NONPROPER INTERSECTION THEORY AND POSITIVE CURRENTS I, LOCAL ASPECTS

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Abstract. We introduce a current calculus to deal with (local) non-proper intersection theory, especially construction of local cycles of Stückrad-Vogel type (Vogel cycles). Given a coherent ideal sheaf $\mathcal{J}$, generated by a tuple of functions f semiglobally on a reduced analytic space X, we construct a current M^f, obtained as a limit of explicit expressions in f, whose Lelong numbers at each point of its components of various bidegrees are precisely the Segre numbers associated to $\mathcal{J}$ at the point. The precise statement is a generalization of the classical King formula. The current M^f can be interpreted, at each point, as a mean value of various local Vogel cycles. Our current calculus also admits a convenient approach to Tworzewski's locally defined invariant intersection theory.

1. Introduction

Let Y be a complex manifold and let $Z_1, \ldots, Z_r$ be (effective) analytic cycles in Y of pure codimensions p_j, $j = 1, ..., r$, that intersect properly, i.e., the intersection V of their supports has codimension $p_1 + \cdots + p_r$. There is a well-defined cycle, called the (proper) intersection of the Z_j,

$$Z_r \cdots Z_1 = \sum m_j V_j, \tag{1.1}$$

where V_j are the irreducible components of V and m_j are certain positive integers. One can obtain these numbers m_j by defining the intersection number $i(x)$, algebraically or geometrically, at each fixed point x of V and prove that $i(x)$ is generically constant on each V_j, see, e.g., [7]. However, by means of currents, (1.1) can be obtained in a more direct way: Let $[Z_j]$ be the Lelong currents associated with Z_j. One can define the wedge product $[Z_r]\wedge \ldots \wedge[Z_1]$ by an appropriate regularization, see, e.g., [7, 9], and this current indeed coincides with the Lelong current associated with $Z_r \cdots Z_1$. In particular, if the Z_j are (effective) divisors defined by holomorphic functions h_j, then the Lelong current of the intersection can be obtained explicitly as

$$[Z_r \cdots Z_1] = \lim_{\epsilon \to 0} \bigwedge dd^c \log(|h_j|^2 + \epsilon). \tag{1.2}$$

There are analogous formulas for cycles of higher codimension, see, e.g., Section 9 below. Though less natural at first sight it is often more convenient to use regularization with analytic continuation: Notice that

$$\lambda \mapsto \bigwedge_j \bar\partial |h_j|^{2\lambda} \wedge \frac{dh_j}{2\pi i h_j} \tag{1.3}$$

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is a well-defined form-valued function for $\mathrm{Re}\,\lambda >> 0$. It turns out that it has a current-valued analytic continuation to a neighborhood of 0 and that the value at the origin is again $[Z_r \cdots Z_1]$.

The overall aim of this paper and the forthcoming paper [6] is to develop a similar current formalism for non-proper intersection theory, i.e., representation of intersections by currents that are limits of explicit forms. We also introduce *generalized cycles*, by means of which we can tie together the local intersection theory in [11] and [28] (corresponding to the intersection numbers $i(x)$ in the proper case) with the global constructions of Fulton-MacPherson, [10], and Stückrad-Vogel, [27]. A key result is a generalized version of the classical King formula, [14, 13]; our formula provides a representation of the Segre class of a coherent ideal sheaf on an analytic space by a current (a generalized cycle) whose Lelong numbers at each point are precisely the so-called Segre numbers of the sheaf, see below. In this paper we focus on the local and semiglobal aspects and postpone the global results to [6]. The semiglobal version of our generalized King formula is given in Theorem 1.4 below.

A standard way to define an intersection of $Z_j \subset Y$ is to form the intersection of $Z_1 \times \cdots \times Z_r$ with the diagonal Δ_Y in $Y \times \cdots \times Y$ [1]. Therefore it is enough to understand the intersection of a complex manifold A and an analytic variety X of pure dimension n, both sitting in some larger complex manifold Y. In the global intersection theories mentioned above, the result only depends on the pullback to X of the sheaf that defines A.

One is therefore led to find a reasonable definition of the intersection of a coherent ideal sheaf $\mathcal{J}$ on an analytic space X of pure dimension n. To describe the local intersection, Tworzewski, [28] and Gaffney–Gassler, [11], independently introduced a list of numbers for each point x that we will call the Segre numbers, following [11]; Tworzewski uses the term *extended index of intersection*[2]. The definition goes via a local variant of the Stückrad-Vogel construction, [27], introduced in [28, 19], that we will now describe; a closely related procedure is used in [11]. A sequence $h = (h_1, h_2, \ldots, h_n)$ in the local ideal $\mathcal{J}_x$ is called a *Vogel sequence of $\mathcal{J}$ at x* if there is a neighborhood $\mathcal{U} \subset X$ of x where the h_j are defined, such that

$$\operatorname{codim}\left[(\mathcal{U} \setminus Z) \cap (|H_1| \cap \ldots \cap |H_k|)\right] = k \text{ or } \infty,\ \ k = 1, \ldots, n; \tag{1.4}$$

here Z is the (reduced) zero set of $\mathcal{J}$ and $|H_\ell|$ are the supports of the divisors H_ℓ defined by h_ℓ. Notice that if $f_0, \ldots, f_m$ generate $\mathcal{J}_x$, any generic sequence of n linear combinations of the f_j is a Vogel sequence at x. Let $X_0 = X$ and let X_0^Z denote the irreducible components of X_0 that are contained in Z and let $X_0^{X\setminus Z}$ be the remaining components[3], so that

$$X_0 = X_0^Z + X_0^{X\setminus Z}.$$

By the Vogel condition (1.4), H_1 intersects $X_0^{X\setminus Z}$ properly. Set

$$X_1 = H_1 \cdot X_0^{X\setminus Z}$$

[1] It is readily checked that if Z_j intersect properly, then the intersection of $Z_1 \times \cdots \times Z_r$ and the diagonal is proper as well.

[2] It is not clear to us whether the coincidence of the two definitions has been noticed in the literature before. In [2], both notions are discussed, and the coincidence follows, but this is not explicitly stated.

[3] In [11], X_0^Z is empty by assumption, but for us it is convenient not to exclude the possibility that $\mathcal{J}$ vanishes identically on some irreducible component of X.

and decompose analogously X_1 into the components X_1^Z contained in Z and the remaining components $X_1^{X\setminus Z}$, so that $X_1 = X_1^Z + X_1^{X\setminus Z}$. Define inductively $X_{k+1} = H_{k+1}\cdot X_k^{X\setminus Z}$, X_{k+1}^Z, and $X_{k+1}^{X\setminus Z}$. Then

$$V^h := X_0^Z + X_1^Z + \cdots + X_n^Z$$

is the *Vogel cycle* associated with the Vogel sequence h[4]. Let V_k^h denote the components of V^h of codimension k, i.e., $V_k^h = X_k^Z$. The irreducible components of V^h that appear in any Vogel cycle, associated with a generic Vogel sequence at x, are called *fixed* components in [11]. The remaining ones are called *moving*. It turns out that the fixed Vogel components of $\mathcal{J}$ coincide with the *distinguished varieties of* $\mathcal{J}$ in the sense of Fulton-MacPherson, see [11] and Section 8.

Recall that the multiplicity of a cycle at a point x is precisely the Lelong number at x of the associated Lelong current, see, e.g., [7]. It is proved in [11] (and will be reproved below) that the multiplicities $e_k(x) = \text{mult}_x V_k^h$ and $m_k(x) = \text{mult}_x X_k^{X\setminus Z}$ are independent of h for a generic h, where however "generic" depends on x, cf., Remark 1.5; these numbers are the *Segre numbers* and *polar multiplicities*, respectively. Theorem 7.1 below asserts that for each fixed x,

$$(e_0(x), e_1(x), \ldots, e_n(x)) = \min_{lex}(\text{mult}_x V_0^h, \text{mult}_x V_1^h, \ldots, \text{mult}_x V_n^h), \tag{1.5}$$

where the $\min_{lex}$ is taken over all Vogel sequences h in $\mathcal{J}_x$. This equality is proved in [28] in case $\mathcal{J}$ is obtained from a smooth analytic set A[5].

Remark 1.1. If $\mathcal{J}_x$ has support at x, then $e_k(x) = 0$ for $k < n$ and $e_n(x)$ is the classical Hilbert-Samuel multiplicity of the ideal $\mathcal{J}_x$. □

Remark 1.2. An algebraic definition of the Segre numbers is given in [2], as generalized Hilbert-Samuel multiplicities (in the sense of [1]) associated to the bigrading $G_{\mathfrak{M}_x}[G_{\mathcal{J}_x}(\mathcal{O}_{X,x})]$ with respect to the ideal $\mathcal{J}_x$ in the local ring $\mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{M}_x$. □

Remark 1.3. If $\mathcal{J}_x$ is generated by $m < n$ functions, then $V_k^h = 0$, $k > m$, for a generic Vogel sequence h. If in addition $\text{codim}\, Z_x = m$, i.e., $\mathcal{J}_x$ is a complete intersection, then for a generic h, $V^h = V_m^h$ is the proper intersection of the divisors of the m generators, and hence $e_m(x)$ is the only nonzero entry in $e(x)$. This number is the classical intersection number $i(x)$ of the proper intersection of the divisors of the m generators of $\mathcal{J}_x$, see, e.g., [7]. □

We introduce a current calculus that is well suited to deal with Vogel sequences. For example we can express (the Lelong current of) a Vogel cycle V^h as a certain product of currents; in fact, we even get (the Lelong current of) V_k^h as the value at $\lambda = 0$, cf., (1.3) above, of

$$\bigwedge_{j=1}^{k} \bar\partial |h_j|^{2\lambda^{k+1-j}} \wedge \frac{dh_j}{2\pi i h_j},$$

[4]If $\mathcal{J}$ is the pullback to X of the radical sheaf of an analytic set A, this is precisely Tworzewski's algorithm, [28]. The notion Vogel cycle was introduced by Massey [18, 19]. For a generic choice of Vogel sequence the associated Vogel cycle coincides with the *Segre cycle* introduced by Gaffney-Gassler, [11], see Lemma 2.2 in [11].

[5]In fact, Tworzewski takes the right hand side of (1.5) as the definition of his extended index of intersection.

see Example 5.5. The different powers of λ are crucial here. Our current calculus is also useful for concrete computations of Segre numbers, see Section 11.

Now assume that $\mathcal{J}$ is generated semi-globally by $f = (f_0, \dots, f_m)$. Taking mean values of (the Lelong currents of) the Vogel cycles associated to (almost) all linear combinations of the f_j, it turns out that we get a positive current whose component of bidegree (k,k) is equal to

$$M_k^f = \mathbf{1}_Z (dd^c \log |f|^2)^k := \mathbf{1}_Z \lim_{\epsilon \to 0} \big(dd^c \log(|f|^2 + \epsilon)\big)^k. \tag{1.6}$$

Here $\mathbf{1}_Z$ means restriction to Z. For practical reasons we will rely on a definition of M_k^f via analytic continuation, see Section 4; for the coincidence with (1.6), see [4][6].

Recall that the *integral closure* of $\mathcal{J}$ (or $\mathcal{J}_x$) generated by $f = (f_0, \dots, f_m)$, consists of all sections ϕ such that $|\phi| \le C|f|$ for some $C > 0$. The following formula in particular provides a semiglobal representation of the Segre numbers associated to $\mathcal{J}$, cf., Remark 1.5.

Theorem 1.4 (Generalized King's formula)**.** *Let X be a reduced analytic space of pure dimension n and let $\mathcal{J}$ be a coherent ideal sheaf over X generated by $f_0, \dots, f_m$. Let Z be the variety of $\mathcal{J}$ and Z_j^k the distinguished varieties of $\mathcal{J}$ of codimension k. Then*

$$M_k^f = \mathbf{1}_Z (dd^c \log |f|^2)^k = \sum_j \beta_j^k [Z_j^k] + N_k^f =: S_k^f + N_k^f, \tag{1.7}$$

where the β_j^k are positive integers and the N_k^f are positive closed currents. The Lelong numbers $n_k(x) = \ell_x(N_k^f)$ are nonnegative integers that only depend on the integral closure class of $\mathcal{J}$ at x, and the set where $n_k(x) \ge 1$ has codimension at least $k+1$.

The Lelong number of M_k^f at x is precisely the Segre number $e_k(x)$ of $\mathcal{J}_x$ on X. The fixed Vogel components of $\mathcal{J}$ are precisely the S_k^f. Finally, the polar multiplicity $m_k(x)$ coincides with the Lelong number at x of the current $\mathbf{1}_{X \setminus Z}(dd^c \log |f|^2)^k$.

When $k = 0$, $(dd^c \log |f|^2)^k$ shall be interpreted as 1 and $M_0^f = \mathbf{1}_Z$ is the current of integration over the components of X that are contained in Z.

Notice that $M_k^f = 0$ if $k < \operatorname{codim} Z$ and that $N_{\operatorname{codim} Z}^f = 0$, cf., Lemma 2.2. Notice that (1.7) is the Siu decomposition, [22], of M_k^f. King's formula in [14, 13] is precisely the case $k = \operatorname{codim} Z$ of (1.7).

Remark 1.5. Assume that x is a point where $n_k(x) \ge 1$ for some k and let V^h be a generic Vogel cycle such that $\operatorname{mult}_x V_k^h = e_k(x)$. Then $V_k^h = S_k^f + W$ where W is a positive cycle of codimension k, such that $\operatorname{mult}_x W = n_k(x)$. Since $n_k(y) \ge 1$ only on a set of codimension $\ge k+1$, at most points y on V_k^h we have that $e_k(y) = \operatorname{mult}_y(S_k^f)$ and hence $\operatorname{mult}_y V_k^h > e_k(y)$. As soon as there is a moving component at x it is thus impossible to find a Vogel cycle that realizes the Segre numbers in a whole neighborhood of x. □

By Siu's theorem [22], the super level sets $V_\ell = \{n_k(x) \ge \ell\}$ are analytic for each integer $\ell \ge 1$. Since $n_k(x)$ is integer valued, it is easy to see, cf., Proposition 2.1 in [28], that there is a unique cycle T_k^f consisting of components of various codimension $> k$ such that the multiplicity at each point coincides with $n_k(x)$. Thus the cycle

[6]For $k \le \operatorname{codim} Z$ there are more elementary ways to define $(dd^c \log |f|^2)^k$, see, e.g., [9].

$T(\mathcal{J}) := \sum_k (S_k^f + T_k^f)$ has total multiplicity precisely equal to $e = e_0 + \cdots + e_n$ at each point x; in case $\mathcal{J}$ is the radical sheaf of a complex manifold A, this is precisely the intersection cycle of A and X defined in [28]. Note that moving components of codimension k are represented by lower dimensional cycles.

Following Tworzewski, [28], given analytic cycles $Z_1, \ldots, Z_r$ in Y, the intersection of $Z_1 \times \cdots \times Z_r$ and the diagonal Δ_Y in $Y \times \cdots \times Y$ provides an intersection product $Z_1 \bullet \cdots \bullet Z_r$. This elegant construction is locally defined and biholomorphically invariant. From the global point of view, however, it is in general "too small". For instance, the self-intersection in the Tworzewski sense of any smooth manifold is just the manifold itself; therefore the self-intersection of a smooth algebraic variety $A \subset \mathbb{P}^N$ cannot satisfy the Bézout equation unless A is linear. The reason is that, in general, there are moving components in global Vogel cycles that are not attached to a fixed point, and therefore are not caught by the Tworzewski intersection. In [6] we will represent the global intersection of arbitrary cycles Z_j in $\mathbb{P}^N$ by a positive current that is invariant, in the sense that it only depends on the standard metric structure of $\mathbb{P}^N$; though moving components are represented by terms that are not Lelong currents of any analytic cycle.

The basic current calculus for Vogel cycles is introduced in Section 3 and the calculus for our currents M^f is developed in the Sections 4 and 5. In Section 6 we show that M^f can be represented as mean values of (Lelong currents of) Vogel cycles. We introduce the Segre numbers in Section 7 and prove formula (1.5). Theorem 1.4 is proved in Section 8. In Sections 9 and 10 we show how proper intersections and the Tworzewski intersections can be represented by our current calculus. Finally we provide various examples in Section 11.

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2. Preliminaries

Let us fix some notation. Given a tuple f of holomorphic functions on an analytic space X we will use $\mathcal{J}(f)$ to the denote the sheaf it generates. Similarly if $A \subset X$ is a submanifold we will use $\mathcal{J}(A)$ to the denote the radical sheaf. We will denote the local ring of germs of holomorphic functions at x in X by $\mathcal{O}_{X,x}$. We say that a sequence $g_1, \ldots g_m$ of functions on an analytic space X is a *geometrically regular sequence* if $\operatorname{codim}\{g_1 = \ldots = g_k = 0\} = k$ for $1 \le k \le m$. If X is smooth (or Cohen-Macaulay) a sequence is geometrically regular if and only if it is regular.

If $\alpha(\lambda)$ is a current valued function, defined in a neighborhood of the origin, we let $\alpha(\lambda)|_{\lambda=0}$ to denote the value at $\lambda = 0$.

2.1. **Positive currents.** Let $d^c = (4\pi i)^{-1}(\partial - \bar\partial)$ so that $dd^c = (2\pi i)^{-1}\bar\partial\partial$. We briefly recall some basic facts about positive currents, referring to [7, 9] for details. Let μ be a positive (k,k)-current defined in some open set $\Omega \subset \mathbb{C}^N$. Then μ has order zero, so that the restriction $\mathbf{1}_S\mu$ is well-defined for any Borel set $S \subset \Omega$. If in addition μ is closed and S is analytic, then the Skoda-El Mir Theorem asserts that $\mathbf{1}_S\mu$ is closed as well. If μ is closed then one can define inductively

$$(dd^c \log|z-x|^2)^{j+1} \wedge \mu = dd^c\big(\log|z-x|^2 dd^c((\log|z-x|^2)^j \wedge \mu)\big),$$

$(dd^c \log|z-x|^2)^{N-k}\wedge\mu$ is a (N,N)-current. Its mass at x is the *Lelong number* $\ell_x(\mu)$ at x of μ, which depends semi-continuously of μ, in the sense that

$$\ell_x(\mu) \geq \limsup_{j\to\infty} \ell_x(\mu_j) \tag{2.1}$$

if $\mu_j \to \mu$. It follows that $x \mapsto \ell_x(\mu)$ is upper semi-continuous.

Lemma 2.1. *If μ is a closed positive (k,k)-current in $\Omega\subset\mathbb{C}^N$, then*

$$\ell_0(\mu)[\{0\}] = \lim_{\lambda\to 0^+} \Big(\bar\partial|z|^{2\lambda}\wedge\frac{\partial|z|^2}{2\pi i|z|^2}\wedge(dd^c\log|z|^2)^{N-k-1}\wedge\mu\Big). \tag{2.2}$$

If $k=N$, then the right hand side of (2.2) shall be interpreted as

$$\lim_{\lambda\to 0^+}\Big(1-|z|^{2\lambda}\Big)\,\mu = \mathbf{1}_{\{0\}}\mu,$$

so Lemma 2.1 is trivially true in this case.

Sketch of proof. If ξ is a test function, then

$$\int (dd^c\log|z|^2)^{N-k}\wedge\mu\,\xi = \lim_{\lambda\to 0^+}\int \frac{|z|^{2\lambda}-1}{\lambda}\wedge(dd^c\log|z|^2)^{N-k-1}\mu\wedge dd^c\xi. \tag{2.3}$$

After an integration by parts, the right-hand side of (2.3) may be rewritten as

$$\lim_{\lambda\to 0^+}\int \bar\partial|z|^{2\lambda}\wedge\frac{\partial|z|^2}{2\pi i|z|^2}\wedge(dd^c\log|z|^2)^{N-k-1}\wedge\mu\,\xi$$
$$+\lim_{\lambda\to 0^+}\int |z|^{2\lambda}(dd^c\log|z|^2)^{N-k}\wedge\mu\xi.$$

The second term is precisely the action of $\mathbf{1}_{\mathbb{C}^N\setminus\{0\}}(dd^c\log|z|^2)^{N-k}\wedge\mu$ on ξ, and consequently the point mass at 0 of $(dd^c\log|z|^2)^{N-k}\wedge\mu$ is the same as the point mass at 0 of the first term, which proves (2.2). $\square$

2.2. **Currents on an analytic space.** Let X be an analytic space of dimension n. Given a local embedding $i\colon X\hookrightarrow\mathbb{C}^N$, we let $\mathcal{E}_X$ be the sheaf of smooth forms on X, obtained from the sheaf of smooth forms in the ambient space, where two forms are identified if their pullbacks to X_{reg} coincide; it is well-known that this definition does not depend on the particular embedding. We say that μ is a *current on X of bidegree* (p,q) if it acts on test forms on X of bidegree $(n-p,n-q)$. Such currents μ are naturally identified with currents $\tau=i_*\mu$ of bidegree $(N-n+p,N-n+q)$ in the ambient space such that τ vanish on the kernel of i^*. Observe that the d-operator is well-defined on currents on X. If W is a subvariety of X of pure codimension $p\geq 0$, then

$$\phi\mapsto [W].\phi = \int_{W_{\rm reg}}\phi$$

is a closed (p,p)-current on X; this is the *current of integration* over W.

Recall that a current ν is *normal* if both ν and $d\nu$ have order zero. The following lemma follows immediately from the corresponding one in $\mathbb{C}^N$.

Lemma 2.2. *Suppose that μ is a normal current of bidegree (p,p) on X that has support on a subvariety W of codimension k. If $k>p$ then $\mu=0$. If $k=p$ and μ is closed, then $\mu=\sum_j\alpha_j[W_j]$ for some numbers α_j, where W_j are the irreducible components of W of codimension p.*

It is readily checked that if we have a proper holomorphic mapping $\nu\colon X' \to X$ between analytic spaces, then the push-forward ν_* is well-defined on currents on X'.

Assume that μ is a positive closed current on the analytic space X. Fix $x \in X$ and let $i\colon X \hookrightarrow \mathbb{C}^N$ be a local embedding. We define the Lelong number $\ell_x(\mu)$ as $\ell_x(i_*\mu)$. After a suitable change of coordinates i can be factorized as $i = j \circ i'$, where $i'\colon X \to \mathbb{C}^M$ is a minimal embedding and j is the natural embedding $\mathbb{C}^M \to \mathbb{C}^M \times \mathbb{C}^{N-M}$. Since the Lelong number is invariant under holomorphic changes of coordinates, all minimal embeddings are equal up to a holomorphic change of variables, and $\ell_x\tau = \ell_x(j_*)$, it follows that $\ell_x(\mu)$ is well-defined. Thus if Z is a subvariety of an analytic space X and we have an embedding $X \subset \mathbb{C}^N$, then the number $\ell_x[Z]$ is indepenent of whether we consider Z as the Lelong current of Z on X or on $\mathbb{C}^N$.

Recall that if Z is a variety in $\mathbb{C}^N$, then the *multiplicity* $\mathrm{mult}_x Z$ of Z at x coincides with the Lelong number $\ell_x([Z])$ of the Lelong integration current $[Z]$, see [7, Prop. 3.15.1.2]; here $\mathrm{mult}_x Z$ is defined as in [7, Ch. 2.11.1]. In particular, the Lelong number of the function 1, considered as a current on an analytic space X, at x is precisely $\mathrm{mult}_x X$.

The classical *Siu decomposition*, [22], of positive closed currents extends immediately to currents on our analytic space X. Let μ be a positive closed (p,p)-current on X; then there is a unique decomposition

$$\mu = \sum_i \beta_i [W_i] + N,$$

where W_i are irreducible analytic varieties of codimension p, $\beta_i \geq 0$, and, for each $\delta > 0$, the set where $\ell_x(N) \geq \delta$ is analytic and has codimension strictly larger than p.

2.3. **Cycles and Lelong currents.** Given an analytic cycle $Z = \sum \alpha_j W_j$, where W_j are varieties, we let $[Z] = \sum \alpha_j [W_j]$ be the associated Lelong current. We will sometimes identify analytic cycles with their Lelong currents. We let $|Z|$ denote the *support* of Z. Sometimes we will be sloppy and identify $|Z|$ with Z; in particular, we will write $\mathbf{1}_Z$ for $\mathbf{1}_{|Z|}$. If H is a Cartier divisor defined by (a germ of) a holomorphic function h, we will (sometimes) use the notation $[h]$ for $[H]$ and $\mathbf{1}_h$ for $\mathbf{1}_{|H|}$. Given an analytic cycle $Z = \sum \alpha_i [W_i]$ of pure dimension, the *multiplicity* of Z at x is defined as $\sum \alpha_i \mathrm{mult}_x W_i$ (this definition follows [11, p. 704]). It follows that

$$\mathrm{mult}_x Z = \ell_x[Z].$$

If $Z = \sum_{k=0}^n Z_k$, where Z_k is an analytic cycle of codimension k we define

$$\mathrm{mult}_x Z := (\mathrm{mult}_x Z_0, \dots, \mathrm{mult}_x Z_n) \tag{2.4}$$

Throughout this paper all analytic cycles are effective, unless otherwise stated.

3. Multiplying a Lelong current by a Cartier divisor

In this section we will describe how the inductive construction of a Vogel cycle V^h can be expediently expressed as certain products of Lelong currents. First, note that if Z, Z' are analytic cycles in some analytic space X, then

$$\mathbf{1}_{Z'}[Z] = [Z^{Z'}], \tag{3.1}$$

so $[Z] \mapsto \mathbf{1}_{Z'}[Z]$ is a linear operator on Lelong currents. To see (3.1), by linearity, we can assume that Z is irreducible. If $|Z|$ is contained in $|Z'|$, then $\mathbf{1}_{Z'}[Z] = [Z]$. Otherwise, $|Z| \cap |Z'|$ has higher codimension than $|Z|$, and thus $\mathbf{1}_{Z'}[Z]$ vanishes by

Lemma 2.2. Notice that $\mathbf{1}_Z := \mathbf{1}_Z 1$ is 1 on the components of X that are contained in Z and 0 otherwise, i.e., it is the Lelong current of X^Z.

If h is a non-vanishing holomorphic function on (each irreducible component of) the analytic space Z, then $\log|h|^2$ is a well-defined $(0,0)$-current on Z. This is clear if Z is smooth and follows in general, e.g., by means of a smooth resolution $\widetilde{Z} \to Z$, cf., the proof below.

Lemma 3.1. *Let Z be an analytic cycle in X, h be a holomorphic function, and let u be a nonvanishing smooth function on X. Then*

$$\lambda \mapsto \bar\partial |uh|^{2\lambda} \wedge \frac{\partial \log|uh|^2}{2\pi i} \wedge [Z], \tag{3.2}$$

a priori defined when $\operatorname{Re}\lambda$ *is large, has an analytic continuation to a half-plane* $\operatorname{Re}\lambda > -\epsilon$*, where* $\epsilon > 0$*. The value at* $\lambda = 0$ *is independent of* u*.*

If h does not vanish identically on any irreducible component of (the support of) Z, then this value is equal to $dd^c(\log|h|^2\,[Z])$*.*

Notice that $v^\lambda := \bar\partial|uh|^{2\lambda} \wedge \partial \log|uh|^2/(2\pi i)$ is smooth when $\operatorname{Re}\lambda$ is large so the product in (3.2) is then well-defined.

Proof. First assume that $Z = X = \mathbb{C}^N$ and h is a monomial $h = z_1^{a_1}\cdots z_N^{a_N}$. Then (3.2) is equal to

$$v^\lambda = \bar\partial |u z_1^{a_1}\cdots z_N^{a_N}|^{2\lambda} \wedge \frac{1}{2\pi i}\Big[\sum_1^N a_j \frac{dz_j}{z_j} + \frac{\partial |u|^2}{|u|^2}\Big].$$

One can check that the desired analytic continuation exists, and that the value at $\lambda = 0$ is the current $\sum_1^N a_j [z_j]/(2\pi i) = dd^c \log|h|^2$; in particular, it is independent of u.

Consider now the general case. By linearity, we may assume that Z is irreducible. If h vanishes identically on Z and $\operatorname{Re}\lambda$ is large, then $v^\lambda \wedge [Z] = 0$, and thus it trivially extends to $\lambda \in \mathbb{C}$. Assume that h does not vanish identically on Z. Let $i\colon Z \hookrightarrow X$ be an embedding and let $\pi\colon \widetilde{Z} \to Z$ be a smooth modification of Z such that $\pi^* i^* h$ is locally a monomial; such a modification exists due to Hironaka's theorem on resolution of singularities. After a partition of unity we are back to the case above. It follows that $\pi^* i^* v^\lambda$ has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ for some $\epsilon > 0$ and thus $v^\lambda \wedge [Z] = i_*\pi_*(\pi^* i^* v^\lambda)$ has the desired analytic continuation. The value at $\lambda = 0$ is equal to

$$i_*\pi_*(dd^c \log|\pi^* i^* h|^2)$$

which proves the second statement, since $(\log|h|^2)[Z] = i_*\pi_*(\log|\pi^* i^* h|^2)$. $\square$

Let H denote the Cartier divisor defined by h. We define $[H]\wedge[Z]$ as the value of (3.2) at $\lambda = 0$. According to the lemma it does not depend on the particular choice of h defining H; in fact it is the Lelong current of the proper intersection of H and the irreducible components of Z that are not contained in H, i.e.,

$$[H]\wedge[Z] = [H]\wedge \mathbf{1}_{X\setminus H}[Z] = [H\cdot Z^{X\setminus H}]. \tag{3.3}$$

If H and Z intersect properly, thus $[H]\wedge[Z] = [H\cdot Z]$. In fact, we can take this as the definition of the proper intersection $[H\cdot Z]$, cf. Section 2.3.

Remark 3.2. It is important to emphasize that $[H]\wedge[Z]$ is *not* analogous to the intersection $H \cdot Z$ in [10]. In fact, if Z is irreducible and contained in H, then $[H]\wedge[Z] = 0$, whereas in [10] the product is a cycle in Z of codimension 1, that is well-defined up to rational equivalence. □

It follows from the definition that

$$[H]\wedge([Z_1] + [Z_2]) = [H]\wedge[Z_1] + [H]\wedge[Z_2] \tag{3.4}$$

and thus $[Z] \mapsto [H]\wedge[Z]$ is a linear operator on Lelong currents, cf., (3.1).

However, in general it is *not* true that $([H_1] + [H_2]) \wedge [Z] = [H_1]\wedge[Z] + [H_2]\wedge[Z]$ or $[H_1] \wedge [H_2] = [H_2] \wedge [H_1]$.

Example 3.3. Let H_1 and H_2 be Cartier divisors and let $H = H_1 + H_2$. Then $[H_1]\wedge[H] = [H_1]\wedge[H_2]$ but $[H]\wedge[H_1] = 0$. Moreover $[H_1] \wedge \mathbf{1}_{H_1}[H] = [H_1]\wedge[H_1] = 0$ but $\mathbf{1}_{H_1}[H_1]\wedge[H] = \mathbf{1}_{H_1}[H_1]\wedge[H_2] = [H_1]\wedge[H_2]$. □

We can construct Vogel cycles by inductively applying operators $\mathbf{1}_Z$ and $[H]\wedge$.

Proposition 3.4. *Let X be an analytic space of dimension n and let $h = (h_1, \dots, h_n)$ be a Vogel sequence of an ideal $\mathcal{J}$ with variety Z at $x \in X$, with corresponding divisors $H_1, \dots, H_n$. Then on X,*

$$[X_0] = 1, \quad [X_\ell] = [H_\ell]\wedge\cdots\wedge[H_1]\,,\ \ell = 1, \dots, n \tag{3.5}$$

and

$$[X_0^Z] = \mathbf{1}_Z, \quad [X_\ell^Z] = \mathbf{1}_Z[H_\ell]\wedge\cdots\wedge[H_1],\ \ell = 1, \dots, n. \tag{3.6}$$

In particular,

$$[V^h] = \mathbf{1}_Z + \mathbf{1}_Z[H_1] + \mathbf{1}_Z[H_2]\wedge[H_1] + \cdots + \mathbf{1}_Z[H_n]\wedge\cdots\wedge[H_1]. \tag{3.7}$$

If we consider X as embedded in some larger analytic space Y, then we have instead

$$[X_0] = [X], \quad [X_\ell] = [H_\ell]\wedge\cdots\wedge[H_1]\wedge[X],\ \ell = 1, \dots, n$$

and

$$[X_0^Z] = \mathbf{1}_Z[X], \quad [X_\ell^Z] = \mathbf{1}_Z[H_\ell]\wedge\cdots\wedge[H_1]\wedge[X],\ \ell = 1, \dots, n$$

Proof. In view of (3.1), (3.6) follows from (3.5). Using (3.4), we have

$$[X_1] = [H_1]\wedge[X_0^{X\setminus Z}] = [H_1]\wedge([X_0] - [X_0^Z]) = [H_1]$$

since $[H_1]\wedge[X_0^Z] = [H_1]\mathbf{1}_Z = 0$. One obtains (3.5) by induction. □

4. Bochner-Martinelli currents

Let X be an analytic space of pure dimension n, $f = (f_0, \dots, f_m)$ a tuple of holomorphic functions on X, $\mathcal{J} = \mathcal{J}(f)$ the coherent sheaf generated by f, and Z the zero set of $\mathcal{J}$. For $\operatorname{Re}\lambda >> 0$, let

$$M_0^{f,\lambda} := 1 - |f|^{2\lambda}$$

$$M_k^{f,\lambda} := \bar\partial|f|^{2\lambda}\wedge\frac{\partial \log|f|^2}{2\pi i}\wedge(dd^c\log|f|^2)^{k-1} \text{ if } k \geq 1,$$

and

$$M^{f,\lambda} := \sum_{k=0}^{\infty} M_k^{f,\lambda}, \tag{4.1}$$

where $|f|^2 = \sum_{j=0}^m |f_j|^2$. Note that the sum in (4.1) is finite for degree reasons, and as $\operatorname{Re}\lambda >> 0$, M^f is locally integrable. We will show that $\lambda \mapsto M_k^{f,\lambda}$ has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, for some $\epsilon > 0$. We denote the value of $M_k^{f,\lambda}$ at $\lambda = 0$ by M_k^f and we write $M^f := \sum_k M_k^f$. The current M^f and its components M_k^f will be referred to as *Bochner-Martinelli currents*, cf., Remark 4.2.

A computation yields that

$$M_k^{f,\lambda} = \lambda \frac{i}{2\pi} \frac{\partial|f|^2 \wedge \bar\partial|f|^2}{|f|^{4-2\lambda}} \wedge (dd^c \log|f|^2)^{k-1}$$

which is positive when $\lambda > 0$, and thus M_k^f is a positive current. Note that M_0^f is the current of integration over the components of X, on which $f \equiv 0$. In particular, if f does not vanish identically on any component of X, then $M_0^f = 0$.

Let $\pi\colon \widetilde X \to X$ be a normal modification such that the pull-back ideal sheaf $\mathcal{J} \cdot \mathcal{O}_{\widetilde X}$ is principal; for instance oen can take the normalization of the blow-up of X along $\mathcal{J}$. Then $\pi^* f = f^0 f'$ where f^0 is a section of the holomorphic line bundle $L \to \widetilde X$ corresponding to the exceptional divisor D_f of $\pi\colon \widetilde X \to X$, i.e., the divisor defined by $\mathcal{J} \cdot \mathcal{O}_{\widetilde X}$, and f' is a nonvanishing tuple of sections of L^{-1}. Let L be equipped with the metric defined by $|f^0|_L = |\pi^* f| = |f^0 f'|$, and let

$$\omega_f := dd^c \log|f'|^2; \tag{4.2}$$

here the right hand side is computed locally for any local trivialization of L. Then $-\omega_f$ is the first Chern form of $(L, |\cdot|_L)$, and clearly $\omega_f \geq 0$.

Since $\log|\pi^* f|^2 = \log|f_0|^2 + \log|f'|^2$ it follows from Lemma 3.1 that

$$dd^c \log|\pi^* f|^2 = [D_f] + \omega_f. \tag{4.3}$$

In particular, $\pi^*[dd^c \log|f|^2] = \omega_f$ outside $\pi^{-1}\{f = 0\}$. Therefore, for $\operatorname{Re}\lambda >> 0$,

$$\pi^* M_0^{f,\lambda} = 1 - |f^0 u|^{2\lambda} \tag{4.4}$$

$$\pi^* M_k^{f,\lambda} = (2\pi i)^{-1} \bar\partial |f^0 f'|^{2\lambda} \wedge \partial \log|f^0 f'|^2 \wedge \omega_f^{k-1},\ k \geq 1. \tag{4.5}$$

Now Lemma 3.1 asserts that $\lambda \mapsto \pi^* M_k^{f,\lambda}$ has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ and since $M_k^{f,\lambda} = \pi_* \pi^* M_k^{f,\lambda}$ for $\operatorname{Re}\lambda >> 0$, it follows that $\lambda \mapsto M_k^{f,\lambda}$ has the desired analytic continuation. Moreover

$$M_0^f = M_0^{f,\lambda}|_{\lambda=0} = \pi_*(\pi^* M_0^{f,\lambda}|_{\lambda=0}) = \pi_*(\mathbf{1}_{D_f}) = \mathbf{1}_{\{f=0\}}\,. \tag{4.6}$$

$$M_k^f = M_k^{f,\lambda}|_{\lambda=0} = \pi_*(\pi^* M_k^{f,\lambda}|_{\lambda=0}) = \pi_*([D_f] \wedge \omega_f^{k-1}),\ k \geq 1. \tag{4.7}$$

Following for example [4] one can check that for $k \geq 1$

$$M_k^f = \mathbf{1}_Z (dd^c \log|f|^2)^k \tag{4.8}$$

and

$$\mathbf{1}_{X \setminus Z} (dd^c \log|f|^2)^k = \pi_*(\omega_f^k).$$

It is not hard to see that in $M_k^{f,\lambda}$ is locally integrable for $\operatorname{Re}\lambda > 0$ and that $M_k^{f,\lambda} \to M_k^f$ as measures when $\lambda \to 0^+$.

Remark 4.1. For further reference, let g be a tuple of holomorphic functions such that $|g| \sim |f|$, i.e., there exists $C \in \mathbb{R}$ such that $1/C|f| \leq |g| \leq C|g|$, and let $\pi\colon \widetilde X \to X$ be a normal modification such that both $\mathcal{J}(f) \cdot \mathcal{O}_{\widetilde X}$ and $\mathcal{J}(g) \cdot \mathcal{O}_{\widetilde X}$ are principal. Then $|f^0 f'| \sim |g^0 g'|$ and since f' and g' are non-vanishing it follows that f^0 and g^0

define the same divisor on $\widetilde{X}$. Therefore the corresponding negative Chern forms ω_f and ω_g are dd^c-cohomologous, i.e., there is a global smooth function γ such that $dd^c\gamma = \omega_f - \omega_g$. $\square$

Remark 4.2. The current M^f can be written as a product of Bochner-Martinelli residue currents and appropriate differentials df_j. More precisely, let $e_1, \dots, e_m$ be a holomorphic frame for a trivial vector bundle $E \to X$ and let e_j^* be the dual frame for E^*. Consider f as the section $f = f_1e_1^* + \cdots + f_me_m^*$ of E^* and let σ be the section $(\bar{f}_1e_1 + \cdots + \bar{f}_me_m)/|f|^2$ of E over $X \setminus Z$. Then we can define the Bochner-Martinelli residue current $R^f = R_0^f + R_1^f + \cdots + R_n^f$ as the value at $\lambda = 0$ of

$$R^{f,\lambda} = 1 - |f|^{2\lambda} + \sum \bar{\partial}|f|^{2\lambda}\wedge\sigma\wedge(\bar{\partial}\sigma)^{k-1},$$

cf., [21] where this current was first introduced, and [3]. It turns out, [4], Proposition 3.2, that we have the factorization

$$M_k^f = R_k^f \cdot (df/2\pi i)^k/k!,$$

where the dot denotes the natural pairing between Λ^kE^* and Λ^kE; see [4] for details. $\square$

5. Products of Bochner-Martinelli currents

Throughout this section let X be an analytic space of pure dimension n. Given tuples $f_1, \dots, f_r$ of holomorphic functions in X, we will give meaning to the product $M^{f_r}\wedge\cdots\wedge M^{f_1}$ of Bochner-Martinelli currents. The construction is recursive. Assume that $M^{f_\ell}\wedge\cdots\wedge M^{f_1}$ is defined; it follows from the proof of Proposition 5.1 that

$$\lambda \mapsto M^{f_{\ell+1},\lambda}\wedge M^{f_\ell}\wedge\cdots\wedge M^{f_1} \tag{5.1}$$

is holomorphic for $\operatorname{Re}\lambda > -\epsilon$, where $\epsilon > 0$. Set

$$M^{f_{\ell+1}}\wedge M^{f_\ell}\dots\wedge M^{f_1} := M^{f_{\ell+1},\lambda}\wedge M^{f_\ell}\wedge\dots\wedge M^{f_1}\big|_{\lambda=0}. \tag{5.2}$$

We define the products $M_{k_r}^{f_r}\wedge\cdots\wedge M_{k_1}^{f_1}$ in the analogous way so that

$$M^{f_r}\wedge\cdots\wedge M^{f_1} = \sum_{k_r,\dots,k_1\geq 0} M_{k_r}^{f_r}\wedge\cdots\wedge M_{k_1}^{f_1}. \tag{5.3}$$

Proposition 5.1. *Let $f_1, \dots, f_r$ be tuples of holomorphic functions in X, with common zero set $Z = \{f_1 = \dots = f_r = 0\}$. Then the current $M^{f_r}\wedge\cdots\wedge M^{f_1}$, defined by* (5.2)*, is positive and has support on Z.*

Let $\pi\colon \widetilde{X} \to X$ be a normal modification such that the sheaves $\mathcal{J}(f_\ell)\cdot\mathcal{O}_{\widetilde{X}}$ are principal for $\ell = 1, \dots r$. As in Section 4, let D_{f_ℓ} and ω_{f_ℓ} be the corresponding divisors and negative Chern forms, respectively. Then

$$M_{k_r}^{f_r}\wedge\dots\wedge M_{k_1}^{f_1} = \pi_*\big([D_{f_r}]\wedge\cdots\wedge[D_{f_1}]\wedge\omega_{f_r}^{k_r-1}\wedge\cdots\wedge\omega_{f_1}^{k_1-1}\big), \tag{5.4}$$

where, if $k_\ell = 0$, the factor $[D_{f_\ell}]$ shall be replaced by $\mathbf{1}_{D_{f_j}}$ and the factor $\omega_{f_\ell}^{k_\ell-1}$ shall be removed.

Assume that $g_1, \dots g_r$ are tuples of holomorphic functions in X such that $|g_\ell| \sim |f_\ell|$ for $\ell = 1, \dots, r$. Then there is a normal current T with support on Z such that

$$dd^cT = M_{k_r}^{f_r}\wedge\cdots\wedge M_{k_1}^{f_1} - M_{k_r}^{g_r}\wedge\cdots\wedge M_{k_1}^{g_1}. \tag{5.5}$$

Proof. Iteratively using Lemma 3.1, (4.6), and (4.7) we see that the desired analytic continuation of (5.1) exists and that (5.4) holds. It follows that $M^{f_r}_{k_r}\wedge\ldots\wedge M^{f_1}_{k_1}$ has its support contained in $\pi(|D_{f_r}|\cap\cdots\cap|D_{f_1}|)=Z$. Moreover $M^{f_r}_{k_r}\wedge\ldots\wedge M^{f_1}_{k_1}$ is the push-forward of a product of positive $(1,1)$-currents and positive forms, and hence it is positive.

To prove the last part, it suffices to change one of the f_ℓ to g_ℓ with $|g_\ell|\sim|f_\ell|$. First notice that then $M^{f_\ell}_0=\mathbf{1}_{f_\ell}=\mathbf{1}_{g_\ell}=M^{g_\ell}_0$. Let us then assume that $k_\ell\geq 1$, and that the modification π is chosen so that also $\mathcal{J}(g_\ell)\cdot\mathcal{O}_{\widetilde{X}}$ is principal. By Remark 4.1, there is a smooth global function γ on $\widetilde{X}$ such that $\omega_{f_\ell}-\omega_{g_\ell}=dd^c\gamma$ and thus we can find a smooth global form w such that $dd^cw=\omega^{k_\ell-1}_{f_\ell}-\omega^{k_\ell-1}_{g_\ell}$. Let

$$T:=\pi_*\big(\tau_r\wedge\cdots\wedge\tau_{\ell+1}\wedge[D_{f_\ell}]\wedge w\wedge\tau_{\ell-1}\wedge\cdots\wedge\tau_1\big),$$

where $\tau_j=\mathbf{1}_{D_{f_j}}$ if $k_j=0$ and $\tau_j=[D_{f_j}]\wedge\omega^{k_j-1}_{f_j}$ otherwise. Then T satisfies (5.5). Note that $\tau_r\wedge\cdots\wedge\tau_{\ell+1}\wedge[D_{f_\ell}]\wedge w\wedge\tau_{\ell-1}\wedge\cdots\wedge\tau_1$ is normal, and since normality is preserved under push-forward, so is T. □

We also define products of Bochner-Martinelli currents and Lelong currents. If $f_1,\ldots,f_r$ are tuples of holomorphic functions in X and Z is an analytic subset of X, we define recursively $M^{f_1}\wedge[Z]:=M^{f_1,\lambda}\wedge[Z]\big|_{\lambda=0}$, and

$$M^{f_{k+1}}\wedge\cdots\wedge M^{f_1}\wedge[Z]:=M^{f_{k+1},\lambda}\wedge M^{f_k}\wedge\cdots\wedge M^{f_1}\wedge[Z]\big|_{\lambda=0}.$$

By arguments as in the proof of Proposition 5.1 we prove that the desired analytic continuations exist, and thus $M^{f_r}\wedge\cdots\wedge M^{f_1}\wedge[Z]$ is well-defined. It is readily checked that if $i\colon Z\hookrightarrow X$, then, for any $k_1,...,k_r\in\mathbb{N}$,

$$M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}\wedge[Z]=i_*[M^{i^*f_r}_{k_r}\wedge\cdots\wedge M^{i^*f_1}_{k_r}]. \tag{5.6}$$

For further reference, note that if f is a tuple of holomorphic functions on the analytic space X then

$$M^f=M^f\mathbf{1}_X=\sum_j M^f\mathbf{1}_{X_j}, \tag{5.7}$$

where X_j are the irreducible components of X.

Proposition 5.2. *Let $f_1,\ldots,f_r$ be tuples of holomorphic functions in X and let ξ be a tuple of holomorphic functions such that $\{\xi=0\}=\{x\}$, where $x\in X$. Then*

$$M^\xi\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}=M^\xi_{n-k}\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}=\alpha[x], \tag{5.8}$$

where $k=k_1+\cdots+k_r$ and α is a non-negative integer. If ξ generates the maximal ideal at $x\in X$, then $\alpha=\ell_x\big(M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}\big)$.

Remark 5.3. It follows from the second part of Proposition 5.1, applied to $f_1,\ldots,f_r,\xi$, that the Lelong number at x of $M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}$ is unchanged if we replace f_j by g_j such that $|f_j|\sim|g_j|$, since T, which has bidegree $(n-1,n-1)$, must vanish by Lemma 2.2. □

Proof. By Proposition 5.1, $M^\xi_{n-k}\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}$ is positive and has support at x, and thus by Lemma 2.2 it is of the form $\alpha[x]$ for some non-negative α. Let $\pi:\widetilde{X}\to X$ be a normal modification such that $\mathcal{J}(f_\ell)\cdot\mathcal{O}_{\widetilde{X}}$ and $\mathcal{J}(\xi)\cdot\mathcal{O}_{\widetilde{X}}$ are principal. Let us use the notation from Section 4. Then, from (5.4), we see that α is an intersection number and hence an integer.

Now assume that ξ generates the maximal ideal at x and that $i\colon X \hookrightarrow \mathbb{C}^N$ is a local embedding such that $i(x)=0$, so that $i_*[x]=[\{0\}]$. By the second part of Proposition 5.1 we may assume that $f_j = i^*F_j$ and $\xi = i^*z$ for some tuples F_j and the standard coordinate system $z=(z_1,\ldots,z_N)$ in $\mathbb{C}^N$. Then

$$i_*(M^{\xi}_{n-k}\wedge M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1}) = M^{z}_{n-k}\wedge M^{F_r}_{k_r}\wedge\cdots\wedge M^{F_1}_{k_1}\wedge[X], \tag{5.9}$$

cf. (5.6). By Lemma 2.1, the right hand side of (5.9) is precisely the Lelong number of $M^{F_r}_{k_r}\wedge\cdots\wedge M^{F_1}_{k_1}\wedge[X]$ at 0 in $\mathbb{C}^N$ times $[\{0\}]$. □

One can replace all the evaluations in the definition of the product by one single evaluation in the following way.

Proposition 5.4. *Assume that μ_j are strictly positive integers such that $\mu_1>\mu_2>\ldots>\mu_r$. Then $\lambda\mapsto M^{f_r,\lambda^{\mu_r}}_{k_r}\wedge\cdots\wedge M^{f_1,\lambda^{\mu_1}}_{k_1}$ is holomorphic in a neighborhood of the half-axis $[0,\infty)$ in $\mathbb{C}$ and*

$$M^{f_r}_{k_r}\wedge\cdots\wedge M^{f_1}_{k_1} = M^{f_r,\lambda^{\mu_r}}_{k_r}\wedge\cdots\wedge M^{f_1,\lambda^{\mu_1}}_{k_1}\big|_{\lambda=0}\,. \tag{5.10}$$

Example 5.5. If $h_1,\ldots,h_n$ is a Vogel sequence of some ideal at some point x, then, cf., Theorem 7.3 below, the Lelong current of the associated Vogel cycle is given as the value at $\lambda=0$ of the function

$$\bigwedge_{k=1}^{n} M^{h_k,\lambda^{\mu_k}} = \bigwedge_{k=1}^{n}\big(1-|h_k|^{2\lambda^{\mu_k}}+\bar\partial|h_k|^{2\lambda^{\mu_k}}\wedge\partial\log|h_k|^2/2\pi i\big)$$

□

Proof. By Hironaka's theorem we can choose a smooth modification $\pi\colon\widetilde{X}\to X$ such that $\pi^*f_j = f_j^0 f_j'$, $j=1,\ldots,r$, where $f_j'\neq 0$ and each f_j^0 is a monomial $x^{\alpha_j}=x_1^{\alpha_{j1}}\cdots x_n^{\alpha_{jn}}$ in local coordinates on $\widetilde{X}$. Then locally on $\widetilde{X}$, by (4.4) and (4.5),

$$\pi^*M_0^{f_j,\lambda_j} = 1-|u_jx^{\alpha_j}|^{\lambda_j},\quad \pi^*M_k^{f_j,\lambda_j} = \frac{\bar\partial|u_jx^{\alpha_j}|^{2\lambda_j}}{x^{\alpha_j}}\wedge\vartheta_{kj}\ \text{for}\ k\geq 1,$$

where u_j are smooth non-vanishing functions and the ϑ_{kj} are smooth forms. The proposition now follows from Lemma 5.6. □

Lemma 5.6. *Let u_ℓ be smooth non-vanishing functions defined in some neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^n$, with coordinates $x_1,\ldots x_n$. For $\lambda=(\lambda_1,\ldots,\lambda_r)\in\mathbb{C}^r$, $\mathrm{Re}\,\lambda_k>>0$ and $\alpha_1,\ldots,\alpha_r\in\mathbb{N}^n$, let*

$$\Gamma(\lambda) := \frac{|u_rx^{\alpha_r}|^{2\lambda_r}\cdots|u_{p+1}x^{\alpha_{p+1}}|^{2\lambda_{p+1}}\bar\partial|u_px^{\alpha_p}|^{2\lambda_p}\wedge\cdots\wedge\bar\partial|u_1x^{\alpha_1}|^{2\lambda_1}}{x^{\alpha_p}\cdots x^{\alpha_1}};$$

here $x^{\alpha_\ell}=x_1^{\alpha_{\ell,1}}\cdots x_n^{\alpha_{\ell,n}}$ if $\alpha_\ell=(\alpha_{\ell,1},\ldots,\alpha_{\ell,n})$. If σ is a permutation of $\{1,\ldots,r\}$, write $\Gamma^\sigma(\lambda_1,\ldots,\lambda_r):=\Gamma(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(r)})$.

Let $\mu_1,\ldots,\mu_r$ be positive integers. Then $\Gamma^\sigma(\kappa^{\mu_1},\ldots,\kappa^{\mu_r})$ has an analytic continuation to a neighborhood of the half-axis $[0,\infty)$ in $\mathbb{C}$, and if $\mu_1>\ldots>\mu_r$

$$\Gamma^\sigma(\kappa^{\mu_1},\ldots,\kappa^{\mu_r})\,|_{\kappa=0} = \Gamma^\sigma(\lambda_1,\ldots,\lambda_r)\,|_{\lambda_1=0}\cdots|_{\lambda_r=0}\,. \tag{5.11}$$

Proof. To begin with let us assume that all $u_j=1$. A straightforward computation shows that

$$\Gamma(\lambda) = \lambda_1\cdots\lambda_p\frac{\prod_{j=1}^r|x^{\alpha_j}|^{2\lambda_j}}{x^{\sum_{j=1}^p\alpha_j}}\sum_I{}' A_I\frac{d\bar x_{i_1}\wedge\cdots\wedge d\bar x_{i_p}}{\bar x_{i_1}\cdots\bar x_{i_p}} =: \lambda_1\cdots\lambda_p\sum_I{}'\Gamma_I,$$

where the sum is over all increasing multi-indices $I=\{i_1,\dots,i_p\}\subset\{1,\dots,n\}$ and A_I is the determinant of the matrix $(\alpha_{\ell,i_j})_{1\le\ell\le p,1\le j\le p}$.

Pick a non-vanishing summand Γ_I; without loss of generality, assume that $I=\{1,\dots,p\}$ and $A_I=1$. With the notation $b_k(\lambda):=\sum_{\ell=1}^r\lambda_\ell\alpha_{\ell,k}$ for $1\le k\le n$,

$$\Gamma_I=\frac{\prod_{k=1}^n|x_k|^{2b_k(\lambda)}}{x^{\sum_{j=1}^p\alpha_j}}\frac{d\bar{x}_1\wedge\cdots\wedge d\bar{x}_p}{\bar{x}_1\cdots\bar{x}_p}=\\ \frac{1}{b_1(\lambda)\cdots b_p(\lambda)}\frac{\bigwedge_{k=1}^p\bar\partial|x_k|^{2b_k(\lambda)}\prod_{k=p+1}^n|x_k|^{2b_k(\lambda)}}{x^{\sum_{j=1}^p\alpha_j}}.$$

Now the current valued function

$$\widetilde\Gamma:(\lambda_1,\dots,\lambda_r)\mapsto\frac{\bigwedge_1^p\bar\partial|x_j|^{2b_j(\lambda)}\prod_{p+1}^n|x_j|^{2b_j(\lambda)}}{x^{\sum_{j=1}^r\alpha_j}}$$

has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^r$; in fact, it is a tensor product of one-variable currents. In particular, $\widetilde\Gamma(\kappa^{\mu_1},\dots,\kappa^{\mu_r})\mid_{\kappa=0}=\widetilde\Gamma(\lambda)\mid_{\lambda_1=0}\cdots\mid_{\lambda_r=0}$. Let

$$\gamma(\lambda)=\frac{\lambda_1\cdots\lambda_p}{b_1(\lambda)\cdots b_p(\lambda)}$$

and $\gamma^\sigma=\gamma(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(r)})$. We claim that, since $A_I=1\neq0$ and $\mu_1>\ldots>\mu_r$ we have

$$\gamma^\sigma(\lambda)\mid_{\lambda_1=0}\cdots\mid_{\lambda_r=0}=\gamma^\sigma(\kappa^{\mu_1},\dots,\kappa^{\mu_r})|_{\kappa=0},$$

where it is a part of the claim that both sides make sense.

Let us prove the claim. Since $A_I\neq0$, reordering the factors $b_1,\dots b_p$ and multiplying $\gamma(\lambda)$ by a non-zero constant, we may assume that $\alpha_{kk}=1$, $k=1,\dots p$, so that

$$\gamma(\lambda)=\frac{\lambda_1}{\lambda_1+\alpha_{21}\lambda_2+\cdots+\alpha_{r1}\lambda_r}\cdots\frac{\lambda_p}{\alpha_{p1}\lambda_1+\cdots+\lambda_p+\cdots+\alpha_{rp}\lambda_r}.$$

For $j<r$ set $\tau_j:=\lambda_j/\lambda_{j+1}$ and $\widetilde\gamma^\sigma(\tau_1,\dots,\tau_{r-1}):=\gamma^\sigma(\lambda)$; notice that γ^σ is 0-homogeneous, so that $\widetilde\gamma^\sigma$ is well-defined. Then $\lambda_j=\tau_j\cdots\tau_{r-1}\lambda_r$, and therefore $\widetilde\gamma^\sigma$ consists of p factors of the form

$$\frac{\tau_k\cdots\tau_{r-1}}{b_1\tau_1\cdots\tau_{r-1}+\cdots+\tau_k\cdots\tau_{r-1}+\cdots+b_{r-1}\tau_{r-1}+b_r}\tag{5.12}$$

where b_j are among the α_{jk}. Observe that (5.12) is holomorphic in τ in some neighborhood of the origin. Indeed, if $b_r\neq0$, then (5.12) is clearly holomorphic, whereas if $b_r=0$ we can factor out τ_{r-1} from the denominator and numerator. In the latter case (5.12) is clearly holomorphic if $b_{r-1}\neq0$ etc. It follows that $\gamma^\sigma(\kappa^{\mu_1},\dots,\kappa^{\mu_r})=\widetilde\gamma^\sigma(\kappa^{\mu_1-\mu_2},\dots,\kappa^{\mu_{r-1}-\mu_r})$ is holomorphic in a neighborhood of $[0,\infty)$ and moreover that $\gamma^\sigma(\lambda_1,\dots,\lambda_r)$ is holomorphic in $\Delta=\{|\lambda_1/\lambda_2|<\epsilon,\dots,|\lambda_{r-1}/\lambda_r|<\epsilon\}$. Let us now fix $\lambda_2\neq0,\dots,\lambda_r\neq0$ in Δ. Then $\gamma^\sigma(\lambda)$ is holomorphic in λ_1 in a neighborhood of the origin. Next, for $\lambda_3\neq0,\dots,\lambda_r\neq0$ fixed in Δ, $\gamma^\sigma(\lambda)|_{\lambda_1=0}$ is holomorphic in λ_2 in a neighborhood of the origin, etc. It follows that

$$\gamma^\sigma(\lambda)|_{\lambda_1}\cdots|_{\lambda_r=0}=\widetilde\gamma^\sigma(\tau)|_{\tau=0}=\gamma^\sigma(\kappa^{\mu_1},\dots,\kappa^{\mu_r})|_{\kappa=0},$$

which proves the claim. Thus (5.10) follows in the case $u_j=1$.

Now, consider the general case. By arguments as in the proof of Lemma 3.1 one can show that the right hand side of (5.11) is independent of u_j. To see that also the left hand side is independent of u_j, start by replacing each $u_j^{2\lambda_j}$ by $u_j^{2\omega_j}$ in $\Gamma(\lambda)$. Then, by arguments as above, the function $(\kappa, \omega_1, \dots, \omega_r) \mapsto \Gamma^\sigma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r})$ is holomorphic in a neighborhood of the origin in $\mathbb{C}_\kappa \times \mathbb{C}^r_\omega$ since it is analytic in each variable. In particular, $\Gamma(\kappa^{\mu_1}, \dots, \kappa^{\mu_r}) \,|_{\kappa=0}$ is obtained by first setting each $\omega_j = 0$, which corresponds to setting $u_j = 1$ and thus brings us back to the special case treated above. This completes the proof of the lemma. □

One can just as well let the tuples f_k be sections of arbitrary Hermitian holomorphic vector bundles $E_k \to X$ and define (products of) Bochner-Martinelli currents in precisely the same way, just interpreting $|f_k|$ as the norm of the section f_k. Then the statements in this section remain true, except for that the currents then will not necessarily be positive, and follow with only minor modifications of the proofs.

Remark 5.7. In a completely analogous way one can define products $R^{f_r} \wedge \dots \wedge R^{f_1}$ of Bochner-Martinelli currents, cf., Remark 4.2, and the analogue of Proposition 5.4 holds; it follows directly from the proof of Proposition 5.4. □

6. Mean values of products

Let X be an analytic space of pure dimension n. For a tuple $f_0, \dots, f_m$ of functions and $\beta = [\beta_0 : \dots : \beta_m] \in \mathbb{P}^m$ we write $\beta \cdot f := \beta_0 f_0 + \dots + \beta_m f_m$. Note that $M^{\beta \cdot f}$ only depends on $\beta \in \mathbb{P}^m$ and not on the choice of homogeneous coordinates.

Theorem 6.1. *Assume that $f = (f_0, \dots, f_m)$ is a tuple of holomorphic functions on X and that $\nu \geq \min(m+1, n+1)$. Then*

$$\int_{\alpha=(\alpha_1,\dots,\alpha_\nu)\in(\mathbb{P}^m)^\nu} M^{\alpha_\nu \cdot f} \wedge \dots \wedge M^{\alpha_1 \cdot f} = M^f. \tag{6.1}$$

Moreover

$$M^f_k = \mathbf{1}_Z \int_{\alpha\in(\mathbb{P}^m)^k} [\alpha_k \cdot f] \wedge \dots \wedge [\alpha_1 \cdot f], \tag{6.2}$$

where $Z = \{f = 0\}$, and (6.2) is interpreted as $\mathbf{1}_Z$ for $k = 0$.

For the proof we will use the following lemma,

Lemma 6.2. *If ϕ is a non-vanishing holomorphic $(m+1)$-tuple on X, then, in the sense of currents,*

$$\int_{\alpha\in\mathbb{P}^m} [\alpha \cdot \phi] d\sigma(\alpha) = dd^c \log |\phi|^2,$$

where $d\sigma$ is the normalized Fubini-Study metric.

This is a simple variant of Crofton's formula that should be well-known, but for the reader's convenience we include a proof. The lemma is true for an arbitrary tuple; a formal iterated application implies that the integral in (6.2) is equal to $(dd^c \log |f|^2)^k$, so that (6.2) follows.

Proof. For a test form $\zeta \mapsto \xi(\zeta)$, we have by the Poincaré-Lelong formula

$$\int_{\alpha\in\mathbb{P}^m} \int_\zeta [\alpha \cdot \phi] \wedge \xi \; d\sigma(\alpha) = \int_{\alpha\in\mathbb{P}^m} \int_\zeta \log(|\alpha \cdot \phi|^2/|\alpha|^2) \, dd^c\xi \, d\sigma(\alpha),$$

and, since $\log(|\alpha\cdot\phi(\zeta)|^2/|\alpha|^2)$ is integrable in α for each fixed ζ, with uniformly bounded norms, we can apply Fubini's theorem. Write $\kappa := \int_{\alpha\in\mathbb{P}^n}\log(|\alpha_0|^2/|\alpha|^2)d\sigma(\alpha)$. Then

$$\int_\zeta (\log|\phi|^2+\kappa)dd^c\xi = \int_\zeta dd^c\log|\phi|^2\wedge\xi,$$

as wanted. □

Proof of Theorem 6.1. Let $\pi\colon \widetilde{X}\to X$ be a normal modification such that $\mathcal{J}(f)\cdot\mathcal{O}_{\widetilde{X}}$ is principal, and use the notation from Section 4. We claim that for a generic choice of $\alpha\in\mathbb{P}^m$, $\alpha_1\cdot f',\dots,\alpha_k\cdot f'$ is a geometrically regular sequence on $\widetilde{X}$ as well as on each component of $|D_f|$.

In fact, if $\alpha_1\cdot f,\dots,\alpha_j\cdot f$ is a geometrically regular sequence, then $\alpha_1\cdot f,\dots,\alpha_{j+1}\cdot f$ is geometrically regular for α_{j+1} chosen outside a hypersurface in $\mathbb{P}^m$. It follows by induction that $\alpha_1\cdot f',\dots,\alpha_k\cdot f'$ is geometrically regular on an (Zariski) open dense subset $A^k\subset(\mathbb{P}^m)^k$, which proves the claim.

Now consider $\alpha=(\alpha_1,\dots,\alpha_\nu)\in A^\nu$. Since $\pi^*(\alpha_\ell\cdot f)=f^0\,\alpha_\ell\cdot f'$, we have that $[\alpha_\ell\cdot f]=\pi_*\big([D_f]+[\alpha_\ell\cdot f']\big)$, and thus, in light of (3.3),

$$[\alpha_2\cdot f]\wedge[\alpha_1\cdot f]=\pi_*\big([D_f]\wedge[\alpha_1\cdot f']+[\alpha_2\cdot f']\wedge[\alpha_1\cdot f']\big).$$

By induction,

$$[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]= \pi_*\big([D_f]\wedge[\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']+[\alpha_k\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\big), \tag{6.3}$$

and so

$$\mathbf{1}_Z[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]=\pi_*\big([D_f]\wedge[\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\big)\,. \tag{6.4}$$

Here we have used that $\mathbf{1}_{D_f}[\alpha_k\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']$ vanishes by Lemma 2.2, and also that

$$\mathbf{1}_Z\,(\pi_*\tau)=\pi_*(\mathbf{1}_{D_f}\tau). \tag{6.5}$$

For $k=1$, (6.4) should be interpreted as $\pi_*([D_f])$.

In view of Lemma 6.2 and (4.2), we have that

$$\int_{\alpha\in\mathbb{P}^m}[\alpha\cdot f']d\sigma(\alpha)=\omega_f. \tag{6.6}$$

Since all currents involved are positive, we can apply Fubini's theorem and get (6.2) from (6.4) by repeated use of Lemma 6.2, cf., (4.7).

We now prove (6.1). By (4.6) and (4.7),

$$M^{\alpha_\ell\cdot f}=M_0^{\alpha_\ell\cdot f}+M_1^{\alpha_\ell\cdot f}=\mathbf{1}_{\alpha_\ell\cdot f}+[\alpha_\ell\cdot f].$$

Using (5.4), (6.5) and (6.4), we get

$$M^{\alpha_\nu\cdot f}\wedge\cdots\wedge M^{\alpha_1\cdot f}= \pi_*\Big(\sum_{k=0}^{\nu-1}\mathbf{1}_{D_f}[D_f]\wedge[\alpha_k\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']+[\alpha_\nu\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\Big)= \sum_{k=0}^{\nu-1}\mathbf{1}_Z[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]+\pi_*\left([\alpha_\nu\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\right);$$

all other terms vanish by (3.3) or by Lemma 2.2.

Finally, using (6.2) and Lemma 6.2, we conclude that

$$\int_{\alpha=(\alpha_1,\dots,\alpha_\nu)\in(\mathbb{P}^m)^\nu} M^{\alpha_\nu\cdot f}\wedge\cdots\wedge M^{\alpha_1\cdot f} \quad=\quad M^f \;+\; \pi_*(dd^c\log|f'|^2)^\nu \quad=\quad M^f;$$

indeed, $(dd^c\log|f'|^2)^\nu=0$ since $\nu\geq\min(m+1,n+1)$. $\square$

It also follows from the proof of Theorem 6.1 that

$$\mathbf{1}_{X\setminus Z}\int_{\alpha\in(\mathbb{P}^m)^k}[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]=\mathbf{1}_{X\setminus Z}(dd^c\log|f|^2)^k. \tag{6.7}$$

As an immediate consequence of (the proof of) Theorem 6.1, $\ell_x(M^f_k)=\int_\alpha\ell_x(\mathbf{1}_Z[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f])$. In fact, we even have

Theorem 6.3. *Let $f=(f_0,...,f_m)$ be a tuple of holomorphic functions in X, pick $x\in X$, and let $Z=\{f=0\}$. Then for $k\geq 0$, and a generic choice of $\alpha=(\alpha_1,\dots,\alpha_k)\in(\mathbb{P}^m)^k$,*

$$\ell_x\big(\mathbf{1}_Z[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]\big)=\ell_x(M^f_k)\,. \tag{6.8}$$

Here the current on the left hand side of (6.8) *should be interpreted as $\mathbf{1}_Z$ if $k=0$.*

Proof. Choose a normal modification $\pi\colon\widetilde{X}\to X$ such that $\mathcal{J}(f)\cdot\mathcal{O}_{\widetilde{X}}$ is principal; we will use the notation from Section 4. Assume moreover that the pullback of the maximal ideal at x is principal, and let D_ξ and ω_ξ be the corresponding divisor and (negative) Chern form, obtained from a tuple ξ that defines the maximal ideal at x.

By arguments as in the proof of Theorem 6.1 one shows that for a generic choice of $\alpha\in(\mathbb{P}^m)^n$ we have that $\alpha_1\cdot f',\dots,\alpha_n\cdot f'$ is a geometrically regular sequence on $\widetilde{X}$, $|D_f|$, $|D_\xi|$, and on the support of $[D_\xi]\wedge[D_f]$. Choose such an α. For $k=0,1$, Theorem 6.3 follows from (4.6), (4.7) and (6.3); in fact, the currents in (6.8) coincide in these cases.

Let us now assume that $k\geq 2$. We claim that there is a normal current $\mathcal{A}_k$ such that

$$dd^c\mathcal{A}_k=[D_\xi]\wedge\omega_\xi^{n-k-1}\wedge[D_f]\wedge\big(\omega_f^{k-1}-[\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\big). \tag{6.9}$$

For $\ell=1,\cdots,k$, $\log|\alpha_\ell\cdot f'|^2$ defines a singular metric on L^{-1} with first Chern form $[\alpha_\ell\cdot f']$, cf., (4.2), and thus $[\alpha_\ell\cdot f']$ is dd^c-cohomologous to ω_f. More precisely, $c_\ell:=\log(|f'|^2/|\alpha_\ell\cdot f'|^2)$ is a global current on $\widetilde{X}$ and $\omega_f-[\alpha_\ell\cdot f']=dd^c c_\ell$. Now, let

$$\mathcal{A}_k:=[D_\xi]\wedge\omega_\xi^{n-k-1}\wedge[D_f]\wedge\sum_{\ell=1}^{k-1}\omega_f^{k-\ell-1}\wedge c_\ell\wedge[\alpha_{\ell-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\,.$$

Then $\mathcal{A}_k$ is normal. Since $a_\ell\cdot f$ does not vanish identically on any irreducible component of (the support of) $[D_\xi]\wedge[D_f]\wedge[\alpha_{\ell-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']$ it follows from Lemma 3.1 that (6.9) holds.

(The proof of) Proposition 5.2 implies that

$$dd^c\pi_*(\mathcal{A}_k)=\big(\ell_x(M^f_k)-\ell_x(\mathbf{1}_Z[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f])\big)[x]\,; \tag{6.10}$$

here we have used (6.3). On the other hand, $\pi_*\mathcal{A}_k$ is a normal $(n-1,n-1)$-current with support at x, and so it vanishes according to Lemma 2.2. $\square$

7. Segre numbers

Throughout this section, let X be an analytic space of pure dimension n. Given a tuple f of (germs of) holomorphic functions at $x \in X$, let $e_k(x) := \ell_x(M_k^f)$. Theorem 6.3 and Proposition 3.4 assert that $e_k(x) = \text{mult}_x V_k^h$ for a generic Vogel cycle V_k of $\mathcal{J}(f)$; this means that $e_k(x)$ is the kth Segre number of $\mathcal{J}(f)$ as defined by Gaffney-Gassler, [11]. In fact, $e_k(x)$ only depends on the integral closure of $\mathcal{J}(f)$, cf., Proposition 5.1.

Let $e(x) := (e_0(x), e_1(x), \dots, e_n(x))$. We will see that if $\mathcal{J}(f)$ is the pull-back of the radical ideal of a smooth manifold A in some ambient space, then $e(x)$ coincides with Tworzewski's, [28], extended index of intersection. Recall that the *lexicographical order* on $\mathbb{R}^N$ is a total order, defined by $(x_1, \dots, x_N) \leq_{lex} (y_1, \dots, y_N)$ if there is an $1 \leq \ell \leq N$ such that $x_i = y_i$ for $i \leq \ell$ and $x_\ell < y_\ell$. We let $\min_{\text{lex}}$ denote the minimum with respect to the lexicographical order.

Given a tuple of functions $f_0, \dots, f_m$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{P}^m)^n$, we will write $\alpha \cdot f$ for the sequence $\alpha_1 \cdot f, \dots, \alpha_n \cdot f$. Recall (from the introduction) that for a generic choice of α, $\alpha \cdot f$ is a Vogel sequence of the ideal generated by $f_0, \dots, f_m$.

Theorem 7.1. *Let I be a given ideal in $\mathcal{O}_{X,x}$ and let $e(x)$ be the associated Segre numbers. Then*

$$e(x) = \min_{lex} \text{mult}_x V^h,$$

where the $\min_{lex}$ is taken over all Vogel sequences h of ideals with the same integral closure as I.

Moreover, if f is a tuple of generators of I (or any ideal J such that J has the same integral clousure as I) then it suffices to take the $\min_{lex}$ over all Vogel sequences of the form $\alpha \cdot f$, where $\alpha \in (\mathbb{P}^m)^n$.

For the proof we will need the following result; if Z is smooth this is Theorem 3.4 in [28].

Proposition 7.2. *Assume that $(W_j)_{j\in\mathbb{N}}$ and W are subvarieties of X of pure dimension such that $\lim_{j\to\infty}[W_j] = [W]$ as currents on X. Let Z be a fixed subvariety of X, let x be a fixed point in Z, and assume that*

$$\ell_x(\mathbf{1}_Z[W]) \leq \ell_x(\mathbf{1}_Z[W_j]). \tag{7.1}$$

for all j. Then there is a neighborhood U of x in X, in which $\lim_{j\to\infty}(\mathbf{1}_Z[W_j]) = \mathbf{1}_Z[W]$ and $\lim_{j\to\infty}(\mathbf{1}_{X\setminus Z}[W_j]) = \mathbf{1}_{X\setminus Z}[W]$.

Proof. Since the currents $[W_j]$ are positive and locally uniformly bounded, so are the currents $\mathbf{1}_Z[W_j]$. Thus, there is a subsequence of $(\mathbf{1}_Z[W_j])_{j\in\mathbb{N}}$ converging to a positive closed current with support on $W \cap Z$. By Lemma 2.2 this current is the integration current $[V]$ for some cycle V. Since $[W_j] - \mathbf{1}_Z[W_j]$ is positive, so is $[W] - [V] = \lim([W_j] - \mathbf{1}_Z[W_j])$, and since $|V| \subset |Z|$, it follows that

$$[V] = \mathbf{1}_Z[V] \leq \mathbf{1}_Z[W]. \tag{7.2}$$

By (7.1) and semicontinuity, (2.1), we have that

$$\ell_x(\mathbf{1}_Z[W]) \leq \limsup(\ell_x(\mathbf{1}_Z[W_j])) \leq \ell_x([V]) \leq \ell_x(\mathbf{1}_Z[W]).$$

Thus $\ell_x(\mathbf{1}_Z[W]) = \ell_x([V])$, and combined with (7.2) and the fact that V and W are effective cycles, it follows that $[V] = \mathbf{1}_Z[W]$ in some neighborhood of x.

Since each subsequence of $(\mathbf{1}_Z[W_j])_{j\in\mathbb{N}}$ has a subsequence that tends to $\mathbf{1}_Z[W]$, it follows that $\lim_{j\to\infty}(\mathbf{1}_Z[W_j]) = \mathbf{1}_Z[W]$. The last statement follows by complementarity. □

Proof of Theorem 7.1. Since each Vogel sequence h can be realized as $\alpha\cdot f$ for some choice of f and α, it is easy to check that the first statement follows from the second one. Let f be a tuple of generators of I. By Theorem 6.3, $e(x) = \mathrm{mult}_x V^{\alpha\cdot f}$ for almost all α, and thus it is enough to prove that $e(x) \leq_{lex} \min_{lex} \mathrm{mult}_x V^{\alpha\cdot f}$ if α is a Vogel sequence.

Suppose that $e(x) \not\leq_{lex} \min_{lex} \mathrm{mult}_x V^{\alpha\cdot f}$. Then there is an r and a Vogel sequence $\alpha\cdot f$ such that $e_k(x) = \mathrm{mult}_x V_k^{\alpha\cdot f}$ for $k\leq r-1$ but $\mathrm{mult}_x V_r^{\alpha\cdot f} < e_r(x)$. Since $\alpha\cdot f$ is a Vogel sequence of I for a generic choice of α, we can choose $(\alpha^j)_{j\in\mathbb{N}}$ in $(\mathbb{P}^m)^n$ such that $(\alpha^j)_{j\in\mathbb{N}}\to\alpha$ and such that $\alpha^j\cdot f$ is a Vogel sequence of I for each j, and moreover, by Theorem 6.3, such that $\mathrm{mult}_x V^{\alpha^j\cdot f} = e(x)$. It then follows that, for j large enough and $k\leq r$,

$$\ell_x(\mathbf{1}_Z[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]) \;\leq\; e_k(x) \;=\; \ell_x(\mathbf{1}_Z[\alpha_k^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f]), \tag{7.3}$$

We claim that

$$\lim_{j\to\infty}[\alpha_k^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f] = [\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f] \tag{7.4}$$

for $k\leq r$. Clearly (7.4) holds for $k=1$. Assume now that it holds for $k<r$. Then by (7.3) and Proposition 7.2,

$$\lim_{j\to\infty}(\mathbf{1}_{X\setminus Z}[\alpha_k^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f]) = \mathbf{1}_{X\setminus Z}[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]. \tag{7.5}$$

Since $\alpha^j\cdot f$ and $\alpha\cdot f$ are Vogel sequences, the currents in (7.5) intersects properly with $[\alpha_{k+1}^j\cdot f]$ and $[\alpha\cdot f]$, respectively. In light of [7, Chapter 2, Corollary 12.3.4] or [28, Theorem 3.6], (7.4) holds for $k+1$, and the claim follows by induction.

Proposition 7.2 and (7.3) imply that

$$\lim_{j\to\infty}(\mathbf{1}_Z[\alpha_r^j\cdot f]\wedge\cdots\wedge[\alpha_1^j\cdot f]) = \mathbf{1}_Z[\alpha_r\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f]. \tag{7.6}$$

By semicontinuity, (2.1), the Lelong number of the left hand side of (7.6), i.e., $\mathrm{mult}_x V_r^{\alpha\cdot f}$, must be larger than or equal to $e_r(x)$, which gives a contradiction. Hence $\min_{lex}\mathrm{mult}_x V^{\alpha\cdot f} = e(x)$. □

Given a positive closed current v, we define $\ell_x(v) := (\ell_{x0},...,\ell_{xn})$, where ℓ_{xk} denotes the Lelong number at x of the component of v of bidegree (k,k). If v and w positive and closed, we let $v\leq_x w$ mean that $\ell_x(v)\leq_{lex}\ell_x(w)$, and $v =_x w$ means that $\ell_x(v)=\ell_x(w)$. Observe that if h is a Vogel sequence of an ideal $\mathcal{J}_x$, then the zero sets of h and $\mathcal{J}_x$ coincide.

Theorem 7.3. *Let $f_1,\dots,f_s$ be a sequence of elements in $\mathcal{O}_{X,x}$ and let $f=(f_1,\dots,f_s)$. Then*

$$M^f \leq_x M^{f_s}\wedge\ldots\wedge M^{f_1}. \tag{7.7}$$

If $s=n$ and $f_1,\dots,f_s$ is a Vogel sequence of an ideal in $\mathcal{O}_{X,x}$, then the right hand side of (7.7) is the corresponding Vogel cycle.

Proof. Let $Z := \{f = 0\}$. In order to prove (7.7), we proceed by induction on the number s of functions. Clearly (7.7) holds for $s = 1$, so assume that it holds for $s-1$ instead of s. Let $\widetilde{f} := (f_2, \ldots, f_s)$. By (5.7) we may assume that X is irreducible and that f_1 does not vanish identically on X, so that $M^{f_1} = M_1^{f_1} = [f_1]$; otherwise $M^{f_1} = M^0 = \mathbf{1}_X$ and $M^f = M^{\widetilde{f}}$ and we are back in the case $s-1$.

Let $[W] := [f_1]$, and let $i_{W_j} : W_j \hookrightarrow X$ be the irreducible components of $W^{X\setminus Z}$. Theorem 6.3 asserts that for a generic choice of $\alpha \in (\mathbb{P}^{s-2})^{n-1}$, $\alpha \cdot \widetilde{f}$ is a Vogel sequence of $\mathcal{J}(i^*_{W_j}\widetilde{f})$ and $M^{\alpha_{n-1}\cdot\widetilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\widetilde{f}} =_x M^{\widetilde{f}}$ on each W_j, so that $M^{\alpha_{n-1}\cdot\widetilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\widetilde{f}}\wedge[W^{X\setminus Z}] =_x M^{\widetilde{f}}\wedge[W^{X\setminus Z}]$. By the induction hypothesis

$$M^{\widetilde{f}}\wedge[W^{X\setminus Z}] \leq_x M^{f_s}\wedge\cdots\wedge M^{f_2}\wedge[W^{X\setminus Z}].$$

Since $\tilde{f}$ vanishes on Z, by (3.3), we get

$$M^{\alpha_{n-1}\cdot\tilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\tilde{f}}\wedge M^{f_1} \leq_x M^{f_s}\wedge\cdots\wedge M^{f_2}\wedge M^{f_1}. \tag{7.8}$$

For a generic choice of α, the sequence $f_1, \alpha_1\cdot\widetilde{f}, \ldots, \alpha_{n-1}\cdot\widetilde{f}$ is a Vogel sequence of $\mathcal{J}(f)$. Thus, by Theorem 7.1,

$$M^f \leq_x M^{\alpha_{n-1}\cdot\widetilde{f}}\wedge\cdots\wedge M^{\alpha_1\cdot\widetilde{f}}\wedge M^{f_1}. \tag{7.9}$$

Combining (7.8) and (7.9), we get (7.7).

Note that the right hand side of (7.7) is a sum of products of currents $M_0^{f_j} = \mathbf{1}_{f_j}$ and $M_1^{f_j} = [f_j]$. To prove the second statement, assume that $f_1, \ldots, f_n$ is a Vogel sequence of some ideal. Then, in light of Lemma 2.2, $\mathbf{1}_{f_\ell}\cdots\mathbf{1}_{f_{k+1}}[f_k]\wedge\cdots\wedge[f_1] = \mathbf{1}_Z[f_k]\wedge\cdots\wedge[f_1]$, and thus, by (3.3), $[f_{\ell+1}]\wedge\mathbf{1}_{f_\ell}\cdots\mathbf{1}_{f_{k+1}}[f_k]\wedge\cdots\wedge[f_1] = 0$. Hence

$$M^{f_s}\wedge\cdots\wedge M^{f_1} = \sum_{k=0}^{n}\mathbf{1}_{f_n}\cdots\mathbf{1}_{f_{k+1}}[f_k]\wedge\cdots\wedge[f_1] = \sum_{k=0}^{n}\mathbf{1}_Z[f_k]\wedge\cdots\wedge[f_1]; \tag{7.10}$$

here we have used that $[f_n]\wedge\cdots\wedge[f_1]$ has support on Z. Now, Proposition 3.4 asserts that the right hand side of (7.10) is equal to $[V^f]$. □

8. Proof of the generalized King formula (Theorem 1.4)

Let X and $\mathcal{J}$ be as in Theorem 1.4 and let Z be the variety of $\mathcal{J}$. The *(Fulton-MacPherson) distinguished varieties* associated with $\mathcal{J}$ are defined in the following way, cf., [10]: Let $\nu\colon X^+ \to X$ be the normalization of the blow-up of X along $\mathcal{J}$ and let E be the exceptional divisor of ν. Then $Z_j \subset X$ is a distinguished variety if it is the image under ν of an irreducible component of E. Let Z_j^k be the distinguished varieties of codimension k. Also, we define the irreducible components of X contained in Z to be distinguished varieties (of codimension 0).

Let us first consider the case $k = 0$. By (5.7) we may assume that X is irreducible. Then either $\mathcal{J} = (0)$ or Z is a proper subvariety of X. In the first case $M_0^f = M_0^0 = \mathbf{1}_X$ and if h is a Vogel sequence of $\mathcal{J}$, then necessarily $h = (0, \ldots, 0)$ and so $V^h = V_0^h = X$. In the second case $M_0^f = 0$ and if h is a Vogel sequence of $\mathcal{J}$, then $V_0^h = X_0^Z = 0$, since $X \not\subset Z$. It follows that Theorem 1.4 holds for $k = 0$.

Next, consider the case $k \geq 1$. Let $\pi\colon \widetilde{X} \to X$ be a normal modification such that $\mathcal{J}\cdot\mathcal{O}_{\widetilde{X}}$ principal. We use the notation from Section 4, so that $M_k^f = \pi_*([D]\wedge\omega_f^{k-1})$, where $D = D_f$. Moreover, we let D^k denote the components of D that are mapped to sets of codimension k in X. Note that $D = D^p + \ldots + D^n$, if $p = \operatorname{codim} Z$.

If $\ell > k$, then $\pi_*([D^\ell]\wedge\omega_f^{k-1})$ is a positive closed (k,k)-current with support on a variety of codimension $\ell > k$, and hence it must vanish in view of Lemma 2.2. Thus

$$M_k^f = S_k^f + N_k^f, \tag{8.1}$$

where

$$S_k^f = \pi_*\big([D^k]\wedge\omega_f^{k-1}\big), \quad N_k^f = \pi_*\Big(\sum_{\ell<k}\big([D^\ell]\wedge\omega_f^{k-1}\big)\Big). \tag{8.2}$$

Note that $M_k^f = 0$ for $k < p$ and $N_p^f = 0$. We claim that (8.1) is the Siu decomposition of M_k^f, cf., Section 2.2. By Lemma 2.2, S_k^f is the Lelong current of a cycle of codimension k, so it is enough to show that N_k^f does not carry any mass on varieties of codimension k. Let $W \subset X$ be such a variety. By (6.5),

$$\mathbf{1}_W\pi_*([D^\ell]\wedge\omega_f^{k-1}) = \sum_j \pi_*(\mathbf{1}_{\pi^{-1}W}[D_j^\ell]\wedge\omega_f^{k-1}), \tag{8.3}$$

where D_j^ℓ are the irreducible components of D^ℓ. Then $\pi^{-1}(W)$ does not contain any component D_j^ℓ, thus each term in the right hand side of (8.3) vanishes, and thus the claim follows.

Since (8.1) is the Siu decomposition of M_k^f, it follows that S_k^f is independent of $\pi : \widetilde{X} \to X$. If we take π to be the normalization of the blow-up of $\mathcal{J}$, we see that the Z_j^k in (1.7) has to be among the distinguished varieties of $\mathcal{J}$. By Proposition 5.2 (for $r = 1$), the Lelong number of M_k^f is an integer at each point, and since the Lelong number of N_k^f generically vanishes on each Z_j^k, we conclude that the β_j^k and $n_k(x)$ are integers. That $n_k(x)$ is an integer can also be seen directly by copying the proof of Proposition 5.2. Moreover, cf., Remark 5.3, β_j^k and $n_k(x)$ only depend on the integral closure of $\mathcal{J}$ at x.

We shall now see that the coefficients β_j^k of the distinguished varieties are, in fact, ≥ 1, following the proof of Corollary 5.4.19, in [17]. The blow-up $\pi_{\mathcal{J}}\colon \mathrm{Bl}_{\mathcal{J}}X \to X$ of X along $\mathcal{J}$ can be seen as the subvariety of $X \times \mathbb{P}_t^m$ defined by the equations $t_j f_k - t_k f_j = 0$, where $0 \leq j < k \leq m$. Moreover, the line bundle associated with the exceptional divisor is the pullback of $\mathcal{O}_{\mathbb{P}_t^m}(-1)$ from $\mathbb{P}^m$ to $\mathrm{Bl}_{\mathcal{J}}X$, so $\omega_t = dd^c \log|t|^2$ represents minus its first Chern class. This form is strictly positive on the fibers of $\pi_{\mathcal{J}}$, and since the normalization $X^+ \to \mathrm{Bl}_{\mathcal{J}}X$ is a finite map, the pullback ω of ω_t to X^+ remains strictly positive on the fibers of $\nu\colon X^+ \to X$ as well. Let E_j be one of the irreducible component of the exceptional divisor of ν. We conclude that $\nu_*([E_j]\wedge\omega^{k-1})$ is a positive integer times $[Z_j^k]$, where $Z_j^k := \nu(E_j)$. On the other hand, this current is unaffected if we replace ω by ω_f since these two forms are first Chern forms of the same line bundle. It follows that $\beta_j^k \geq 1$.

We saw in (the beginning of) Section 7 that $\ell_x(M_k^f)$ is equal to the k:th Segre number of $\mathcal{J}$ at x. Next, we show that the fixed Vogel components of $\mathcal{J}$ are precisely the S_k^f. Fix a point $x \in X$. As in proof of Theorem 6.3 we can construct, for $k \geq 1$ and a generic $\alpha \in (\mathbb{P}^m)^n$, a normal current $\mathcal{A}_k$ with support on $|D^k|$ such that

$$dd^c\mathcal{A}_k = [D^k]\wedge([\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f'] - \omega_f^{k-1}).$$

Now $\pi_*\mathcal{A}_k$ is a normal $(k-1,k-1)$-current with support on $\bigcup_j Z_j^k$, and thus it vanishes by Lemma 2.2. It follows that $\pi_*\big([D^k]\wedge[\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']\big)=S_k^f$ and hence S_k^f occurs in a generic Vogel cycle at x, meaning that S_k^f is a fixed Vogel cycle. On the other hand, the cycles

$$\pi_*(\sum_{\ell<k}[D^\ell]\wedge[\alpha_{k-1}\cdot f']\wedge\cdots\wedge[\alpha_1\cdot f']) \tag{8.4}$$

must be moving. Indeed, by (the proof of) Theorem 6.1, taking mean values of (8.4) over all $\alpha\in(\mathbb{P}^m)^k$, we get the current N_k^f, which carries no mass on any variety of codimension k, as seen above.

By arguments as in the proof of Theorem 6.3 one shows that for a generic choice of $\alpha\in(\mathbb{P}^m)^k$,

$$\ell_x(\mathbf{1}_{X\setminus Z}[\alpha_k\cdot f]\wedge\cdots\wedge[\alpha_1\cdot f])=\ell_x(\mathbf{1}_{X\setminus Z}(dd^c\log|f|^2)^k) \tag{8.5}$$

cf. (6.7). However, the left hand side of (8.5) is by Proposition 3.4 equal to $m_k(x)$. This concludes the proof of Theorem 1.4.

Remark 8.1. One can see more directly that only the distinguished varieties occur in S_k^f if S_k^f is defined by (8.2) from an arbitrary normal modification $\pi\colon\widetilde{X}\to X$. To begin with, π factors over ν, i.e., there exists a modification $\widetilde{\nu}\colon\widetilde{X}\to X^+$ such that $\pi=\nu\circ\widetilde{\nu}$. If ω_+ is the form associated with $\mathcal{J}\cdot\mathcal{O}_{X^+}$ in X^+, then $\tilde{\nu}^*\omega_+=\omega_f$.

Let D_j^k be an irreducible component of the divisor D^k. Since $|D_j^k|\subset\pi^{-1}(Z)$, it follows that $\widetilde{\nu}(|D_j^k|)$ is contained in one of the components E_j of E in X^+. If $\widetilde{\nu}(|D_j^k|)$ has codimension ≥ 1 in E_j, then $\widetilde{\nu}_*([D_j^k]\wedge\omega_f^{k-1})=(\widetilde{\nu}_*[D_j^k])\wedge\omega_+^{k-1}$ vanishes by Lemma 2.2. Hence $\pi_*([D_j^k]\wedge\omega^{k-1})=\nu_*\tilde{\nu}_*([D_j^k]\wedge\omega^{k-1})$ vanishes unless $\widetilde{\nu}(|D_j^k|)=E_j$, in which case $\pi(|D_j^k|)$ is a distinguished variety. □

9. Proper intersections

In Section 3 we defined the proper intersection of a Cartier divisor H and a pure dimensional cycle Z as the current $[H]\wedge[Z]$. We will now sketch how (the Lelong current of) a general proper intersection can be defined as a limit of "explicit" regular forms.

Assume that Y is a smooth manifold of dimension n, and that $Z_1,\ldots,Z_r$ are analytic cycles in Y of pure codimensions $p_1,\ldots,p_r$, respectively, that intersect properly, i.e., the set-theoretical intersection $V:=\bigcap_j|Z_j|$ has codimension $p:=p_1+\ldots+p_r$. Choose holomorphic tuples f_j such that $M_{p_j}^{f_j}=[Z_j]$. This is always possible semi-globally; for example, if each component of Z_j has multiplicity one, then just take f_j as generators of the radical of Z_j. Then, by Section 5,

$$[Z_r\cdots Z_1]:=M_{p_r}^{f_r}\wedge\cdots\wedge M_{p_1}^{f_1} \tag{9.1}$$

is a closed positive current of bidegree (p,p) with support on V, so it is the Lelong current of a cycle with support on V; we define this cycle, which we denote by $Z_r\cdots Z_1$, to be the *intersection cycle* of $Z_1,\ldots,Z_r$. Notice that, by Proposition 5.2, $Z_r\cdots Z_1$ has integer coefficients. We claim that $[Z_r\cdots Z_1]$ is independent of the choices of f_j and that it is commutative and associative regarded as a product, i.e.,

$[Z_3\cdot(Z_2\cdot Z_1)]=[(Z_3\cdot Z_2)\cdot Z_1]$ provided that all the involved intersections are proper[7]. In fact, since (the right hand side of) (9.1) is defined recursively from the right and $M^{f_1}_{p_1}=[Z_1]$, it is clearly independent of the choice of f_1. The claim then follows if we can show that (9.1) is not affected if we interchange the first and last factor.

Given a tuple of holomorphic functions g, let

$$A_k^{g,\lambda}:=(2\pi i)^{-1}|g|^{2\lambda}\partial\log|g|^2\wedge(dd^c\log|g|^2)^{k-1},$$

so that $dA_k^{g,\lambda}=\bar\partial A_k^{g,\lambda}=M_k^{g,\lambda}$. One can show that $A_k^{g,\lambda}$ has an analytic continuation as a current to $\mathrm{Re}\,\lambda>-\epsilon$, cf. Section 4. Set $A_k^g:=A_k^{g,\lambda}|_{\lambda=0}$. Then $dA_k^g=\bar\partial A_k^g=M_k^g$. Following section 5 we can define

$$\Omega:=A^{f_m}_{p_m}\wedge M^{f_{m-1}}_{p_{m-1}}\wedge\cdots\wedge M^{f_1}_{p_1}-M^{f_1}_{p_1}\wedge M^{f_{m-1}}_{p_{m-1}}\wedge\cdots\wedge M^{f_2}_{p_2}\wedge A^{f_m}_{p_m}. \tag{9.2}$$

The support of Ω is clearly contained in $\bigcap_1^{m-1}|Z_j|$ and since $A^{f_m}_{p_m}$ is a smooth form outside $|Z_m|$, actually $\mathrm{supp}\Omega\subset V$. Now Ω is a pseudomeromorphic current in the sense of [5], and thus, since it has bidegree $(*,p-1)$ and support on a variety of codimension p, it vanishes, see [5, Corollary 2.4]. One can check that the formal Leibniz rule holds for products of the form (9.2). Hence applying $\bar\partial$ to (9.2) we get $M^{f_m}_{p_m}\wedge\cdots\wedge M^{f_1}_{p_1}=M^{f_1}_{p_1}\wedge M^{f_{m-1}}_{p_{m-1}}\wedge\cdots\wedge M^{f_2}_{p_2}\wedge M^{f_m}_{p_m}$, which concludes the proof of the claim.

Remark 9.1. If $Z_1,\dots,Z_r$ intersect properly, one can show that the current valued funtion

$$(\lambda_1,...,\lambda_r)\mapsto M^{f_r,\lambda_r}_{p_r}\wedge\cdots\wedge M^{f_1,\lambda_1}_{p_1},$$

a priori defined when $\mathrm{Re}\,\lambda_j$ are large, can be analytically continued to where $\lambda_j>-\epsilon$, for some $\epsilon>0$, cf. [25]. This gives an alternative proof of that (9.1) is commutative. $\square$

Proposition 9.2. *Assume that $Z_1,...,Z_r$ are analytic cycles in Y of pure dimensions that intersect properly. Then $Z_r\cdots Z_1$ coincides with the (proper) intersection of the product cycle $Z_r\times\cdots\times Z_1$ and the diagonal Δ_Y in $Y\times\cdots\times Y$.*

For the proof we will need the following lemmas, which are of independent interest.

Lemma 9.3. *Let $x\in Y$. Let $h_1,\dots,h_m$ be a regular sequence in $\mathcal{O}_{Y,x}$. Then*

$$M^h=M^{h_m}\wedge\cdots\wedge M^{h_1}=[h_m]\wedge\cdots\wedge[h_1]. \tag{9.3}$$

Remark 9.4. Lemma 9.3 follows with the same proof if Y is an analytic space and $h_1,\dots,h_m$ is a geometrically regular sequence. This version of the lemma implies that if $f_1,\dots,f_m$ are tuples $f_j=f_{j,1},\dots,f_{j,r_j}$ of holomorphic functions on Y such that $f_{1,1},\dots,f_{1,r_1},\dots,f_{m,1},\dots,f_{m,r_m}$ is a geometrically regular sequence, then

$$M^f=M^{f_m}\wedge\cdots\wedge M^{f_1}=M^{f_m}_{r_m}\wedge\cdots\wedge M^{f_1}_{r_1},$$

see [29] and [16] for similar results for residue currents. $\square$

Sketch of proof of Lemma 9.3. For generic choices of α_j, $\alpha_m\cdot h$ is equal to a non-zero constant times h_1 on $\cap_{j=1}^{m-1}\{\alpha_j\cdot h=0\}$. If $\gamma=[\alpha_{m-1}\cdot h]\wedge\ldots\wedge[\alpha_1\cdot h]$ we thus have that $[\alpha_m\cdot h]\wedge\gamma=M_1^{\alpha_m\cdot h}\wedge\gamma=M_1^{h_1}\wedge\gamma=[h_1]\wedge\gamma$. Since the proper intersection is commutative, this is in fact equal to $\gamma\wedge[h_1]$. By induction, $[\alpha_m\cdot h]\wedge\cdots\wedge[\alpha_1\cdot h]=[h_m]\wedge\cdots\wedge[h_1]$ for generic α_k. Now, (9.3) follows from Theorem 6.1. $\square$

[7]Notice that the assumption that $Z_1,\dots,Z_r$ intersect properly, does not imply that the members of a arbitrary subfamily intersect properly.

Lemma 9.5. *Let f be a tuple of holomorphic functions on an analytic space X and let $i : X \hookrightarrow X \times \mathbb{C}_w$ be the trivial embedding. Then $M_0^{(f,w)} = 0$ and*

$$M_{k+1}^{(f,w)} = i_* M_k^f, \quad k \geq 0. \tag{9.4}$$

Moreover, if $W \subset X$ is an analytic variety,

$$M_k^{f\otimes 1} \wedge [W \times \{0\}] = i_*(M_k^f \wedge [W]). \tag{9.5}$$

If we consider X as embedded in some larger analytic space X' and $i\colon X' \to X' \times \mathbb{C}_w$, then (9.4) reads

$$M_{k+1}^{(f,w)} \wedge [X \times \mathbb{C}_w] = i_*(M_k^f \wedge [X]).$$

In particular, if $f = 0$,

$$M_1^w \wedge [X \times \mathbb{C}_w] = i_*[X] = [X \times \{0\}]. \tag{9.6}$$

Proof. Let z be local coordinates on X. Since $(z, w) \mapsto (f(z), w)$ does not vanish identically on $X \times \mathbb{C}_w$, it follows that $M_0^{(f,w)} = 0$.

Let us now prove (9.4). First consider the case when $k = 0$. By (5.7) we may assume that X is irreducible. Then either $f \equiv 0$ on X or the zero set of f has at least codimension 1 in X. In the first case

$$M_1^{(f,w)} = M_1^w = [w] = i_* 1 = i_* M_0^0 = i_* M_0^f.$$

In the latter case the zero set of (f, w) has at least codimension 2 on $X \times \mathbb{C}_w$, and and so both sides of (9.4) vanish by Lemma 2.2. Thus (9.4) holds for $k = 0$.

Next let $\pi\colon \widetilde{X} \to X$ be a smooth modification such that $\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$ is principal and moreover f^0 is locally a monomial; use the notation from Section 4. Observe that then $\pi \otimes \mathrm{id}_w\colon \widetilde{X} \times \mathbb{C}_w \to X \times \mathbb{C}_w$ is a smooth modification with the same properties. It follows that it is enough to prove (9.4) in case X is smooth, $\mathcal{J} = (f^0)$ is principal and f^0 is (in local coordinates) a monomial.

In light of Section 4 we thus have to show that

$$(2\pi i)^{-1} \bar\partial (|f|^2 + |w|^2)^\lambda \wedge \partial \log(|f|^2 + |w|^2) \wedge (dd^c \log(|f|^2 + |w|^2))^k \tag{9.7}$$

is equal to $[f^0] \wedge (dd^c \log |f'|)^{k-1} \wedge [w] = M_k^f$ when $\lambda = 0$. Indeed, at $\lambda = 0$, (9.7) is equal to $M_{k+1}^{(f,w)}$. Note that (9.7) is locally integrable for $\mathrm{Re}\,\lambda > 0$. Moreover, if $\mathrm{Re}\,\lambda < 1$, it is integrable in the w-direction and thus acts on forms that are just bounded in the w-direction. Since $M_{k+1}^{(f,w)}$ is of order zero and $\mathrm{supp} M_{k+1}^{(f,w)} \subset \{w = 0\}$, it follows that to check the action of $M_{k+1}^{(f,w)}$ on test form, it is enough to consider forms $\xi(z, w) = \tilde\xi(z)$, where $\tilde\xi(z)$ is any test form in X. However, after the (generically $1-1$) change of variables $f^0 \omega = w$, so that $|f|^2 + |w|^2 = |f^0|^2(|f'|^2 + |\omega|^2)$, the action of (9.7) on ξ is equal to

$$(2\pi i)^{-1} \int_{z,\omega} \bar\partial |f^0|^{2\lambda} (|f'|^2 + |\omega|^2)^\lambda \wedge \partial \log |f_0|^2 \wedge (dd^c \log(|f'|^2 + |\omega|^2))^k \wedge \tilde\xi(z).$$

Taking $\lambda = 0$, we get

$$\int_z [f_0] \wedge \tilde\xi(z) \wedge \int_\omega (dd^c \log(|f'|^2 + |\omega|^2))^k. \tag{9.8}$$

One can check that the inner integral in (9.8) is equal to $(dd^c\log|f'|^2)^{k-1}$, which proves (9.4). Finally we prove (9.5). Let $j: W\hookrightarrow X$. Then, using (5.6),

$$M^{f\otimes 1}\wedge[W\times\{0\}] = i_*j_*M^{j^*i^*f\otimes 1} = i_*j_*M^{j^*f} = i_*M^f\wedge[W].$$

□

Proof of Proposition 9.2. With no loss of generality we may assume that $Y=\mathbb{C}^n$. Pick coordinates $(z_1,\ldots,z_r)$ on $Y^r=\mathbb{C}^{rn}$. That the Z_j intersect properly implies that $\Delta_{\mathbb{C}^n}$ and $Z_r\times\cdots\times Z_1$ intersect properly in $\mathbb{C}^{rn}$. It follows that $z_2-z_1,\ldots,z_r-z_1$, is a geometrically regular sequence on $Z_r\times\cdots\times Z_1$; indeed, note that $\Delta_{\mathbb{C}^n}=\{z_2-z_1,\ldots,z_r-z_1\}$.

Let $p_j:=\operatorname{codim} Z_j$, and let f_j be holomorphic tuples in $\mathbb{C}^n$ such that $M^{f_j}_{p_j}=[Z_j]$. Then $[Z_r\times\cdots\times Z_1]=M^{f_r(z_r)}_{p_r}\wedge\cdots\wedge M^{f_1(z_1)}_{p_1}$, so that

$$[\Delta_{\mathbb{C}^n}\cdot Z_r\times\cdots\times Z_1] = M^{(z_2-z_1,\ldots,z_r-z_1)}_{(r-1)n}\wedge M^{f_r(z_r)}_{p_r}\wedge\cdots\wedge M^{f_1(z_1)}_{p_1}. \tag{9.9}$$

Introducing new sets of variables $w:=z_1,\eta_2:=z_2-z_1,\ldots,\eta_r:=z_r-z_1$, the right hand side of (9.9) is equal to

$$M^{(\eta_2,\ldots\eta_r)}_{(r-1)n}\wedge M^{f_r(w+\eta_r)}_{p_r}\wedge\cdots\wedge M^{f_2(w+\eta_2)}_{p_2}\wedge M^{f_1(w)}_{p_1}. \tag{9.10}$$

Note that the factors in (9.10) correspond to cycles

$$\Delta_{\mathbb{C}^n},\ \mathbb{C}^{(r-1)n}_{z_1,\ldots,z_{r-1}}\times Z_r,\ \ldots,\ \mathbb{C}^n_{z_1}\times Z_2\times\mathbb{C}^{(r-2)n}_{z_3,\ldots,z_r},\ Z_1\times\mathbb{C}^{(r-1)n}_{z_2,\ldots,z_r}, \tag{9.11}$$

respectively. Since the Z_j intersect properly, the cycles (9.11) intersect properly in $\mathbb{C}^{rn}$, and thus we are free to move the left hand factor in (9.10) to the right. After that we can replace $f_j(w+\eta_j)$ with $f_j(w)$, since they coincide when $\eta=0$. After moving back the factor $M^{\eta_2,\ldots\eta_r}_{(r-1)n}$, (9.10) is equal to

$$M^{(\eta_2,\ldots\eta_r)}_{(r-1)n}\wedge M^{f_r(w)}_{p_r}\wedge\cdots\wedge M^{f_1(w)}_{p_1} = M^{(\eta_2,\ldots\eta_r)}_{(r-1)n}\wedge[Z\times\mathbb{C}^{(r-1)n}_{\eta_2,\ldots,\eta_r}] = [Z\times\{0\}],$$

where $Z=Z_r\cdots Z_1$. Here we have used Lemma 9.3 and (9.6) for the last equality. □

Remark 9.6. Observe that by Lemma 9.3,

$$[\Delta_{\mathbb{C}^n}\cdot Z_r\times\cdots\times Z_1] = \bigwedge_{j=1}^{(r-1)n}[h_j]\wedge[Z_r\times\cdots\times Z_1],$$

where h_j are appropriate hyperplanes. Thus general proper intersections of cycles can be reduced to the proper intersections of cycles and smooth divisors. □

Let us remark that several results in this paper for Bochner-Martinelli currents and their products follow from corresponding results for the residue current, cf., Remarks 4.2, 9.1, and 9.4.

10. The Tworzewski product

We now turn our attention to nonproper intersections. The aim of this section is to reconstruct Tworzewski's intersection cycles of arbitrary analytic cycles by means of currents. Throughout this section Y is a manifold of dimension n. We first define the intersection $A\circ Z$ of a smooth manifold $A\subset Y$, or, more generally, a coherent ideal sheaf $\mathcal{J}$ on Y, and an analytic cycle Z in Y.

As in the introduction we may assume that $\mathcal{J}$ is a sheaf on the analytic space Z. If the generic Vogel cycle has no moving components at any point, then, assuming that f generates $\mathcal{J}$, $N^f = 0$, we have a well-defined intersection cycle (whose Lelong current is) $M^f = S^f$, in view of Theorem 1.4. If there are moving components, the situation is more complicated. It is tempting to define the intersection as the positive current M^f, although M^f is not in general the Lelong current of a cycle and it depends on the choice of generators f of $\mathcal{J}$. This viewpoint will be further exploited in [6].

Tworzewski's idea is that one can anyway associate a cycle, as described in the introduction: We define $A \circ X$ as the unique cycle whose total multiplicity at x is equal to $\sum e_k^{A,Z}(x)$, where $e_k^{A,Z}(x)$ is the Segre numbers of the sheaf $\mathcal{J}(A)$ on Z. It is not clear to us that $A \circ Z$ is always an effective cycle, although we believe it is true.

Note that $A \circ Z = A \cdot Z$ if A and Z intersect properly. Moreover, if $|Z| \subset A$, then $A \circ Z = Z$. If A is a divisor, then $[A \circ Z] = \mathbf{1}_A[Z] + [A]\wedge[Z]$, i.e., $A \circ Z$ consists of the irreducible components of Z contained in A plus the proper intersection of A and the remaining components of Z.

Proposition 10.1. *Let A be a submanifold of Y and Z an analytic cycle in Y. Then*

$$(A \times \mathbb{C}_w) \circ (Z \times \{0\}) = (A \circ Z) \times \{0\} \tag{10.1}$$

and

$$(A \times \{0\}) \circ (Z \times \mathbb{C}_w) = (A \circ Z) \times \{0\}. \tag{10.2}$$

Proof. Let f be a tuple that defines the ideal sheaf $\mathcal{J}(A)$ in Y. Then $f \otimes 1$ defines $\mathcal{J}(A \times \mathbb{C}_w)$ in $Y \times \mathbb{C}_w$ and

$$e_k^{A\times\mathbb{C}_w, Z\times\{0\}}(x) = \ell_x(M_k^{f\otimes 1}\wedge[Z \times \{0\}]) = \ell_x(M_k^f\wedge[Z]) = e_k^{A,Z}(x),$$

where we have used (9.5) for the second equality. This proves (10.1).

Next, note that f, w defines $\mathcal{J}(A \times \{0\})$ in $Y \times \mathbb{C}_w$. Thus, using (5.6) and (9.4) we have

$$e_{k+1}^{A\times\{0\}, Z\times\mathbb{C}_w}(x) = \ell_x(M_{k+1}^{(f,w)}\wedge[Z \times \mathbb{C}_w]) = \ell_x(M_k^f\wedge[Z]) = e_k^{A,Z}(x),$$

which proves (10.2); the shift $k+1$ to k is because the index k in the expression $e_k^{A,Z}$ refers to the codimension on Z. $\square$

Now, let us consider intersections of general analytic cycles. If Z is an analytic cycle in Y, let $i_{\Delta_Y} Z$ denote its image in the diagonal $\Delta_Y \subset Y \times \cdots \times Y$ under

$$i_{\Delta_Y} : Y \hookrightarrow Y \times \cdots \times Y,\ x \mapsto (x, \ldots, x).$$

Given analytic cycles $Z_1, \ldots, Z_r$ in Y, following Tworzewski [28], we define the *Tworzewski product*, $Z_1 \bullet \cdots \bullet Z_r$, by

$$i_{\Delta_Y}\big(Z_1 \bullet \cdots \bullet Z_r\big) := \Delta_Y \circ (Z_1 \times \cdots \times Z_r).$$

This product is clearly commutative in each entry and locally defined.

If $Z_1, \ldots, Z_r$ are cycles in $Y \subset Y'$, where Y is a submanifold of the manifold Y', then the definition of $Z_1 \bullet \cdots \bullet Z_r$ does not depend on whether we consider the Z_j as sitting in Y or in Y'. Let Δ_Y be the diagonal in $Y \times \cdots \times Y$ and let $\Delta_{Y'}$ be the diagonal in $Y' \times \cdots \times Y'$. Then one just has to show that the pullbacks to $Z_1 \times \cdots \times Z_r$ of $\mathcal{J}(\Delta_{Y'})$ and $\mathcal{J}(\Delta_Y)$ coincide. In fact, since the definition is local, we can reduce to the case when $Y = \mathbb{C}^n$ and $Y' = Y \times \mathbb{C}_w = \mathbb{C}^n \times \mathbb{C}_w$. Then what we claim is straightforward.

Finally, consider analytic cycles $Z_1, \dots, Z_r$ in an analytic (not necessarily smooth) space X. A local minimal embedding $i\colon X \hookrightarrow \mathbb{C}^N$ is unique up to a (local) biholomorphism on $\mathbb{C}^N$, and any embedding is like $(i, 0)\colon X \hookrightarrow Y = \mathbb{C}^n \times \mathbb{C}^m_w$. Hence there is a well-defined product $Z_1 \bullet \cdots \bullet Z_r$.

Assume that $Z_1, \dots, Z_r$ are of pure dimensions. Then it is natural to consider the tuple

$$(Z_1 \bullet \cdots \bullet Z_r)_k(x) := e^{\Delta, Z_1 \times \cdots \times Z_r}_{\sum \dim Z_i - k}(x);$$

here the entry corresponding to k corresponds to the Segre number for the "dimension" k. It is clear that $(Z_1, ..., Z_r) \mapsto (Z_1 \bullet \cdots \bullet Z_r)_k(x)$ is monotonous in each Z_j-entry. In the same way we introduce, for any smooth submanifold $A \subset Y$ and any cycle Z in Y, the tuple $(A \circ Z)_k(x) := e^{A,Z}_{\dim Z - k}(x)$.

If $Z_1, \dots, Z_r$ intersect properly, then $Z_1 \bullet \cdots \bullet Z_r$ coincides with the classical proper intersection $Z_1 \cdots Z_r$ as defined in Section 9; this follows immediately from Proposition 9.2.

Proposition 10.2. *Assume that A is a smooth submanifold of Y and that Z is an analytic cycle in Y. Then $A \bullet Z = A \circ Z$.*

Proof. Assume without loss of generality that $Y = \mathbb{C}^n$. Choose local coordinates $z = (z', z'')$ on $\mathbb{C}^n$ so that $A = \{z' = 0\}$, and local coordinates (z, w) on $\mathbb{C}^n \times \mathbb{C}^n$. We will show that

$$(A \bullet Z)_j(x) = e^{\Delta, Z \times A}_{\dim A + \dim Z - j}(x) = \ell_x(M^{z-w}_{\dim A + \dim Z - j} \wedge [Z \times A]) \tag{10.3}$$

coincides with

$$(A \circ Z)_j(x) = e^{A,Z}_{\dim Z - j}(x) = \ell_x(M^{z'}_{\dim Z - j} \wedge [Z]). \tag{10.4}$$

Note that $M^{z-w}_k \wedge [Z \times A] = M^{(z', z'' - w'')}_k \wedge [Z \times \{w' = 0\}]$. Let $z', z'', w', \eta'' := z'' - w''$ be new coordinates on $\mathbb{C}^n \times \mathbb{C}^n$. Then (9.4) implies that $M^{(z', \eta'')}_{k + \dim A} \wedge [Z \times \{w' = 0\}] = i_* M^{z'}_k \wedge [Z \times \{w' = 0\}]$, where $i : \mathbb{C}^{2n - \dim A}_{z', z'', w'} \hookrightarrow \mathbb{C}^{2n}_{z', z'', w', \eta''}$. Moreover, by (9.5), $M^{z'}_k \wedge [Z \times \{w' = 0\}] = j_* M^{z'}_k \wedge [Z]$, where $j : \mathbb{C}^n_{z', z''} \hookrightarrow \mathbb{C}^{2n - \dim A}_{z', z'', w'}$. Hence

$$M^{z-w}_{\dim A + \dim Z - j} \wedge [Z \times A] = i_* j_* M^{z'}_{\dim Z - j} \wedge [Z]$$

and thus (10.3) is equal to (10.4). $\square$

Corollary 10.3. *Assume that A and B are smooth analytic subsets in Y. Then*
(i) $A \circ B = A \bullet B = B \circ A$.
(ii) *If $A \subset B$, then $A \bullet B = A$.*

The first statement follows since $Z_1 \bullet \cdots \bullet Zr$ is commutative. For the second statement we have used that $A \circ Z = Z$ if $|Z| \subset A$.

Corollary 10.4. *For any point $a \in Y$, $\{a\} \bullet Z = \mathrm{mult}_a(Z)\{a\}$.*

Indeed if a is a point, then $\{a\} \bullet Z = \{a\} \circ Z = \ell_a[Z]\{a\} = \mathrm{mult}_a(Z)\{a\}$.

Most of the results in this section, or similar statements, can be found in either [28] or [23], but with other proofs. However, we have not found Proposition 10.2 in the literature.

11. Examples

The following simple lemma is useful for computations.

Lemma 11.1. *Let X and X' be two analytic spaces, let $\tau\colon X'\to X$ be a holomorphic map, and let f be a tuple of holomorphic functions on X. Assume that τ is proper, surjective, and generically r to 1. Then*

$$rM_k^f=\tau_*M_k^{\tau^*f}. \tag{11.1}$$

Moreover, if ξ is a tuple that defines the maximal ideal at $x\in X$, then the Segre numbers at x associated with $\mathcal{J}=\mathcal{J}(f)$ on X are given by

$$e_k(x)=\frac{1}{r}\int_{X'}M_{n-k}^{\tau^*\xi}\wedge M_k^{\tau^*f}, \tag{11.2}$$

where $n=\dim X$.

Proof. Since $\tau^*M_k^{f,\lambda}=M_k^{\tau^*f,\lambda}$ if $\text{Re}\,\lambda>>0$, we have that then

$$\int_X M_k^{f,\lambda}\wedge\psi=\frac{1}{r}\int_{X'}M_k^{\tau^*f,\lambda}\wedge\tau^*\psi$$

for test forms ψ. Taking analytic continuations to $\lambda=0$, we get (11.1). In view of Proposition 5.4, we have

$$e_k(x)=\ell_xM_k^f=\int_X M_{n-k}^{\xi,\lambda}\wedge M_k^{f,\lambda^2}\big|_{\lambda=0}=$$
$$\frac{1}{r}\int_{X'}M_{n-k}^{\tau^*\xi,\lambda}\wedge M_k^{\tau^*f,\lambda^2}\big|_{\lambda=0}=\frac{1}{r}\int_{X'}M_{n-k}^{\tau^*\xi}\wedge M_k^{\tau^*f}.$$

□

In particular,

$$\text{mult}_xX=\int_X M_n^{\xi}=\frac{1}{r}\int_{X'}M_n^{\tau^*\xi}. \tag{11.3}$$

Example 11.2. Let r,s be relatively prime and consider the cusp $Z=\{z_1^r-z_2^s=0\}$ in $\mathbb{C}^2_z$. Since we have the parametrization $\tau\colon t\mapsto(t^s,t^r)$ of Z, using (11.3) we get

$$\text{mult}_0Z=\int_Z M_1^{(z_1,z_2)}=\int_{\mathbb{C}_t}M_1^{(t^s,t^r)}=\int_{\mathbb{C}_t}M_1^{t^{\min(s,r)}}=\min(s,r).$$

□

Example 11.3. Let $Z=\{x_2x_1^m-x_3^2=0\}\subset\mathbb{C}^3_x$, where $m\geq 1$, and let $A=\{x_2=x_3=0\}$. Since A is smooth and contained in Z, and Z is smooth outside the origin in $\mathbb{C}^3$, we must have that $A\bullet Z=A$ outside the origin, cf., Corollary 10.3. Thus $A\bullet Z=A+a\{0\}$.

To determine a let us consider a generic Vogel sequence of $\mathcal{J}(A)$ on Z at the origin and let us compute the corresponding Vogel cycle. Let H_1 be a generic hyperplane that contains A, defined by $h_1=\alpha x_2-x_3$. Then $Z_1=H_1\cdot Z$ is the curve $\{x_2x_1^m-(\alpha x_2)^2=0,\alpha x_2-x_3=0\}$. It follows that Z_1^A is equal to A, whereas $Z_1^{Z\setminus A}$ is the curve $\{x_1^m-\alpha^2x_2=0,\alpha x_2-x_3=0\}$. Next, let $h_2=\beta x_2-x_3$. Then $Z_2=H_2\cdot Z_1^{Z\setminus A}$ is the cycle $\{x_3=x_2=0,\ x_1^m=0\}$. Since its support is contained in A, it is equal to Z_2^A and it has order m at the origin. We conclude that $[V^h]=[A]+m[\{0\}]$. In particular $a=m$.

We can also compute a, which is the second Segre number $e_2^{A,Z}(0)$ of $\mathcal{J}(A)$ on Z, as the Lelong number of a certain Bochner-Martinelli current. Notice that $\tau:(t_1,t_2)\mapsto(t_1^2,t_2^2,t_1^m t_2)$ is a surjective, generically $2-1$, mapping $\mathbb{C}_t^2\to Z$. If $i\colon Z\hookrightarrow\mathbb{C}^3$ is the identity map we have by Lemma 11.1 that

$$e_2^{A,Z}(0)=\ell_x(M_2^{(x_2,x_3)}\wedge[Z])=\int_{\mathbb{C}^3}M_0^{(x_1,x_2,x_3)}\wedge M_2^{(x_2,x_3)}\wedge[x_2x_1^m-x_3^2]=$$
$$\int_Z M_0^{(i^*x_1,i^*x_2,i^*x_3)}\wedge M_2^{(i^*x_2,i^*x_3)}=\frac{1}{2}\int_{\mathbb{C}^2_{t_1,t_2}}M_0^{(t_1^2,t_2^2,t_1^m t_2)}\wedge M_2^{(t_2^2,t_1^m t_2)}.$$

According to Theorem 6.1, $M_2^{(t_2^2,t_1^m t_2)}$ is the mean value of all

$$[(\beta t_2-t_1^m)t_2]\wedge[(\alpha t_2-t_1^m)t_2]$$

for generic choices of $\alpha,\beta\in\mathbb{C}$. For generic α,β, using the new variables $v_1=t_1,v_2=\alpha t_2-t_1^m$, we get

$$[\beta t_2-t_1^m]\wedge[\alpha t_2-t_1^m]=[\beta' v_2-\alpha' v_1^m]\wedge[v_2]=[v_1^m]\wedge[v_2]=m[\{0\}]$$

for some $\alpha',\beta'\in\mathbb{C}$. Since $[(\alpha t_2-t_1^m)t_2]=[t_2]+[\alpha t_2-t_1^m]$, by (3.4) and (3.3), we thus have that

$$[(\beta t_2-t_1^m)t_2]\wedge[(\alpha t_2-t_1^m)t_2]=\big([\beta t_2-t_1^m]+[t_2]\big)\wedge[\alpha t_2-t_1^m]=2m[\{0\}].$$

Now, $M_0^{(t_1^2,t_2^2,t_1^m t_2)}=\mathbf{1}_{(0,0)}$, so $e_2^{A,Z}(0)=m$ as expected. $\square$

The next example shows that the Tworzewski product is not associative in general.

Example 11.4. Let A and Z be as in Example 11.3. According to Corollaries 10.3 and 10.4, $(\{0\}\bullet A)\bullet Z=\{0\}\bullet Z=2\{0\}$, since $\text{mult}_0 Z=2$, which can be seen using (11.3). However, by Corollary 10.3 and Example 11.3,

$$\{0\}\bullet(A\bullet Z)=\{0\}\bullet(A+m\{0\})=(m+1)\{0\},$$

so that $(\{0\}\bullet A)\bullet Z\neq\{0\}\bullet(A\bullet Z)$ unless $m=1$. Notice that $m+1$ is the total degree of $A\bullet Z$ at 0.

Let us also compute $\{0\}\bullet A\bullet Z$. Since $\{0\}\times A\times Z\subset(\mathbb{C}^3)^3_{x,y,z}$ has dimension 3, $\{0\}\bullet A\bullet Z=\alpha\{0\}$, where α is the Lelong number of

$$M_3^{(y-x,z-x)}\wedge[\{0\}\times A\times Z]=M_3^{(y-x,z-x)}\wedge[\{x=0,y_2=y_3=0\}\times Z]$$
$$=M_3^{(y_1,z_1,z_2,z_3)}\wedge[\mathbb{C}_{y_1}\times Z(z)]=M_2^z\wedge[Z];$$

here we have used Lemma 9.5 for the last two equalities. Now, by Corollary 10.4, $M_2^z\wedge[Z]=(\text{mult}_0 Z)[\{0\}]=2[\{0\}]$, so that $\alpha=2$. $\square$

Example 11.5. Consider the tuple $f=t_3(t_1,t_2,t_3)=t_3t$ in $X=\mathbb{C}_t^3$, with zero set $Z=\{t_3=0\}$. Let h be a Vogel sequence of $\mathcal{J}(f)$ at 0 of the form $h_1=\alpha_1\cdot f,\dots,h_3=\alpha_3\cdot f$. Let us compute the corresponding Vogel cycle V^h. First note that $X_0^{X\setminus Z}=X_0=X$. Thus, by Proposition 3.4,

$$[X_1]=M_1^{t(\alpha_1\cdot t)}=[t_3]+[\alpha_1\cdot t]=:[X_1^Z]+[X_1^{X\setminus Z}].$$

Furthermore, using (3.4) and (3.3), we obtain

$$[X_2]=M_1^{t_3(\alpha_2\cdot t)}\wedge M_1^{t_3(\alpha_1\cdot t)}=[t_3]\wedge[\alpha_1\cdot t]+[\alpha_2\cdot t]\wedge[\alpha_1\cdot t]=:[X_2^Z]+[X_2^{X\setminus Z}].$$

and

$[X_3] = M_1^{t_3(\alpha_3\cdot t)}\wedge M_1^{t_3(\alpha_2\cdot t)}\wedge M_1^{t(\alpha_1\cdot t)} = ([t_3]+[\alpha_3\cdot t])\wedge[\alpha_2\cdot t]\wedge[\alpha_1\cdot t] = 2[\{0\}] =: [X_3^Z]$,

for a generic α_j. Hence

$$[V^h] = [V_1^h] + [V_2^h] + [V_3^h] = [t_3] + [t_3]\wedge[\alpha_1\cdot t] + 2[\{0\}]$$

and, in particular, $e_0(0)=0$, $e_1(0)=1$ $e_2(0)=1$, and $e_3(0)=2$. Observe that V_1^h and V_3^h are fixed, whereas V_2^h is moving. A computation, using Theorem 6.1 and Lemma 6.2, yields

$$M_0^f = 0,\ M_1^f = [t_3],\ M_2^f = [t_3]\wedge dd^c\log(|t_1|^2+|t_2|^2),\ M_3^f = 2[\{0\}].$$

□

Example 11.6. The mapping $\gamma\colon \mathbb{C}_t^3 \to \mathbb{C}_z^6$ defined by

$$(t_1,t_2,t_3)\mapsto \gamma(t) = (t_1,t_2,t_3t_1,t_3t_2,t_3^2,t_3^3)$$

is proper and injective, so that $Z := \gamma(\mathbb{C}^3)$ is a subvariety of $\mathbb{C}^6$. Let $A=\{z_3=z_4=z_5=z_6=0\}$. Then A is smooth and contained in Z and, since Z is smooth outside 0, it follows from Corollary 10.3 that $A\bullet Z = A+\alpha\{0\}$. Here $\alpha = e_2^{A,Z}(0)+e_3^{A,Z}(0)$ since $e_0^{A,Z}(0)=0$ and $e_1^{A,Z}(0)$ is precisely the multiplicity of A at 0. By Lemma 11.1,

$$e_k^{A,Z}(0) = \int_{\mathbb{C}_t^3} M_{3-k}^{\gamma^*z}\wedge M_k^{\gamma^*(z_3,z_4,z_5,z_6)} = \int_{\mathbb{C}_t^3} M_{3-k}^{(t_1,t_2,t_3^2)}\wedge M_k^{t_3(t_1,t_2,t_3)},$$

where we have used that the ideal γ^*z is generated by t_1,t_2,t_3^2, that the ideal $\gamma^*\mathcal{J}(A)$ is generated by $t_3(t_1,t_2,t_3)$, and that $e_k^{A,Z}(0)$ only depends on the ideals.

Thus in light of Example 11.5,

$$e_2^{A,Z}(0) = \int_{\mathbb{C}_t^3} M_1^{(t_1,t_2,t_3^2)}\wedge[t_3]\wedge dd^c\log|t|^2 = \int_{\mathbb{C}_t^3} M_1^{(t_1,t_2)}\wedge[t_3]\wedge dd^c\log|t|^2$$
$$= \int_{\mathbb{C}^2_{(t_1,t_2)}} M_1^{t_1,t_2}\wedge dd^c\log|t'|^2 = \ell_0(dd^c\log|t'|^2) = 1,$$

where $t'=(t_1,t_2)$. To see the last equality, by Theorem 6.3, one can replace $dd^c\log|t|^2$ with a generic hyperplane $[\alpha\cdot t]$. In a similar way one concludes that $e_3^{A,Z}(0)=2$. Hence $A\bullet Z = A+3\{0\}$. □

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