# EXAMPLES OF NON-POLYGONAL LIMIT SHAPES IN I.I.D. FIRST-PASSAGE PERCOLATION AND INFINITE COEXISTENCE IN SPATIAL GROWTH MODELS

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Abstract. We construct an edge-weight distribution for i.i.d. first-passage percolation on $\mathbb{Z}^2$ whose limit shape is not a polygon and has extreme points which are arbitrarily dense in the boundary. Consequently, the associated Richardson-type growth model can support coexistence of a countably infinite number of distinct species, and the graph of infection has infinitely many ends.

## 1. Introduction

Throughout this note $\mu$ denotes a Borel probability measure on $\mathbb{R}^+$ with finite mean, and $\mathcal{M}$ is the family of such measures. Let $\mathbb{E}$ denote the set of nearest-neighbor edges of the lattice $\mathbb{Z}^2$, and let $\{\tau_e : e \in \mathbb{E}\}$ be a family of i.i.d. random variables with marginal $\mu$ and joint distribution $\mathbb{P} = \mu^{\mathbb{E}}$. The *passage time* of a path $\gamma = (e_1, \ldots, e_n) \in \mathbb{E}^n$ in the graph $(\mathbb{Z}^2, \mathbb{E})$ is $\tau(\gamma) = \sum_{i=1}^n \tau_{e_i}$, and for $x, y \in \mathbb{Z}^2$ the *passage time* from $x$ to $y$ is

$$\tau(x, y) = \min_{\gamma} \tau(\gamma),$$

where the minimum is over all paths $\gamma$ joining $x$ to $y$. A minimizing path is called a *geodesic* from $x$ to $y$.

The theory of first passage percolation (FPP) is concerned with the large-scale geometry of the metric space $(\mathbb{Z}^2, \tau)$. The following fundamental result concerns the asymptotic geometry of balls. Write $B(t) = \{x \in \mathbb{Z}^2 \ : \ \tau(0, x) \leq t\}$ for the ball of radius $t$ at the origin, and for $S \subseteq \mathbb{R}^2$ and $a \geq 0$, write $aS = \{ax \ : \ x \in S\}$.

**Theorem 1.1** (Cox-Durrett [1]). *For each $\mu \in \mathcal{M}$ there exists a deterministic, convex, compact set $B_\mu$ such that for any $\varepsilon > 0$,*

$$\mathbb{P}\left( (1 - \varepsilon) B_\mu \subseteq \frac{1}{t} B(t) \subseteq (1 + \varepsilon) B_\mu \text{ for all large } t \right) = 1.$$

Little is known about the geometry of $B_\mu$, which is called the *limit shape*. It is conjectured to be strictly convex when $\mu$ is non-atomic, and non-polygonal in all but the most degenerate cases, but, in fact, there are currently no known examples of $\mu$ for which these properties are verified (see [9]). For a compact, convex set $C \subseteq \mathbb{R}^2$ write $\text{ext}(C)$ for the set of extreme points and $\text{sides}(C) = |\,\text{ext}(C)|$, so that $C$ is a polygon if and only if $\text{sides}(C) < \infty$. The best result to date, due to Marchand [7], is that under mild

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assumptions, $\mathrm{sides}(B_\mu) \geq 8$. Building on results of Marchand, our purpose of this note is to give the first examples of distributions for which the limit shape is not a polygon.

**Theorem 1.2.** *For every $\varepsilon > 0$ there exists $\mu \in \mathcal{M}$ (with atoms) such that $B_\mu$ is not a polygon, i.e., $\mathrm{sides}(B_\mu) = \infty$, and $\mathrm{ext}(B_\mu)$ is $\varepsilon$-dense in $\partial B_\mu$. There exist continuous $\mu$ such that $\mathrm{sides}(B_\mu) > 1/\varepsilon$ and $\mathrm{ext}(B_\mu)$ is $\varepsilon$-dense in $\partial B_\mu$.*

It is tempting to try to obtain a strictly convex limit shape by taking a limit of measures $\mu_n$ such that $B_{\mu_n}$ have progressively denser sets of extreme points, but unfortunately the limit one gets in our example is the unit ball of $\ell^1$.

We also obtain examples of measures $\mu$ such that, at the points $v \in \mathrm{ext}(B_\mu)$ which lie on the boundary of the $\ell^1$-unit ball, $\partial B_\mu$ is infinitely differentiable. This should be compared with the work of Zhang [10], where such behavior was ruled out for certain $\mu$.

Theorem 1.2 has implications for the Richardson growth model, whose definition we recall next. Fix $x_1, \ldots, x_k \in \mathbb{Z}^2$ and imagine that at time 0 the site $x_i$ is inhabited by a species of type $i$. Each species spreads at unit speed, taking time $\tau_e$ to cross an edge $e \in \mathbb{E}$. An uninhabited site is exclusively and permanently colonized by the first species that reaches it, i.e., $y \in \mathbb{Z}^2$ is occupied at time $t$ by the $i$-th species if $\tau(y, x_i) \leq t$ and $\tau(y, x_i) < \tau(y, x_j)$ for all $j \neq i$. This is well-defined when there are *unique geodesics*, i.e., $\mathbb{P}$-a.s. no two paths have the same passage time, as is the case when $\mu$ is continuous, but we shall also want to consider measures $\mu$ with atoms, so we require a mechanism to break ties. For simplicity we introduce a worst-case model: if two species $i \neq j$ reach an unoccupied site $x$ at the same instant then $x$ is colonized by a species of type $-1$, which spreads according to the same rules as the others. Under this convention if a site is occupied by the species $i \neq -1$ then it would be so occupied under any other tie-breaking rule.

Given initial sites $x_1, \ldots, x_k$, consider the set colonized by the $i$-th species:

$$C_i = \{y \in \mathbb{Z}^2 : y \text{ is eventually occupied by } i\}.$$

One says that $\mu$ *admits coexistence of $k$ species* if for some choice of $x_1, \ldots, x_k$,

$$\mathbb{P}(|C_i| = \infty \text{ for all } i = 1, \ldots, k) > 0.$$

Coexistence of infinitely many species is defined similarly.

It is not known, even in simple examples, how many species can coexist. When $\mu$ is the exponential distribution, Häggström and Pemantle [5] proved coexistence of 2 species, and for a broad class translation-invariant measures on $(0, \infty)^{\mathbb{E}}$, including some non-i.i.d. ones, Hoffman [6] demonstrated coexistence of 8 species by establishing a relation with the number of sides of the limit shape in the associated FPP. Using the same relation we obtain the following:

**Theorem 1.3.** *There exists $\mu \in \mathcal{M}$ (with atoms) which admits coexistence of infinitely many species. For each $k$ there exist continuous $\mu \in \mathcal{M}$ admitting coexistence of $k$ species.*

Finally, the *graph of infection* $K \subseteq \mathbb{E}$ is the union over $x \in \mathbb{Z}^d$ of the edges of geodesics from 0 to $x$. If $\mu$ is continuous this is a.s. a tree. A graph has $m$ *ends* if, after removing a finite set of vertices, the induced graph contains at least $m$ infinite connected components, and, if there are $m$ ends for every $m \in \mathbb{N}$, we say there are infinitely many ends. Newman [8] has conjectured for a broad class of $\mu$ that $K$ has infinitely many ends. Hoffman [6] showed for continuous distributions that in general there are a.s. at least 4 ends.

**Theorem 1.4.** *There exist $\mu \in \mathcal{M}$ (with atoms) such that $\mathbb{P}$-a.s., $K$ has infinitely many ends. For each $k$ there exist continuous $\mu \in \mathcal{M}$ such that $\mathbb{P}$-a.s., $K$ has at least $k$ ends.*

When $\mu$ is continuous Theorems 1.3 and 1.4 follow, respectively, from Theorem 1.2 and from Theorems 1.4 and 1.6 of Hoffman [6]. For the non-continuous case we provide the necessary modifications of Hoffman's arguments in Section 4.

## 2. Background on the limit shape

Endow $\mathcal{M}$ with the topology of weak convergence and for convenience fix a compatible metric $d(\cdot,\cdot)$ on $\mathcal{M}$. Next, fix the $\ell^1$-metric on $\mathbb{R}^2$, and write $A^{(\varepsilon)}$ for the $\varepsilon$-neighborhood of $A \subseteq \mathbb{R}$. Let $\mathcal{C}$ denote the space of non-empty, closed, convex subsets of $\mathbb{R}^2$ endowed with the Hausdorff metric $d_H$:

$$d_H(A, B) = \inf\{\varepsilon \ : \ A \subseteq B^{(\varepsilon)} \text{ and } B \subseteq A^{(\varepsilon)}\}.$$

**Theorem 2.1** (Cox-Kesten [2])**.** *The map $\mu \mapsto B_\mu$ from $\mathcal{M}$ to $\mathcal{C}$ is continuous.*

It is elementary to verify that for $A \in \mathcal{C}$, the map $A \mapsto \mathrm{ext}(A)$ is semi-continuous in the sense that, given $x \in \mathrm{ext}(A)$ and $\varepsilon > 0$, there is a $\delta > 0$ such that if $A' \in \mathcal{C}$ and $d_H(A, A') < \delta$ then there exists $x' \in \mathrm{ext}(A')$ with $\|x - x'\|_1 < \varepsilon$. Combined with the continuity theorem above, we have:

**Corollary 2.2.** *Let $\mu \in \mathcal{M}$. For every $x_1, \dots, x_k \in \mathrm{ext}(B_\mu)$ and $\varepsilon > 0$ there is a $\delta > 0$ such that, if $\nu \in \mathcal{M}$ and $d(\nu, \mu) < \delta$ then there are $y_1, \dots, y_k \in \mathrm{ext}(B_\nu)$ such that $\|x_i - y_i\| < \varepsilon$ for $i = 1, \dots, k$.*

We next recall some results about limit shapes for a special class of measures. Given $0 < p < 1$, let $\mathcal{M}_p \subseteq \mathcal{M}$ denote the set of measures $\mu \in \mathcal{M}$ with an atom of mass $p$ located at $x = 1$, i.e., $\mu(\{1\}) = p$, and no mass to the left of 1, i.e., $\mu((-\infty, 1)) = 0$. Limit shapes for $\mu$ of this form were first studied in Durrett and Liggett [4]. Writing $\vec{p}_c$ for the critical parameter of oriented percolation on $\mathbb{Z}^2$ (see Durrett [3] for background), it was shown that when $p > \vec{p}_c$ and $\mu \in \mathcal{M}_p$, the limit shape $B_\mu$ contains a "flat edge", or, more precisely, $\partial B_\mu$ has sides which lie on the boundary of the $\ell^1$-unit ball. The nature of this edge was fully characterized in [7]. For $p \geq \vec{p}_c$, let $\alpha_p$ be the asymptotic speed of super-critical oriented percolation on $\mathbb{Z}^2$ with parameter $p$ (see [3]). Define points

$w_p, w'_p \in \mathbb{R}^2$ by

$$\begin{aligned} w_p &= (1/2 + \alpha_p/\sqrt{2}\ ,\ 1/2 - \alpha_p/\sqrt{2}), \\ w'_p &= (1/2 - \alpha_p/\sqrt{2}\ ,\ 1/2 + \alpha_p/\sqrt{2}). \end{aligned}$$

Let $[w_p, w'_p] \subseteq \mathbb{R}^2$ denote the line segment with endpoints $w_p$ and $w'_p$. It will be important to note that $\alpha_p$ is strictly increasing in $p > \vec{p}_c$, so the same is true of $[w_p, w'_p]$.

**Theorem 2.3** (Marchand [7]). *Let $\mu \in \mathcal{M}_p$. Then*

*(1) $B_\mu \subseteq \{x \in \mathbb{R}^2 \ :\ \|x\|_1 \leq 1\}$.*
*(2) If $p < \vec{p}_c$, then $B_\mu \subseteq \{x \in \mathbb{R}^2 \ :\ \|x\|_1 < 1\}$.*
*(3) If $p > \vec{p}_c$, then $B_\mu \cap [0,\infty)^2 = [w_p, w'_p]$.*
*(4) If $p = \vec{p}_c$, then $B_\mu \cap [0,\infty)^2 = \{(1/2, 1/2)\}$.*

As noted by Marchand, this implies $\text{sides}(B_\mu) \geq 8$ for $\mu \in \mathcal{M}_p$ and $\vec{p}_c < p < 1$, since $w_p, w'_p$ and their reflections about the axes are extreme points.

## 3. Proof of Theorem 1.2

Our aim is to construct a $\mu \in \mathcal{M}$ with $\text{sides}\, B_\mu = \infty$. Fix any $p_0 > \vec{p}_c$, $\mu_0 \in \mathcal{M}_{p_0}$ and $\delta_0 > 0$. We will inductively define a sequence $p_1 > p_2 > \ldots > \vec{p}_c$, measures $\mu_1 \in \mathcal{M}_{p_1}$, $\mu_2 \in \mathcal{M}_{p_2}, \ldots$, and $\delta_1, \delta_2, \ldots > 0$ such that for every $n \geq 0$ and all $k \leq n$,

(1) If $\nu \in \mathcal{M}$ and $d(\nu, \mu_k) < \delta_k$ then $\text{sides}(B_\nu) \geq k$, and
(2) $d(\mu_k, \mu_n) < \frac{1}{2}\delta_k$.

Note that (1) implies that $\text{sides}(B_{\mu_k}) \geq k$. Assuming $p_k, \mu_k$ and $\delta_k$ are defined for $k \leq n$, we define them for $n+1$. Fix $p_{n+1} \in (\vec{p}_c, p_n)$ and set $r = p_n - p_{n+1} > 0$. For $y > 1$ construct $\mu^y_{n+1}$ from $\mu_n$ by moving an amount $r$ of mass from the atom at 1 to $y$, that is,

$$\mu^y_{n+1} = \mu_n - r\delta_1 + r\delta_y.$$

We claim that for small enough $y > 1$ and any sufficiently small choice of $\delta_{n+1} > 0$ the measure $\mu_{n+1} = \mu^y_{n+1}$ has the desired properties. First, $\mu^y_{n+1} \to \mu_n$ weakly as $y \downarrow 1$, so, since $\mu_n$ satisfies (2), so does $\mu^y_{n+1}$ for all sufficiently small $y$.

Second, we claim that $\text{sides}(B_{\mu^y_{n+1}}) \geq n+1$ for $y$ close enough to 1. Indeed, since $r > 0$ we have $w_{p_{n+1}} \neq w_{p_n}$. Using (1), choose $n$ extreme points $x_1, \ldots, x_n \in \text{ext}(B_{\mu_n})$ and let

$$a = \min\left\{\|x_i - x_j\|_1, \left\|x_i - w_{p_{n+1}}\right\|_1 \ :\ i \neq j\right\}.$$

Note that $a > 0$ by Marchand's theorem. By Corollary 2.2, for $y$ close enough to 1, for each $i = 1, \ldots, n$ we can choose an extreme point $x'_i$ of $B_{\mu^y_{n+1}}$ with $\|x'_i - x_i\| < a/2$. By Marchand's theorem, $B_{\mu^y_{n+1}}$ also has an extreme point at $w_{p_{n+1}}$. By definition of $a$, these extreme points are distinct, giving $\text{sides}(B_{\mu^y_{n+1}}) \geq n+1$.

Finally, by Corollary 2.2, $\mu^y_{n+1}$ satisfies (1) for any sufficiently small choice of $\delta_{n+1}$.

Let $\mu$ be a weak limit of $\mu_n$. Then $d(\mu, \mu_n) \leq \frac{1}{2}\delta_n$ for all $n$, so by (2), $\text{sides}(B_\mu) = \infty$.

At each step, rather than creating a new atom at $y$, one can instead add, e.g., Lebesgue measure on a small interval around $y$. In this way one can make the atom at 1 be the only atom of $\mu$.

Regarding the degree of denseness of the extreme points in the boundary, note that at each stage if $y$ is small enough and $p_{n+1}$ is close enough to $p_n$, the new extreme point we introduce can be made arbitrarily close to $w_{p_n}$ (here we use that $\alpha_p$ is continuous in $p > \vec{p}_c$, from [3]), and in the limit we can ensure an extreme point close to it. Thus, if we begin from $\mu_0 = \delta_1$ and choose $p_n$ so that $\lim p_n = \vec{p}_c$, and using Marchand's result that the flat edge in $B_{\mu_n}$ then shrinks to a point (and symmetry of the limit shape about the axes), we can ensure $\varepsilon$-density of the extreme points of $B_\mu$.

For the second part of Theorem 1.2, choose a sequence $\nu_n \in \mathcal{M}$ of continuous measures converging weakly to $\mu$. By Corollary 2.2, $\mathrm{sides}(B_{\nu_n}) \to \infty$, and if $\mathrm{ext}(B_\mu)$ is $\varepsilon$-dense in $\partial B_\mu$ then the same holds for $B_{\nu_n}$ for sufficiently large $n$.

Regarding the remark after the theorem, one may verify that if at each stage $y$ is chosen close enough to 1 and $p = \lim p_n$, then $w_p$ is a $C^\infty$-point of $\partial B_\mu$.

## 4. Proof of Theorem 1.3.

Let us recall Hoffman's argument relating coexistence to the geometry of the limit shape for continuous $\mu$ (Theorem 1.6 of [6]). Extend $\tau$ to $\mathbb{R}^2 \times \mathbb{R}^2$ by $\tau(x, y) = \tau(x', y')$ where $x'$ is the unique lattice point in $x + [-1/2, 1/2)^2$. Similarly, a geodesic between $x, y$ is a geodesic between $x', y'$. For $S \subseteq \mathbb{R}^2$, the Busemann function $B_S : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$B_S(x, y) = \inf_{z \in S} \tau(x, z) - \inf_{w \in S} \tau(y, w).$$

For $v \in \mathbb{R}^2$, write $S + v = \{s + v \,:\, s \in S\}$. If $v \in \partial B_\mu$ is a point of differentiability and $w$ is a tangent vector at $v$, let $\pi_v$ denote the linear functional $av + bw \mapsto a$. Define the lower density of a set $A \subseteq \mathbb{N}$ by $\underline{d}(A) = \liminf \frac{1}{N}|A \cap \{1, \ldots, N\}|$.

**Theorem 4.1** (Hoffman [6])**.** *Let $\mu \in \mathcal{M}$ and let $v \in B_\mu$ be a point of differentiability of $\partial B_\mu$ with tangent line $L \subseteq \mathbb{R}^2$. Then for every $\varepsilon > 0$ there exists an $M = M(v, \varepsilon) > 0$ such that, if $x, y \in \mathbb{R}^2$ satisfy $\pi_v(x - y) > M$, then*

$$\mathbb{P}\Big(\underline{d}\Big(n \in \mathbb{N} \,:\, B_{L+nv}(y, x) > (1 - \varepsilon)\pi_v(x - y)\Big) > 1 - \varepsilon\Big) > 1 - \varepsilon.$$

Hoffman's proof of this result does not use unique passage times.

Theorem 4.1 is related to coexistence as follows. Suppose $\mathrm{sides}(B_\mu) \geq k$. We can then find $k$ points of differentiability $v_1, \ldots, v_k \in \partial B_\mu$ with distinct tangent lines $L_i$, and in particular $\pi_{v_i}(v_i - v_j) > 0$ for all $j \neq i$. Fix $\varepsilon > 0$ and choose $R > 0$ large enough so that the points $x_i = Rv_i$ satisfy $\pi_{v_i}(x_i - x_j) > M(v_i, \varepsilon/k^2)$. Using the elementary relation $\underline{d}(\cap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n (1 - \underline{d}(A_i))$, for each $i$ we have

$$\mathbb{P}\Big(\underline{d}\Big(n \in \mathbb{N} \,:\, B_{L+nv_i}(x_j, x_i) > 0 \text{ for all } j \neq i\Big) > 1 - \frac{\varepsilon}{k}\Big) > 1 - \frac{\varepsilon}{k},$$

and hence with positive probability (which can be made arbitrarily close to 1 by decreasing $\varepsilon$), for each $i$ there is a positive density of $n$ such that $B_{L_i+nv_i}(x_j, x_i) > 0$ for all $j \neq i$. For such an $n$, take $y_{i,n} \in L_i + nv_i$ to be the closest point (in the sense of passage times) to $x_i$; assuming unique geodesics, by definition $y_{i,n}$ is reached first by species $i$. The points $y_{i,n}$ are in $C_i$, so $|C_i| = \infty$ for $i = 1, \ldots, k$, i.e., coexistence occurs.

If geodesics are not unique the argument still applies, using the following observation. If $B_{L_i+nv_i}(x_j, x_i) > 0$, $j \neq i$, then the only way that $y_{i,n}$ could be colonized by a species other than $i$ is if it is colonized by $-1$. This can occur only if there is a site $z$ on a geodesic from $x_i$ to $y_{i,n}$ which is reached simultaneously by species $i$ and $j \neq i$. If this happens then by concatenating the geodesic from $x_j$ to $z$ with the geodesic from $z$ to $y$ we get a path from $x_j$ to $L_i + nv_i$ with passage time equal to $\tau(x_i, y_{i,n})$, so $\tau(x_j, L_i + nv_i) \leq \tau(x_i, y_{i,n})$ and hence $B_{L_i+nv_i}(x_j, x_i) \leq 0$, contrary to assumption.

When $\text{sides}(B_\mu) = \infty$, one proves coexistence of infinitely many types similarly. Choose a sequence $\{v_i\}_{i=1}^{\infty} \subseteq \partial B_\mu$ of points of differentiability of the boundary, ordered clockwise, say. Given $\varepsilon > 0$, define the points $x_i$ inductively by $x_{i+1} = x_i + R_i(v_{i+1} - v_i)$ for a sufficiently large $R_i > 0$ so as to ensure that for $i \neq j$, $\pi_{v_i}(x_i - x_j) > M(v_i, \varepsilon_{i,j})$, where $\sum_{i,j} \varepsilon_{i,j} < \varepsilon$. The rest of the argument is the same as above.

## 5. Proof of Theorem 1.4.

Recall that $K$ denotes the graph of infection. When there are unique geodesics, Hoffman's results imply that the number of ends of $K$ is at least $\text{sides}(B_\mu)/2$ (Theorem 1.4 of [6]), establishing Theorem 1.4 for non-atomic $\mu$. In this section we deal with the presence of atoms, proving the following result:

**Theorem 5.1.** *If $\mu$ is not purely atomic and has at least $s \in \mathbb{N}$ sides, then the number of ends in $K$ is $\mathbb{P}$-a.s. at least*

$$k = 4 \left\lfloor \frac{s-4}{12} \right\rfloor . \tag{5.1}$$

See below Theorem 5.3 for an explanation of this bound.

For simplicity, in the following discussion we fix a measure $\mu \in \mathcal{M}_p$ with $p < 1$, and make two further assumptions: (a) the only atom of $\mu$ is the atom at 1, and (b) $\mu$ is supported on a bounded interval $[1, R]$. The argument can be modified to deal with the general case of the theorem.

**Proposition 5.2.** *Suppose that $\mathbb{P}$-a.s. there exist $k$ infinite geodesics $\gamma_1, \ldots, \gamma_k$ starting at 0, edges $e_1, \ldots, e_k$, and a finite set $V \subseteq \mathbb{Z}^2$, such that (a) $e_i$ lies on $\gamma_i$ but not on $\gamma_j$ for $i \neq j$, (b) the endpoints of $e_i$ are in $V$, (c) $\tau_{e_i} > 1$, and (d) each pair of geodesics is disjoint outside of $V$. Then $\mathbb{P}$-a.s., $K$ has $k$ ends.*

*Proof.* Under our assumptions on $\mu$, with probability 1 every pair of edges with passage times $> 1$ has distinct passage times. Consequently, geodesics between $x, y \in \mathbb{Z}^2$ can differ only in edges $e$ with $\tau_e = 1$, and must share edges with $\tau_e > 1$. We assume we are in this probability-1 event.

We claim that no two of the given geodesics are connected in $K \setminus V$. Suppose for instance that $\gamma_1, \gamma_2$ were connected in $K \setminus V$ by a path $\sigma$ which we may assume is simple (non self-intersecting) and with endpoints $y_1 \in \gamma_1$ and $y_2 \in \gamma_2$. Denote the sequence of vertices in $\sigma$ by $y_1 = v_1, v_2, \dots, v_k = y_2$. Write $e = e_1$ and let $J \subseteq \{1, \dots, k\}$ denote the set of $j$ such that there exists a geodesic $\sigma_j$ from 0 to $v_j$ which contains $e$. We claim that $k \in J$. This leads to a contradiction because then $\sigma_k$ and $\gamma_2$ are both geodesics connecting 0 and $y_2$, but only one of them, $\sigma_k$, contains $e$.

Clearly $1 \in J$. Suppose now that $j \in J$ with corresponding geodesic $\sigma_j$. Write $f$ for the edge between $v_j$ and $v_{j+1}$, and note that $f \neq e$ because the endpoints of $e$ are in $V$ while those of $f$ are not. Also, $\tau(0, v_j) \neq \tau(0, v_{j+1})$, since $f \in K$. If $\tau(0, v_{j+1}) > \tau(0, v_j)$ then we adjoin $f$ to $\sigma_j$ and obtain a geodesic $\sigma_{j+1}$ with the desired properties. If $\tau(0, v_{j+1}) < \tau(0, v_j)$ and $v_{j+1}$ lies on $\sigma_j$ we remove $f$ from $\sigma_j$ to obtain $\sigma_{j+1}$. On the other hand, if $v_{j+1}$ does not lie on $\sigma_j$ but $\tau(0, v_{j+1}) < \tau(0, v_j)$, then there is a geodesic $\sigma'_{j+1}$ from 0 to $v_j$ whose last edge is $f$. Because $\sigma'_{j+1}$ must reach $v_j$ in the same time as $\sigma_j$ does, it must pass through each of those edges of $\sigma_j$ which have passage times $> 1$, and in particular through $e$. We remove $f$ from $\sigma'_{j+1}$ to obtain $\sigma_{j+1}$. $\square$

Our goal is to establish the hypotheses of the proposition for $k$ as in (5.1). It is enough to show that there exist random variables $m < M$ such that with probability one,

(1) There exist $k$ geodesics $\gamma_1, \dots, \gamma_k$ which are disjoint outside of $mB_\mu$.
(2) There are edges $e_i$ in $\gamma_i$, with endpoints in $MB_\mu \setminus mB_\mu$, such that $\tau_{e_i} > 1$.

This suffices because we can then set $V = MB_\mu$ in the proposition. To show that such $m, M$ exist, it is enough to show that for every $\varepsilon > 0$ there exists deterministic $m < M$ such that each of the conditions above holds on an event of probability $> 1 - \varepsilon$.

Given $u, v, w \in \partial B_\mu$ which are points of differentiability of $\partial B_\mu$, let $C(u, v, w)$ denote the open arc in $\partial B_\mu$ from $u$ to $w$ containing $v$. We rely on the following result, whose proof does not require unique geodesics:

**Theorem 5.3** (Hoffman [6]). *Let $u, v, w \in \partial B_\mu$ be points of differentiability of $\partial B_\mu$, let $L$ be the tangent line at $v$, and write $C = C(u, v, w)$. Then for every $\varepsilon > 0$ there is an $M_0 = M_0(\varepsilon)$ such that for every $M > M_0$, with probability $> 1 - \varepsilon$ the set*

$$I = \{n \in \mathbb{N} : \gamma \cap M\partial B_\mu \subseteq MC \text{ for all geodesics } \gamma \text{ from } 0 \text{ to } L + nv\}$$

*satisfies $\underline{d}(I) > 1 - \varepsilon$.*

Henceforth fix $k$ as in (5.1) and $\varepsilon > 0$ and for $i = 1, \dots, k$ choose points $u_i, v_i, w_i \in \partial B_\mu$ and lines $L_i$ as in the theorem, and such that the closed sets $C_i = \overline{C(u_i, v_i, w_i)}$ are pairwise disjoint and do not intersect the boundary of the $\ell^1$ unit ball; write $C = \bigcup_{i=1}^k C_i$. Note that $k$ was picked so that such a choice is possible: note that $\frac{1}{4}(\text{sides}(B_\mu) - 4)$ is the number of distinct sides on each of the four curves in $\partial B_\mu$ which constitute the complement of the $\ell^1$ unit ball; dividing this number by 3 gives an upper bound on the number of triples we can choose in each of these curves. Taking integer part and multiplying by 4 gives $k$.

**Claim 5.4.** *There exists $M_0$ and $\rho > 0$ such that with probability at least $1-\varepsilon$, for all $M > M_0$, every $x \in MC$ and every geodesic $\gamma$ from 0 to $x$, at least a $\rho$-fraction of the edges of $\gamma$ have passage times $> 1$.*

*Proof.* Pick $\delta > 0$ and define edge weights $\{\tau'_e : e \in \mathbb{E}\}$ by the rule that if $\tau_e > 1$ then $\tau'_e = \tau_e + \delta$ and $\tau'_e = \tau_e$ otherwise. Let $\mu'$ denote the marginal distribution of $\tau'_e$.

Choose $\eta > 0$ so that $(1-\eta)C \cap B_{\mu'} = \emptyset$ (we can do so by a theorem of Marchand [7, Theorem 1.5] and the fact that $C$ is disjoint from the $\ell^1$ unit ball). For a path $\sigma$ let $f(\sigma)$ denote the fraction of edges of $\sigma$ with passage time $> 1$. By Theorem 1.1, there is an event $A$ with $\mathbb{P}(A) > 1-\varepsilon$ and an $M_0$ such that for all $M > M_0$ and $y \in \partial B_\mu$, the $\tau$-geodesic $\gamma$ from 0 to $y$ satisfies $(1-\eta^2)M < \tau(\gamma) < (1+\eta^2)M$, and similarly for $y' \in M\partial B'_\mu$ and the $\tau'$-length of $\tau'$-geodesics from 0 to $y'$. We claim that $A$ is the desired event. Indeed, let $M > M_0$ and let $\gamma$ be a $\tau$-geodesic from 0 to some $x \in MC$. Since $\tau_e \geq 1$ for all $e \in \mathbb{E}$, the number of edges in $\gamma$ is at most $\tau(\gamma)$. Hence

$$\begin{aligned}\tau'(\gamma) &= \tau(\gamma) + \delta f(\gamma)\#\{\text{edges of } \gamma\} \\ &\leq (1+\delta f(\gamma))\tau(\gamma) \\ &\leq (1+\delta f(\gamma))(1+\eta^2)M.\end{aligned}$$

On the other hand $x = \frac{M}{s}y$ for some $y \in \partial B_{\mu'}$ and $s < 1-\eta$, so

$$\tau'(\gamma) \geq (1-\eta^2)\frac{M}{1-\eta}.$$

Combining these we find that $f(\gamma) \geq \frac{\eta}{\delta} \cdot \frac{1-\eta}{1+\eta^2}$. This lower bound on $f$ can serve as $\rho$. $\square$

Let $\alpha > 0$ be the quantity

$$\alpha = \frac{1}{2}\min\{\pi_{v_i}(x_i - x_j) : x_i \in C_i\, , \, x_j \in C_j\, , \, i \neq j\}. \tag{5.2}$$

and fix finite, $\frac{\alpha}{100R}$-dense sets $D_i \subseteq C_i$. Let $\rho$ be as in the claim and $L \gg m/\rho$. By Theorems 5.3 and 1.1, we can choose an integer $m$ and $M = Lm$ such that, with probability $> 1-\varepsilon$, there is a set $I \subseteq \mathbb{N}$ of density $> 1-\varepsilon$ such that, for $n \in I$,

(A) Every geodesic $\gamma_{i,n}$ from 0 to $L_i + nv_i$ intersects $m\partial B_\mu$ in $mC_i$ and intersects $M\partial B_\mu$ in $MC_i$.
(B) If $i \neq j$ then $B_{L_i+nv_i}(x_j, x_i) > m\alpha$ for all $x_i \in mD_i$ and $x_j \in mD_j$.
(C) $|\tau(0,x) - m| < \frac{m\alpha}{10}$ for all $x \in mD_i$.
(D) At least a $\rho/2$-fraction of edges on $\gamma_{i,n} \cap (MB_\mu \setminus mB_\mu)$ have passage time $> 1$.

Fix $\gamma_{i,n}$ as in (A). We may choose an infinite $J \subseteq I$ such that $\lim_{n\in J}\gamma_{i,n} \to \gamma_i$ for some infinite geodesics $\gamma_i$ originating at 0, i.e., for every $r > 0$ we have $\gamma_i \cap [-r,r]^2 = \gamma_{i,n} \cap [-r,r]^2$ for all large enough $n \in J$. Henceforth we only consider such $n$.

Let $y_{i,n}$ be the first intersection point of $\gamma_{i,n}$ with $mC_i$, and choose $x_{i,n} \in D_i$ such that $|x_{i,n} - y_{i,n}| \leq \frac{m\alpha}{R10}$. Since $\mu$ is supported on $[0, R]$, we conclude that $\tau(x_{i,n}, y_{i,n}) \leq \frac{m\alpha}{10}$, so

$$|B_{L_i+nv_i}(y_{j,n}, y_{i,n}) - B_{L_i+nv_i}(x_{j,n}, x_{i,n})| < \frac{2m\alpha}{10} \text{ for } i \neq j. \tag{5.3}$$

**Claim 5.5.** *The $\gamma_i$'s are disjoint outside of $mB_\mu$.*

*Proof.* Suppose for example that $\gamma_1, \gamma_2$ intersect at some point $z$ outside of $mB_\mu$. Then for large enough $n \in J$ the same is true of $\gamma_{1,n}$ and $\gamma_{2,n}$. Then

$$\tau(0, y_{1,n}) + \tau(y_{1,n}, z) = \tau(0, y_{2,n}) + \tau(y_{2,n}, z).$$

By (C) we have $|\tau(0, y_{1,n}) - \tau(0, y_{2,n})| < \frac{2m\alpha}{10}$, so

$$|\tau(y_{1,n}, z) - \tau(y_{2,n}, z)| < \frac{2m\alpha}{10}.$$

Write $\sigma_1$ for the part of $\gamma_{1,n}$ from $y_{1,n}$ to $L_1 + nv_1$. Let $\sigma_2$ be path which starts at $y_{2,n}$, follows $\gamma_{2,n}$ until $z$, and then follows $\gamma_{1,n}$ until $L_1 + nv_1$. We find that $|\tau(\sigma_1) - \tau(\sigma_2)| < \frac{2m\alpha}{10}$. But $\gamma_{1,n}$ is a shortest path from 0 to $L_1 + nv_1$, so $\sigma_1$ is a shortest path from $y_{1,n}$ to $L_1 + nv_1$. Hence $B_{L_1+nv_1}(y_{2,n}, y_{1,n}) \leq \frac{2m\alpha}{10}$. By (5.3), this contradicts (B). □

Finally, combining the last claim with (D) establishes the two claims stated after Proposition 5.2. This completes the proof of Theorem 1.4.

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