On some properties of local determinantal representations

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Abstract. Matrices of locally analytic functions in two variables correspond to the maximal Cohen-Macaulay modules over plane curve singularities and to the local determinantal representations.

We study particularly nice classes of these representations: the class of weakly maximal representations (corresponding to the Ulrich-maximal modules) and a more restricted class of maximal representations.

When the curve singularity is locally reducible we obtain various decomposability criteria for weakly-maximal/maximal determinantal representations. Namely, the criteria for the corresponding module to be decomposable or an extension.

Further we relate the weak maximality/maximality to the minimal modification of the curve such that the appropriate lifting of the module becomes free.

Contents

1. Introduction

Many of the results in §3 are valid over arbitrary algebraically closed field of zero characteristic, but to use the approximation theorems we restrict to the complex case.

In this paper by a plane curve/hypersurface we always mean the germ at the singular point, which is assumed to be the origin $0 \in \mathbb{C}^n$.

1.1. **Setup.** Let $\mathcal{M}$ be a $d \times d$ matrix with the entries in $\mathbb{C}\{x_1, .., x_n\}$ or $\mathbb{C}[[x_1, .., x_n]]$, the rings of locally analytic or formal power series. We always assume $f = \det(\mathcal{M}) \not\equiv 0$ and $d > 1$. In addition we usually assume that the matrix vanishes at the origin, $\mathcal{M}|_0 = \mathbb{0}$.

Such objects were studied classically in various fields. In geometry they are called local/global determinantal representations of the (complex, possibly singular and non-reduced) hypersurface $\{\det(\mathcal{M}) = 0\}$. In singularity theory and linear algebra they are called matrix functions/matrix families. In algebra they are known as matrix factorizations and maximal Cohen Macaulay modules over singular hypersurfaces. For a short mixture of classical results cf. §1.3.

The representations are studied up to the *local equivalence* $\mathcal{M} \sim A\mathcal{M}B$ for $A, B \in GL(d, \mathbb{C}\{x_1, .., x_n\})$. This equivalence preserves the embedded hypersurface pointwise. Restrict $\mathcal{M}$ to the hypersurface $V = \{\det \mathcal{M} = 0\} \subset \mathbb{C}^n$.

Note that at each point $corank\mathcal{M}|_{pt} \leq mult(V, pt)$ (cf. property 2.15). This motivates the following

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Definition 1.1. *The representation is called* <u>*weakly maximal*</u> *at the point* $0 \in V \subset \mathbb{C}^n$, *or* <u>*Ulrich-maximal*</u> [Ulrich84], *if* $corank\mathcal{M}|_0 = mult(V,0)$. *The representation is called weakly maximal near the point* $0 \in V \subset \mathbb{C}^n$ *if it is weakly maximal in some neighborhood of* $0 \in \mathbb{C}^n$.

The representation of a reduced plane curve $(C,0)\subset(\mathbb{C}^2,0)$ *is called* <u>*maximal*</u> *at the point if any entry of the adjoint matrix* $\mathcal{M}^\vee$ *belongs to the adjoint ideal* $Adj_{(C,0)}$ *(cf.§2.2.3).*

So, any determinantal representation of a reduced smooth curve is weakly maximal and maximal. Weak maximality at the point and weak maximality near the point coincide for reduced curves.

Maximality for non-reduced curves is defines as follows. Let $f = \prod f_i^{p_i}$ be the decomposition into branches (with multiplicities). Then $\mathcal{M}$ is called maximal if any entry of the adjoint matrix $\mathcal{M}^\vee$ is of the form $g \prod f_i^{p_i-1}$ for g in the adjoint ideal of the reduced curve (C^{red}, pt) (this definition is motivated by the property 2.14). Note that weak maximality and maximality are invariant with respect to the local equivalence.

In this paper we study the properties of weakly maximal and maximal determinantal representations. They happen to be particularly important in applications. On the other hand such determinantal representations are easy to work with. Maximality appears to be the suitable strengthening of weak maximality. For reduced curves with only ordinary multiple points (e.g. nodes), maximality coincides with weak maximality.

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1.2. **Contents of the paper.** As the paper aims for broad audience a brief intro and overview are given in §1.3. Throughout the paper we repeat some known facts.

In §2 we recall some notions and fix the notations. In particular we discuss singularities of plane curves, sheaves on singular curves, conductor and adjoint ideals.

In §3 we study various notions of local decomposability. Suppose $(C,0)$ is locally reducible: $(C_1,0) \cup (C_2,0)$, here $(C_i,0)$ can be further reducible or non-reduced, but without common components (i.e. the intersection $C_1 \cap C_2$ is finite). A natural question is whether $\mathcal{M}$ is locally decomposable: $\mathcal{M} \sim \mathcal{M}_1 \oplus \mathcal{M}_2$, here $det(\mathcal{M}_i)$ defines C_i. Or at least is locally equivalent to an upper-block-triangular form $\begin{pmatrix} \mathcal{M}_1 & * \\ \mathbb{0} & \mathcal{M}_2 \end{pmatrix}$. In other words, whether the kernel E_C of $\mathcal{M}|_C$, a module over the local ring $\mathcal{O}_{(C,0)}$, is decomposable. Or at least an extension $0 \to E_1 \to E \to E_2 \to 0$.

An arbitrary determinantal representation cannot be brought to an upper-block-triangular form, even when the singularity $(C,0)$ is an ordinary multiple point. However in theorem 3.5 we prove that E is an extension for weakly maximal representations. Namely, $\mathcal{M}$ is locally equivalent to an upper block triangular matrix with blocks of a very special form. If the germs C_1, C_2 have no common tangent lines then E is decomposable for any weakly maximal representation, theorem 3.1.

These results reduce the study of weakly maximal determinantal representations of $(C,0)$ to those of the tangent components, i.e. the union of all the branches with the common tangent line.

As a simple application in §3.3 we study weakly maximal determinantal representations of the singularity $\prod(y + \alpha_i x^{l_i})$. In particular in examples 3.3 and 3.6 we re-derive some of the results of [Bruce-Tari04].

In §3.5 various aspects of local maximality are studied. Maximality implies weak maximality and complete local decomposability (theorem 3.10). Hence a curve singularity with only smooth branches has *only one* maximal determinantal representation.

Conversely, weak maximality and local decomposability reduce the check of maximality to per-branch consideration. For example, for curves with smooth branches weak maximality and local decomposability imply maximality. Example 3.11 gives weakly maximal but non-maximal determinantal representation for a singular branch.

Maximal determinantal representations are related to the curve normalization in the following way. For a given $(C,0)$ and the kernel module E of $\mathcal{M}$, a natural question is the minimal modification $C' \stackrel{\nu}{\to} C$ such that the lifting $\nu^*(E)/Torsion$ is locally free. In lemma 4.1 we prove that $\mathcal{M}$ is maximal iff the minimal lifting is normalization. Some related results are obtained too.

1.3. **A brief introduction and overview.** We recall here the local aspects of determinantal representations only, for some reference on the global aspect cf.[Kerner-Vinnikov2010].

1.3.1. *A view from singularities.* The modern study started probably from the seminal paper [Arnol'd1971] (cf. the citing papers) and is mentioned in [Arnol'd-problems, 1975-26,pg.23]. It seems the main type of questions considered was to write down the miniversal deformation of a constant matrix for various equivalences (i.e. to write a normal form for a linear family of matrices), cf. e.g. [Tannenbaum81, Chapter5], [Khabbaz-Stengle70] or [Lancaster-Rodman05].

Various examples of miniversal deformations were also considered by van-Straten's students, e.g. [Meyer-Brandis98].

Some recent works (for the arbitrary number of variables) from the singularity side are: [Bruce-Tari04, Bruce-Goryunov-Zakalyukin02, Goryunov-Zakalyukin03, Goryunov-Mond05], in particular [Bruce-Tari04] in the introduction describe the applications of determinantal representations.

1.3.2. *A view from algebra.* The case of matrix family depending on one variable is trivial (e.g. [Gantmakher-book, chapter VI]): $\mathcal{M}(x)$ is locally equivalent to the diagonal matrix $\begin{pmatrix} x^{k_1} & 0 & .. & \\ .. & .. & .. & .. \\ 0 & .. & 0 & x^{k_n} \end{pmatrix}$ where $k_i \le k_{i+1}$. In more modern language: any $k\{x\}$ module is the direct sum of cyclic modules. For an introduction to the case of more variables see [Yoshino-book].

Let $\mathcal{M}$ be a local determinantal representation of $f \in k\{x_1..x_n\}$, or $f \in k[[x_1..x_n]]$. Let E be its kernel spanned by the columns of $\mathcal{M}^\vee$ as a module over $R := k\{x_1..x_n\}/(f)$ or $k[[x_1..x_n]]/(f)$. Then E has a resolution by free R modules of period two:

$$... \stackrel{\mathcal{M}^\vee}{\to} R^{\oplus d} \stackrel{\mathcal{M}}{\to} R^{\oplus d} \stackrel{\mathcal{M}^\vee}{\to} R^{\oplus d} \to E \to 0 \qquad (1)$$

One can show that $depth(E) = n-1 = dim R$ hence it is a maximally Cohen Macaulay (MCM) module.

Vice-versa [Eisenbud80]: any maximally Cohen Macaulay (MCM) module E over the ring R as above has a periodic resolution:

$$... \stackrel{\mathcal{M}_1}{\to} R^{\oplus d} \stackrel{\mathcal{M}_2}{\to} R^{\oplus d} \stackrel{\mathcal{M}_1}{\to} R^{\oplus d} \to E \to 0 \qquad (2)$$

corresponding to the matrix factorization: $f\mathbb{1} = \mathcal{M}_1\mathcal{M}_2$. Note that here the dimensions of $\{\mathcal{M}_i\}$ are not fixed so in general $\mathcal{M}_1, \mathcal{M}_2$ are determinantal representations of $\prod f_i^{p_i}$, for $f_{reduced} = \prod f_i$.

Note, that the module E_R determines the resolution, and hence the determinantal representation, up to isomorphisms. Indeed, the isomorphism $E \approx E'$ of two R modules is naturally extended to the isomorphisms of resolutions:

$$\begin{array}{ccccccccc} ... \overset{A_1}{\to} & R^{\oplus d} & \overset{A_2}{\to} & R^{\oplus d} & \overset{A_1}{\to} & R^{\oplus d} & \to & E & \to 0 \\ & \approx & & \approx & & \approx & & \approx & \\ ... \overset{A'_1}{\to} & R^{\oplus d} & \overset{A'_2}{\to} & R^{\oplus d} & \overset{A'_1}{\to} & R^{\oplus d} & \to & E' & \to 0 \end{array} \tag{3}$$

Suppose f above is homogeneous, of degree d. Then, by [Backelin-Herzog-Sanders88], f admits a matrix factorization in linear matrices: $f\mathbb{1} = \mathrm{A_1...A_d}$, i.e. all the entries of $\{A_i\}$ are homogeneous linear.

For an MCM module E over the local ring R of a hypersurface the minimal number of generators of E is not bigger than $multiplicity(R) \times rank(E)$ [Ulrich84, §3]. Modules for which the equality occurs are called Ulrich's modules, in our case they are precisely the weakly maximal determinantal representations, where $multiplicity(R)$ is taken as the multiplicity for reduced hypersurface. For an arbitrary algebraic hypersurface Ulrich modules exist [Backelin-Herzog89, Theorem 1]. Hence, for any $f \in \mathbb{C}\{x_1, .., x_n\}$ its multiples f^p, for p high enough, have weakly maximal determinantal representations.

MCM modules have been classified in many particular cases. A ring is called of finite/tame CM-representation type if it has a finite/countable number of indecomposable MCM's up to isomorphism.

- A series of papers resulted in [Buchweitz-Greuel-Schreyer87]: a hypersurface ring is of finite CM-representation type iff it is a ring of simple (ADE) singularity.
- For the singularity $\sum_{i=1}^n x_i^3$ the MCM modules and their factorizations were classified in [LazaPfisterPopescu02] for n=3 and in [Baciu-Ene-Pfister-Popescu05] for n=4.
- The MCM modules over the ring $k[[x, y]]/(x^n)$ were classified in [Ene-Popescu08].
- The MCM modules over the ring $k[[x, y]]/(xy^2)$ were classified in [Buchweitz-Greuel-Schreyer87]
- The MCM modules over Thom-Sebastiani rings, i.e. $k[[x_1..x_k, y_1..y_n]]/(f(x) \oplus g(y))$, were studied in [Herzog-Popescu97]. In particular the modules over $k[[x, y]]/(x^a + y^3)$ were classified.

Let $\mathcal{M} \in Mat(d \times d)$ be a matrix of d^2 *indeterminates*, providing the polynomial factorization $\mathcal{M}\mathcal{M}^\vee = det(\mathcal{M})\mathbb{1} = \mathcal{M}^\vee\mathcal{M}$. Are $(\mathcal{M}, \mathcal{M}^\vee)$ and $(\mathcal{M}^T, \mathcal{M}^{\vee T})$ the only polynomial factorizations (up to equivalence)? The answer is yes for d odd and no for d even [Buchweitz-Leuschke07].

2. Preliminaries and notations

2.1. **On the relevant rings.** When studying the local determinantal representations several rings appear naturally: the ring of polynomials (or its localization at the the origin $\mathbb{C}[x_1..x_n]_{(m)}$), the ring of locally analytic functions $\mathbb{C}\{x_1..x_n\}$ and the ring of formal power series $\mathbb{C}[[x_1..x_n]]$.

The ring $\mathbb{C}\{x_1..x_n\}$ comes inevitably from the local equivalence. For example, when bringing a determinantal representation to some useful form (block-diagonal, upper-block-triangular etc.) the result is almost never a matrix of polynomials. Thus for local questions we work usually in $\mathbb{C}\{x_1..x_n\}$.

However in some inductive arguments the formal power series $\mathbb{C}[[x_1..x_n]]$ appear, fortunately just as an intermediate step. The final result (the determinantal representation and the matrices of equivalence $GL(d) \times GL(d)$) are always locally analytic due to the approximation theorems:

Theorem 2.1. [Artin68][GLS-book1, pg.32] *Let $\underline{x}, \underline{y}$ be the multi-variables and $f_1, .., f_k \in \mathbb{C}\{\underline{x}, \underline{y}\}$ the locally analytic series. Suppose there exist formal power series $\bar{Y}_1(\underline{x}), .., \bar{Y}_l(\underline{x}) \in$*

$\mathbb{C}[[\underline{x}]]$ *solving the equations:*

$$f_i\Big(\underline{x}, \bar{Y}_1(\underline{x}), .., \bar{Y}_l(\underline{x})\Big) \equiv 0, \ i = 1..k$$

Then there exists a locally analytic solution $Y_1(\underline{x}), .., Y_l(\underline{x}) \in \mathbb{C}\{\underline{x}\}$*:*

$$f_i\Big(\underline{x}, Y_1(\underline{x}), .., Y_l(\underline{x})\Big) \equiv 0, \ i = 1..k$$

Theorem 2.2. [Pfister-Popescu75] *Let* $F_1 = 0, F_2 = 0, ..F_k = 0$ *be a system of polynomial equations over a complete local ring* (R, m). *The system has a solution in* R *iff it has a solution in* R/m^N *for any* N.

These results are used as follows. Suppose, for the function decomposition $f = \prod f_i$ a given matrix $M \in Mat(d \times d, \mathcal{O})$ is to be transformed to a block-diagonal (or some other) specified form. So we consider the system of equations:

$$(4) \quad (\mathbb{1} + \mathrm{A})\mathrm{M}(\mathbb{1} + \mathrm{B}) = \begin{pmatrix} \mathcal{M}_1 & \mathbb{0} & .. & & \mathbb{0} \\ \mathbb{0} & \mathcal{M}_2 & \mathbb{0} & .. & \mathbb{0} \\ .. & .. & .. & .. & .. \\ \mathbb{0} & \mathbb{0} & .. & \mathbb{0} & \mathcal{M}_k \end{pmatrix}, \qquad \begin{array}{l} A|_{(0,0)} = \mathbb{0} = B|_{(0,0)}, \\ \det(\mathcal{M}_1) = f_1, \ .. \ , \det(\mathcal{M}_k) = f_k \end{array}$$

with the unknowns $A, B, \mathcal{M}_i$ (matrices whose entries are locally analytic functions).

By some inductive procedure we find a solution in R/m^N for each N. These are polynomial equations in matrix entries. Then, a locally analytic solution is guaranteed by the two approximation theorems.

2.2. **Singular curves and sheaves.**

2.2.1. *Plane curve singularities.* For local considerations we always assume the (singular) point to be at the origin and use the ring of locally convergent power series $\mathbb{C}\{x, y\}$.

Associated to any germ $(C, 0)$ is the branch decomposition $(C, 0) = \cup(p_i C_i, 0)$ where each $(C_i, 0)$ is reduced and locally irreducible. The (reduced) tangent cone $T_{(C,0)} = \{l_1..l_k\}$ traces all the tangents of the branches. To this cone is associated the tangential decomposition: $(C, 0) = \cup(C_\alpha, 0)$. Here $C_\alpha = \{f_\alpha = 0\}$ consists of all the branches with the tangent line l_α, in general C_α is reducible and non-reduced. Let $mult(C, 0) = m$ and $mult(C_\alpha, 0) = m_\alpha$.

For any reduced curve-germ $(C, 0)$ the normalization $(\tilde{C}, 0)$ is a multi-germ, corresponding to the branches.

2.2.2. *The normalization and its factorization.* Given a branch $(C, 0)$, its normalization is the morphism $(\tilde{C}, \tilde{0}) \to (C, 0)$ that is an isomorphism over $C \setminus \{0\}$ with $\tilde{C}$ smooth. For a reduced reducible germ $(C, 0) = \cup_i (C_i, 0)$ the normalization is the combination of morphisms $(\tilde{C}, \tilde{0}) := \coprod(\tilde{C}_i, \tilde{0}_i) \to (C, 0)$.

Usually the normalization $(\tilde{C}, \tilde{0}) \xrightarrow{\nu} (C, 0)$ can be (nontrivially) factorized: $(\tilde{C}, 0) \to (C', 0) \to (C, 0)$. Here both maps are bi-rational morphisms. Usually this can be done in many distinct ways. All the possible intermediate steps form an oriented graph, usually not a tree. Algebraically, the intermediate steps correspond to embeddings of the local rings:

$$(5) \qquad \mathcal{O}_{(C,0)} \overset{\nu^*_{\tilde{C}/C}}{\hookrightarrow} \mathcal{O}_{(C',0)} \overset{\nu^*_{\tilde{C}/C'}}{\hookrightarrow} \mathcal{O}_{(\tilde{C},0)}$$

Example 2.3. Consider the ordinary triple point i.e. the germ of the type $xy(x - y) = 0$. Here the tangent cone consists of three lines $\{x = 0\}$, $\{y = 0\}$ and $\{x - y\}$ and the tangential decomposition coincides with the branch decomposition. The normalization is defined by the embedding of local rings:

$$(6) \qquad \mathbb{C}[x, y]\big/\big(xy(x - y)\big) \overset{i}{\hookrightarrow} \mathbb{C}[t_1] \times \mathbb{C}[t_2] \times \mathbb{C}[t_3], \qquad \begin{array}{l} 1_{xy} \to 1_1 + 1_2 + 1_3 \\ x \to t_1 + t_2 \\ y \to t_2 + t_3 \end{array}$$

Hence, in this case the graph of the possible modifications is:

$$
\begin{tikzcd}[column sep=small]
 & Spec\left({}^{\mathbb{C}[t_1,t_2]}\big/_{(t_1,t_2)} \times\mathbb{C}[t_3]\right) \arrow[dr] & & \\
Spec\Big(\mathbb{C}[t_1]\times\mathbb{C}[t_2]\times\mathbb{C}[t_3]\Big) \arrow[ur] \arrow[r] \arrow[dr] & Spec\left({}^{\mathbb{C}[t_1,t_3]}\big/_{(t_1,t_3)} \times\mathbb{C}[t_2]\right) \arrow[r] & Spec\left(\frac{\mathbb{C}[t_1,t_2,t_3]}{\langle t_1t_2,t_2t_3,t_3t_1\rangle}\right) \arrow[r] & Spec\left(\frac{\mathbb{C}[x,y]}{\langle xy(x-y)\rangle}\right) \\
 & Spec\left({}^{\mathbb{C}[t_2,t_3]}\big/_{(t_3,t_2)} \times\mathbb{C}[t_1]\right) \arrow[ur] & &
\end{tikzcd}
\tag{7}
$$

Note that in this example we have at an intermediate step a *non-planar* singularity whose embedding dimension is 3.

Example 2.4. A particular kind of modification is the separation of all the branches: $\coprod(C_i,0_i)\to\cup(C_i,0)$. It is isomorphism when restricted to each particular branch. Then for the rings: $\mathcal{O}_{\cup(C_i,0)}\overset{i}{\hookrightarrow}\prod\mathcal{O}_{(C_i,0_i)}$. If E is a module over $\mathcal{O}_{\cup(C_i,0)}$ then it is lifted to the collection of modules $\{E\underset{\mathcal{O}_{\cup(C_i,0)}}{\otimes}\mathcal{O}_{\coprod(C_i,0)}\}/Torsion$, defined by the diagonal embedding $1_{xy}\to\oplus 1_i$.

2.2.3. *Adjoint and conductor ideals.* Let $(C,0)=\{f(x,y)=0\}\subset(\mathbb{C}^2,0)$ be a reduced singular curve and $(C',0)\overset{\nu}{\to}(C,0)$ a birational morphism (e.g. the normalization). As $(C,0)$ is usually reducible, $(C',0)=\coprod(C'_i,0_i)$ is usually a multi-germ with the morphisms $(C'_i,0)\overset{\nu_i}{\to}(C,0)$.

Definition 2.5. *The relative conductor ideal*

$$
\mathcal{O}_{(C,0)}\supset I^{cd}_{C'/C}:=Ann_{\mathcal{O}_{(C,0)}}(\mathcal{O}_{(C',0)})=\{g|\ \forall i:\ \nu^*_i(g)\mathcal{O}_{(C'_i,0)}\subset\nu^*_i(\mathcal{O}_{(C,0)})\}
\tag{8}
$$

Consider the ideals in $\mathbb{C}\{x,y\}=\mathcal{O}_{(\mathbb{C}^2,0)}$ whose restriction to $(C,0)$ gives $I^{cd}_{C'/C}$. Call the maximal among them: the relative adjoint ideal $Adj_{C'/C}$.

In the case of normalization $(\tilde{C},0)\overset{\nu}{\to}(C,0)$, by duality [Serre-book, §IV.11], the adjoint ideal $Adj_{\tilde{C}/C}$ can be also defined as follows. Let $\vec{v}$ be the generic tangent direction, not tangent to any of the branch.

Let $\mu=\mu(C,0)=\dim_{\mathbb{C}}\left({}^{\mathbb{C}[x,y]}\big/_{(\partial_x f,\partial_y f)}\right)$ be the Milnor number. Let $\delta=\delta(C,0)=\dim_{\mathbb{C}}\left(\nu_*\mathcal{O}_{(\tilde{C},0)}/\mathcal{O}_{(C,0)}\right)$ be the delta invariant (also known as the genus defect or the virtual number of nodes). They are basic invariants of singular curves, cf. [GLS-book1, §I.3.4].

Definition-Proposition 2.6. *1. The adjoint divisor of* $(C,0)$ *on* $\tilde{C}$ *is* $D:=\sum m_i\tilde{0}_i$ *where* $m_i=-ord(\nu^*_i\frac{d\vec{v}}{\partial_{\vec{v}}f(x,y)})=\sum\limits_{j\neq i}\langle C_j,C_i\rangle+\mu(C_i,0)$. *In particular* $\sum m_i=2\delta(C)$.
2. The adjoint ideal is $Adj_{\tilde{C}/C}:=\{g\mid\nu^*div(g_C)\geq D\}\subset\mathcal{O}_{(\mathbb{C}^2,0)}$

Proof. (For the full detail cf. [GLS-book1, §I.3.4].) Note that

$$
\partial_{\vec{v}}f|_{C_i}=(\prod_{j\neq i}f_j)(\partial_{\vec{v}}f_i)|_{C_i}\ \Rightarrow\ ord(\partial_{\vec{v}}f|_{C_i})=\sum_{j\neq i}(C_j,C_i)+\kappa(C_i)
\tag{9}
$$

where $\kappa(C_i)$ is the classical invariant, in particular $\kappa(C_i)=\mu(C_i)+mult(C_i)-1$. Also $ord(\partial_{\vec{v}}f|_{C_i})=mult(C_i)-1$, hence: $m_i=\sum\limits_{j\neq i}(C_j,C_i)+\mu(C_i)$. ■

Traditionally the conductor and adjoint ideals are considered for the normalization $\tilde{C}\to(C,0)$ and then denoted by I^{cd} and Adj. They have various nice properties

Property 2.7. *1.*[GLS-book1, I.3.4, pg 214]
$I^{cd}=\Big\{g|\ \forall i:\ div\ \nu^*_i(g)\geq 2\delta(C_i)+\sum\limits_{j\neq i}(C_j,C_i)\Big\}$, $\dim\frac{\mathcal{O}_{(C,0)}}{I^{cd}}=\delta$, $\dim\frac{\mathcal{O}_{\tilde{C}}}{I^{cd}}=2\delta$
2. Let $f=\prod f_i$. *Then* $g\in Adj_{\tilde{C}/C}$ *iff* $g=\sum\frac{f}{f_i}g_i$ *with* $g_i\in Adj_{\tilde{C}_i/C_i}$. *In particular, if all the branches are smooth then* $Adj_{\tilde{C}/C}=\langle\frac{f}{f_1}...\frac{f}{f_r}\rangle$
3. [GLS-book2, Lemma 1.26] *The adjoint ideal is a cluster ideal and* $Adj_{\tilde{C}/C}\subset m^{p-1}_{x,y}$ *for* $p=mult(C,0)$ *and* $m_{x,y}\subset\mathbb{C}\{x,y\}$ *the maximal ideal.*

Example 2.8. Let $C=\{y^{ar}-x^{br}=0\}\subset(\mathbb{C}^2,0)$ where $a\le b$ and $gcd(a,b)=1$. For $a=1=b$ this is an ordinary multiple point, other choices give e.g. A_k singularity (y^2-x^{k+1}) etc. For $a<b$ the (reduced) tangent cone consists of one line $\{y=0\}$.

The normalization $\coprod\limits_{i=1}^{r}(\tilde{C}_i,\tilde{0}_i)\to(C,0)$ is defined by: $t_i\to(t_i^a,\omega_i t_i^b)$ where ω_i is an appropriate root of unity. Alternatively, for the corresponding local rings: $\mathcal{O}_{(C,0)}$ and $\mathcal{O}_{\tilde{C}}=\mathbb{C}[t_1..t_r]/\langle t_it_j|i\neq j\rangle$ the homomorphism is defined by: $x\to\sum t_i^a\alpha_i$ and $y\to\sum t_i^b\beta_i$ (where $\alpha_i\beta_i$ are some numbers).

By choosing the generic coordinates of $(\mathbb{C}^2,0)$ and calculating the order of the pole of $\frac{dx}{\partial_y f(x,y)}$ one has: $D=(abr+1-a-b)\sum\tilde{0}_i\subset\coprod(\tilde{C}_i,\tilde{0}_i)$. Therefore the adjoint ideal is generated by $\{x^iy^j\}$ for $\frac{i+1}{b}+\frac{j+1}{a}\ge r+\frac{1}{ab}$. So, e.g.
• for an ordinary multiple point $(x^r+y^r=0)$ have: $I_D=m_{xy}^{r-1}$ =all functions with vanishing order at the origin at least $(r-1)$.
• for an $A_k:y^2+x^{k+1}$ have: $I_D=\langle y,x^{\lfloor k\rfloor}\rangle\mathbb{C}[x,y]$
• For the cusp $y^{d-1}+x^d$ the adjoint ideal is $m_{x,y}^{d-2}$ where $m_{x,y}$ is the local maximal ideal (of the reduced point).

Example 2.9. For $(y-x^{\frac{d}{2}})^2-y^d$ with d even one has: $D=\frac{d^2}{4}(0_1+0_2)$ and thus $Adj=\langle y-x^{\frac{d}{2}},y^{\frac{d}{2}},x^{\frac{d^2}{4}}\rangle$.

Note that the multiplicities $\{m_i\}$ from the definition 2.6 of $D\subset\coprod(\tilde{C}_i,0_i)$ depend on the topological type of the singularity only, while $Adj_{\tilde{C}/C}\subset\mathcal{O}_{(\mathbb{C}^2,0)}$ depends essentially on the particular germ $(C,0)$.

Adjoint and conductor ideals behave well with respect to blowup. Consider the morphisms $(C_2,0)\to(C_1,0)\overset{\pi}{\to}(C,0)$. Here all the curve singularities are planar, $(C_2,0)$, $(C_1,0)$ are possibly multigerms and π is the morphism induced by the blowup of origin in $(\mathbb{C}^2,0)$ (so that $(C_1,0)\overset{\pi}{\to}(C,0)$ is the strict transform).

We want to compare $I^{cd}_{C_2/C_1}$ to the "strict transform" of $I^{cd}_{C_2/C}$. Consider the pullbacks $\pi^*(I^{cd}_{C_2/C})=\pi^{-1}(I^{cd}_{C_2/C})\otimes\mathcal{O}_{(C_1,0)}$ and $\pi^*(Adj_{C_2/C})=\pi^{-1}(Adj_{C_2/C})\otimes\mathcal{O}_{(Bl(\mathbb{C}^2),0)}$ locally at the points of the multigerm $\cup(C_1,0_i)$. At each such point let g be the local equation of the exceptional divisor. Let the multiplicity of $(C,0)$ be p.

Definition-Proposition 2.10. *$I^{cd}_{C_2/C_1}=\widetilde{I^{cd}_{C_2/C}}:=\bigcap\limits_{0_i\in(C_1,0)}\frac{1}{g_i^{p-1}}\pi^*(I^{cd}_{C_2/C})$. Similarly $Adj_{C_2/C_1}=\widetilde{Adj_{C_2/C}}:=\bigcap\limits_{0_i\in\pi^{-1}(0)}\frac{1}{g_i^{p-1}}\pi^*(Adj_{C_2/C})$.*

Proof. The $\supset$ part.
First observe that any element of $I^{cd}_{C_2/C}$ vanishes at 0 to the order at least $(p-1)$, property 2.7. Hence $\frac{1}{g^{p-1}}\pi^*(I^{cd}_{C_2/C})$ is a well defined ideal in $\mathcal{O}_{(C_1,0)}$.

Let $h\in\mathbb{C}\{x,y\}$ such that $h|_C\in I^{cd}_{C_2/C}$. Then for the intersection multiplicity have: $(h^{-1}(0),C)\ge 2\delta_{C_2/C}$ where $\delta_{C_2/C}=dim\mathcal{O}_{C_2}/\mathcal{O}_C$. Hence for the strict transforms (globally on the exceptional divisor E):

$$\begin{aligned}(10)\qquad & ord(\tfrac{1}{g^{p-1}}\pi^*h|_{C_1})=\Big(\pi^*h^{-1}(0)-(p-1)E,\pi^*C-pE\Big)=(h^{-1}(0),C)-p(p-1)\ge\\ &\ge 2\delta_{C_2/C}-p(p-1)=2\textstyle\sum_i\delta_{(C_2/C_1,0_i)}\end{aligned}$$

The last equality follows from the formula for δ-invariant in terms of the resolution tree multiplicities: [GLS-book1, pg.207, prop 3.34]

$$(11)\qquad \delta=\sum_{j\in\Gamma_{C,0}}\binom{p_j}{2}$$

So, $\frac{1}{g^{p-1}}\pi^*h|_{C_1}\in I^{cd}_{C_2/C_1}$.
The $\subset$ *part.*
Let $h\in I^{cd}_{C_2/C}$ such that $ordh|_C=c(I^{cd}_{C_2/C})$, the conductor. Then $ord\frac{1}{g^{p-1}}\pi^*(h)|_{C_1}=c(I^{cd}_{C_2/C_1})$. Hence for any $q\in I^{cd}_{C_2/C_1}$ there exists $r\in\mathcal{O}_{C_1}$ such that

$$ord(q-r\frac{1}{g^{p-1}}\pi^*(h)|_{C_1})>ord(q|_{C_1}) \tag{12}$$

Note that $r\frac{1}{g^{p-1}}\pi^*(h)\in\widetilde{I^{cd}_{C_2/C}}$, so now it's enough to prove the statement for $(q-r\frac{1}{g^{p-1}}\pi^*(h))$. Iterating this procedure several times get an element with sufficiently high order on C_1, hence in $\widetilde{I^{cd}_{C_2/C}}$.

The statement for adjoint ideals follows now from their definition. ■

2.2.4. *Pullback and pushforward of torsion free modules.* Given a modification $\nu_{C'/C}:C'\to C$ and a torsion free module E_C over $\mathcal{O}_{(C,0)}$ the pullback $\nu^*E:=\nu^{-1}E\otimes\mathcal{O}_{C'}$ usually contains torsion. In this paper we consider the torsion free part: $\nu^{-1}_{C'/C}E_C\underset{\mathcal{O}_{(C,0)}}{\otimes}\mathcal{O}_{(C',0)}/Torsion$.

Example 2.11. Let $C=\{x^p=y^q\}\subset\mathbb{C}^2$ with $(p,q)=1$ and $q>p$. Let $m=\langle x,y\rangle\mathcal{O}_{(C,0)}$ be the maximal ideal. Then $\nu:\mathbb{C}^1\to C$ sends t to (t^q,t^p) and $\nu^*(m)$ contains torsion. For example $\nu^*(x)-t^{q-p}\nu^*(y)$ is annihilated by $t^p=\nu^*(x)\in\mathbb{C}\{t\}$.

Let E_C be a torsion free module and $\tilde{C}\overset{\nu}{\to}C$ the normalization. Then $\nu^*E_{\tilde{C}}/Torsion$ is locally free (being torsion free on a smooth curve). It can happen that already for some intermediate lifting $\tilde{C}\to C'\overset{\nu}{\to}C$ the sheaf $\nu^*_{C'/C}(E)/Torsion$ is locally free. There always exists the *minimal* such lifting.

Lemma 2.12. *Given a torsion free module E_C over $\mathcal{O}_{(C,0)}$ there exists a modification $C'\to C$ such that:*
- *$\nu^*_{C'/C}(E)/Torsion$ is free.*
- *If for some modification $C''\to C$ the pullback $\nu^*_{C''/C}(E)/Torsion$ is free then the modification factorizes as on the diagram.*

$$\begin{array}{ccc} C'' & \to & C' \\ & \searrow & \downarrow \\ & & C \end{array}$$

Proof. Let, E be a torsion free module M over the one-dimensional local ring R with the integral closure $\bar{R}$. We prove that the minimal extension $R\subset R'\subset\bar{R}$, such that $E\underset{R}{\otimes}R'/Torsion$ is free, is unique.

Let $E\overset{i}{\hookrightarrow}E\underset{R}{\otimes}\bar{R}$ be the natural embedding $(e\to e\otimes 1)$. By the assumption $E\underset{R}{\otimes}\bar{R}/Torsion$ is free. We can choose its generators to lie in the image of E:

$$E\underset{R}{\otimes}\bar{R}/Torsion=\langle i(e_1),..,i(e_k)\rangle,\ \text{for } e_1,..,e_q\in E \tag{13}$$

Consider

$$R':=\{r|\ r\times i(E)\subset i(E)\}\subset\bar{R} \tag{14}$$

By the construction: $R\subset R'\subset\bar{R}$ and $E\underset{R}{\otimes}R'/Torsion$ is generated by $e_1,..,e_k$, i.e. is a free module. And $R\subset R'$ is obviously the minimal extension by torsion freeness of E. ■

Note that in the decomposable case, $E=E_1\oplus E_2$ the minimal modification for E is constructed from those of E_1, E_2 by the minimal completion of the diagram.

$$\begin{array}{ccc} C'_{E_1\oplus E_2} & \to & C'_{E_2} \\ \downarrow & & \downarrow \\ C'_{E_1} & \to & C \end{array}$$

Suppose a rank 1 module E_C is embedded into $\mathcal{O}_{(C,0)}$. Given a modification $C'\overset{\nu}{\to}C$ we can consider $\nu^*(E_C)/Torsion$ as a $\nu^{-1}\mathcal{O}_{(C,0)}\approx\mathcal{O}_{(C,0)}$ module. Particularly important modules are those that lift "isomorphically": $\nu^*(E_C)/Torsion \approx \nu^{-1}(E_C)$. Alternatively: $\nu_*\nu^*(E_C)/Torsion \equiv E_C$.

Property 2.13. *Let E_C torsion free module of rank=1, embedded into $\mathcal{O}_{(C,0)}$. Then $\nu_*\nu^*(E_C) \equiv E_C$ iff $E \subset I^{cd}_{C'/C}$*

This is just the definition of conductor: $s \in H^0(U, I^{cd}_{C'/C})$ iff $\nu^*(s)\mathcal{O}_{C'} \subset \nu^*\mathcal{O}_C$.

2.3. **The matrix and its adjoint.** We work with (square) matrices, their sub-blocks and particular entries. Sometimes to avoid confusion we emphasize the dimensionality, e.g. $\mathcal{M}_{d\times d}$. Then $\mathcal{M}_{i\times i}$ denotes an $i \times i$ block in $\mathcal{M}_{d\times d}$ and $\det(\mathcal{M}_{i\times i})$ the corresponding minor. On the contrary by $\mathcal{M}_{ij}$ we mean a particular entry.

Let $\mathcal{M}$ be a determinantal representation of $(V,0) \subset (\mathbb{C}^n, 0)$, let $\mathcal{M}^\vee$ be the adjoint matrix of $\mathcal{M}$ (so $\mathcal{M}\mathcal{M}^\vee = \det(M)\mathbb{1}_{\mathrm{d}\times\mathrm{d}}$). Then $\mathcal{M}$ is non-degenerate outside the hypersurface V and the corank over the hypersurface satisfies:

$$1 \leq corank(\mathcal{M}|_{0\in V}) \leq mult(V,0) \tag{15}$$

(as is checked e.g. by taking derivatives of the determinant). Hence any determinantal representation of a smooth hypersurface is weakly maximal , cf. definition 1.1. The adjoint matrix $\mathcal{M}^\vee$ is not zero at smooth points of V. As $\mathcal{M}^\vee|_V \times \mathcal{M}|_V = \mathbb{0}$ the rank of $\mathcal{M}^\vee$ at any smooth point of V is 1. If $corank(\mathcal{M}|_{pt}) \geq 2$ then $\mathcal{M}^\vee|_{pt} = \mathbb{0}$. Note that $\mathcal{M}^{\vee\vee} = (\det \mathcal{M})^{d-2}\mathcal{M}$ and $\det \mathcal{M}^\vee = (\det \mathcal{M})^{d-1}$.

Property 2.14. *(cf. e.g.* [Vinnikov89, lemma on pg. 114]*)*
1. Let $\mathcal{M} \in Mat(d\times d)$ and f irreducible. Suppose any $i \times i$ minor $\det(\mathcal{M}_{i\times i})$ *is divisible by f^l. Then any $(i+1)\times(i+1)$ minor* $\det(\mathcal{M}_{(i+1)\times(i+1)})$ *is divisible by f^{l+1}.*
2. Consider the hypersurface germ $\{\prod f_i^{p_i} = 0\} \subset (\mathbb{C}^n, 0)$ for $\{f_i\}$ reduced. Let $\mathcal{M}$ be its determinantal representation, $\mathcal{M}|_0 = \mathbb{0}$, weakly maximal near the origin. Then all entries of $\mathcal{M}^\vee$ are divisible by $\prod f_i^{p_i-1}$.

This motivates the definition of maximality (after definition 1.1) for non-reduced curves.

Proof. 1. Let $\mathcal{M}_{(i+1)\times(i+1)}$ be a submatrix of $\mathcal{M}$. By the assumption, any entry of ${\mathcal{M}^\vee}_{(i+1)\times(i+1)}$ is divisible by f^l, thus $\det({\mathcal{M}^\vee}_{(i+1)\times(i+1)})$ is divisible by $f^{l(i+1)}$. But $\det({\mathcal{M}^\vee}_{(i+1)\times(i+1)}) = (\det \mathcal{M}_{(i+1)\times(i+1)})^i$. Hence $(\frac{\det \mathcal{M}_{(i+1)\times(i+1)}}{f^l})^i$ is divisible by f^l. As f is irreducible the statement follows.

2. By the assumption, for any point of $pt \in \{f_i = 0\}$ near the origin we have: $corank(\mathcal{M}|_{pt}) \geq p_i$. So any $(d-p_i+1)\times(d-p_i+1)$ minor of $\mathcal{M}$ is divisible by f_i. By the first part of the statement we get: any $(d-1)\times(d-1)$ minor of $\mathcal{M}$ is divisible by $f_i^{p_i-1}$. Hence the statement. ■

Note that in the second case the assumption of weak maximality *near the point* is important. For example $\begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$ is weakly maximal at the origin but not near the origin. And for it the statement does not hold.

When working with matrix of polynomials/locally analytic functions, several natural notions are:
- $deg_{x_i}(\mathcal{M})$=the maximal degree of x_i in the entries of $\mathcal{M}$. This is infinity unless all the entries of $\mathcal{M}$ are polynomials in x_i. Similarly for $deg(\mathcal{M})$, the total degree.
- $ord_{x_i}(\mathcal{M})$=the minimal degree of x_i appearing in $\mathcal{M}$. If an entry of $\mathcal{M}$ does not depend on

x_i the order is zero, if $A \equiv \mathbb{0}$ then $ord_{x_i}(A) := \infty$. Similarly $ord(\mathcal{M})$ and $ord_x(\mathcal{M}_{ij})$ for a particular entry. So, e.g. $ord(\mathcal{M}) \geq 1$ iff $\mathcal{M}|_{(0,0)} = \mathbb{0}$
• $jet_k(\mathcal{M})$ is obtained from $\mathcal{M}$ by truncation of all the monomials with total degree higher than k.

2.3.1. *Reduction to a minimal form.* Let $\mathcal{M}_{d\times d}$ be a locally analytic or formal matrix, without the assumption $\mathcal{M}|_0 = \mathbb{0}$. Let the multiplicity of the hypersurface germ $\{\det(\mathcal{M}) = 0\} \subset (\mathbb{C}^n, 0)$ be $m \geq 1$.

Property 2.15. *1. Locally $\mathcal{M}_{d\times d}$ is equivalent to $\begin{pmatrix} \mathbb{1}_{(d-p)\times(d-p)} & \mathbb{0} \\ \mathbb{0} & \mathcal{M}_{p\times p} \end{pmatrix}$ with $\mathcal{M}_{p\times p}|_{(0,0)} = \mathbb{0}$ and $1 \leq p \leq m$.*
2. The stable equivalence (i.e. $\mathbb{1} \oplus \mathcal{M}_1 \sim \mathbb{1} \oplus \mathcal{M}_2$) implies the ordinary local equivalence ($\mathcal{M}_1 \sim \mathcal{M}_2$).
3. If the representation is weakly maximal , i.e. $rank(\mathcal{M}_{d\times d}|_0) = d - m$ then $p = m$ and $det(jet_1\mathcal{M}_{p\times p}) \not\equiv 0$ and $det(jet_{p-1}\mathcal{M}^\vee{}_{p\times p}) \not\equiv 0$.

Proof. From the algebraic point of view the first statement is the reduction to a minimal presentation of the module. The second is the uniqueness of such a reduction. Both are proved e.g. in [Yoshino-book, pg. 58].

The first statement is proved for the symmetric case in [Piontkowski2006, lemma 1.7]. Both bounds are sharp, regardless of the singularities.

The third claim is immediate. ■

2.3.2. *Fitting ideals.*

Definition-Proposition 2.16. *The fitting ideal $I_k(M) \subset \mathbb{C}\{x_1, .., x_n\}$, generated by all the $k\times k$ minors of M, is invariant under the local equivalence.*

Proof. First consider the case $k = 1$, i.e. the ideal $I_1(M)$ is generated by the entries of M. Then immediately: $I_1(AMB) \subset I_1(M)$. As A, B are locally invertible the opposite inclusion holds too.

For arbitrary k note that the wedge $\wedge^k M$ is the collection of all the $k \times k$ minors, hence continue as for $k = 1$. ■

Remark 2.17. An elementary observation about fitting ideals. Suppose $\mathcal{M}_{p\times p}$ is locally decomposable as $\begin{pmatrix} \mathcal{M}_{p_1\times p_1} & \mathbb{0} \\ \mathbb{0} & \mathcal{M}_{p_2\times p_2} \end{pmatrix}$. Then $I_1(\mathcal{M})$ is generated by at most $p^2 - 2p_1p_2$ elements.

Similarly, if $\mathcal{M}$ can be locally brought to an upper-block-triangular form then $I_1(\mathcal{M})$ is generated by at most $p^2 - p_1p_2$ elements.

2.4. **Kernel modules.** Recall that a module E over a CM ring R is called maximally Cohen-Macaulay (MCM) if $depth(E) = dim(R)$. If $dim(R) = 1$, and R is reduced then E is MCM iff it is torsion-free.

Given a local determinantal representation $\mathcal{M}|_{(0,..,0)} = \mathbb{0}$ define the kernel module over $\mathbb{C}\{x_1, .., x_n\}$ as follows. Let $E \subset \mathbb{C}\{x_1, .., x_n\}^{\oplus d}$ be the collection of all the kernel vectors, i.e. $\mathcal{M}v \sim det(M)$...

Lemma 2.18. *1. E is a module over $\mathbb{C}\{x_1, .., x_n\}$, minimally generated by the columns of $\mathcal{M}^\vee$.*
2. Its restriction to the hypersurface (i.e. $E \otimes \mathcal{O}_{(V,0)}$) is a torsion free module.
3. For a reduced hypersurface the module $E \otimes \mathcal{O}_{(V,0)}$ is free iff $\mathcal{M}$ is a 1×1 matrix.

Proof. 1. (This statement is also proved in [Yoshino-book, pg.56].) Let E' be the $\mathbb{C}\{x_1, .., x_n\}$

module generated by the columns of $\mathcal{M}^\vee$. So $E'\subset E$. Let $v\in E$ and $\mathcal{M}v=\det(\mathcal{M})\begin{pmatrix} a_1\\ ..\\ a_p\end{pmatrix}$. Let $v_1..v_p$ be the columns of $\mathcal{M}^\vee$, then $\mathcal{M}(v-\sum a_i v_i)=0\in\mathbb{C}\{x_1,..,x_n\}$. As $\mathcal{M}$ is non-degenerate on $\mathbb{C}^n$ get $v\in E'$, hence $E'=E$. By linear independence, the columns of $\mathcal{M}^\vee$ form a minimal set of generators.
2. The module E is torsion free as a submodule of a free module $\mathbb{C}\{x_1,..,x_n\}^{\oplus p}$.
3. Suppose E is free and $v_1..v_p$ are the columns of $\mathcal{M}^\vee$, then $E\approx\oplus\mathcal{O}_{(C,0)}v_i$. Hence $\mathcal{M}$ decomposes and $\det(\mathcal{M}_i)\mathcal{O}_{(C,0)}v_i=0$ contradicting the freeness, unless $\mathcal{M}$ has only one component. Then E is generated by one element, so $\mathcal{M}$ is a 1×1 matrix. ∎
By its definition the kernel module has a natural basis $\{v_1..v_d\}$=the columns of $\mathcal{M}^\vee$. The embedded kernel with its basis determines the determinantal representation:

Property 2.19. *1. Let $\mathcal{M}_1,\mathcal{M}_2\in Mat(d\times d,\mathbb{C}\{x_1,..,x_n\})$ be two local determinantal representation of the same hypersurface and E_1,E_2 the corresponding kernel modules. Then $\mathcal{M}_1=\mathcal{M}_2$ or $\mathcal{M}_1=A\mathcal{M}_2$ or $\mathcal{M}_1=A\mathcal{M}_2B$ (for A,B locally invertible) iff $\Big(E_1,\{v^1_1..v^1_d\}\Big)=\Big(E_2,\{v^2_2..v^2_d\}\Big)\subset\mathbb{C}\{x_1,..,x_n\}^{\oplus d}$ or $E_1=E_2\subset\mathbb{C}\{x_1,..,x_n\}^{\oplus d}$ or $E_1\approx E_2$.*
2. In particular: $\mathcal{M}$ is decomposable (or locally equivalent to an upper-block-triangular form) iff E is a direct sum (or an extension).

Here in the first statement we mean the coincidence of the natural bases/the coincidence of the embedded modules/the embedded isomorphism of modules.

Proof. 1. The direction $\Rrightarrow$ in all the statements is immediate. The converse follows from the uniqueness of minimal free resolution [Eisenbud-book].
2. Suppose $E=E_1\oplus E_2$, let $F_2\stackrel{\mathcal{M}}{\to}F_1\to E\to 0$ be the minimal resolution. Let $F^{(i)}_2\stackrel{\mathcal{M}_i}{\to}F^{(i)}_1\to E_i\to 0$ be the minimal resolutions of E_1,E_2. Consider their direct sum:

$$F^{(1)}_2\oplus F^{(2)}_2\stackrel{\mathcal{M}_1\oplus\mathcal{M}_2}{\to}F^{(1)}_1\oplus F^{(1)}_1\to E_1\oplus E_2=E\to 0 \tag{16}$$

This resolution of E is minimal. Indeed, by the decomposability assumption the number of generators of E is the sum of those of E_1,E_2, hence $rank(F_1)=rank(F^{(2)}_1)+rank(F^{(1)}_1)$. Similarly, any linear relation between the generators of E (i.e. a syzygy) is the sum of relations for E_1 and E_2. Hence $rank(F_2)=rank(F^{(2)}_2)+rank(F^{(1)}_2)$.

Finally, by the uniqueness of the minimal resolution we get that the two proposed resolutions of E are isomorphic, hence the statement.

Similarly for the extension of modules. ∎

For various applications one needs some simple conditions that are necessary or sufficient for decomposability of determinantal representation. Sometimes we impose the following conditions of linear independence. Given a point $pt\in C$, let $\{C_i\}$ be the local branches of (C,pt). Let $E_i=\overline{E|_{C_i\setminus pt}}$ i.e. restrict the kernel to a branch outside the singular point, then extend to the singular point by direct image. Let $\{E_i|_{pt}\}$ be the reduced fibres of the corresponding (embedded) kernel bundles.

We often ask for the "weak" independence: $Span(\cup E_i|_{pt})=\oplus E_i|_{pt}$ or for the "strong" independence: $Span(\cup E_i)=\oplus Span(E_i)$. In the last case the span means the minimal linear subspace (of the ambient space) into which the kernels embed, when restricted to some neighborhood of the point. The second notion is stronger, cf. remark 3.9.

3. Local decomposability and extension criteria for determinantal representations

Suppose the curve is locally reducible $(C,0)=(C_1,0)\cup(C_2,0)$, where $(C_i,0)$ can be further reducible. Let E and $E_i=E|_{(C_i,0)}$ be the kernels of determinantal representations of

$(C_i,0)$. If both E_i are maximally generated, i.e. correspond to weakly maximal determinantal representations, then so are their extensions, corresponding to weakly maximal determinantal representations of $(C,0)$. We prove the converse: if E is maximally generated then it is an extension of maximally generated. Sometimes it is even decomposable.

3.1. **Decomposability according to the tangential decomposition.** For the tangent cone $T_{(C,0)} = \{l_\alpha\}$ of the germ of curve, consider the local *tangential decomposition*: $(C,0) = \bigcup\limits_\alpha (C_\alpha,0)$. Here $C_\alpha = \{f_\alpha = 0\}$ consists of all the branches with the tangent line l_α, in general C_α can be reducible and non-reduced. Let $mult(C,0) = m$ and $mult(C_\alpha,0) = m_\alpha$. As always we assume $\mathcal{M}|_0 = \mathbb{0}$.

Theorem 3.1. *Let $\mathcal{M}_{m\times m}$ be a weakly maximal determinantal representation of $(C,0)$. Corresponding to the tangential decomposition of $(C,0)$, the representation $\mathcal{M}$ is locally equivalent to:*

$$\begin{pmatrix} \mathcal{M}_{m_1\times m_1} & \mathbb{0} & .. & \mathbb{0} \\ \mathbb{0} & \mathcal{M}_{m_2\times m_2} & \mathbb{0} & .. \\ \mathbb{0} & .. & .. & \mathcal{M}_{m_k\times m_k} \end{pmatrix}$$

Here $\mathcal{M}_{m_\alpha\times m_\alpha}$ is a weakly maximal determinantal representation of $(C_\alpha,0)$.

Proof. The theorem states that there exists a locally analytic solution to the problem:
(17)
$$(\mathbb{1}+\mathrm{A})\mathcal{M}(\mathbb{1}+\mathrm{B}) = \begin{pmatrix} \mathcal{M}_{m_1\times m_1} & \mathbb{0} & .. & \mathbb{0} \\ \mathbb{0} & .. & & \\ \mathbb{0} & .. & \mathbb{0} & \mathcal{M}_{m_k\times m_k} \end{pmatrix}, \ \mathrm{A}|_{(0,0)} = \mathbb{0} = \mathrm{B}|_{(0,0)}, \ \det(\mathcal{M}_{\mathrm{m}_\alpha\times \mathrm{m}_\alpha}) = \mathrm{f}_\alpha$$

for the unknowns $A, B, \{\mathcal{M}_{m_\alpha\times m_\alpha}\}_\alpha$. Using Artin's and Pfister-Popescu theorems (from §1.3.1)it is enough to prove that the solution exists in $\mathbb{C}\{x,y\}/m^N$ for any N.

By the assumption $\mathcal{M}$ vanishes at the origin, while the property 2.15 gives: $\det(jet_1\mathcal{M}) \not\equiv 0$.
Part 1. By $GL(m,\mathbb{C})\times GL(m,\mathbb{C})$ bring $jet_1(\mathcal{M})$ to the Jordan form. For that, let $jet_1(\mathcal{M}) = xP + yQ$ with P,Q constant matrices. Then P can be assumed as $\begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}$. The remaining transformation of $GL(m,\mathbb{C})\times GL(m,\mathbb{C})$ preserving this form of P include the conjugation: $\mathcal{M} \to U\mathcal{M}U^{-1}$. Hence Q can be assumed in the Jordan form.

Part 2. The matrix $\mathcal{M}$ is naturally subdivided into the blocks B_{ij}, which are $m_i\times m_j$ rectangles (corresponding to the fixed eigenvalues of $jet_1(\mathcal{M})$). We should remove the off-diagonal blocks, B_{ij} for $i\neq j$. We do this by induction, at the N'th step removing all the terms whose order is $\leq N$.

Let $N = min_{i\neq j}(ord\mathcal{M}_{ij})$ for (ij) not in a diagonal block (thus $N>1$). Consider $jet_N(\mathcal{M})$, i.e. truncate all the monomials whose total degree is bigger than N. Suppose the block $B_{12}\subset jet_N(\mathcal{M})$ is non-zero, i.e. there is an entry of order N.

As l_1,l_2 are linearly independent, by a linear change of coordinates in $(\mathbb{C}^2,0)$ can assume $l_1 = x$, $l_2 = y$. Decompose: $B_{12} = xT + yR$, where T,R are $m_1\times m_2$ matrices, with $ord(T)\geq N-1$ and $ord(R)\geq N-1$. From the last row of B_{12} subtract the rows

$$jet_N\mathcal{M}_{m_1+1,*},\ jet_N\mathcal{M}_{m_1+2,*}, .., jet_N\mathcal{M}_{m_1+m_2,*} \tag{18}$$

of $jet_N(\mathcal{M})$ multiplied by $R_{m_11}, R_{m_12}..R_{m_1m_2}$. By the assumptions this doesn't change $jet_N(\mathcal{M})$ outside the block B_{12}. After this procedure every entry of the last row of B_{12} is divisible by x. Thus subtract from the columns of B_{12} the column $jet_N\mathcal{M}_{*,m_1}$ multiplied by the appropriate factors.

Now the last row of B_{12} consists of zeros, while $jet_N(\mathcal{M})$ is unchanged outside B_{12}. Do the same procedure for the row $jet_N\mathcal{M}_{m_1-1,*}$ of B_{12} (using the rows $jet_N\mathcal{M}_{m_1+1,*}, jet_N\mathcal{M}_{m_1+2,*}, .., jet_N\mathcal{M}_{m_1+m_2,*}$ and the column $jet_N\mathcal{M}_{*,m_1-1}$). And so on.

Part 3. After the last step one has the refined matrix $jet_N(\mathcal{M}')$ which coincides with $jet_N(\mathcal{M})$ outside the block B_{12} and has zeros inside this block. Do the same thing for all other (off-diagonal) blocks. Then one has a block diagonal matrix $jet_N(\mathcal{M}')$.

Now repeat all the computation starting from non-truncated version $\mathcal{M}$. This results in the increase of N. Continue by induction. Thus, for each N can bring $\mathcal{M}$ to such a form that the $jet_N(\mathcal{M})$ is block diagonal. ■

Example 3.2. Any weakly maximal determinantal representation of an ordinary multiple point, the curve singularity of the type $f = \prod l_i$ with l_i linearly independent linear forms, is diagonalizable: $\mathcal{M} \sim \begin{pmatrix} l_1 & 0 & .. & 0 \\ 0 & l_2 & .. & \\ 0 & .. & & \\ 0 & .. & & l_p \end{pmatrix}$

Example 3.3. The theorem reduces the classification of local weakly maximal determinantal representations of plane curve singularities (i.e. families of matrices depending on 2 parameters) to the case of singularity with one tangent line.

Consider the singularity of D_k type: $\{y^2x + x^{k-1} = 0\}$, the union of an A_{k-3} part $(y^2 + x^{k-2})$ and a non-tangent smooth branch. The (non-trivial) local determinantal representations of such singularity are either 2×2 or 3×3. We get that in the later case the representation is decomposable and the classification problem is reduced to that of A_{k-3}. Compare to the classification in [Bruce-Tari04, table 4].

Remark 3.4. • The assumption of weak maximality is necessary. Consider $\begin{pmatrix} x^{p-1}y & x^p - y^p \\ x^p + y^p & xy^{p-1} \end{pmatrix}$ for $p > 2$. This determinantal representation of an ordinary multiple point is not locally equivalent to an upper triangular form.

Indeed, observe that $I(\mathcal{M})$ is minimally generated by 4 elements, apply remark 2.17.

• It is not clear how to generalize the statement to weakly maximal determinantal representation of more variables. For example, the matrix $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ is a weakly maximal determinantal representation of two transversely intersecting planes. The matrix is not equivalent to a diagonal one, e.g. by comparing the fitting ideals.

3.2. **Extensions for the case of singularity with one tangent line.** Theorem 3.1 reduces the problem of local decomposition to the case of curves $(C, 0)$ whose tangent cone is just one line. Suppose $(C, 0)$ is such, choose $\hat{x}$ axis as the unique tangent line, so $(C, 0) = \{y^p + .. = 0\}$, with the dots for higher order terms. Let $(C, 0) = \bigcup_{i=1}^{k} (C_i, 0)$ be the decomposition into (irreducible) branches. Here some $(C_i, 0)$ can coincide (if the germ $(C, 0)$ is non-reduced). A weakly maximal determinantal representation of such a curve can be brought to a particular upper block triangular form.

Theorem 3.5. *Let $\mathcal{M}$ be a weakly maximal local determinantal representation of the curve-germ as above. Then $\mathcal{M}$ is locally equivalent to an upper-block-triangular matrix:*

$$\begin{pmatrix} \mathcal{M}_{m_1 \times m_1}(x, y) & \mathcal{M}_{m_1 \times m_2}(x) & * & \\ \mathbb{0} & \mathcal{M}_{m_2 \times m_2}(x, y) & \mathcal{M}_{m_2 \times m_3}(x) & * \\ .. & .. & .. & .. \\ \mathbb{0} & \mathbb{0} & .. & \mathcal{M}_{m_k \times m_k}(x, y) \end{pmatrix}$$

Here the blocks $\{\mathcal{M}_{m_i \times m_i}(x, y)\}_i$ are local determinantal representations of $\{(C_i, 0)\}$, while the blocks $\mathcal{M}_{m_i \times m_j}(x)$ for $i < j$ depend on x only.

Moreover, the blocks $\{\mathcal{M}_{m_i \times m_i}(x, y)\}_i$ can be assumed in the form: $Diag_{m_i \times m_i}(x, y) + \mathcal{N}_{m_i \times m_i}(x)$, where $\mathcal{N}_{m_i \times m_i}(x)$ depends on x only.

Proof. As in the previous theorem, it is enough to prove the statement module the maximal ideal m^N for any N.

The proof consists of several parts. Choose some $(C_j,0)$.

Part 1. We bring the kernel over $(C_j,0)$ to a specific form. Adjust the coordinates in $(\mathbb{C}^2,0)$ to bring the defining equation of $(C_j,0)$ to the Weierstraß form: $f_j=y^{m_j}+g_1(x)y^{m_j-1}+...+g_{m_j}(x)=0$. Here $g_i(x)$ are locally analytic functions. This transformation certainly lifts to the transformation of $\mathcal{M}_{(m\times m)}$.

Let $E_j=Ker\mathcal{M}_{(m\times m)}|_{C_j}$ and s_j a local section. So, s_j is a vector of functions in (x,y) such that $\mathcal{M}_{(m\times m)}s_j$ is divisible by f_j. Can assume that the entries of s_j have no common divisor, thus at least one entry contains the monomial y^{n-1}. Reduce $s_j mod f_j$, so can assume that the only powers of y appearing in s_j are $<m_j$. Thus by local $GL(d,\mathcal{O})$ transformations can bring s_j to the form $(y^{n-1}+xh_1(x,y),y^{n-2}x^a+h_2(x,y),..,..,h_n(x),0,..,0)$ where h_i are some locally analytic functions with $deg_y(h_i)\le n-1-i$.

Part 2. Now several first columns of $\mathcal{M}$ are brought to a specific form.

According to the choice of s_j, consider the first n columns of $\mathcal{M}$ (with $n\le m_j$), denote this matrix by $\mathcal{M}_{m\times n}$. Let s be the corresponding truncation of s_j, so that $\mathcal{M}_{m\times n}s$ is divisible by f_j.

By lemma 2.15 one can choose n rows in $\mathcal{M}_{m\times n}$ to form a submatrix $A_{n\times n}$ such that $\det(jet_1 A_{n\times n})\not\equiv 0$. As $A_{n\times n}s$ is divisible by f_j, get: $\det A_{n\times n}$ is divisible by f_j. But $ord\det A_{n\times n}=n\le m_j$. Therefore: $n=m_j$ and $\det A_{m_j\times m_j}=f_j$ (up to a constant) and $\det(jet_1 A_{m_j\times m_j})=y^{m_j}$. So, by $GL(m_j,\mathbb{C})_L$ can assume (cf. the beginning of the proof of proposition 3.1) that $jet_1 A_{m_j\times m_j}=y1\!\!1+\mathrm{x}\tilde{\mathrm{A}}$, for $\tilde{A}$ strictly upper triangular (i.e. with zeros on or below the diagonal).

Return now to $\mathcal{M}_{m\times m_j}=\begin{pmatrix}A_{m_j\times m_j}\\ C_{(m-m_j)\times m_j}\end{pmatrix}$. By $GL(m,\mathbb{C})_L$, acting on $\mathcal{M}_{m\times m_j}$ from the left, can assume that $C_{(m-m_j)\times m_j}$ has no linear y-terms. We show that by the action of $GL(m,\mathcal{O})_L$ all the y-dependence of $C_{(m-m_j)\times m_j}$ can be removed. It is enough to prove this for any particular row.

Consider the row $(\beta_1(x)+y\gamma_1(x,y),...\beta_{m_j}(x)+y\gamma_{m_j}(x,y))$. Let $q=\min_j(ord(\gamma_i(x,y)))$. By the assumption $q\ge 1$. By a permutation of the indices can assume: $ord(\gamma_1(x,y))=q$. Subtract from this row the first row of $A_{m_j\times m_j}$ multiplied by γ_1. Recall that $A_{m_j\times m_j}=y1\!\!1+\mathrm{x}\tilde{\mathrm{A}}+\mathrm{B(x,y)}$, where $\tilde{A}$ is a constant strictly upper triangular matrix and $ordB(x,y)\ge 2$. Thus one gets the row (omitting the monomials containing x only):

$$(-B_{11}\gamma_1,y\gamma_2-B_{12}\gamma_1-x\gamma_1\tilde{A}_{12},..y\gamma_{m_j}-B_{1m_j}\gamma_1-x\gamma_1\tilde{A}_{1m_j}) \tag{19}$$

Again, omit all the monomials containing x only, then the row is: $(y\tilde{\gamma}_1,..y\tilde{\gamma}_{m_j})$ where: $ord\tilde{\gamma}_i\ge ord\gamma_i$ and $ord\tilde{\gamma}_1>ord\gamma_1$. Continue by induction.

So, by the $GL(m,\mathcal{O})_L$ action all the monomials $x^a y^b$ with $b<N$ can be removed, for any given N. By taking the limit of this procedure (i.e. taking the product of all the $GL(m,\mathcal{O})_L$ actions), one has:

$\mathcal{M}_{m\times m_j}$ is locally equivalent to $\begin{pmatrix}A_{m_j\times m_j}(x,y)\\ C_{(m-m_j)\times m_j}(x)\end{pmatrix}$.

Note that this is achieved by multiplication from the left only (the permutations of columns can be undone at the end), so the form of s is not changed.

Part 3. By now $C_{(m-m_j)\times m_j}$ depends on x only. We prove that in fact $C_{(m-m_j)\times m_j}(x)\equiv \mathbb{0}$. Indeed, by construction $C_{(m-m_j)\times m_j}(x)s=f_j..$ and y appears in s only in powers $<m_j$. Thus, in fact $C_{(m-m_j)\times m_j}(x)s\equiv 0$ and the claim follows by considering the highest power of y, then the next etc.

So, $\mathcal{M}_{m\times m_j}$ is now in the needed form. By applying the procedure as above for each j we arrive at the upper-block-triangular matrix:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{m_1\times m_1}(x,y) & * & * & .. \\ \mathbb{0} & \mathcal{M}_{m_2\times m_2}(x,y) & * & .. \\ \mathbb{0} & .. & .. & \mathcal{M}_{m_k\times m_k}(x,y) \end{pmatrix} \tag{20}$$

Part 4. Now we bring each $\mathcal{M}_{m_i\times m_i}$ to the needed form: $Diag_{m_i\times m_i}(x,y) + \mathcal{N}_{m_i\times m_i}(x)$. This is done by induction, at k'th step we do this for $jet_k(\mathcal{M}_{m_i\times m_i})$.

First observe that the linear part of each $\mathcal{M}_{m_i\times m_i}$ is non-degenerate and can be brought to the form: $y\mathbb{1}$ + xT.. where T is a constant strictly upper triangular matrix whose only possibly non-zero values are right over the diagonal. So, assume $\mathcal{M}_{m_i\times m_i}$ in this form, hence in $jet_1\mathcal{M}_{m_i\times m_i}$ the y dependence in on the diagonal only.

Suppose $U_2 jet_2(\mathcal{M}_{m_i\times m_i})V_2$ satisfies the assumption, denote $\mathcal{M}^{(2)} := U_2\mathcal{M}_{m_i\times m_i}V_2$. In general, if $\mathcal{M}^{(n)}$ satisfies the assumption for jet_n and $U_{n+1}\mathcal{M}_n V_{n+1}$ satisfies the assumption for jet_{n+1} define $\mathcal{M}^{(n+1)} := U_{n+1}\mathcal{M}_n V_{n+1}$. Consider the limit $\lim_N (U_N U_{N-1}...U_2\mathcal{M}_{m_i\times m_i}V_2V_3..V_N)$. The limit exists (at least as a matrix of formal power series) because at k'th step only monomials of the total degree at least $k-1$ are involved.

To show the k'th step, assume $jet_{k-1}(\mathcal{M}^{(k-1)})$ satisfies the condition. Let $\mathcal{M}^{(k-1)} = A(x) + yB(x,y)$, by the assumption the non-diagonal entries of B are of order at least k, the under diagonal entries of $\mathcal{M}^{(k-1)}$ have order at least 2. Apply to $\mathcal{M}^{(k-1)}$ the following row subtractions:
$\star$ subtract from the second row the first row multiplied by B_{21},
$\star$ subtract from the third row the first row multiplied by B_{31},
$\star$ ...

Note that these operations do not change the $jet_{k-1}(\mathcal{M}^{(k-1)})$. But now the first column of the so obtained matrix satisfies the condition for jet_k. Now act on the second column (below the diagonal): subtract from the third row the second row multiplied by B_{32}, subtract from the fourth row the third row multiplied by B_{42} etc.

By the assumption these operations do not change jet_{k-1} of the matrix, do not change jet_k of the first column. After these operations jet_k of the under diagonal part of the second column satisfies the condition. Continue for other columns, till one gets a matrix whose jet_{k-1} equals $jet_{k-1}(\mathcal{M}^{(k-1)})$, while jet_k of the elements below the diagonal does not depend on y.

Now do the same for the elements above the diagonal, but in the following order:
$\star$ bring the first row (its part above the diagonal) to the needed form, by subtracting rows multiplied by $B_{1,i}$,
$\star$ bring the second row (its part above the diagonal) to the needed form, by subtracting rows multiplied by $B_{2,i}$,
$\star$ ...

The so obtained matrix satisfies the condition for jet_k, i.e. we have constructed the $(k-1)\to k$ step of induction. As explained above the limit of such a process exists, at least as a matrix of formal power series. Now invoke Artin's approximation lemma from §1.3.1 to obtain: there exists a transformation $\mathcal{M}_{m_i\times m_i} \to U\mathcal{M}_{m_i\times m_i}V$, such that $U\mathcal{M}_{m_i\times m_i}V = Diag_{m_i\times m_i}(x,y) + \mathcal{N}_{m_i\times m_i}(x)$ and all the matrices have locally analytic entries.

Part 5. Finally we kill all the y-dependence in the blocks above the diagonal.

Apply the induction similar to the one above, at k'th step we obtain a matrix whose jet_k has no y-dependence above the diagonal. As the diagonal blocks $\mathcal{M}_{m_i\times m_i}$ already have the needed form, we consider the non-diagonal blocks: $\mathcal{M}_{m_i\times m_j}$ for $j>i$.

Suppose $\mathcal{M}^{(k-1)}$ satisfies the assumption for jet_{k-1}, present its entries as $\alpha_{ij}(x) + y\beta_{ij}(x,y)$. Consider the part of the column $\mathcal{M}_{i,m_1+1}$ for $1\le i\le m_1$. To kill the y-dependence subtract from the rows $1,2...m_1$ the row m_1 multiplied by $\beta_{im_1}(x,y)$.

Do the same for the part of the column $\mathcal{M}_{i,m_1+2}$ for $1\le i\le m_1$, subtracting the row m_1+1 multiplied by $\beta_{i,m_1+1}(x,y)$, etc.

After these operations are done for all the blocks above the diagonal we get the matrix $\mathcal{M}^{(k)}$ with the properties:
$\star$ $\mathcal{M}^{(k)}$ is upper block-diagonal, its diagonal blocks $\mathcal{M}_{m_i,m_i}$ are of the form $Diag_{m_i\times m_i}(x,y)+\mathcal{N}_{m_i\times m_i}(x)$.
$\star$ The off diagonal entries of $jet_k(\mathcal{M}^{(k)})$ do not depend on y. ■

3.3. An application: weakly maximal determinantal representations of the reduced singularity $\prod\limits_{i=1}^{k}(y+\alpha_i x^{l_i})$. These are k smooth branches with various pairwise tangency.

Corollary 3.6. Any weakly maximal determinantal representation of $\prod\limits_{i=1}^{k}(y+\alpha_i x^{l_i})$ is locally equivalent to:

$$\begin{pmatrix} y+\alpha_1 x^{l_1} & \beta_1 x^{n_1} & h_{13}(x) & .. & .. & h_{1n}(x)\\ 0 & y+\alpha_2 x^{l_2} & \beta_2 x^{n_2} & h_{24}(x) & .. & h_{2n}(x)\\ .. & .. & .. & .. & & \\ 0 & .. & .. & .. & 0 & y+\alpha_k x^{l_k}\end{pmatrix} \tag{21}$$

with $1\leq n_i<\min(l_i,l_{i+1})$ and $\beta_i\in\{0,1\}$
and either $h_{ij}(x)\equiv 0$ or $h_{ij}(x)$ a polynomial in x such that $ord_x(h_{ij})\geq 2$ and $deg(h_{ij})<\min(l_i,l_j)$.

Example 3.7. • For $k=2$ we get the possible determinantal representations of $(y+\alpha_1x^{l_1})(y+\alpha_2x^{l_2})$:

$$\mathcal{M}\sim\begin{pmatrix} y+\alpha_1x^l & x^l\\ 0 & y+\alpha_2x^{l_2}\end{pmatrix},\ 1\leq l<\min(l_1,l_2) \text{ or } \mathcal{M}\sim\begin{pmatrix} y+\alpha_1x^l & 0\\ 0 & y+\alpha_2x^{l_2}\end{pmatrix} \tag{22}$$

Compare to [Bruce-Tari04, Table 2].
• For $k=3$ consider the weakly maximal determinantal representations of $y(y+x^2)(y+\alpha x^l)$, for $l>1$. Suppose in the corollary $\beta_1\neq 0\neq\beta_2$, so $\mathcal{M}\sim\begin{pmatrix} y & x & h(x)\\ 0 & y+x^2 & x\\ 0 & 0 & y+\alpha x^l\end{pmatrix}$ with $ord_xh(x)\geq 2$.
Subtracting from the third column the second multiplied by $h(x)/x$, to kill $h(x)$. Now multiply the matrix from the right by a diagonal matrix, to get:

$$\mathcal{M}\sim\begin{pmatrix} y & x & 0\\ 0 & y+x^2 & x\\ 0 & 0 & y+\alpha x^l\end{pmatrix} \tag{23}$$

Proof. of corollary 3.6. Use the proposition 3.5 for the curve decomposition $\prod\limits_{i=1}^{k}(y+\alpha_i x^{l_i})$ to achieve the upper triangular form, such that elements over the main diagonal depend on x only.

Consider the diagonal $(i,i+1)$. Represent each nonzero element $\mathcal{M}_{i,i+1}(x)$ as $x^{n_i}\tilde{\mathcal{M}}_{i,i+1}$, where $\tilde{\mathcal{M}}_{i,i+1}|_{(0,0)}\neq 0$, i.e. is locally invertible. If $n_i\geq l_i$ then by adding the i'th column to the column $(i+1)$ and subtracting the row $(i+1)$ from the row i the $x-$order can be increased. Continue this process inductively, as in proofs of the previous propositions. Hence, if for some element $\mathcal{M}_{i,i+1}$ the $x-$order is at least l_i or l_{i+1} the element can be just set to zero. The remaining non-zero elements $x^{n_i}\tilde{\mathcal{M}}_{i,i+1}$ are set to x^{n_i} by the conjugation $\mathcal{M}\to U^{-1}\mathcal{M}U$ with

$$U=\begin{pmatrix} \prod\limits_{i\geq 1}\tilde{\mathcal{M}}_{i,i+1} & 0 & .. & 0\\ 0 & \prod\limits_{i\geq 2}\tilde{\mathcal{M}}_{i,i+1} & .. & 0\\ 0 & .. & .. & 0 & \tilde{\mathcal{M}}_{k-1,k}\end{pmatrix} \tag{24}$$

Regarding the remaining entries $h_{ij}(x)$ with $j-i\geq 2$, bring them to the needed form diagonal-by-diagonal. This is again done by the standard procedure: add $y+x^{l_i}$, subtract $y+x^{l_j}$ etc. ■

3.4. **Some additional criteria.** Weakly maximal determinantal representations are *not completely decomposable* in general, cf. example 3.11. We give some criteria.

Proposition 3.8. *1. Suppose $(C,0)=\cup(C_i,0)$ has only smooth reduced branches and the (normalized) limits of the kernel sections are linearly independent: $Span(\cup\lim s_i)=\oplus Span(\lim s_i)$. Then $\mathcal{M}$ is decomposable.*
2. Let $(C,0)=\cup(p_iC_i,0)$ be the local decomposition into branches $C_i=\{f_i=0\}$ with $f=\prod f_i^{p_i}$. A weakly maximal determinantal representation decomposes locally ($\mathcal{M}\sim\oplus\mathcal{M}_i$) iff the adjoint matrix can be written as $\mathcal{M}^\vee=\sum\frac{f}{f_i^{p_i}}\mathcal{M}^\vee{}_i$.

Proof. **1.** By the assumption $\mathcal{M}|_{(0,0)}=\mathbb{0}$ and the dimensionality equals the number of (smooth) branches, i.e. the multiplicity of the singularity. So, $\mathcal{M}$ is weakly maximal , hence is decomposable corresponding to the tangential decomposition (theorem 3.1). Thus can assume $(C,0)$ has only one tangent line, i.e. all the branches are tangent.

Now, by corollary 3.6 can bring $\mathcal{M}$ to the upper triangular form, such that the entries over the diagonal depend on x only and their degrees are bounded from above.

But by the assumption $Span(\cup\lim s_i)=\oplus Span(\lim s_i)$, this bounds the degrees from below, contradicting the upper bounds as above. Hence there are no terms over the diagonal.

2. It is enough to prove the decomposability for the case $(C,0)=(C_1,0)\cup(C_2,0)$, where $(C_i,0)$ are possibly reducible, non-reduced, but without common components (i.e. $C_1\cap C_2$ is finite). Let f, f_1, f_2 be the defining functions of the germs, so f_1, f_2 are relatively prime and $f=f_1f_2$. Let m,m_1,m_2 be the corresponding multiplicities at the origin, so by weak maximality $ord(\mathcal{M}^\vee)\geq(m-1)$ and $ord(\mathcal{M}^\vee{}_i)\geq(m_i-1)$.

Multiply $\mathcal{M}^\vee=\frac{f}{f_1}\mathcal{M}^\vee{}_1+\frac{f}{f_2}\mathcal{M}^\vee{}_2$ by $\mathcal{M}$, then one has:

$$f\mathbb{1}=\mathcal{M}\mathcal{M}^\vee=\frac{\mathrm{f}}{\mathrm{f}_1}\mathcal{M}\mathcal{M}^\vee{}_1+\frac{\mathrm{f}}{\mathrm{f}_2}\mathcal{M}\mathcal{M}^\vee{}_2 \tag{25}$$

So, $\mathcal{M}\mathcal{M}^\vee{}_i$ is divisible by f_i. Therefore can define the matrices $\{A_i\}$, $\{B_i\}$ by $f_iA_i:=\mathcal{M}\mathcal{M}^\vee{}_i$ and $f_iB_i:=\mathcal{M}^\vee{}_i\mathcal{M}$. By definition: $\sum A_i=\mathbb{1}$ and $\sum B_i=\mathbb{1}$. We prove that in fact $\oplus A_i=\mathbb{1}$ and $\oplus B_i=\mathbb{1}$. The key ingredient is the identity:

$$\mathcal{M}^\vee{}_jf_iA_i=\mathcal{M}^\vee{}_j\mathcal{M}\mathcal{M}^\vee{}_i=f_jB_j\mathcal{M}_i \tag{26}$$

It follows that $\mathcal{M}^\vee{}_jA_i$ is divisible by f_j and thus $jet_{m_j-1}(\mathcal{M}^\vee{}_jA_i)=0$ for $i\neq j$. Hence, due to the orders of $\mathcal{M}^\vee{}_j$,$\mathcal{M}$ we get: $jet_{m_j}(\mathcal{M}\mathcal{M}^\vee{}_jA_i)=jet_{m_j}(f_jA_jA_i)=0$, implying:

$$jet_0(A_j)jet_0(A_i)\overset{for\ i\neq j}{=}\mathbb{0},\ \text{and}\ \sum jet_0(A_i)=\mathbb{1}\ \Rrightarrow\ \mathbb{1}=\oplus\mathrm{jet_0(A_i)} \tag{27}$$

The equivalence $\mathcal{M}\to U\mathcal{M}V$ results in: $A_i\to UA_iU^{-1}$ and $B_j\to VB_jV^{-1}$. So, by the conjugation by (constant) matrices can assume the block form: $jet_0(A_1)=\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{0}\end{pmatrix}$ and $jet_0(A_2)=\begin{pmatrix}\mathbb{0}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}$.

Apply further conjugation to remove the terms of A_i in the columns of the i'th block to get:

$$A_1=\begin{pmatrix}\mathbb{1}&*\\\mathbb{0}&*\end{pmatrix},\ A_2=\begin{pmatrix}*&\mathbb{0}\\*&\mathbb{1}\end{pmatrix} \tag{28}$$

Finally, use $A_1+A_2=\mathbb{1}$ to obtain $A_1=\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{0}\end{pmatrix}$ and $A_2=\begin{pmatrix}\mathbb{0}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}$.

Do the same procedure for B_i's, this keeps A_i's intact. Now use the original definition, to write: ${\mathcal{M}^\vee}_i = \frac{1}{f}\mathcal{M}^\vee f_i A_i$ and ${\mathcal{M}^\vee}_i = f_i B_i \frac{1}{f}\mathcal{M}^\vee$. This gives:

$$\mathcal{M}^\vee = \oplus \frac{f}{f_i} {\mathcal{M}^\vee}_i \tag{29}$$

■

Remark 3.9. The smoothness of the branches in the first statement is important. For example, consider $\mathcal{M} = \begin{pmatrix} x^a & y^{d+1} \\ y^c & x^b y \end{pmatrix}$, a determinantal representation of $y(x^{a+b} - y^{c+d})$ for $(a+b, c+d) = 1$. Assume also $c > 1$ and $d > 0$. This determinantal representation is not equivalent to an upper-triangular. Otherwise one would have $I_1(\mathcal{M}) \ni y$.

On the other hand the limits of the kernel sections are linearly independent. $\mathcal{M}^\vee = \begin{pmatrix} x^b y & -y^{d+1} \\ -y^c & x^a \end{pmatrix}$. So, on $y = 0$ the kernel is generated by $\begin{pmatrix} 0 \\ x^a \end{pmatrix}$, whose limit is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. On $x^{a+b} = y^{c+d}$ both columns of $\mathcal{M}^\vee$ are non-zero, but linearly dependent. So, for $a > d+1$ or $c - 1 > b$ their (normalized) limit at the origin is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3.5. **Maximal determinantal representations.**

Theorem 3.10. *Let $\mathcal{M}_{p\times p}$ be maximal determinantal representation of (possibly non-reduced) $(C,0)$. As always assume $\mathcal{M}|_{(0,0)} = \mathbb{0}$. Then:*
1. The representation is weakly maximal , i.e. $p = mult(C,0)$.
2. The representation is completely decomposable, i.e. for the branch decomposition $(C,0) = \cup(p_i C_i, 0)$ one has: $\mathcal{M} \approx \oplus \mathcal{M}_i$ with $\mathcal{M}_i$ the (maximal) representation of $(p_i C_i, 0)$.
3. In particular, if $(C,0) = f^{-1}(0)$ is a reduced curve with smooth branches, and $f = \prod f_i$ is the branch decomposition, any maximal determinantal representation is locally equivalent to the diagonal $\begin{pmatrix} f_1 & 0 & .. & \\ 0 & f_2 & 0 & .. \\ .. & .. & .. & \\ 0 & .. & 0 & f_k \end{pmatrix}$.
4. Conversely, if a determinantal representation is completely decomposable ($\mathcal{M} \sim \oplus \mathcal{M}_i$ according to $(C,0) = \cup(p_i C_i, 0)$) then it is maximal iff each $\mathcal{M}_i$ is. In particular if $(C,0)$ is reduced and all the branches are smooth then the complete local decomposability implies maximality.

Proof. 1. By the maximality assumption and property 2.7: $ord(\mathcal{M}^\vee) \geq mult(C,0) - 1 = m - 1$.
Thus $ord(\det \mathcal{M}^\vee) \geq p(m-1)$. But $\det \mathcal{M}^\vee = f^{p-1}$, giving: $p(m-1) \leq (p-1)m$, i.e. $m \leq p$. As $\mathcal{M}|_{(0,0)} = \mathbb{0}$ have $m \geq p$ and the claim follows.
2. Immediate corollary of the property 2.7 and the proposition 3.8.
The other statements are immediate. ■
The following example shows that weak maximality implies neither local decomposability nor local maximality (even for a branch).

Example 3.11. *The case of two branches.* Consider the weakly maximal local representation of $\{y^2 - x^4 = 0\}$

$$\mathcal{M} = \begin{pmatrix} y - x^2 & x \\ 0 & y + x^2 \end{pmatrix} \tag{30}$$

To see that $\mathcal{M}$ is not locally decomposable note that $I_1(\mathcal{M}) =< y, x >$. If $\mathcal{M} \sim \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ then f_i are the equations of branches and $x \notin < f_1, f_2 >$.

The case of one branch. Consider the following weakly maximal representation of $\det \mathcal{M} = y^2 - x^d$, for d odd: $\mathcal{M}_k = \begin{pmatrix} y & x^k \\ x^{d-k} & y \end{pmatrix}$. It is not locally maximal for $k < \frac{d}{2} - 1$ as $I_1(\mathcal{M}^\vee{}_k) \not\subset Adj_{\tilde{C}/C}$.

A natural question is the existence of maximal determinantal representations or even the description of all of them. By the complete decomposability as above the question is completely reduced to the case of a multiple branch, i.e. $\{f^p = 0\} \subset (\mathbb{C}^2, 0)$ for $f \in \mathbb{C}\{x, y\}$ irreducible. We answer this in the reduced case.

Theorem 3.12. • *If a locally irreducible, reduced curve $(C,0) \subset (\mathbb{C}^2, 0)$ admits a maximal determinantal representation then $(C, 0)$ is equisingular to $x^p + y^q = 0$*
• *If $(C, 0)$ is equisingular to $x^p + y^q = 0$ then it admits precisely one maximal determinantal representation, up to equivalence, and this determinantal representation can be chosen symmetric.*

4. The minimal liftings

Recall that any torsion free module on a reduced curve has the minimal lifting 2.2.4. This is tightly related to (weak) maximality and decomposability.

Theorem 4.1. *1. Suppose $(C, 0)$ is reduced. Let s_1, s_2 be two local sections of the kernel, i.e. vectors in $\mathcal{O}^{\oplus p}_{(C,0)}$. Consider the normalized (non-zero) limits at the origin: $\lim s_i \in \mathbb{C}^p$. If $\lim s_1 \neq \lim s_2$ then $(C, 0) = (C_1, 0) \cup (C_2, 0)$ and the minimal lifting separates these components.*
Vice versa, if the minimal lifting separates the branches $(C_1, 0)$ and $(C_2, 0)$ then the (normalized) limits of s_1, s_2 are distinct.
2. Let $(C, 0) = (C_1, 0) \cup (C_2, 0)$ be a local decomposition, here $(C_i, 0)$ can be further reducible but without common components. Suppose $\mathcal{M}$ can be brought to the corresponding upper-block-triangular form $\mathcal{M} \sim \begin{pmatrix} \mathcal{M}_1 & .. \\ \mathbb{0} & \mathcal{M}_2 \end{pmatrix}$. Let $(C', 0) \stackrel{\nu}{\to} (C, 0)$ be the minimal modification such that the pullback $\nu^*(E_C)/Torsion$ is locally free. Then ν separates C_1 and C_2: $(C', 0) = \nu^{-1}(C_1, 0) \coprod \nu^{-1}(C_2, 0)$.
3. In particular the minimal lifting of a weakly maximal determinantal representation separates all the distinct branches.
4. Suppose $\mathcal{M}$ is local maximal and $(C, 0)$ reduced. The minimal morphism $(C', 0) \stackrel{\nu}{\to} (C, 0)$, for which the lifting $\nu^* E/Torsion$ is locally free, is the normalization.

Proof. 1. If a sheaf of rank 1 is locally free then any two sections have the same normalized limit.
Vice versa, if $\lim s_1 = \lim s_2$ then the lifting of $(C_1, 0) \cup (C_2, 0)$ to $(\tilde{C}_1, 0) \cup (\tilde{C}_2, 0)$ (two normalized branches, intersecting transversally) makes the kernel locally free.
2. The adjoint matrix of $\begin{pmatrix} \mathcal{M}_1 & A \\ \mathbb{0} & \mathcal{M}_2 \end{pmatrix}$ is: $\begin{pmatrix} \mathcal{M}^\vee{}_1 \det(\mathcal{M}_2) & -\mathcal{M}^\vee{}_1 A \mathcal{M}^\vee{}_2 \\ \mathbb{0} & \det(\mathcal{M}_1)\mathcal{M}^\vee{}_2 \end{pmatrix}$. Here $A|_{(0,0)} = \mathbb{0}$, i.e. $ord(A) > 0$.
Let $C' \stackrel{\nu}{\to} C$ be the minimal modification such that $\nu^*_{C'/C}(E)/Torsion$ is locally free. Assume $(C_1, 0)$ and $(C_2, 0)$ do not get separated by ν, then $\nu^*(\det \mathcal{M}_1) \neq 0 \in \mathcal{O}_{C'}$. So $\nu^* \det(\mathcal{M}_1)$ cannot annihilate the generators of $\nu^*_{C'/C}(E)/Torsion$.

No combination of the columns of $\nu^* \begin{pmatrix} \mathcal{M}^\vee{}_1 \det(\mathcal{M}_2) \\ \mathbb{0} \end{pmatrix}$ can be a generator of $\nu^*_{C'/C}(E)$ as they all are annihilated by $\det \mathcal{M}_1$. Hence all the generators of the free module $\nu^*_{C'/C}(E)$ are some combinations of the columns of $\nu^* \begin{pmatrix} -\mathcal{M}^\vee{}_1 A \mathcal{M}^\vee{}_2 \\ \det(\mathcal{M}_1)\mathcal{M}^\vee{}_2 \end{pmatrix}$. Then in particular:

$$\nu^* \begin{pmatrix} \mathcal{M}^\vee{}_1 \det(\mathcal{M}_2) \\ \mathbb{0} \end{pmatrix} = \nu^* \begin{pmatrix} -\mathcal{M}^\vee{}_1 A \mathcal{M}^\vee{}_2 \\ \det(\mathcal{M}_1)\mathcal{M}^\vee{}_2 \end{pmatrix} B \tag{31}$$

for some matrix B. Thus $\nu^*\mathcal{M}^\vee{}_2B$ is divisible by $\nu^*\det(\mathcal{M}_2)$, contradicting $-\nu^*(\mathcal{M}^\vee{}_1A\mathcal{M}^\vee{}_2B) = \nu^*(\mathcal{M}^\vee{}_1\det(\mathcal{M}_2))$, because $ord(A) > 0$.

3. By the proposition 3.5 each weakly maximal determinantal representation can be brought to the upper triangular form $\begin{pmatrix} \mathcal{M}_1 & .. & \\ .. & .. & .. \\ \mathbb{0} & \mathbb{0} & \mathcal{M}_k \end{pmatrix}$ corresponding to the branch decomposition $(C,0) = \cup_{i=1}^k (p_iC_i, 0)$.

■

Remark 4.2.

• If the minimal lifting is the normalization, the determinantal representation is not necessarily weakly maximal or equivalent to upper triangular, even if $(C,0)$ is an ordinary multiple point. Recall the example from remark 3.4. The determinantal representation $\begin{pmatrix} x^{p-1}y & x^p - y^p \\ x^p + y^p & xy^{p-1} \end{pmatrix}$ defines the ordinary multiple point: $\{x^py^p + y^{2p} - x^{2p} = 0\}$, the union of lines.

Note that both $\mathcal{M}$ and $\mathcal{M}^\vee$ are homogeneous, so the kernel sections are proportional to constant vectors. The limits of any two kernel sections are distinct, so the minimal lifting separates all the branches, i.e. is the normalization. But the determinantal representation is not weakly maximal (e.g. because $I(M)$is generated by four elements). The problem here is that the limits, though distinct, are not linearly independent.

• For a locally irreducible curve (a branch) the weak maximality does not seem to impose restrictions on the minimal lifting. For example consider the representation $\begin{pmatrix} x & y^k \\ y^l & x \end{pmatrix}$ with $k < l$ and $(k,l) = 1$. Here $\mathbb{C}[x,y] = \mathbb{C}[t^{k+l}, t^2] \subset \mathbb{C}[t]$ and the kernel is generated by $\begin{pmatrix} x \\ -y^l \end{pmatrix}$, $\begin{pmatrix} -y^k \\ x \end{pmatrix}$. Hence the minimal lifting is $\mathbb{C}[t^2, t^{k+l}] \subset \mathbb{C}[t^2, t^{k-l}] = \mathbb{C}[x, y, \frac{x}{y^k}]$, which can be the normalization or just one blowup.

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