 9, which is a contradiction. \square

5.3. Localization Theorem. With the background of the previous sections, we can now give a proof of the localization property for the sheaf $D^{\lambda, \chi}$. Throughout this section, we make the following assumption: the cohomology groups $H^i(\tilde{\mathcal{S}}_{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{S}}_{\mathcal{N}}})$ and $H^i(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}})$ vanish for $i > 0$. This is true in characteristic zero by the Grauert-Riemenschneider vanishing theorem, and hence it is true in sufficiently large positive characteristic. We do not know an explicit bound.

Now, we recall the general yoga of localization in positive characteristic, which is proposition 2.2 in [BK], and is also presented in chapter 3 of [BMR]:

Theorem 12. *Let X be a smooth algebraic variety with trivial canonical class, and suppose that X is proper over an affine scheme S . Let \mathcal{A} be an Azumaya algebra over X , and suppose that $R\Gamma(\mathcal{A}) = \Gamma(\mathcal{A})$. Then we have functor*

$$R\Gamma : D^b(\text{Coh}(\mathcal{A})) \rightarrow D^b(\text{Mod}^{f.g.}(\Gamma(\mathcal{A})))$$

Let us further suppose that the functor

$$\text{Loc} : D^-(\text{Mod}^{f.g.}(\Gamma(\mathcal{A}))) \rightarrow D^-(\text{Coh}(\mathcal{A}))$$

given by $\text{Loc}(M) = \mathcal{A} \otimes_{\Gamma(\mathcal{A})}^L M$ has finite homological dimension, and hence restricts to a functor

$$\mathcal{L} : D^b(\text{Mod}^{f.g.}(\Gamma(\mathcal{A}))) \rightarrow D^b(\text{Coh}(\mathcal{A}))$$

Then $R\Gamma$ and \mathcal{L} are inverse equivalences of categories.

The requirement that X have trivial canonical class is satisfied by any algebraic symplectic variety. The second requirement will be fulfilled by any algebra with finite homological dimension. It is not quite obvious that our algebras will satisfy this condition. But, following [BMR], chapter 3, we will get around this by using a slight generalization for the sheaf $\tilde{D}(\chi)$.

We should also make a few remarks about the proof. The fact that the two functors are adjoint is general nonsense. The homological assumption on the Azumaya algebra implies $R\Gamma(\text{Loc}(\mathcal{A})) = R\Gamma(\mathcal{A}) = \mathcal{A}$. This implies, by using free resolutions, that Loc is a fully faithful functor. Thus the difficulty is in showing that it is essentially surjective. This is accomplished by using the assumption on the triviality of the canonical class. This assumption implies that the Grothendieck-Serre duality is simply given by the shift functor $[\dim X]$. This means that the essential image of Loc , which is a triangulated subcategory, is closed under the action of this functor. This is a very strong categorical condition, which can be used to show that in fact the essential image of Loc is the entire category.

In addition, we would like to have a description of the algebra $\Gamma(\mathcal{A})$ in our case. We shall show in the next section that it agrees with the modular version of the W -algebra described by Premet. In particular, we have the following

Proposition 13. *We have isomorphisms of algebras:*

$$\Gamma(\tilde{D}(\chi)) \cong U(\mathfrak{g}, e) \otimes_{\mathcal{O}(\mathfrak{h}^*)^W} \mathcal{O}(\mathfrak{h}^*)$$

and

$$\Gamma(D^{\lambda, \chi}) \cong U^\lambda(\mathfrak{g}, e)$$

where these algebras are Premet's modular W -algebras, defined over an algebraically closed field of positive characteristic. The tensor product in the first line makes sense because $O(\mathfrak{h}^*)^W \cong Z_{HC}(U(\mathfrak{g}))$ is a central subalgebra of $U(\mathfrak{g}, e)$.

The reason we need to delay the proof is that these algebras are actually defined in terms of reduction mod p of the analogous characteristic zero objects. Thus we have to use the discussion of the reduction procedure in the next section. Although we do use this proposition later in this section, the results of the next section are completely independent of this one.

In addition, without describing the multiplication on these algebras explicitly, we can give the following consequences of this definition:

Lemma 14. *The algebras $\Gamma(\tilde{D}(\chi))$ and $\Gamma(D^{\lambda, \chi})$ carry natural filtrations, and we have that $gr\Gamma(\tilde{D}(\chi)) = O(S) \otimes_{S(\mathfrak{h})^W} S(\mathfrak{h})$ and $gr(\Gamma(D^{\lambda, \chi})) = O(S_{\mathcal{N}})$. Further, we have that $\Gamma(\tilde{D}(\chi)) \cong R\Gamma(\tilde{D}(\chi))$ and that $\Gamma(D^{\lambda, \chi}) \cong R\Gamma(D^{\lambda, \chi})$.*

Proof. The fact that the algebras are filtered follows by taking global sections of the natural filtrations on $\tilde{D}(\chi)$ and $D^{\lambda, \chi}$, respectively (we recall here that these sheaves of algebras are defined by taking a quantized algebra mod $\hbar - 1$; thus they carry filtrations). By definition, we have that

$$gr(D^{\lambda, \chi}) \cong O(S_{\mathcal{N}})$$

and

$$gr(\tilde{D}(\chi)) \cong O(\tilde{S})$$

The cohomology vanishing assumption at the beginning of this section now shows the cohomology vanishing for $D^{\lambda, \chi}$ and $\tilde{D}(\chi)$ by a standard spectral sequence argument. Further, we then see that the sequences

$$0 \rightarrow \Gamma(\tilde{S}_{\mathcal{N}}, D_i^{\lambda, \chi}) \rightarrow \Gamma(\tilde{S}_{\mathcal{N}}, D_{i+1}^{\lambda, \chi}) \rightarrow \Gamma(\tilde{S}_{\mathcal{N}}, O(\tilde{S}_{\mathcal{N}})_{i+1}) \rightarrow 0$$

(and the analogous one for \tilde{S}) are exact for all i . This shows the statements about the associated graded algebras. \square

Now we are almost in a position to state and prove the localization theorems which are relevant to this paper. The only remaining obstacle is the issue of the algebras having finite homological dimension. We shall get around this in the same way as [BMR], section 3, whose notation and proofs we follow very closely.

First we define the localization functor

$$\mathcal{L} : D^b(mod^{f.g.}(U(\mathfrak{g}, e))) \rightarrow D^b(mod^{coh}(\tilde{D}(\chi)))$$

as $\mathcal{L}(M) = \tilde{D}(\chi) \otimes_{U(\mathfrak{g}, e)}^L M$. We note that the above proposition makes $U(\mathfrak{g}, e)$ a subalgebra of $\tilde{W}(\chi)$, and further that $U(\mathfrak{g}, e)$ is a filtered algebra whose associated graded is isomorphic to $O(S)$ (the coordinate ring of affine space). Thus this algebra has finite homological dimension. Therefore this definition makes sense.

Next, we note that there is an action of $O(\mathfrak{h}^*)$ on the sheaf $\mathcal{L}(M)$ (via its action on $\tilde{D}(\chi)$), while there is only an action of $O(\mathfrak{h}^*)^W$ on M . So, for $\lambda \in \mathfrak{h}^*$, we can define the category $mod_{\lambda}^{f.g.}(U(\mathfrak{g}, e))$ of modules such that the algebra $O(\mathfrak{h}^*)^W$ acts by a generalized central character λ (i.e., the image of λ in \mathfrak{h}^*/W), and there is a decomposition

$$\mathcal{L}(M) \cong \bigoplus_{\mu \in W \cdot \lambda} \mathcal{L}^{\mu \rightarrow \lambda}(M)$$

via the action of the generalized characters in $O(\mathfrak{h}^*)$. We wish to study the functor $\mathcal{L}^{\lambda \rightarrow \hat{\lambda}}(M)$ when $\hat{\lambda}$ is a regular element of \mathfrak{h}^* .

If we define the category $\text{mod}_{\lambda}^{\text{coh}}(\tilde{D}(\mathcal{X}))$ to be the full subcategory of $\text{mod}^{\text{coh}}(\tilde{D}(\mathcal{X}))$ on objects such that $O(\mathfrak{h}^*)$ acts with generalized central character λ , then we note that the image of $\mathcal{L}^{\lambda \rightarrow \hat{\lambda}}$ lands in $D^b(\text{mod}_{\lambda}^{\text{coh}}(\tilde{D}(\mathcal{X})))$. We shall denote this functor by

$$\mathcal{L}^{\hat{\lambda}} : D^b(\text{mod}_{\lambda}^{\text{f.g.}}(U(\mathfrak{g}, e))) \rightarrow D^b(\text{mod}_{\lambda}^{\text{coh}}(\tilde{D}(\mathcal{X})))$$

We note right off the bat that this functor takes makes sense on bounded derived categories since it is defined as a summand of a functor which does the same.

Our other important functor will be the functor

$$\mathcal{L}^{\lambda} : D^-(\text{mod}^{\text{f.g.}}(U(\mathfrak{g}, e)^{\hat{\lambda}})) \rightarrow D^-(\text{mod}^{\text{coh}}(D^{\lambda, \mathcal{X}}))$$

defined as $\mathcal{L}^{\lambda}(M) = D^{\lambda, \mathcal{X}} \otimes_{U(\mathfrak{g}, e)^{\hat{\lambda}}}^L M$. Our aim is to show that in fact this functor has finite homological dimension. This will be accomplished once we prove

Lemma 15. *Suppose λ is regular. Then we have a compatibility between \mathcal{L}^{λ} and $\mathcal{L}^{\hat{\lambda}}$; in other words, if we consider the inclusions $i : D^-(\text{mod}^{\text{f.g.}}(U(\mathfrak{g}, e)^{\hat{\lambda}})) \rightarrow D^-(\text{mod}_{\lambda}^{\text{f.g.}}(U(\mathfrak{g}, e)))$ and $\iota : D^-(\text{mod}^{\text{coh}}(D^{\lambda, \mathcal{X}})) \rightarrow D^-(\text{mod}_{\lambda}^{\text{coh}}(\tilde{D}(\mathcal{X})))$ then we have*

$$\iota \mathcal{L}^{\lambda} \simeq \mathcal{L}^{\hat{\lambda}} i$$

Obviously, this lemma proves the needed claim that \mathcal{L}^{λ} preserves the bounded derived categories.

Proof. (of the lemma). The key point is to rewrite the functor $\mathcal{L}^{\hat{\lambda}}$ in a way that makes it closer to \mathcal{L}^{λ} . To do that, we first define the sheaf

$$\tilde{D}(\mathcal{X})^{\hat{\lambda}} := \tilde{D}(\mathcal{X}) \otimes_{O(\mathfrak{h}^*)} O(\mathfrak{h}^*)^{\hat{\lambda}}$$

where $O(\mathfrak{h}^*)^{\hat{\lambda}}$ is the completion of the ring $O(\mathfrak{h}^*)$ at the ideal generated by λ . Then for any $M \in \text{mod}_{\lambda}^{\text{f.g.}}(U(\mathfrak{g}, e))$, the definitions yield

$$(5.1) \quad \mathcal{L}^{\hat{\lambda}}(M) \simeq \tilde{D}(\mathcal{X})^{\hat{\lambda}} \otimes_{U(\mathfrak{g}, e)} M$$

On the other hand, we have by definition

$$D^{\lambda, \mathcal{X}} = \tilde{D}(\mathcal{X}) \otimes_{O(\mathfrak{h}^*)} k_{\lambda}$$

(where k_{λ} is the one dimensional $O(\mathfrak{h}^*)$ -module corresponding to the maximal ideal λ). Now, since λ is regular, the projection $\mathfrak{h}^* \rightarrow \mathfrak{h}^*/W$ is etale at λ , and so there is an isomorphism

$$O(\mathfrak{h}^*)^{\hat{\lambda}} \otimes_{O(\mathfrak{h}^*/W)} k_{\lambda} \simeq k$$

and so we deduce

$$\tilde{D}(\mathcal{X})^{\hat{\lambda}} \otimes_{U(\mathfrak{g}, e)}^L U(\mathfrak{g}, e)^{\hat{\lambda}} \simeq (\tilde{D}(\mathcal{X}) \otimes_{O(\mathfrak{h}^*)} O(\mathfrak{h}^*)^{\hat{\lambda}}) \otimes_{U(\mathfrak{g}, e)}^L U(\mathfrak{g}, e)^{\hat{\lambda}} \simeq D^{\lambda, \mathcal{X}}$$

because of the isomorphism $U(\mathfrak{g}, e)^{\hat{\lambda}} \simeq U(\mathfrak{g}, e) \otimes_{O(\mathfrak{h}^*/W)} k_{\lambda}$. But this equivalence is precisely the isomorphism of functors that we wanted, after writing out the definitions of the localization functors, and using the realization 6.1. \square

With this out of the way, we can now state our localization theorem for modular W -algebras:

Theorem 16. *The functors*

$$R\Gamma : D^b(\text{mod}_{\lambda}^{\text{coh}}(\tilde{D}(\mathcal{X}))) \rightarrow D^b(\text{mod}_{\lambda}^{\text{f.g.}}(U(\mathfrak{g}, e)))$$

and

$$R\Gamma : D^b(\text{mod}^{\text{coh}}(D^{\lambda, \mathcal{X}})) \rightarrow D^b(\text{mod}^{\text{f.g.}}(U(\mathfrak{g}, e)^{\lambda}))$$

are equivalences of categories, with the inverse functors given by $\mathcal{L}^{\hat{\lambda}}$ and \mathcal{L}^{λ} , respectively.

The proof of this theorem is a direct application of the general theory recalled above (in the case of the second functor), and a slight generalization in the case of the first functor, as in [BMR], chapter 3.

5.4. Restriction to a Springer Fibre. Analogously to [BMR], chapter 4, (c.f. section 4 above) we can consider the above equivalence of categories after restriction to the springer fibre $\mathcal{B}_{\chi}^{(1)}$. In particular, we can define categories $\text{mod}_{\mathcal{B}_{\chi}^{(1)}}^{\text{coh}}(\tilde{D}(\mathcal{X}))$ and $\text{mod}_{\mathcal{B}_{\chi}^{(1)}}^{\text{coh}}(D^{\lambda, \mathcal{X}})$ for sheaves which are set theoretically supported on the variety $\mathcal{B}_{\chi}^{(1)}$. On the representation-theoretic side, we should restrict to those representations on which the central subalgebra $O(\tilde{S}^{(1)})$ acts via the generalized character $\chi^{(1)}$. We denote this category by $\text{mod}_{\chi}(U(\mathfrak{g}, e))$. Then from the above equivalences of categories we immediately deduce the following

Theorem 17. *We have the following equivalences of categories:*

$$R\Gamma : D^b(\text{mod}_{\lambda, \mathcal{B}_{\chi}^{(1)}}^{\text{coh}}(\tilde{D}(\mathcal{X}))) \rightarrow D^b(\text{mod}_{\lambda, \chi}^{\text{f.g.}}(U(\mathfrak{g}, e)))$$

$$R\Gamma : D^b(\text{mod}_{\mathcal{B}_{\chi}^{(1)}}^{\text{coh}}(D^{\lambda, \mathcal{X}})) \rightarrow D^b(\text{mod}_{\chi}^{\text{f.g.}}(U^{\lambda}(\mathfrak{g}, e)))$$

5.5. Azumaya Splitting. Our aim in this section is to give a brief explanation of the structure of our Azumaya algebras upon restriction to the springer fibre $\mathcal{B}_{\chi}^{(1)}$. The result, which will follow from the analogous one in [BMR], is the following:

Theorem 18. *a) For all $\lambda \in \mathfrak{h}^*$, the Azumaya algebra $\tilde{D}(\mathcal{X})$ splits on the formal neighborhood of $\mathcal{B}_{\chi}^{(1)} \times_{\mathfrak{h}^{*(1)}} \lambda$ in $\tilde{S}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$.*

b) Let M_{χ}^{λ} be the vector bundle appearing in [BMR], theorem 5.1.1, and E_{χ}^{λ} be the vector bundle appearing in part a). Let i denote the inclusion map $\tilde{S}^{(1)} \times_{\mathfrak{h}^{(1)}} \mathfrak{h}^* \rightarrow \tilde{\mathfrak{g}} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$. Then there is a vector space V , of rank $p^{\dim \mathcal{B} - \dim \mathcal{B}_e}$, such that $i^* M_{\chi}^{\lambda} \cong E_{\chi}^{\lambda} \otimes_k V$.*

The proof of this theorem relies on an examination of the proof of theorem 5.1.1 in [BMR]. In particular, the argument there relies on an analysis of the generic structure of the Azumaya algebra \tilde{D} , which works as follows:

We let $\mathfrak{h}_{\text{unr}}^*$ denote the open subset of \mathfrak{h}^* consisting of those λ such that for any coroot α we have either $\langle \alpha, \lambda + \rho \rangle = 0$ or $\langle \alpha, \lambda \rangle \notin \mathbb{F}_p$. These are called the unramified weights. Then Brown and Gordon [BG] described the structure of $U(\mathfrak{g})$ as an algebra over the scheme $\mathfrak{Z}_{\text{unr}} := \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}_{\text{unr}}^*$ as follows:

Proposition 19. *The algebra $U(\mathfrak{g}) \otimes_{\mathfrak{Z}} \mathfrak{Z}_{\text{unr}}$ is Azumaya over $\mathfrak{Z}_{\text{unr}}$.*

In [BMR], chapter 3, they deduce the following (which is recorded there as proposition 5.2.1b))

Proposition 20. $U(\mathfrak{g}) \otimes_{\mathfrak{Z}} O(\tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}_{\text{unr}}^*) \xrightarrow{\sim} \tilde{D}|_{\tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}_{\text{unr}}^*}$.

Thus, if the weight λ is unramified, the restriction of \tilde{D} to the formal neighborhood $\mathcal{B}_\chi^{(1)} \times_{\mathfrak{h}^*(1)} \lambda$, is the pullback of an Azumaya algebra on the formal neighborhood of the point $\chi^{(1)}$ in $\mathfrak{g}^*(1)$. Since every Azumaya algebra over the formal neighborhood of a point is split, this implies the existence of M_χ^λ in this case. One can deduce the general case from this one, by noting that there is a functor of “twist by a line bundle” which interchanges different weights.

To see how to deduce the result in our case, we apply Hamiltonian reduction to both sides of proposition 19. We immediately arrive at the isomorphism

$$U(\mathfrak{g}, e) \otimes_{S^*(1) \times_{\mathfrak{h}^*(1)/W} \mathfrak{h}^*} \mathcal{O}(\tilde{S}^{*(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}_{unr}^*) \xrightarrow{\sim} \tilde{D}(\chi)|_{\tilde{S}^{*(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}_{unr}^*}$$

Thus we will be able to finish the argument the same way if we can show that the algebra appearing on the left hand side is Azumaya, at least upon restriction to the formal neighborhood of the point $\chi^{(1)}$. This is indeed the case, and we can argue as follows: let $U(\mathfrak{g}, e)_\chi$ denote the quotient of the algebra $U(\mathfrak{g}, e)$ by the ideal generated by point $\chi^{(1)} \in \tilde{S}^{(1)}$, and let $U(\mathfrak{g})_\chi$ denote the enveloping algebra at the p -character χ . Then Premet in [P1] has given an isomorphism

$$U(\mathfrak{g})_\chi \cong \text{Mat}_{p^{d(e)}}(k) \otimes_k U(\mathfrak{g}, e)_\chi$$

where $d(e)$ is the number $\dim(\mathcal{B}) - \dim(\mathcal{B}_e)$. Thus the restriction $U(\mathfrak{g}, e)_\chi^\lambda$ (with λ unramified) is indeed a matrix algebra over k , as required. The general theorem (part a) now follows by the same “twist by a line bundle” argument as in [BMR]. Part b) follows immediately from the above isomorphism.

Finally, combining the last two sections we arrive at the following equivalences:

Theorem 21. *There are equivalences of categories, for regular integral λ :*

$$\begin{aligned} D^b(\text{Coh}_{\mathcal{B}_\chi^{(1)} \times \lambda}(S^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*)) &\xrightarrow{\sim} D^b(\text{mod}_{\lambda, \mathcal{B}_\chi^{(1)}}^{\text{coh}}(\tilde{D}(\chi))) \xrightarrow{\sim} D^b(\text{mod}_{\lambda, \chi}^{f.g.}(U(\mathfrak{g}, e))) \\ D^b(\text{Coh}_{\mathcal{B}_\chi^{(1)}}(\tilde{S}^\lambda)) &\xrightarrow{\sim} D^b(\text{mod}_{\mathcal{B}_\chi^{(1)}}^{\text{coh}}(D^{\lambda, \chi})) \xrightarrow{\sim} D^b(\text{mod}_\chi^{f.g.}(U^\lambda(\mathfrak{g}, e))) \end{aligned}$$

Remark 22. In a later version of the preprint, we shall give some applications of this theorem to the representation theory of modular lie algebras. In particular, we shall use it to give a new proof of a theorem of Jantzen and Soergel concerning the endomorphism algebra of a projective object in the case that χ is a regular nilpotent element.

6. REDUCTION MOD P

Since our problem involves relating certain constructions over \mathbb{C} with those over fields of positive characteristic, we shall have to give a construction of all our objects over a ring A which is finitely generated over \mathbb{Z} . We shall have to follow very closely Premet’s work [P2] on the modular W -algebras, since part of our goals involve relating our constructions to his.

We begin by recalling the sheaf $D_h(\lambda)(0)_{\mathbb{Z}}$, which is a quantization of the scheme $T^*\mathcal{B}(\mathbb{Z})$. By base extension, this yields a sheaf $D_h(\lambda)(0)_A$ for an arbitrary ring A , which is a quantization of $T^*\mathcal{B}(A)$.

At this point, we would like to ape the construction given in the previous section, and define a quantization of $\tilde{S}_e(A)$ via Hamiltonian reduction of the sheaf $D_h(\lambda)(0)_A$. As it turns out, we will have to do a little more work than that to get a reasonable object.

As a first approximation to what we want, recall from the previous section that all of the objects needed to define the finite W -algebra- the enveloping algebra, the nilpotent subalgebra m_l , its associated group M_l , are defined and free over a ring $A = \mathbb{Z}[S^{-1}]$ where S is a finite set of primes which depends on the type of \mathfrak{g} and the choice of nilpotent element. Then we can define an ideal sheaf of $D_h(\lambda)(0)_A$

$$I = D_h(\lambda)(0)_A \cdot \langle m - \chi(m) \mid m \in m_{l,A} \rangle$$

where an element $m \in m_{l,A}$ acts on $D_h(\lambda)(0)_A$ through the natural action of \mathfrak{g}_A (c.f. the definition of $D_h(\lambda)(0)_A$ given above). From here, the naive Hamiltonian reduction can be defined as:

$$D_h(\lambda, \chi)(0)_A := p_* \mathcal{E}nd_{D_h(\lambda)(0)_A}(D_h(\lambda)(0)_A/I)$$

where $p : \mu^{-1}(\chi) \rightarrow \tilde{S}_e$ is the morphism of Hamiltonian reduction, as above. When $A = \mathbb{C}$ then this is just the usual reduction of differential operators, defined, e.g., in [KR] and used in [DK]. Unfortunately, for an arbitrary ring A , this object might not be flat over the variety $\tilde{S}_{e,A}$, and hence is unsatisfactory for use in arguments where we have to reduce mod p .

To get around this problem, we shall use the technique of defining an A -lattice of the sheaf $D_h(\lambda, \chi)_{\mathbb{C}}$ (for suitable A) which will be flat by construction. For reasonable rings A , this construction will agree with the naive one introduced above.

We start with the ‘‘global’’ case, which is Premet’s construction.

6.1. Premet’s Construction. In this subsection, we give a slight variant of the construction of Premet. We start with the finite W -algebra over \mathbb{C} , $U(\mathfrak{g}, e)$. We know (c.f. section 2 above) that $U(\mathfrak{g}, e)$ is a filtered algebra, and it satisfies $gr(U(\mathfrak{g}, e)) \cong O(S)$. Let’s consider the Rees algebra of $U(\mathfrak{g}, e)$ with respect to this filtration, which we shall denote $U_h(\mathfrak{g}, e)$ (this is uncompleted with respect to h). Then we have that $U_h(\mathfrak{g}, e)/h \cong O(S)$.

So, we choose homogeneous elements $\{X_i\}$ which generate $U_h(\mathfrak{g}, e)$ as a $\mathbb{C}[h]$ -module and whose images mod h form a homogeneous algebraically independent generating set for $O(S)$. The relations for the elements X_i involve finitely many complex numbers. Therefore, we can choose a ring A , finitely generated over \mathbb{Z} , which contains all constants for these relations. Thus we define a ring $U_{h,A}(\mathfrak{g}, e)$, which is a finitely generated graded $A[h]$ -algebra and which satisfies $U_{h,A}(\mathfrak{g}, e) \otimes_A \mathbb{C} \cong U_h(\mathfrak{g}, e)^4$. Then this ring is clearly a free $A[h]$ -module by construction, since $U_h(\mathfrak{g}, e)$ is a free $\mathbb{C}[h]$ -module.

By possibly making a finite extension of A , we can also demand something more. Recall that there is an injection of algebras $\mathbb{C}[\mathfrak{h}^*]^W \rightarrow U_h(\mathfrak{g}, e)$ via the action of the center $Z(U(\mathfrak{g}))$ and the Harish-Chandra isomorphism. We choose the ring A so that there is an injection $A[\mathfrak{h}^*]^W \rightarrow U_{h,A}(\mathfrak{g}, e)$, which, when base-changed to \mathbb{C} , becomes the action of the center.

Next, we define the algebra $U_h(\mathfrak{g}, e)_A$ as the ‘‘naive’’ Hamiltonian reduction $(U_h(\mathfrak{g}_A)/I)^{M_{l,A}}$ as in the previous section. Then we have the standard realization

$$U_h(\mathfrak{g}, e)_A \cong \mathcal{E}nd_{U_h(\mathfrak{g}_A)}(Q_{h,\chi}(A))$$

where $Q_{h,\chi}(A) = U_h(\mathfrak{g}_A)/I$ as a left $U_h(\mathfrak{g}_A)$ -module. Then $Q_{h,\chi}(A)$ is an A -lattice in $Q_{h,\chi}(\mathbb{C})$. So, enlarging A if necessary, we may assume that each of the X_i preserve $Q_{h,\chi}(A)$. This yields a map $U_{h,A}(\mathfrak{g}, e) \rightarrow U_h(\mathfrak{g}, e)_A$.

Now, let k be any algebraically closed field of positive characteristic such that there is a morphism $A \rightarrow k$ (there are infinitely many characteristics possible since A is finitely generated over \mathbb{Z}). Then we make the

⁴We should note that this algebra depends on the choice of the $\{X_i\}$, which is far from unique. Premet [P] makes a specific choice. This won’t be necessary for us, since the claim below implies that the resulting algebras are well defined after base-changing to an algebraically closed field.

Claim 23. We have $U_{h,A}(\mathfrak{g}, e) \otimes_A k \cong U_h(\mathfrak{g}, e)_k$

Proof. First, we have the natural map $U_{h,A}(\mathfrak{g}, e) \otimes_A k \rightarrow U_h(\mathfrak{g}, e)_k$ obtained by base change of the map in the previous paragraph. The right hand side is now a (Frobenius-constant) quantization of the variety S (by the theory of Hamiltonian reduction presented in the previous section), and hence satisfies $U_h(\mathfrak{g}, e)_k/h \cong O(S)$. But the left hand algebra satisfies the same relation by construction. So the natural map mod h is an isomorphism, and hence so is the map itself. \square

Now we shall generalize all this to the local case.

To do this, we momentarily work over \mathbb{C} again: we have by section 6.4 above that the quantization $D_h(\lambda, \chi)(0)_{\mathbb{C}}$ was the restriction of a sheaf of algebras $D_h^{\lambda, \chi}(\mathbb{C})$ on the scheme $\tilde{S}_{\mathcal{N}} \times \mathbb{A}_{\mathbb{C}}^1$, which of course satisfies

$$D_h^{\lambda, \chi}(\mathbb{C})/hD_h^{\lambda, \chi}(\mathbb{C}) \cong O_{\tilde{S}_{\mathcal{N}}}$$

It is well-known that the scheme $\tilde{S}_{\mathcal{N}}$ is defined and flat over A for suitable A (in particular $\mathbb{Z}[S^{-1}]$ where S is a finite set of primes will suffice). For a given finite open affine cover of $\tilde{S}_{\mathcal{N}, A}$, denoted U_i , we can choose, consistently with the cover, generators of $O(\tilde{S}_{e,A}, U_i)$ (as A -algebras), and then choose (consistently with the cover) lifts of these to $D_h^{\lambda, \chi}(\mathbb{C})(U_i)$.

Now we proceed exactly as above: we regard these elements as living inside $p_* \mathcal{E}nd_{D_h^{\lambda}}(D_h^{\lambda}/I)$, we extend A so that they preserve the subspace $(D_h^{\lambda}(A)/I_A)$. Because $D_h^{\lambda}(A)/I_A$ is finitely generated over $D_h^{\lambda}(A)$, the resulting A can be chosen to be finitely generated over \mathbb{Z} . Then, we can look at the $A[h]$ algebra generated by these elements inside each $D_h^{\lambda, \chi}(\mathbb{C})(U_i)$. Because of the consistency conditions specified above, these glue together to form a sheaf on $\tilde{S}_{e,A}$ which will be denoted $D_{h,A}^{\lambda, \chi}$. It is clearly a subsheaf of $D_h^{\lambda, \chi}(A)$. In addition, since $D_h^{\lambda, \chi}(\mathbb{C})$ is a free finitely generated $\mathbb{C}[h]$ -algebra, we clearly have that $D_{h,A}^{\lambda, \chi}$ is free and finitely generated over $A[h]$. By the construction we see that

$$D_{h,A}^{\lambda, \chi}/hD_{h,A}^{\lambda, \chi} \cong O(\tilde{S}_{\mathcal{N}, A})$$

and

$$D_{h,A}^{\lambda, \chi} \otimes_A \mathbb{C} \cong D_h^{\lambda, \chi}(\mathbb{C})$$

In a completely parallel fashion, we can define the sheaf $\tilde{D}_{h,A}(\chi)$ as a sheaf of algebras on the variety \tilde{S}_A . We choose A so that there is an injection of algebras $A[\hbar^*] \rightarrow \Gamma(\tilde{D}_{h,A}(\chi))$, which base changes to the corresponding map over \mathbb{C} .

So, we can argue exactly as in the claim above to show

Claim 24. Let k be any algebraically closed field of positive characteristic such that there is a morphism $A \rightarrow k$. Then we have an isomorphism

$$D_{h,A}^{\lambda, \chi} \otimes_A k \cong D_h^{\lambda, \chi}(k)$$

there is also an isomorphism

$$\tilde{D}_{h,A}(\chi) \otimes_A k \cong \tilde{D}_{h,k}(\chi)$$

6.2. Global Sections. In this section, we shall prove the unproved assertions of the previous chapter. Above we have defined algebras $U_{h,A}(\mathfrak{g}, e)$.

Definition 25. The algebra $U(\mathfrak{g}_A, e)$ is defined to be $U_{h,A}(\mathfrak{g}, e)/(h-1)$. By the discussion in the previous section, this agrees (for some choice of basis) with the definition given in [P2].

We wish to compare these algebras to the global sections of the ‘‘local’’ versions that we also introduced. This is a straightforward matter given the discussion above- we simply choose A large enough so that there are isomorphisms $\Gamma(D_h^\lambda(A)/I_A) \cong U^\lambda(\mathfrak{g}_A)/I_A$ and $\Gamma(\tilde{D}_h(A)/I_A) \cong U(\mathfrak{g}_A) \otimes_{A[[\hbar^*]]^W} A[[\hbar^*]]/I_A$. Then, since everything is defined as operators on these spaces, we immediately deduce isomorphisms of naive Hamiltonian reductions

$$\Gamma(\tilde{D}_h(\mathcal{X})(A)) \cong U_h(\mathfrak{g}, e)_A \otimes_{A[[\hbar^*]]^W} A[[\hbar^*]]$$

and

$$\Gamma(D_h^{\lambda, \mathcal{X}}(A)) \cong U_h(\mathfrak{g}, e)_A \otimes_{A[[\hbar^*]]^W} A_\lambda := U_h^\lambda(\mathfrak{g}, e)_A$$

where by A_λ we mean the $A[[\hbar^*]]^W$ -module corresponding to the maximal ideal generated by the integral weight λ . By choosing appropriate bases, we then get isomorphisms of our flat quantizations

$$\Gamma(\tilde{D}_{h,A}(\mathcal{X})) \cong U_{h,A}(\mathfrak{g}, e) \otimes_{A[[\hbar^*]]^W} A[[\hbar^*]]$$

and

$$\Gamma(D_{h,A}^{\lambda, \mathcal{X}}) \cong U_{h,A}(\mathfrak{g}, e) \otimes_{A[[\hbar^*]]^W} A_\lambda := U_{h,A}^\lambda(\mathfrak{g}, e)$$

By base-change to k , we see that these isomorphisms yield immediately proposition 13 in the previous section.

6.3. Localization over A . In this subsection, we shall describe a localization functor which lives over the ring A . In particular, the isomorphisms of the previous section allow us to define

$$\mathcal{L}_A^\lambda : D^b(\text{mod}^{f.g.}(U_{h,A}(\mathfrak{g}, e)^\lambda)) \rightarrow D^b(\text{mod}^{coh}(D_{h,A}^{\lambda, \mathcal{X}}))$$

via

$$\mathcal{L}_A^\lambda(M) = M \otimes_{U_{h,A}(\mathfrak{g}, e)^\lambda}^L D_{h,A}^{\lambda, \mathcal{X}}$$

⁵For the application we have in mind, we will start with an $M \in \text{mod}^{f.d.}(U^\lambda(\mathfrak{g}, e)_\mathbb{C})$. We choose a good filtration F on M , and using it we arrive at a module $\text{Rees}(M_A)$ over the ring $U_{h,A}^\lambda(\mathfrak{g}, e)$, and hence a complex of sheaves, its localization, which by abuse of notation we denote $\mathcal{L}_A^\lambda(M_A)$.

From here, we can define the complex $CS(M_A) := \mathcal{L}_A^\lambda(M_A)/h$, which is a complex of coherent sheaves on $\tilde{S}_{\mathcal{N}, A}$. This object will interpolate between the characteristic cycle in characteristic zero and the localization in characteristic p . Although this sheaf depends on the choice of a good filtration, the underlying cycles will not (c.f. [HTT], appendix D).

Let us note a few things about the sheaf $\mathcal{L}_A^\lambda = M_A \otimes_{U_{h,A}(\mathfrak{g}, e)^\lambda}^L D_{h,A}^{\lambda, \mathcal{X}}$. Upon taking the (derived) base change to \mathbb{C} , and inverting h , we arrive at the localization functor of [DK]. In particular, we have a class $[CS(M_A)] \in K(\tilde{S}_A)$, whose base change to \mathbb{C} is the class $[CS(M)]$.

We can also take the (derived) base change to any algebraically closed field k (of positive characteristic), taking the quotient of this by $h-1$ then yields the localization functor

⁵It might not be immediately obvious that this functor lands in the bounded derived category. But we can argue as in the previous section (reducing to the whole ring $U_{h,A}(\mathfrak{g}, e)$ to see that it does).

$\mathcal{L}_k^\lambda(M_k)$ discussed above (we note that taking the quotient by $h-1$ is actually an equivalence of categories in positive characteristic by [BK], lemma 3.4). From this we deduce immediately the following compatibility: $CS(\mathcal{L}^\lambda(M_k)) = \mathcal{L}_A^\lambda(M_A) \otimes_A k/(h)$ where CS denotes the sheaf we get after taking associated graded with respect to the induced filtration (as a $D^{\lambda, \mathcal{X}}$ -module).

We wish to analyze the support of the sheaf $\mathcal{L}_A^\lambda(M_A) \otimes_A k$. To this end, we have the

Proposition 26. *The support of $\mathcal{L}_A^\lambda(M_A) \otimes_A k/(h)$, as a closed subset of \tilde{S}_k , is simply the image under Fr of the support of $\mathcal{L}^\lambda(M_k)$ in $\tilde{S}_k^{(1)}$. In fact, we can even say that $Fr_*[CS(\mathcal{L}^\lambda(M_k))] = [\mathcal{L}^\lambda(M_k)]$ in $K(\tilde{S}_k^{(1)})$.⁶*

Proof. This is essentially just a restatement of the fact that $D^{\lambda, \mathcal{X}}$ is a Frobenius constant quantization. For this tells us that $F : O(\tilde{S}_k^{(1)}) \rightarrow D^{\lambda, \mathcal{X}}$ becomes the Frobenius morphism after taking the associated graded. On the other hand, pulling back under F is exactly how we arrive at the support of $\mathcal{L}^\lambda(M_k)$ as a sheaf on $\tilde{S}_k^{(1)}$. The refined result follows from the rational invariance of K -theory. We are comparing the classes of two coherent sheaves on $\tilde{S}^{(1)}$. Both are obtained from the sheaf $Rees(\mathcal{L}_k^\lambda(M_k))$ on $\tilde{S}^{(1)} \times \mathbb{A}^1$, the first by restriction to $h=0$, the second by restriction to $h=1$. \square

Now, since M is a finite dimensional module over $U^\lambda(\mathfrak{g}, e)$, its support over S is simply the point χ . After reduction mod p , for p sufficiently large, this implies that M_k is a module in $mod_\chi(U^\lambda(\mathfrak{g}, e))$ - simply because the map $O(\tilde{S}_k^{(1)}) \rightarrow U(\mathfrak{g}, e)_k$ becomes the Frobenius morphism after taking gr (as in the proposition).

So, combining this discussion with the theorems in the previous section, we see that $\mathcal{L}_A^\lambda(M_A)/(h)$ is supported, set theoretically, on \mathcal{B}_χ - and hence the same is true of the base change $\mathcal{L}_\mathbb{C}^\lambda(M)$. So now we can state definitively our ‘‘base change’’ lemma:

Lemma 27. *The class $[CS(M_A)]$ actually lives in $K(Coh_{\mathcal{B}_{\chi, A}}(\tilde{S}_A) = K(\mathcal{B}_{\chi, A}))$. Its pullback to $K(\mathcal{B}_{\chi, k})$ induces the class $[CS(\mathcal{L}^\lambda(M_k))]$, and the pullback to \mathbb{C} induces $[CS(M)]$.*

Since the ‘‘specialization morphism’’ $K(\mathcal{B}_{\chi, \mathbb{C}}) \rightarrow K(\mathcal{B}_{\chi, k})$ (c.f. [BMR], chapter 7) is an isomorphism (and the same is true of the induced map $H_*(\mathcal{B}_{\chi, k}) \rightarrow H_*(\mathcal{B}_{\chi, \mathbb{C}})$), and the map $K(mod^{f.d.}(U^\lambda(\mathfrak{g}, e))) \rightarrow K(mod_\chi^{f.s.}(U^\lambda(\mathfrak{g}, e)))$ is injective, see that we have reduced the injectivity problem to the following

Theorem 28. *The morphism $K(mod_\chi^{f.s.}(U^\lambda(\mathfrak{g}, e))) \rightarrow K(\mathcal{B}_{\chi, k}) \rightarrow H_{top}(\mathcal{B}_{\chi, k})$ is injective.*

Remark 29. We should comment here that for all p sufficiently large, there are isomorphisms between the groups $K(\mathcal{B}_{\chi, k})$ obtained via the comparison with characteristic zero. If we choose a ring A which works simultaneously for all simple finite dimensional $U^\lambda(\mathfrak{g}, e)$ -modules, then we see that we can choose any algebraically closed field of large positive characteristic to show injectivity. We make such a choice from now on.

The proof of this theorem will occupy the next section.

7. K-THEORY

We shall need to recall a few facts from the algebraic K -theory developed in [F] and [BFM]. In particular, recall that if we have a proper scheme Y , and a closed embedding

⁶Here we are simply regarding the sheaf $\mathcal{L}^\lambda(M_k)$ as a coherent sheaf on $\tilde{S}^{(1)}$ in the naive way; we are not invoking any Azumaya splitting.

$Y \rightarrow X$, where X is smooth, then we have a ‘‘localized chern character’’ map

$$ch_Y^X : K_{\mathbb{Q}}(Y) \rightarrow A_{\mathbb{Q}}(Y)$$

where $A_{\mathbb{Q}}(Y)$ denotes the rational chow ring of algebraic cycles. The map is obtained by realizing $A(Y)$ as a ring of cycles on X which are supported on Y . This map has the following functorial property: if $Z \rightarrow W$ is another inclusion of a proper into a smooth variety, and $f : X \rightarrow W$ restricts to a morphism from Y to Z , then we have

$$f^* ch_Z^W = ch_Y^X f^*$$

There is also the ‘‘Riemann-Roch’’ morphism $\tau : K_{\mathbb{Q}}(Y) \rightarrow A_{\mathbb{Q}}(Y)$; this morphism respects proper pushforward. The two morphisms are of course quite different, but in both cases the projection to the ‘‘top’’ piece $A_{\dim(Y)}(Y)$ yields the same algebraic cycle.

As $A(Y)$ is a graded vector space, we shall denote by $(ch_X^Y)_i$ and τ_i the maps obtained after projection to the degree i cycles.

All of the varieties we will consider below (namely, the springer fibres) have the property that their cohomology is spanned by the classes of algebraic cycles (c.f. the main results of [DLP]). Thus we have a degree doubling isomorphism

$$A_{\mathbb{Q}_l}(Y) \rightarrow H_*(Y)$$

where we must now consider the etale Borel-Moore homology group. Given this, we shall mainly work with the groups $A_{\mathbb{Q}}(Y)$ from now on. From this condition on our varieties it also follows that the morphisms ch_Y^X and τ are isomorphisms.

With all this in hand, we can proceed to the proof of theorem 28. First of all, we have a functor $mod_{\chi}^{coh}(D^{\lambda, \chi}) \rightarrow mod_{\chi}^{coh}(D^{\lambda})$, given by $\mathcal{F} \rightarrow V \otimes \mathcal{F}$ (c.f. section 6.7 for the vector space V of dimension $p^{d(e)}$).

Next, we recall the Azumaya splitting of section six. Let us denote the coherent sheaf obtained from M_k via this splitting $Coh(M_k)$. So we have that

$$E_{\chi}^{\lambda} \otimes Coh(M_k) \cong \mathcal{L}^{\lambda}(M_k)$$

and therefore (by section 5.5) that

$$M_{\chi}^{\lambda} \otimes Coh(M_k) \cong V \otimes \mathcal{L}^{\lambda}(M_k)$$

where M_{χ}^{λ} is the splitting bundle of [BMR] (note here that $Coh(M_k)$ is scheme theoretically supported on $\tilde{S}_{\mathcal{N}}$). But the class in K -theory of this bundle has already been studied. If we make the normalization following [BR], then by [BMR] section 6, we in fact have

$$[M_{\chi}^{\lambda}] = [((Fr_{\mathcal{B}})_* \mathcal{O}_{\mathcal{B}})|_{\mathcal{B}_{\chi}^{(1)}}]$$

where we are now considering \mathcal{B}_{χ} as a subvariety of \mathcal{B} (i.e., the class on the right lives in $K_{\mathcal{B}_{\chi}^{(1)}}(\mathcal{B}^{(1)})$ which is isomorphic to $K(\mathcal{B}_{\chi}^{(1)})$). Combining these equalities with push-pull now yields the following:

Lemma 30. *We have the following equalities in $K(\mathcal{B}_{\chi}^{(1)})$:*

$$[\mathcal{L}^{\lambda}(M_k)] = p^{-d(e)} [((Fr_{\mathcal{B}})_* \mathcal{O}_{\mathcal{B}})|_{\mathcal{B}_{\chi}^{(1)}}][Coh(M_k)] = p^{-d(e)} (Fr_{\mathcal{B}})_* (Fr_{\mathcal{B}})^*[Coh(M_k)]$$

where we use the fact that the action of Fr on \mathcal{B} takes \mathcal{B}_{χ} to $\mathcal{B}_{\chi}^{(1)}$.

Now, we would like to combine this information with our equality from the previous section

$$Fr_*[CS(\mathcal{L}^\lambda(M_k))] = [\mathcal{L}^\lambda(M_k)]$$

In this equality however, we were considering the frobenius with respect to scheme $\tilde{S}_{\mathcal{N}}^{(1)}$. However, the discrepancy is rectified by the following

Claim 31. $(Fr_{\tilde{S}_{\mathcal{N}}})_*[CS(\mathcal{L}^\lambda(M_k))]$ and $(Fr_{\mathcal{B}})_*[CS(\mathcal{L}^\lambda(M_k))]$ agree as classes in $K(\mathcal{B}_\chi^{(1)})$.

Proof. As $CS(\mathcal{L}^\lambda(M_k))$ is set-theoretically supported on \mathcal{B}_χ , it has a finite filtration by sheaves which are scheme-theoretically supported there. This filtration means that we can write the class $[CS(\mathcal{L}^\lambda(M_k))]$ as a sum of classes $[A_i]$ of sheaves on the scheme \mathcal{B}_χ .

However, the proof for sheaves of this type is simply an examination of the definition of the frobenius morphism, which makes it clear that the restriction of $Fr_{\tilde{S}_{\mathcal{N}}}$ to \mathcal{B}_χ and $Fr_{\mathcal{B}}$ to \mathcal{B}_χ coincide- in fact they are both $Fr_{\mathcal{B}_\chi}$. \square

So now we are free to compare the previous inequalities and deduce that

$$(Fr_{\mathcal{B}})_*[CS(\mathcal{L}^\lambda(M_k))] = p^{-d(e)}(Fr_{\mathcal{B}})_*(Fr_{\mathcal{B}})^*[Coh(M_k)]$$

in $K(\mathcal{B}_\chi^{(1)})$.

To prove the theorem, it remains to study the action of the operators $(Fr_{\mathcal{B}})_*$ and $(Fr_{\mathcal{B}})^*$ on K -theory. To start, we have

Proposition 32. *The map $(Fr_{\mathcal{B}})_* : K(\mathcal{B}_\chi) \rightarrow K(\mathcal{B}_\chi^{(1)})$ is injective.*

Proof. The map Fr is finite, flat, and bijective on closed points. After application of the map τ , which commutes with proper pushforward, we see that it is enough to check our claim at the level of algebraic cycles. But it is immediate from the aforementioned properties of the map Fr that it takes a set of linearly independent cycles to another. Further, we know that the chow groups of springer fibres are spanned by such cycles. \square

Combining the proposition with the previous equality, we arrive an equality in $K(\mathcal{B}_\chi)$:

$$[CS(\mathcal{L}^\lambda(M_k))] = p^{-d(e)}(Fr_{\mathcal{B}})^*[Coh(M_k)]$$

Now, we must evaluate the operator $(Fr_{\mathcal{B}})^*$. To this end, we shall translate the problem to $A(\mathcal{B}_\chi)$ via the localized chern character $ch_{\mathcal{B}_\chi}^{\mathcal{B}}$. We see that we have equalities

$$(ch_{\mathcal{B}_\chi}^{\mathcal{B}})_i(Fr_{\mathcal{B}})^* = (Fr_{\mathcal{B}})^*(ch_{\mathcal{B}_\chi^{(1)}}^{\mathcal{B}^{(1)}})_i$$

in $A_i(\mathcal{B}_\chi)$.

Further, we can make an identification $A(\mathcal{B}_\chi) \cong A(\mathcal{B}_\chi^{(1)})$ by the isomorphism of abstract varieties $\mathcal{B}_\chi \cong \mathcal{B}_\chi^{(1)}$. Then we have the

Proposition 33. *The map $(Fr_{\mathcal{B}})^*|_{A_i(\mathcal{B}_\chi^{(1)})}$, followed by the identification $A_i(\mathcal{B}_\chi) \cong A_i(\mathcal{B}_\chi^{(1)})$ is simply multiplication by $p^{\dim \mathcal{B} - i}$.*

Proof. As above, since Fr is finite, flat, and bijective on points, it is clear that the pull-back takes a cycle to a multiple of itself. Thus, it suffices to check the multiplicity locally, on an open subset of a cycle (which is equivalent to checking at the generic point). So, we let $V^{(1)}$ be an algebraic subvariety of $\mathcal{B}_\chi^{(1)}$ of dimension i , and $x \in V^{(1),sm}$ its smooth locus.

We consider the completed local ring at such a point, $\widehat{\mathcal{O}}_{x,V(1)} \cong k[[x_1^p, \dots, x_i^p]]$. Then we can compute the pullback under Fr as:

$$\widehat{\mathcal{O}}_{x,\mathcal{B}} \otimes_{\widehat{\mathcal{O}}_{x,\mathcal{B}(1)}} \widehat{\mathcal{O}}_{x,V(1)} \cong k[[x_1, \dots, x_{dimB}]] / (x_{i+1}^p, \dots, x_{dimB}^p)$$

which is $\widehat{\mathcal{O}}_{x,V}$ -module of rank p^{dimB-i} . The identification $A_i(\mathcal{B}_\chi) \cong A_i(\mathcal{B}_\chi^{(1)})$ simply sends the cycle V to $V^{(1)}$. This proves the proposition. \square

Now, we let $[Coh(M_k)]^{(-1)}$ be the class in $K(\mathcal{B}_\chi)$ which corresponds to $[Coh(M_k)]$ under the identification $K(\mathcal{B}_\chi) \cong K(\mathcal{B}_\chi^{(1)})$ (we choose the identification of these groups which is compatible under $ch_{\mathcal{B}_\chi}^{\mathcal{B}}$ and $ch_{\mathcal{B}_\chi^{(1)}}^{\mathcal{B}^{(1)}}$ with the identification $A(\mathcal{B}_\chi) \cong A(\mathcal{B}_\chi^{(1)})$ used above). Then the proposition combined with our previous equality yields

$$(ch_{\mathcal{B}_\chi}^{\mathcal{B}})_i [CS(\mathcal{L}^\lambda(M_k))] = p^{-d(e)} p^{dim(\mathcal{B})-i} (ch_{\mathcal{B}_\chi}^{\mathcal{B}})_i [Coh(M_k)]^{(-1)} = p^{dim(\mathcal{B}_\chi)-i} (ch_{\mathcal{B}_\chi}^{\mathcal{B}})_i [Coh(M_k)]^{(-1)}$$

(the last equality is simply because $d(e) = dim(\mathcal{B}) - dim(\mathcal{B}_\chi)$).

Now we finish the proof of the theorem: for $char(k)$ sufficiently large, the class $[CS(\mathcal{L}^\lambda(M_k))]$ is independent of p (it is defined as the reduction of a class over A). Further, the main result of [BM], chapter 5, asserts that the same is true of the class $[Coh(M_k)]^{(-1)}$. So we deduce immediately the equalities

$$(ch_{\mathcal{B}_\chi}^{\mathcal{B}})_i [CS(\mathcal{L}^\lambda(M_k))] = 0$$

for $i \neq dim(\mathcal{B}_\chi)$ and

$$(ch_{\mathcal{B}_\chi}^{\mathcal{B}})_{dim(\mathcal{B}_\chi)} [CS(\mathcal{L}^\lambda(M_k))] = (ch_{\mathcal{B}_\chi}^{\mathcal{B}})_{dim(\mathcal{B}_\chi)} [Coh(M_k)]^{(-1)}$$

So from this we conclude that

$$(ch_{\mathcal{B}_\chi}^{\mathcal{B}})_{dim(\mathcal{B}_\chi)} [CS(\mathcal{L}^\lambda(M_k))] = (ch_{\mathcal{B}_\chi}^{\mathcal{B}})_{dim(\mathcal{B}_\chi)} [Coh(M_k)]^{(-1)} = (ch_{\mathcal{B}_\chi}^{\mathcal{B}}) [Coh(M_k)]^{(-1)}$$

The map on the left is exactly the characteristic cycle map. In addition, the map on the right is the image in K -theory of the equivalence of categories

$$D^b(mod_{\mathcal{X}}^{f,g}(U(\mathfrak{g}, e)^\lambda)) \rightarrow D^b(Coh_{\mathcal{B}_\chi}(\tilde{\mathcal{S}}_{\mathcal{N}}))$$

and thus is obviously injective. This proves Theorem 27, and thus the injectivity in characteristic zero.

8. W-EQUIVARIANCE

This section is devoted to a discussion of the equivariance of the characteristic cycle with respect to the action of the Weyl group W^7 . To see the underlying reasoning, let us recall the following

Theorem 34. ([B], [R]) *For any algebraically closed field k of sufficiently large positive characteristic (or characteristic zero), the variety $\tilde{\mathcal{S}}_{\mathcal{N},k}$ admits a (weak) action of the braid group B of type \mathfrak{g} . This action induces an action of the Weyl group W on $K_{\mathbb{Q}}(\mathcal{B}_\chi)$, where it is equivalent to Springer's representation.*

Thus, combining this with the result of the previous section, we see that it suffices to show that the W action on $K(mod_{\mathcal{X}}^{f,g}(U^\lambda(\mathfrak{g}, e)))$, upon reduction mod p , agrees with the above defined action on the category of coherent sheaves. Since it is already known (c.f. [R]) that this braid group action agrees with the one coming from the translation functors on $U(\mathfrak{g})$ -modules, the problem then boils down to showing that the translation functor

⁷This is only a brief sketch of an argument. A more complete proof will appear in a later version of the paper

theory for W -algebras is compatible with the one for lie algebras. This essentially follows from the definitions.

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