Properties of the extremal solution for a fourth-order elliptic problem *†‡§¶‖

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Abstract Let $\lambda^* > 0$ denote the largest possible value of λ such that

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^p} & \text{in } B, \\ 0 < u \leq 1 & \text{in } B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B. \end{cases}$$

has a solution, where B is the unit ball in R^n centered at the origin, $p > 1$ and n is the exterior unit normal vector. We show that for $\lambda = \lambda^*$ this problem possesses a unique weak solution u^*. We prove that u^* is smooth if $n \leq 12$ and singular when $n \geq 13$ for p large enough, in which case $1 - C_0 r^{\frac{4}{p+1}} \leq u_{\lambda^*}(x) \leq 1 - r^{\frac{4}{p+1}}$ on the unit ball, where $C_0 := (\lambda^*/\bar{\lambda})^{\frac{1}{p+1}}$ and $\bar{\lambda} := \frac{8(p-1)}{(p+1)^2}[n - \frac{2(p-1)}{p+1}][n - \frac{4p}{p+1}]$.

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1. Introduction and main result

The main purpose of this paper is to investigate regularity of the extremal solution for a class of fourth-order problem

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^p} & \text{in } B, \\ 0 < u \leq 1 & \text{in } B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B. \end{cases} \qquad (1.1)_\lambda$$

Here B denotes the unit ball in R^n $(n \geq 2)$ centered at the origin, $\lambda > 0, p > 1$ and $\frac{\partial}{\partial n}$ the differentiation with the respect to the exterior unit normal, i.e., in radial direction. We consider only radial solutions, since all positive smooth solutions of $(1.1)_\lambda$ are radial, see Berchio et al. [3].

The motivation for studying $(1.1)_\lambda$ stems from a model for the steady sates of a simple micro electromechanical system (MEMS) which has the general form (see for example [19],

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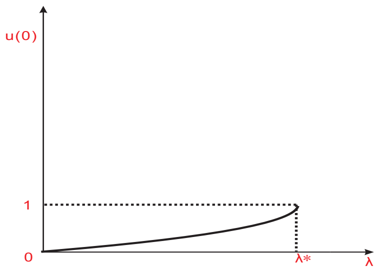

Figure 1 (The bifurcation diagram in the case the extremal solution is singular)

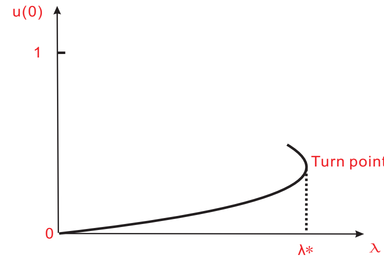

Figure 2 (The local bifercation diangram in the case the extremal solution is regular)

[22])

$$\begin{cases} \alpha\Delta^2 u = (\beta\int_\Omega |\nabla u|^2 dx + \gamma)\Delta u + \frac{\lambda f(x)}{(1-u)^2(1+\chi\int_\Omega \frac{dx}{(1-u)^2})} & \text{in } \ \Omega, \\ 0 < u < 1 & \text{in } \ \Omega, \\ u = \alpha\frac{\partial u}{\partial n} = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.2}$$

where $\alpha, \beta, \gamma, \chi \geq 0$, are fixed, $f \geq 0$ represents the permittivity profile, Ω is a bounded domain in R^n and $\lambda > 0$ is a constant which is increasing with respect to the applied voltage.

Recently, Equation (1.2) posed in $\Omega = B$ with $\beta = \gamma = \chi = 0, \alpha = 1$ and $f(x) \equiv 1$, which is reduced to

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } \ B, \\ 0 < u < 1 & \text{in } \ B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \ \partial B, \end{cases} \tag{1.3}$$

has been studied extensively in [8]. For convenience, we now give the following notion of solution.

Definition 1.1. If u_λ is a solution of $(1.1)_\lambda$ such that for any other solution v_λ of $(1.1)_\lambda$ one has

$$u_\lambda \leq v_\lambda, \qquad \text{a.e. } x \ \in B,$$

we say that u_λ is a minimal solution of $(1.1)_\lambda$.

It is shown that there exists a critical value $\lambda^* > 0$ (pull-in voltage) such that if $\lambda \in (0, \lambda^*)$ the problem (1.3) has a smooth minimal solution , while for $\lambda > \lambda^*$ (1.3) has no solution even in a weak sense. Moreover, the branch $\lambda \to u_\lambda(x)$ is increasing for each $x \in B$, and therefore the function $u^*(x) := \lim_{\lambda\to\lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the issue of the regularity of this extremal solution-which, by elliptic regularity theory, is equivalent to whether $\sup_\Omega u^* < 1$- is an important question for many reasons. For example, one of the reason is that it decides whether the set of solutions stops there, or whether a new branch of solutions emanates from a bifurcation state (u^*, λ^*) (see Figures 1,2). This issue turned out to depend closely on the dimension. Indeed by the key uniform estimate of $\|(1-u)^{-3}\|_{L^1}$, Guo and Wei [16] obtained the regularity of the extremal solution for small dimensions and they proved that for dimension $n = 2$ or $n = 3, u^*$ is smooth. But from their result, the regularity of extremal solution of (1.3) is unknown for $n \geq 4$. Recently, using certain improved Hardy-Rellich inequalities, Cow-Esp-Gho-Mor [8] improved the above result and they obtained that u^* is regular in dimensions $1 \leq n \leq 8$, while it is singular for $n \geq 9$, i.e., the critical dimension is 9. So the issue of the regularity of the extremal solution of $(1.1)_\lambda$ for power $p = -2$ is completely solved, but the critical dimension for generally negative power is **unknown**.

In this paper, we investigated the relation between p and critical dimension of of the extremal solution of the equation $(1.1)_\lambda$, we find that the critical dimension $n(p)$ increase for p increase, and if p is large enough, the the critical dimension is 13, which is independent of p, i.e., the extremal solution of $(1.1)_\lambda$ is singular ($\sup_\Omega u^* = 1$) for $n \geq 13$. Our result is stated as follows:

Theorem 1.1 (i) There exists $p_0 > 1$ large enough such that for $p \geq p_0$, the unique extremal solution of $(1.1)_{\lambda^*}$ is regular for $n < 13$; while it is singular for $n \geq 13$;
(ii) For any $p > 1$, the unique extremal solution of $(1.1)_{\lambda^*}$ is regular for $n \leq 4$.

From the technical point of view, one of the basic tools in the analysis of nonlinear second order elliptic problems in bounded and unbounded domains of $R^n (n \geq 2)$ is the maximum principle. However, for high order problems, such principle dose not normally hold for general domains (at least for the clamped boundary conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$), which causes several technical difficulties. One of reasons to the study $(1.1)_\lambda$ in a ball is that a maximum principle holds in this situation, see [1], [5]. The second obstacle is the well-known difficulty of extracting energy estimates for solutions of fourth order problems from their stability properties. Besides, for the corresponding second order problem, the starting point was an explicit singular solution for a suitable eigenvalue parameter λ which turned out to play a fundamental role for the shape of the corresponding bifurcation diagram, see [4]. When turning to the biharmonic problem $(1.1)_\lambda$ the second boundary condition $\frac{\partial u}{\partial n} = 0$ prevents to find an explicit singular solution. This means that the method used to analyze the regularity of the extremal solution for second order problem could not carry to the corresponding problem for $(1.1)_\lambda$. In this paper, we, in order to overcome the third obstacle, use improved and non standard Hardy-Rellich inequalities recently established by Ghoussoub-Moradifam in [13] to construct a semi-stable singular $H^2(B)-$ weak sub-solution of $(1.1)_\lambda$.

This paper is organized as follows. In the next section, some preliminaries are reviewed. In Section 3, we give the uniform estimate of $\|(1-u)^{p+1}\|_{L^1}$ according to the stability of the minimal solutions. We study the regularity of the extremal solution of $(1.1)_\lambda$ and the Theorem 1 (ii) is established in Section 4. Finally, we will show that the extremal solution u^* in dimensions $n \geq 13$ is singular by constructing a semi-stable singular $H^2(B)-$ weak sub-solution of $(1.1)_\lambda$.

2. Preliminaries

First we give some comparison principles which will be used throughout the paper

Lemma 2.1. (Boggio's principle, [5]) If $u \in C^4(\bar{B_R})$ satisfies

$$\begin{cases} \Delta^2 u \geq 0 & \text{in } B_R, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B_R, \end{cases}$$

then $u \geq 0$ in B_R.

Lemma 2.2. Let $u \in L^1(B_R)$ and suppose that

$$\int_{B_R} u \Delta^2 \varphi \geq 0$$

For a proof see Lemma 17 in [1].

Lemma 2.3. If $u \in H^2(B_R)$ is radial, $\Delta^2 u \geq 0$ in B_R in the weak sense, that is

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \geq 0$$

and $u|_{\partial B_R} \geq 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$ then $u \geq 0$ in B_R.

Proof. We only deal with the case $R = 1$ for simplicity. Solve

$$\begin{cases} \Delta^2 u_1 = \Delta^2 u & \text{in } B \\ u_1 = \frac{\partial u_1}{\partial n} = 0 & \text{on } \partial B \end{cases}$$

in the sense $u_1 \in H_0^2(B)$ and $\int_B \Delta u_1 \Delta \varphi = \int_B \Delta u \Delta \varphi$ for all $\varphi \in C_0^\infty(B)$. Then $u_1 \geq 0$ in B by lemma 2.2.

Let $u_2 = u - u_1$ so that $\Delta^2 u_2 = 0$ in B. Define $f = \Delta u_2$. Then $\Delta f = 0$ in B and since f is radial we find that f is a constant. It follows that $u_2 = ar^2 + b$. Using the boundary conditions we deduce $a + b \geq 0$ and $a \leq 0$, which imply $u_2 \geq 0$.

As in [8], we are now led here to examine problem $(1.1)_\lambda$ with non-homogeneous boundary conditions such as

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^p} & \text{in } B, \\ \alpha < u \leq 1 & \text{in } B, \\ u = \alpha, \frac{\partial u}{\partial n} = \gamma & \text{on } \partial B, \end{cases} \tag{$(2.1)_{\lambda,\alpha,\gamma}$}$$

where α, γ are given.

Let Φ denote the unique solution of

$$\begin{cases} \Delta^2 \Phi = 0 & \text{in } B, \\ \Phi = \alpha, \frac{\partial \Phi}{\partial n} = \gamma & \text{on } \partial B. \end{cases} \tag{2.2}$$

We will say that the pair (α, γ) is admissible if $\gamma \leq 0$, and $\alpha - \frac{\gamma}{2} < 1$. We now introduce a notion of weak solution.

Definition 2.1. We say that u is a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$, if $\alpha \leq u \leq 1$ a.e. in Ω, $\frac{1}{(1-u)^p} \in L^1(\Omega)$ and if

$$\int_\Omega (u - \Phi)\Delta^2 \varphi = \lambda \int_\Omega \frac{\varphi}{(1-u)^p} \quad \forall \varphi \in C^4(\bar{B}) \bigcap H_0^2(B),$$

where Φ is given in (2.1). We say u is a weak super-solution (resp. weak sub-solution) of $(2.1)_{\lambda,\alpha,\gamma}$, if the equality is replaced with $\geq$ (resp.$\leq$) for $\varphi \geq 0$.

Definition 2.2. We say a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$ is regular (resp. singular) if $\|u\|_\infty < 1$ (resp. $\|u\| = 1$) and stable (resp. semi-stable) if

$$\mu_1(u) = \inf\{\int_B (\Delta\varphi)^2 - p\lambda \int_B \frac{\varphi^2}{(1-u)^p} : \phi \in H_0^2(B), \|\phi\|_{L^2} = 1\}$$

is positive (resp. non-negative).

We now define

and

$$\lambda_*(\alpha,\gamma) := \sup\{\lambda > 0 : (2.1)_{\lambda,\alpha,\gamma} \text{ has a weak soltion}\}.$$

Observe that by Implicit Function Theorem, we can classically solve $(2.1)_{\lambda,\alpha,\gamma}$ for small $\lambda's$. Therefore, $\lambda^*(\alpha,\gamma)$ and $\lambda_*(\alpha,\gamma)$ are well defined for any admissible pair (α,γ). To cut down notations we won't always indicate α and γ.

Let now give the following standard existence result.

Theorem 2.1. For every $0 \le f \in L^1(\Omega)$ there exists a unique $0 \le u \in L^1(\Omega)$ which satisfies

$$\int_\Omega u\Delta^2\varphi dx = \int_\Omega f\varphi dx$$

for all $\varphi \in C^4(\bar{B}) \bigcap H_0^2(B)$.

The proof is standard, please see [14], here we omit it. From this Theorem, we immediately have the following result.

Proposition 2.1. Assume the existence of a weak super-solution U of $(2.1)_{\lambda,\alpha,\gamma}$. Then there exists a weak solution u of $(2.1)_{\lambda,\alpha,\gamma}$ so that $\alpha \le u \le U$ a.e in B.

For the sake of completeness, we include a brief proof here, which be called "weak" iterative scheme: $u_0 = U$ and (inductively) let $u_n, n \ge 1$, be the solution of

$$\int_\Omega (u_n - \Phi)\Delta^2\varphi = \lambda\int_\Omega \frac{\varphi}{(1-u_{n-1})^p} \quad \forall\varphi \in C^4(\bar{B})\bigcap H_0^2(B),$$

given by Theorem 2.2. Since α is a sub-solution of $(2.1)_{\lambda,\alpha,\gamma}$, inductively it is easily shown by Lemma 2.2 that $\alpha \le u_{n+1} \le u_n \le U$ for every $n \ge 0$. Since

$$(1-u_n)^{-p} \le (1-U)^{-p} \quad \in L^1(B),$$

by Lebesgue Theorem the function $u = \lim_{n\to\infty} u_n$ is a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$ so that $\alpha \le u \le U$.

In particular, for every $\lambda \in (0,\lambda_*)$, we can find a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$. In the same range of $\lambda's$, this is still true for regular weak solutions as shown in the following lemma.

Lemma 2.4. Let (α,γ) be an admissible pair and u be a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$. Then, there exists a regular solution for every $0 < \mu < \lambda$.

Proof. Let $\epsilon \in (0,1)$ be given and let $\bar{u} = (1-\epsilon)u + \epsilon\Phi$, where Φ is given in (2.2). by lemma 2.2 $\sup_B \Phi < \sup_B u \le 1$. Hence

$$\sup_B \bar{u} \le (1-\epsilon) + \epsilon\sup_B \Phi < 1, \quad \inf \bar{u} \ge (1-\epsilon)\alpha + \epsilon\inf_B \Phi = \alpha$$

$$\begin{aligned}\int_B (\bar{u}-\Phi)(\beta\Delta^2\varphi - \tau\Delta\varphi) &= (1-\epsilon)\int_B (u-\Phi)(\beta\Delta^2\varphi - \tau\Delta\varphi) = (1-\epsilon)\lambda\int_B \frac{\varphi}{(1-u)^p} \\ &= (1-\epsilon)^{p+1}\lambda\int_B \frac{\varphi}{(1-\bar{u}+\epsilon(\Phi-1))^p} \ge (1-\epsilon)^{p+1}\lambda\int_B \frac{\varphi}{(1-\bar{u})^p}.\end{aligned}$$

Note that $0 \le (1-u)(1-\epsilon) = 1-\bar{u}+\epsilon(\Phi-1) \le 1-\bar{u}$. So $\bar{u}$ is a weak super-solution

solution B of $(2.1)_{(1-\epsilon)^{p+1}\lambda,\alpha,\gamma}$ so that $\alpha \le \omega \le \bar{u}$. In particular, $\sup_B \bar{u} < 1$ and ω is a regular weak solution. Since $\epsilon \in (0,1)$ is arbitrarily chosen, the proof is done.

Now we recall some basic facts about the minimal branch

Theorem 2.2. $\lambda^* \in (0, +\infty)$ and the following holds:

1. For each $0 < \lambda < \lambda^*$ there exists a regular and minimal solution u_λ of $(2.1)_{\lambda,\alpha,\gamma}$;
2. For each $x \in B$ the map $\lambda \to u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$;
3. For $\lambda > \lambda^*$ there are no weak solutions of $(2.1)_{\lambda,\alpha,\gamma}$.

The proof is standard, see [8], here we omit it.

3. Stability of the minimal solutions

In this section we shall show that the extremal solution is regular in small dimensions. Let us begin with the following priori estimates along the minimal branch u_λ. In order to achieve this, we shall need yet another notion of H^2(B)- weak solutions, which is an intermediate class between classical and weak solutions.

Definition 3.1. We say that u is a H^2(B)- weak solution of $(2.1)_{\lambda,\alpha,\beta}$ if $u - \Phi \in H_0^2$(B), $\alpha \le u \le 1 \in B$, $\frac{1}{(1-u)^p} \in L^1(B)$ and if

$$\int_B \Delta u \Delta \phi = \lambda \int_B \frac{\phi}{(1-u)^p}, \quad \forall \phi \in C^4(\bar{B}) \bigcap H_0^2(B),$$

where Φ is given in (2.2). We say that u is a H^2(B)- weak super-solution (resp. H^2(B)- weak sub-solution) of $(2.1)_{\lambda,\alpha,\beta}$ if for $\phi \ge 0$ the equality is replaced with $\ge$ (resp.$\le$) and $u \ge \alpha$ (resp. $\le$), $\partial_v u \le \beta$ (resp. $\ge$) on ∂B.

Theorem 3.1. Suppose that (α, γ) is an admissible pair.

1. The minimal solution u_λ is stable, and is the unique semi-stable $H-$ weak solution of $(2.1)_{\lambda,\alpha,\gamma}$;
2. The function $u^* := \lim_{\lambda \to \lambda^*} u_\lambda$ is a well-defined semi-stable $H-$ weak solution of $(2.1)_{\lambda^*,\alpha,\gamma}$;
3. u^* is the unique $H-$ weak solution of $(2.1)_{\lambda^*,\alpha,\gamma}$, and when u^* is classical solution, then $\mu_1(u^*) = 0$;
4. If v is a singular, semi-stable $H-$ weak solution of $(2.1)_{\lambda,\alpha,\gamma}$, then $v = u^*$ and $\lambda = \lambda^*$.

The main tool which we use to prove the theorem 3.1 is the following comparison lemma which is valid exactly in the class H.

Lemma 3.2. Let (α, γ) is an admissible pair and u be a semi-stable $H-$ weak solution of $(P_{\lambda,\alpha,\gamma})$. Assume U is a $H-$ weak super-solution of $(2.1)_{\lambda,\alpha,\gamma}$. Then

1. $u \le U$ a.e. in Ω;
2. If u is a classical solution and $\mu_1(u) = 0$ then $U = u$.

A more general version of Lemma 3.2 is available in the following.

Lemma 3.3. Let (α, γ) is an admissible pair and $\gamma' \le 0$. Let u be a semi-stable $H-$ weak sub- solution of $(2.1)_{\lambda,\alpha,\gamma}$ with $u = \alpha' \le \alpha, \Delta u = \beta' \ge \beta$ on $\partial\Omega$. Assume that U is a $H-$ weak super-solution of $(2.1)_{\lambda,\alpha,\gamma}$ with $U = \alpha, \Delta U = \beta$ on $\partial\Omega$. Then $U \ge u$ a.e. in Ω.

We need also some a priori estimates along the minimal branch u_λ.

Lemma 3.4. Let (α, γ) be an admissible pair. Then for every $\lambda \in (0, \lambda^*)$, we have

$$p \int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} \leq \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^p},$$

where Φ is given by (2.1). In particular, there is a constant C independent of λ so that

$$\int_B |\Delta u_\lambda|^2 dx + \int_B \frac{1}{(1-u_\lambda)^{p+1}} \leq C. \tag{3.1}$$

Proof. Testing $(2.1)_{\lambda,\alpha,\gamma}$ on $u_\lambda - \Phi \in W^{4,2}(B) \cap H_0^2(B)$. We see that

$$\lambda \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^p} = \int_B (\Delta(u_\lambda - \Phi))^2 dx \geq p\lambda \int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} dx$$

in the view of $\beta\Delta^2\Phi - \tau\Delta\Phi = 0$. In particular, for $\delta > 0$ small we have that

$$\int_{|u_\lambda - \Phi| \geq \delta} \frac{1}{(1-u_\lambda)^{p+1}} \leq \frac{1}{\delta^2} \int_{|u_\lambda - \Phi| \geq \delta} \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} \quad \leq \quad \frac{1}{\delta^2} \int_B \frac{1}{(1-u_\lambda)^p}$$
$$\leq \delta^{p-1} \int_B \frac{1}{(1-u_\lambda)^{p+1}} + C_\delta$$

by means of Young's inequality. Since for δ small

$$\int_{|u_\lambda - \Phi| \leq \delta} \frac{1}{(1-u_\lambda)^{p+1}} \leq C$$

for some $C > 0$, we get that

$$\int_B \frac{1}{(1-u_\lambda)^{p+1}} \leq C$$

for some $C > 0$ and for every $\lambda \in (0, \lambda^*)$. By Young' s and Hölder's inequalities, we have

$$\begin{aligned} \int_B |\Delta u_\lambda|^2 dx &= \int_B \Delta u_\lambda \Delta\Phi dx + \lambda \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^p} dx \\ &\leq \delta \int_B |\Delta u_\lambda|^2 dx + C_\delta + C\left(\int_B \frac{1}{(1-u_\lambda)^{p+1}}\right)^{\frac{p}{p+1}}. \end{aligned}$$

So we have

$$\int_B |\Delta u_\lambda|^2 dx + \int_B \frac{1}{(1-u_\lambda)^{p+1}} \leq C$$

where C is absolute constant.

Proof of the Theorem 3.1. (1) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0, \lambda^*)$. since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in B$, it is easily seen that $\lambda \to \mu_1(u_\lambda)$ is a decreasing and continuous function on $(0, \lambda^*)$. Define

$$\lambda_{**} := \sup\{0 < \lambda < \lambda^* : \mu_1(u_\lambda) > 0\}.$$

We have that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have $\mu_1(u_{\lambda_{**}}) = 0$, and for every since Lemma 3.2 implies $u_\mu = u_{\lambda_{**}}$. Finally, Lemma 3.2 guarantees the uniqueness in the class of semi-stable $H-$ weak solutions.

(2) It follows from (3.1) that $u_\lambda \to u^*$ in a pointwise sense and weakly in $H^2(B)$, and $\frac{1}{1-u^*} \in L^{p+1}$. In particular, u^* is a $H-$ weak solution of $(P_{\lambda^*,\alpha,\gamma})$ which is also semi-stable as the limiting function of the semi-stable solutions $\{u_\lambda\}$.

(3) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the Implicit Function Theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By Lemma 3.2, u^* is then the unique $H-$ weak solution of $(P_{\lambda^*,\alpha,\gamma})$.

(4) If $\lambda < \lambda^*$, we get by uniqueness that $v = u_\lambda$. So v is not singular and a contradiction arises. Now by theorem 2.5 (3) we have that $\lambda = \lambda^*$. Since v is a semi-stable $H-$ weak solution of $(P_{\lambda^*,\alpha,\gamma})$ and u^* is a $H-$ weak super-solution of $(P_{\lambda^*,\alpha,\gamma})$, we can apply Lemma 3.2 to get $v \le u^*$ a.e. in Ω. Since u^* is also a semi-stable solution, we can reverse the roles of v and u^* in Lemma 3.2 to see that $v \ge u^*$ a.e. in Ω. So equality $v = u^*$ holds and the proof is done

4. Regularity of the extremal solutions and the Proof of the Theorem 1.1 (ii)

In this section we first show that the extremal solution is regular in small dimensions by the uniformly bounded of u_λ in $H_0^2(B)$. Second, using the refined version of Hardy-Rellich inequality, we prove the extremal solution is singular for $n \ge 13$ and p large enough. Now we give the proof of Theorem 1.1 (ii).

Proof of Theorem 1.1 (ii). As already observed, estimate (3.1) implies that $f(u^*) = (1-u^*)^{-p} \in L^{\frac{p+1}{p}}(B)$. Since u^* is radial and radially decreasing. We need to show that $u^*(0) < 1$ to get the regularity of u^*. In fact, on the contrary suppose that $u^*(0) = 1$. By the standard elliptic regularity theory shows that $u^* \in W^{4,\frac{p+1}{p}}$. By the Soblev imbedding theorem, i.e. $W^{4,\frac{p+1}{p}} \hookrightarrow C^m(0 < m \le 4 - \frac{pn}{p+1}, 1 \le n \le 4)$. We have u^* is a Lipschitz function in B for $1 \le n \le 3$.

Now suppose $u^*(0) = 1$ and $1 \le n \le 2$. Since

$$\frac{1}{1-u^*} \ge \frac{C}{|x|} \quad \text{in} \quad B$$

for some $C > 0$. One see that

$$+\infty = C^{p+1} \int_B \frac{1}{|x|^{p+1}} \le \int_B \frac{1}{(1-u^*)^{p+1}} < +\infty.$$

A contradiction arises and hence u^* is regular for $1 \le n \le 2$.

For $n = 3$, by the Sobolev imbedding theorem, we have $u^* \in C^{\frac{p+4}{p+1}}(\bar{B})$, if $\frac{p+4}{p+1} \ge 2$, then $u^*(0) = 1, Du^*(0) = 0$ and

$$|Du^*(\varepsilon) - Du^*(0)| \le M|\varepsilon| \le M|x|$$

where $0 < |\varepsilon| < |x|$. Thus

$$|u(x) - u(0)| \le |Du(\varepsilon)||x| \le M|x|^2.$$

This inequality shows that

A contradiction arises and hence u^* is regular for $n=3$; if $\frac{p+4}{p+1}<2$, then

$$|Du(\varepsilon)-Du(0)|\le M|\varepsilon|^{\frac{4}{p-1}-1}\le M|x|^{\frac{3}{p+1}}$$

where $0<|\varepsilon|<|x|$. Thus

$$|u(x)-u(0)|\le |Du(\varepsilon)||x|\le M|x|^{\frac{4+p}{p+1}},$$

and a contradiction is obtained as above.

For $n=4$, by the Sobolev imbedding theorem, we have $u^*\in C^{\frac{4}{p+1}}(\bar{B})$. If $1<\frac{4}{p+1}<2$, then $u^*(0)=1, Du^*(0)=0$ and

$$|Du(\varepsilon)-Du(0)|\le M|\varepsilon|^{\frac{4}{p+1}-1}\le M|x|^{\frac{4}{p+1}-1}$$

where $0<|\varepsilon|<|x|$. Thus

$$|u(x)-u(0)|\le |Du(\varepsilon)||x|\le M|x|^{\frac{4}{p+1}}.$$

If $\frac{4}{p+1}\le 1$, then u^* is a Hölder's continues and

$$1-u^*(x)\le M|x|^{\frac{4}{p+1}},$$

and we obtain a contradiction as above.

We now tackle the regularity of u^* for $5\le n\le 12$. We start with the following crucial result.

Lemma 4.1. Let $n\ge 5$ and (u^*,λ^*) be the extremal pair of $(1.1)_\lambda$, when u^* is singular, then

$$1-u^*\le C_0|x|^{\frac{4}{p+1}},$$

where $C_0:=(\lambda^*/\bar{\lambda})^{\frac{1}{p+1}}$ and $\bar{\lambda}:=\frac{8(p-1)}{(p+1)^2}[n-\frac{2(p-1)}{p+1}][n-\frac{4p}{p+1}]$.

In order to prove the Lemma 4.1, we need the lower bounds of λ^* and state as follows

Lemma 4.2. λ^* satisfies the following lower bounds for $n\ge 4$

$$\lambda^*\ge\bar{\lambda}$$

where $\bar{\lambda}=\frac{8(p-1)}{(p+1)^2}[n-\frac{2(p-1)}{p+1}][n-\frac{4p}{p+1}]$.

Proof. the proof is standard, here we include the proof for the sake of completeness. Notice that for $n\ge 4$ the function $\bar{u}=1-|x|^{\frac{4}{p+1}}$ satisfies

$$\frac{1}{(1-\bar{u})^p}\in L^1(B)$$

and $\bar{u}$ is a weak solution of

$$\Delta^2\bar{u}=\frac{\bar{\lambda}}{(1-\bar{u})^p},$$

and $\bar{u}(1)=0=u_\lambda(1)$; $\frac{\partial u_\lambda}{\partial n}(1)\ge\frac{\partial\bar{u}_\lambda}{\partial n}(1)$. Therefore, $\bar{u}$ turns out to be a weak super-solution of $(1.1)_\lambda$ provided $\lambda<\bar{\lambda}$. Thus necessarily, we have

The proof is done.

Proof of Lemma 4.1. First note that Lemma 4.2 gives the lower bound:

$$\lambda^* \geq \bar{\lambda}.$$

For $\delta > 0$, we define $u_\delta(x) := 1 - C_\delta |x|^{\frac{4}{p+1}}$ with $C_\delta := (\frac{\lambda^*}{\bar{\lambda}} + \delta)^{\frac{1}{p+}} > 1$. Since $n \geq 5$. we have that $u_\delta \in H^2_{loc}(R^n)$, $\frac{1}{1-u_\delta} \in L^3_{loc}(R^n)$ and u_δ is a H^2-weak solution of

$$\Delta^2 u_\delta = \frac{\lambda^* + \delta\bar{\lambda}}{(1-u_\delta)^p} \quad \text{in} \quad R^n$$

We claim that $u_\delta \leq u^*$ in B, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set

$$\Gamma := \{r \in (0,1) : u_\delta(r) > u^*(r)\}$$

is non-empty, and let $r_1 = \sup \Gamma$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$ and one infers that

$$\alpha := u^*(r_1) = u_\delta(r_1), \quad \beta = (u^*)'(r_1) \geq u'_\delta(r_1).$$

Setting $u_{\delta,r_1}(r) = r_1^{-\frac{4}{p+1}}(u_\delta(r_1 r) - 1) + 1$, we easily see that the function $u_{\delta,r_1}(r)$ is a $H^2(B)$-weak super-solution of $(2.1)_{\lambda^*+\delta\bar{\lambda},\alpha',\beta'}$, where

$$\alpha' := r_1^{-\frac{4}{p+1}}(u^*(r_1 r) - 1) + 1, \quad \beta' := r_1^{\frac{p-3}{p+1}}\beta.$$

Similarly, define $u^*_{r_1} = r_1^{-\frac{4}{p+1}}(u^*(r_1 r) - 1) + 1$, we have $u^*_{r_1}$ is singular semi-stable $H^2(B)$-weak solution of $(2.1)_{\lambda^*,\alpha',\beta'}$.

Now we claim that (α', β') is an admissible pair. Since u^* is radially decreasing, we have that $\beta' \leq 0$. Define the function

$$\omega(r) := (\alpha' - \frac{\beta'}{2}) + \frac{\beta'}{2}|x|^2 + \gamma(x),$$

where $\gamma(x)$ is a solution of $\Delta^2\gamma = \lambda^*$ in B with $\gamma = \partial_v\gamma = 0$ on ∂B. Then ω is a classical solution of

$$\begin{cases} \Delta^2\omega = \lambda^* & \text{in} \quad B \\ \omega = \alpha', \partial_v\omega = \beta' & \text{on} \quad \partial B. \end{cases}$$

Since $\frac{\lambda^*}{(1-u^*_{r_1})^p} \geq \lambda^*$, by Lemma 2.1 we have

$$u^*_{r_1} \geq \omega \quad \text{a.e.} \quad \text{in} B$$

Since $\omega(0) = \alpha' - \frac{\beta'}{2} + \gamma(0)$ and $\gamma(0) > 0$, we have

$$\alpha' - \frac{\beta'}{2} < 1$$

So (α', β') is an admissible pair and by Theorem 2.1 (4) we get that $(u^*_{r_1}, \lambda^*)$ coincide

Since (α', β') is an admissible pair and u_{δ,r_1} is a $H^2(B)$-weak super-solution of $(2.1)_{\lambda^*+\delta\bar{\lambda},\alpha',\beta'}$. We get from Proposition 2.1, the existence of a weak solution of $(2.1)_{\lambda^*+\delta\bar{\lambda},\alpha',\beta'}$. Since

$$\lambda^* + \delta\bar{\lambda} > \lambda^*,$$

we contradict the fact that λ^* is the extremal parameter of $(2.1)_{\lambda,\alpha',\beta'}$.

Thanks to the lower estimate on u^*, we get the following regular result.

Theorem 4.1. If p is large enough, the extremal solution u^* is regular in dimensions for $5 \le n \le 12$.

Proof. Assume u^* is singular. For $\varepsilon > 0$, define $\varphi(x) := |x|^{\frac{4-n}{2}+\varepsilon}$ and note that:

$$(\Delta\varphi)^2 = (H_n + O(\varepsilon))|x|^{-n+2\varepsilon},$$

where $H_n = \frac{n^2(n-4)^2}{16}$. Given $\eta \in C_0^\infty(B)$ and since $n \ge 5$, using a standard approximation argument, we can use the test function $\eta\varphi \in H_0^2$ into the stability inequality to obtain

$$\int_B (\Delta\varphi)^2 + O(1) \ge p\lambda^* \int_B \frac{\varphi^2}{(1-u^*)^{p+1}},$$

since the contribution of the integrals outside a fixed ball around the origin remains bounds as $\varepsilon \to 0$ (Here $O(1)$ denotes a bounded function as $\varepsilon \to 0$). By Lemma 4.2 and Rellich's inequality, we find

$$\begin{aligned} p\bar{\lambda}B_n\frac{1}{2\varepsilon} &= p\bar{\lambda}\int_B \frac{\varphi^2}{|x|^4}dx \le \int_B (\Delta\varphi)^2 dx + O(1) \\ &\le (H_n + O(\varepsilon))\int_B |x|^{-n+2\varepsilon}dx = B_n\frac{H_n}{2\varepsilon} + O(1), \end{aligned}$$

where B_n is the surface area of the unit n-1 dimensional sphere S^{n-1}. Obviously we have

$$p\bar{\lambda} \le H_n + O(\varepsilon).$$

Letting $\varepsilon \to 0$, we get $p\bar{\lambda} \le H_n$, i.e.,

$$\frac{n^2(n-4)^2}{16} \ge \frac{8p(p-1)}{(p+1)^2}(n - \frac{2(p-1)}{p+1})(n - \frac{4p}{p+1}). \tag{4.1}$$

As $p \to +\infty$, we have

$$\frac{n^2(n-4)^2}{16} \ge 8(n-2)(n-4) + o(\frac{1}{p}).$$

Graphing this relation one see (4.1) holds only for $n \ge 13$. So the extremal solution u^* is regular in dimensions for $5 \le n \le 12$ and the proof is done.

We can now slightly improve the lower bound of λ^*

Corollary 4.1. In any dimension $n \ge 1$, we have

$$\lambda^* \ge \bar{\lambda} = \frac{8(p-1)}{\cdots}[n - \frac{2(p-1)}{\cdots}][n - \frac{4p}{\cdots}].$$

Proof. The function $\bar{u} := 1 - |x|^{\frac{4}{p+1}}$ is a $H^2(B)$-weak solution of $(2.1)_{\bar{\lambda},0,-\frac{4}{p+1}}$. If by contradiction $\lambda^* = \bar{\lambda}$, then $\bar{u}$ is a $H^2(B)$-weak super-solution of $(1.1)_\lambda$ for every $\lambda \in (0, \lambda^*)$. By Lemma 3.2 we get that $u_\lambda \leq \bar{u}$ for all $\lambda < \lambda^*$, and then $u^* \leq \bar{u}$ a.e. in B.

If $1 \leq n \leq 4$, u^* is then regular by Theorem (i). By Theorem 3.1 (3) there holds $\mu_1(u^*) = 0$. By Lemma 3.2 then yields that $u^* = \bar{u}$, which is a contradiction since then u^* will not satisfy the boundary conditions.

If now $n \geq 5$ and $\lambda^* = \bar{\lambda}$, then $C_0 = 1$ in Lemma 4.1, and we then have $u^* \geq \bar{u}$. It means again that $u^* = \bar{u}$, a contradiction that completes the proof.

In what follows, we will show that the extremal solution u^* of $(1.1)_\lambda$ in dimensions $n \geq 13$ is singular. To do this, first we give the following lemmas which are the key for the proof of the singularity of u^* in higher dimensions.

5. The extremal solution is singular for $n \geq 13$

We prove in this section that the extremal solution is singular for $n \geq 13$ and p large enough. For that we will need a a suitable Hardy-Rellich type inequality which was established by Ghoussoub-Moradifam in [13]. As in the previous section (u^*, λ^*) denotes the extremal pair of $(2.1)_\lambda$

Lemma 5.1. Let $n \geq 5$ and B be the unit ball in R^n. Then there exists $C > 0$, such that the following improved Hardy-Rellich inequality holds for all $\varphi \in H_0^2(B)$:

$$\int_B (\Delta\varphi)^2 dx \geq \frac{n^2(n-4)^2}{16} \int_B \frac{\varphi^2}{|x|^4} dx + C \int_B \varphi^2 dx$$

Lemma 5.2. Let $n \geq 5$ and B be the unit ball in R^n. Then the following improved Hardy-Rellich inequality holds for all $\varphi \in H_0^2(B)$:

$$\begin{aligned} \int_B (\Delta\varphi)^2 dx \ &\geq \ \frac{(n-2)^2(n-4)^2}{16} \int_B \frac{\varphi^2 dx}{(|x|^2 - 0.9|x|^{\frac{n}{2}+1})(|x|^2 - |x|^{\frac{n}{2}})} \\ &+ \ \frac{(n-1)(n-4)^2}{4} \int_B \frac{\varphi^2 dx}{|x|^2(|x|^2 - |x|^{\frac{n}{2}})}. \end{aligned} \tag{5.0}$$

As a consequence, the following improvement of the classical Hardy-Rellich inequality holds:

$$\int_B (\Delta\varphi)^2 dx \geq \frac{n^2(n-4)^2}{16} \int_B \frac{\varphi^2}{|x|^2(|x|^2 - |x|^{\frac{n}{2}})}.$$

Lemma 5.3. If $n \geq 13$, then $u^* \leq 1 - |x|^{\frac{4}{p+1}}$.

Proof. Recall from Corollary 4.1 that $\bar{\lambda} < \lambda^*$. Let $\bar{u} = 1 - |x|^{\frac{4}{p+1}}$, we now claim that $u_\lambda \leq \bar{u}$ for all $\lambda \in (\bar{\lambda}, \lambda^*)$. Indeed, fix such a λ and assume by contradiction that

$$R_1 := \inf\{0 \leq R \leq 1 : u_\lambda < \bar{u} \ \text{ in the interval } (R,1)\} > 0.$$

From the boundary condition, one has that $u_\lambda < \bar{u}(r)$ as $r \to 1^-$. Hence, $0 < R_1 < 1, \alpha := u_\lambda(R_1) = \bar{u}(R_1)$ and $\beta := u_\lambda'(R_1) \leq \bar{u}'(R_1)$. The same as the proof of Lemma 4.1, we have $u_\lambda \geq \bar{u}$ in B_{R_1} and a contradiction arises in view of the fact that $\lim_{x\to 0} \bar{u}(x) = 1$ and $\|u_\lambda\|_\infty < 1$. It follows that $u_\lambda \leq \bar{u}$ in B for every $\lambda \in (\bar{\lambda}, \lambda^*)$ and in particular $u^* \leq \bar{u}$

Lemma 5.4. Let $n \geq n(p)$. Suppose there exists $\lambda' > 0$ and a singular radial function $\omega(r) \in H^2(B)$ with $\frac{1}{1-\omega} \in L^{\infty}_{loc}(\bar{B} \setminus \{0\})$ such that

$$\begin{cases} \Delta^2 \omega \leq \frac{\lambda'}{(1-\omega)^p} & \text{for } \ 0 < r < 1, \\ \omega(1) = 0, \quad \omega'(1) = 0, \end{cases} \tag{5.1}$$

and

$$p\beta \int_B \frac{\varphi^2}{(1-\omega)^{p+1}} \leq \int_B (\Delta\varphi)^2 \quad \text{for all } \varphi \in H_0^2(B) \tag{5.2}$$

1. If $\beta \geq \lambda'$, then $\lambda^* \leq \lambda'$.
2. If either $\beta > \lambda'$ or $\beta = \lambda' = \frac{H_n}{p}$, then the extremal solution u^* is necessarily singular.

Proof. (1). First, note that (5.2) and $\frac{1}{1-\omega} \in L^{\infty}_{loc}(\bar{B} \setminus \{0\})$ yield to

$$\frac{1}{1-\omega} \in L^1(B).$$

At the same time, (5.1) implies that $\omega(r)$ is a $H^2(B)$- weak stable sub-solution of $(1.1)_{\lambda'}$. If now $\lambda' < \lambda^*$, then by Lemma 3.3, we have

$$\omega(r) < u_{\lambda'},$$

which ia a contradiction since ω is singular while $u_{\lambda'}$ is regular.

(2) Suppose first that $\beta = \lambda' = \frac{H_n}{p}$ and that $n \geq 13$. Since by part (1) we have $\lambda^* \leq \frac{H_n}{p}$, we get from Lemma 5.3 and improved Hardy-Rellich inequality that there exists $C > 0$ so that for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2 - p\lambda^* \int_B \frac{\phi^2}{(1-u^*)^{p+1}} \geq \int_B (\Delta\phi)^2 - H_n \int_B \frac{\phi^2}{|x|^4} \geq C \int_B \phi^2.$$

It follows that $\mu_1(u^*) > 0$ and u^* must therefore be singular since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*.

Suppose now that $\beta > \lambda'$, and let $\frac{\lambda'}{\beta} < \gamma < 1$ in such a way that

$$\alpha := (\frac{\gamma\lambda^*}{\lambda'})^{\frac{1}{p+1}} < 1.$$

Setting $\bar{\omega} := 1 - \alpha(1-\omega)$, we claim that

$$u^* \leq \bar{\omega} \quad \text{in} \quad B. \tag{5.3}$$

Note that by the choice of α we have $\alpha^{p+1}\lambda' < \lambda^*$, and therefore to prove (3.4) it suffices to show that for $\alpha^{p+1}\lambda' \leq \lambda < \lambda^*$, we have $u_\lambda \leq \bar{\omega}$ in B. Indeed, fix such λ and note that

$$\Delta^2 \bar{\omega} = \alpha \Delta^2 \omega \leq \frac{\alpha\lambda'}{(1-\omega)^p} = \frac{\alpha^{p+1}\lambda'}{(1-\bar{\omega})^p} \leq \frac{\lambda}{(1-\bar{\omega})^p}.$$

Assume that $u_\lambda \leq \bar{\omega}$ dose not hold in B, and consider

Since $\bar{\omega}(1) = 1 - \alpha > 0 = u_\lambda(1)$, we then have

$$R_1 < 1, u_\lambda(R_1) = \bar{\omega}(R_1) \text{ and } u'_\lambda(R_1) \leq \bar{\omega}'(R_1).$$

Introduce, as in the proof of the Lemma 4.1, the functions u_{λ,R_1} and $\bar{\omega}_{R_1}$. We have that u_{λ,R_1} is a classical solution of $(2.1)_{\lambda,\alpha',\beta'}$, where

$$\alpha' := R_1^{-\frac{4}{p+1}}(u_\lambda(R_1) - 1) + 1, \beta' := R_1^{\frac{p-3}{p+1}}(u_\lambda)'(R_1).$$

Since $\lambda < \lambda^*$ and then

$$\frac{p\lambda}{(1-\bar{\omega})^p} \leq \frac{p\lambda^*}{\alpha^{p+1}(1-\omega)^{p+1}} < \frac{p\beta}{(1-\omega)^{p+1}},$$

by (3.3) $\bar{\omega}_{R_1}$ is a stable $H^2(B)$-weak sub-solution of $(2.1)_{\lambda,\alpha',\beta'}$. By Lemma 3.3, we deduce that $u_\lambda \geq \bar{\omega}$ in B_{R_1} which is impossible, since $\bar{\omega}$ is singular while u_λ is regular. This establishes claim (3.4) which, combined with the above inequality, yields

$$\frac{p\lambda^*}{(1-u^*)^{p+1}} \leq \frac{p\lambda^*}{\alpha^{p+1}(1-\omega)^{p+1}} < \frac{p\beta}{(1-\omega)^p},$$

and Thus

$$\inf_{\varphi\in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \frac{p\lambda^*\varphi^2}{(1-u^*)^{p+1}}}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth function, since otherwise, one could use the Implicit function Theorem to continue the minimal branch beyond λ^*.

Proof Theorem 1.1 (i).

We have proven that the u^* is regular for $n \leq 12$. Now we only prove that u^* is a singular solution of $(1.1)_{\lambda^*}$ for $n \geq 13$, in order to achieve this, we shall find a singular $H-$weak sub-solution of $(1.1)_{\lambda'}$, denote by $\omega_m(r)$, which is stable, according to the Lemma 5.4.

Choosing

$$\omega_m(r) = 1 - a_1 r^{\frac{4}{p+1}} + a_2 r^m, \quad \bar{\lambda} = \frac{8(p-1)}{(p+1)^2}[n - \frac{2(p-1)}{p+1}][n - \frac{4p}{p+1}],$$

since $\omega(1) = \omega'(1) = 0$, we have

$$a_1 = \frac{m}{m - \frac{4}{p+1}}; \quad a_2 = \frac{\frac{4}{p+1}}{m - \frac{4}{p+1}}.$$

For any m fixed, when $p \to \infty$, we have

$$a_1 = 1 + \frac{4}{(p+1)m} + o(p^{-1}) \quad \text{and} \quad a_2 = a_1 - 1 = \frac{4}{(p+1)m} + o(p^{-1})$$

and

$$\bar{\lambda} = \frac{8(n-2)(n-4)}{p} + o(p^{-1}).$$

Note that

$$\begin{aligned}
\frac{\lambda_n'\bar{\lambda}}{(1-\omega_m(r))^p} &- \Delta^2\omega_m(r) = \frac{\lambda_n'\bar{\lambda}}{(1-\omega_m(r))^p} \\
&- a_1\bar{\lambda}r^{-\frac{4p}{p+1}} - a_2\frac{m(m-2)(m+n-2)(m+n-4)}{r^{4-m}} \\
&= \frac{\lambda_n'\bar{\lambda}}{(a_1r^{\frac{4}{p+1}} - a_2r^m)^p} - a_1\bar{\lambda}r^{-\frac{4p}{p+1}} \\
&- a_2\frac{m(m-2)(m+n-2)(m+n-4)}{r^{4-m}} \\
&= \bar{\lambda}r^{-\frac{4p}{p+1}}\Big[\frac{\lambda_n'}{(a_1-a_2r^{m-\frac{4}{p+1}})^p} - a_1 \\
&- \frac{a_2m(m-2)(m+n-2)(m+n-4)}{\bar{\lambda}}r^{\frac{4p}{p+1}+m-4}\Big] \\
&= \bar{\lambda}r^{-\frac{4p}{p+1}}\Big[\frac{\lambda_n'}{(a_1-a_2r^{m-\frac{4}{p+1}})^p} - a_1 \\
&- \frac{a_2m(m-2)(m+n-2)(m+n-4)}{\bar{\lambda}}r^{m-\frac{4}{p+1}}\Big] \\
&= \frac{\bar{\lambda}r^{-\frac{4p}{p+1}}}{(a_1-a_2r^{m-\frac{4}{p+1}})^p}\left[\lambda_n' - H(r^{m-\frac{4}{p+1}})\right] \qquad (5.4)
\end{aligned}$$

with

$$H(x) = (a_1-a_2x)^p\left[a_1 + \frac{a_2m(m-2)(m+n-2)(m+n-4)}{\bar{\lambda}}x\right]. \qquad (5.5)$$

(1) Let $m=2$, then we can prove that

$$\sup_{[0,1]} H(x) = H(0) = a_1^{p+1} \longrightarrow e^2 \quad \text{as} \quad p \longrightarrow +\infty.$$

So (5.4) ≥ 0 is valid as long as

$$\lambda_n' = e^2.$$

At the same time, we have (since $a_1 - a_2r^{2-\frac{4}{p+1}} \geq a_1 - a_2 \geq 1$ in $[0,1]$)

$$\frac{n^2(n-4)^2}{16}\frac{1}{r^4} - \frac{p\beta_n}{r^4(a_1-a_2r^{2-\frac{4}{p+1}})^{p+1}} \geq r^{-4}\left[\frac{n^2(n-4)^2}{16} - p\beta_n\right]. \qquad (5.6)$$

Let $\beta_n = (\lambda_n'+\varepsilon)\bar{\lambda}$, where ε is arbitrary sufficient small, we need finally here

$$\frac{n^2(n-4)^2}{16} - p\beta_n = \frac{n^2(n-4)^2}{16} - p\lambda_n'\bar{\lambda} > 0.$$

For that, it is sufficient to have for $p \longrightarrow +\infty$

$$\frac{n^2(n-4)^2}{16} - 8(e^2+\varepsilon)(n-2)(n-4) + o(\frac{1}{p}) > 0.$$

So (5.6) ≥ 0 holds only for $n \geq 32$ when $p \longrightarrow +\infty$. Moreover, for p large enough

Thus it follows from Lemma 5.4 that u^* is singular and $\lambda^* \le e^2\bar{\lambda}$.

(2) Assume $13 \le n \le 31$. We shall show that $u = \omega_{3.5}$ satisfies the assumptions of Lemma 5.4 for each dimension $13 \le n \le 31$. Using Maple, for each dimension $13 \le n \le 31$ one can verify that inequality (5.4) ≥ 0 holds for the λ'_n given by Table 1. Then, by using Maple again, we show that there exists $\beta_n > \lambda'_n$ such that

$$\frac{(n-2)^2(n-4)^2}{16}\frac{1}{(|x|^2-0.9|x|^{\frac{n}{2}+1})(|x|^2-|x|^{\frac{n}{2}})} + \frac{(n-1)(n-4)^2}{4}\frac{1}{|x|^2(|x|^2-|x|^{\frac{n}{2}})} \ \ge\ \frac{p\beta_n}{(1-w_{3.5})^p}.$$

The above inequality and and improved Hardy-Rellich inequality (5.0) guarantee that the stability condition (5.2) holds for $\beta_n > \lambda'_n$. Hence by Lemma 5.4 the extremal solution is singular for $13 \le n \le 31$ the value of λ'_n and β_n are shown in Table 1.

Remark 1 *The values of λ'_n and β_n in Table 1 are not optimal.*

Table1

n	λ'_n	β_n
31	$3.15\bar{\lambda}$	$4\bar{\lambda}$
30-19	$4\bar{\lambda}$	$10\bar{\lambda}$
18	$3.19\bar{\lambda}$	$3.22\bar{\lambda}$
17	$3.15\bar{\lambda}$	$3.18\bar{\lambda}$
16	$3.13\bar{\lambda}$	$3.14\bar{\lambda}$
15	$2.76\bar{\lambda}$	$3.12\bar{\lambda}$
14	$2.34\bar{\lambda}$	$2.96\bar{\lambda}$
13	$2.03\bar{\lambda}$	$2.15\bar{\lambda}$

Remark 2 *The improved Hardy-Rellich inequality (5.0) is crucial to prove that u^* is singular in dimensions $n \ge 13$. Indeed by the classical Hardy-Rellich inequality and $u := w_2$, Lemma 5.4 only implies that u^* is singular n dimensions $n \ge 32$.*

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Properties of the extremal solution for a fourth-order elliptic problem *†‡§¶‖

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Abstract Let $\lambda^* > 0$ denote the largest possible value of λ such that

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^p} & \text{in } B, \\ 0 < u \leq 1 & \text{in } B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B. \end{cases}$$

has a solution, where B is the unit ball in R^n centered at the origin, $p > 1$ and n is the exterior unit normal vector. We show that for $\lambda = \lambda^*$ this problem possesses a unique weak solution u^*, called the extremal solution. We prove that u^* is singular when $n \geq 13$ for p large enough and $1 - C_0 r^{\frac{4}{p+1}} \leq u^*(x) \leq 1 - r^{\frac{4}{p+1}}$ on the unit ball, where $C_0 := (\lambda^*/\bar{\lambda})^{\frac{1}{p+1}}$ and $\bar{\lambda} := \frac{8(p-1)}{(p+1)^2}[n - \frac{2(p-1)}{p+1}][n - \frac{4p}{p+1}]$. Our results actually complete part of the open problem which [11] left

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1. Introduction and main result

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^p} & \text{in } B, \\ 0 < u \leq 1 & \text{in } B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B. \end{cases} \quad (1.1)_\lambda$$

Here B denotes the unit ball in R^n $(n \geq 2)$ centered at the origin, $\lambda > 0, p > 1$ and $\frac{\partial}{\partial n}$ the differentiation with the respect to the exterior unit normal, i.e., in radial direction. We consider only radial solutions, since all positive smooth solutions of $(1.1)_\lambda$ are radial, see Berchio et al. [3].

The motivation for studying $(1.1)_\lambda$ stems from a model for the steady sates of a simple micro electromechanical system (MEMS) which has the general form (see for example [20], [23])

$$\begin{cases} \alpha\Delta^2 u = (\beta \int_\Omega |\nabla u|^2 dx + \gamma)\Delta u + \frac{\lambda f(x)}{(1-u)^2(1+\chi \int_\Omega \frac{dx}{(1-u)^2})} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = \alpha \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

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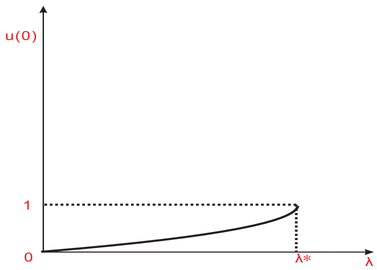

Figure 1 (The bifurcation diagram in the case the extremal solution is singular)

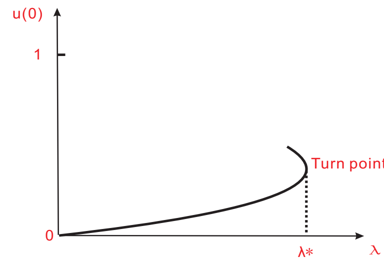

Figure 2 (The local bifercation diangram in the case the extremal solution is regular)

where $\alpha, \beta, \gamma, \chi \geq 0$, are fixed, $f \geq 0$ represents the permittivity profile, Ω is a bounded domain in R^n and $\lambda > 0$ is a constant which is increasing with respect to the applied voltage.

Recently, Equation (1.2) posed in $\Omega = B$ with $\beta = \gamma = \chi = 0, \alpha = 1$ and $f(x) \equiv 1$, which is reduced to

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in} \quad B, \\ 0 < u < 1 & \text{in} \quad B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial B, \end{cases} \tag{1.3}$$

has been studied extensively in [8]. For convenience, we now give the following notion of solution.

Definition 1.1. If u_λ is a solution of $(1.1)_\lambda$ such that for any other solution v_λ of $(1.1)_\lambda$ one has

$$u_\lambda \leq v_\lambda, \qquad \text{a.e. } x \in B,$$

we say that u_λ is a minimal solution of $(1.1)_\lambda$.

It is shown that there exists a critical value $\lambda^* > 0$ (pull-in voltage) such that if $\lambda \in (0, \lambda^*)$ the problem (1.3) has a smooth minimal solution , while for $\lambda > \lambda^*$ (1.3) has no solution even in a weak sense. Moreover, the branch $\lambda \to u_\lambda(x)$ is increasing for each $x \in B$, and therefore the function $u^*(x) := \lim_{\lambda \to \lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the issue of the regularity of this extremal solution-which, by elliptic regularity theory, is equivalent to whether $\sup_\Omega u^* < 1$- is an important question for many reasons. For example, one of the reason is that it decides whether the set of solutions stops there, or whether a new branch of solutions emanates from a bifurcation state (u^*, λ^*) (see Figures 1,2). This issue turned out to depend closely on the dimension. Indeed by the key uniform estimate of $\|(1-u)^{-3}\|_{L^1}$, Guo and Wei [17] obtained the regularity of the extremal solution for small dimensions and they proved that for dimension $n = 2$ or $n = 3, u^*$ is smooth. But from their result, the regularity of extremal solution of (1.3) is unknown for $n \geq 4$. Recently, using certain improved Hardy-Rellich inequalities, Cow-Esp-Gho-Mor [8] improved the above result and they obtained that u^* is regular in dimensions $1 \leq n \leq 8$, while it is singular for $n \geq 9$, i.e., the critical dimension is 9. So the issue of the regularity of the extremal solution of $(1.1)_\lambda$ for power $p = -2$ is completely solved, but the critical dimension for generally negative power is **unknown**.

Recently, Juan Dàvila etal [11] gave the deep research about the multiplicity phenomenon of $(1.1)_\lambda$ radial solutions of and the regularity of the extremal solution of $(1.1)_\lambda$ for a large range of negative powers.

For convenience, we now define:

$$p_c = \frac{n+2-\sqrt{4+n^2-4\sqrt{N^2+H_n}}}{N-6-\sqrt{4+N^2-4\sqrt{N^2+H_n}}} \qquad \text{for } \ n \geq 3;$$

$$p_c^+ = \frac{n+2+\sqrt{4+n^2-4\sqrt{n^2+H_n}}}{N-6+\sqrt{4+n^2-4\sqrt{n^2+H_n}}} \qquad \text{for } \ n \geq 3, \quad n \neq 4$$

with $H_n = (n(n-4)/4)^2$ and the numbers p_c and p_c^+ are such that when $p = p_c$ or $p = p_c^+$ then

$$(\frac{4}{p-1}+4)(\frac{4}{p-1}+2)(n-2-\frac{4}{p-1})(n-4-\frac{4}{p-1}) = H_n$$

Now we give the main result of [11]

Theorem A Assume

$$n = 3 \ \text{ and } \ p_c^+ < p < p_c, \ \text{ or } \ 4 \leq n \leq 12 \ \text{ and } \ -\infty < p < p_c. \tag{1.5}$$

Then there exist a unique λ_s such that $(1.1)_\lambda$ with $\lambda = \lambda_s$ has infinitely many radial smooth solutions. For $\lambda \neq \lambda_s$ there are finitely many radial smooth solutions and their number goes to infinity as $\lambda \to \lambda_s$. Moreover, $\lambda_s < \lambda^*$ and u^* is regular.

From this Theorem, we known that the extremal solution of $(1.1)_\lambda$ is regular for a certain range of p and n. At the same time, they left a open natural problem: if

$$\begin{cases} N = 3 \text{ and } p \in (-3, p_c^+] \bigcup [p_c, -1) \text{ or} \\ 5 \leq N \leq 12 \ \text{ and } \ p_c \leq p < -1, \ \text{or} \\ N \geq 13 \ \text{ and } p < -1, \end{cases}$$

is u^* singular?

In this paper, by constructing a semi-stable singular $H^2(B)-$ weak sub-solution of $(1.1)_\lambda$, we prove that, if p is large enough, the extremal solution is singular for dimensions $n \geq 13$ and complete part of the above open problem. Our result is stated as follows:

Theorem 1.1 ; (i) For any $p > 1$, the unique extremal solution of $(1.1)_{\lambda^*}$ is regular for dimensions $n \leq 4$;

(ii) There exists $p_0 > 1$ large enough such that for $p \geq p_0$, the unique extremal solution of $(1.1)_{\lambda^*}$ is singular for dimensions $n \geq 13$.

From the technical point of view, one of the basic tools in the analysis of nonlinear second order elliptic problems in bounded and unbounded domains of $R^n (n \geq 2)$ is the maximum principle. However, for high order problems, such principle dose not normally hold for general domains (at least for the clamped boundary conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$), which causes several technical difficulties. One of reasons to the study $(1.1)_\lambda$ in a ball is that a maximum principle holds in this situation, see [1], [5]. The second obstacle is the well-known difficulty of extracting energy estimates for solutions of fourth order problems from their stability properties. Besides, for the corresponding second order problem, the starting point was an explicit singular solution for a suitable eigenvalue parameter λ which turned out to play a fundamental role for the shape of the corresponding bifurcation method used to analyze the regularity of the extremal solution for second order problem could not carry to the corresponding problem for $(1.1)_\lambda$. In this paper, we, in order to overcome the third obstacle, use improved and non standard Hardy-Rellich inequalities recently established by Ghoussoub-Moradifam in [14] to construct a semi-stable singular $H^2(B)-$ weak sub-solution of $(1.1)_\lambda$.

This paper is organized as follows. In the next section, some preliminaries are reviewed. In Section 3, we give the uniform estimate of $\|(1-u)^{p+1}\|_{L^1}$ according to the stability of the minimal solutions. We study the regularity of the extremal solution of $(1.1)_\lambda$ and the Theorem 1 (ii) is established in Section 4. Finally, we will show that the extremal solution u^* in dimensions $n \geq 13$ is singular by constructing a semi-stable singular $H^2(B)-$ weak sub-solution of $(1.1)_\lambda$.

2. Preliminaries

First we give some comparison principles which will be used throughout the paper

Lemma 2.1. (Boggio's principle, [5]) If $u \in C^4(\bar{B}_R)$ satisfies

$$\begin{cases} \Delta^2 u \geq 0 & \text{in } \ B_R, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \ \partial B_R, \end{cases}$$

then $u \geq 0$ in B_R.

Lemma 2.2. Let $u \in L^1(B_R)$ and suppose that

$$\int_{B_R} u \Delta^2 \varphi \geq 0$$

for all $\varphi \in C^4(\bar{B}_R)$ such that $\varphi \geq 0$ in B_R, $\varphi|_{\partial B_R} = \frac{\partial \varphi}{\partial n} = 0$. Then $u \geq 0$ in B_R. Moreover $u \equiv 0$ or $u > 0$ a.e., in B_R.

For a proof see Lemma 17 in [1].

Lemma 2.3. If $u \in H^2(B_R)$ is radial, $\Delta^2 u \geq 0$ in B_R in the weak sense, that is

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \ \ \forall \varphi \in C_0^\infty(B_R), \ \varphi \geq 0$$

and $u|_{\partial B_R} \geq 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$ then $u \geq 0$ in B_R.

Proof. We only deal with the case $R = 1$ for simplicity. Solve

$$\begin{cases} \Delta^2 u_1 = \Delta^2 u & \text{in } \ B \\ u_1 = \frac{\partial u_1}{\partial n} = 0 & \text{on } \ \partial B \end{cases}$$

in the sense $u_1 \in H_0^2(B)$ and $\int_B \Delta u_1 \Delta \varphi = \int_B \Delta u \Delta \varphi$ for all $\varphi \in C_0^\infty(B)$. Then $u_1 \geq 0$ in B by lemma 2.2.

Let $u_2 = u - u_1$ so that $\Delta^2 u_2 = 0$ in B. Define $f = \Delta u_2$. Then $\Delta f = 0$ in B and since f is radial we find that f is a constant. It follows that $u_2 = ar^2 + b$. Using the boundary conditions we deduce $a + b \geq 0$ and $a \leq 0$, which imply $u_2 \geq 0$.

As in [8], we are now led here to examine problem $(1.1)_\lambda$ with non-homogeneous

$$
\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^p} & \text{in } B, \\ \alpha < u \leq 1 & \text{in } B, \\ u = \alpha, \frac{\partial u}{\partial n} = \gamma & \text{on } \partial B, \end{cases} \qquad (2.1)_{\lambda,\alpha,\gamma}
$$

where α, γ are given.

Let Φ denote the unique solution of

$$
\begin{cases} \Delta^2 \Phi = 0 & \text{in } B, \\ \Phi = \alpha, \frac{\partial \Phi}{\partial n} = \gamma & \text{on } \partial B. \end{cases} \qquad (2.2)
$$

We will say that the pair (α, γ) is admissible if $\gamma \leq 0$, and $\alpha - \frac{\gamma}{2} < 1$. We now introduce a notion of weak solution.

Definition 2.1. We say that u is a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$, if $\alpha \leq u \leq 1$ a.e. in Ω, $\frac{1}{(1-u)^p} \in L^1(\Omega)$ and if

$$
\int_\Omega (u - \Phi)\Delta^2 \varphi = \lambda \int_\Omega \frac{\varphi}{(1-u)^p} \quad \forall \varphi \in C^4(\bar{B}) \bigcap H_0^2(B),
$$

where Φ is given in (2.1). We say u is a weak super-solution (resp. weak sub-solution) of $(2.1)_{\lambda,\alpha,\gamma}$, if the equality is replaced with $\geq$ (resp.$\leq$) for $\varphi \geq 0$.

Definition 2.2. We say a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$ is regular (resp. singular) if $\|u\|_\infty < 1$ (resp. $\|u\| = 1$) and stable (resp. semi-stable) if

$$
\mu_1(u) = \inf\{\int_B (\Delta\varphi)^2 - p\lambda \int_B \frac{\varphi^2}{(1-u)^p} : \phi \in H_0^2(B), \|\phi\|_{L^2} = 1\}
$$

is positive (resp. non-negative).

We now define

$$
\lambda^*(\alpha, \gamma) := \sup\{\lambda > 0 : (2.1)_{\lambda,\alpha,\gamma} \text{ has a classical soltion}\}
$$

and

$$
\lambda_*(\alpha, \gamma) := \sup\{\lambda > 0 : (2.1)_{\lambda,\alpha,\gamma} \text{ has a weak soltion}\}.
$$

Observe that by Implicit Function Theorem, we can classically solve $(2.1)_{\lambda,\alpha,\gamma}$ for small $\lambda' s$. Therefore, $\lambda^*(\alpha, \gamma)$ and $\lambda_*(\alpha, \gamma)$ are well defined for any admissible pair (α, γ). To cut down notations we won't always indicate α and γ.

Let now give the following standard existence result.

Theorem 2.1. For every $0 \leq f \in L^1(\Omega)$ there exists a unique $0 \leq u \in L^1(\Omega)$ which satisfies

$$
\int_\Omega u \Delta^2 \varphi dx = \int_\Omega f \varphi dx
$$

for all $\varphi \in C^4(\bar{B}) \bigcap H_0^2(B)$.

The proof is standard, please see [15], here we omit it. From this Theorem, we immediately have the following result.

For the sake of completeness, we include a brief proof here, which be called "weak" iterative scheme: $u_0 = U$ and (inductively) let $u_n, n \geq 1$, be the solution of

$$\int_\Omega (u_n - \Phi)\Delta^2\varphi = \lambda \int_\Omega \frac{\varphi}{(1-u_{n-1})^p} \quad \forall \varphi \in C^4(\bar{B}) \bigcap H_0^2(B),$$

given by Theorem 2.2. Since α is a sub-solution of $(2.1)_{\lambda,\alpha,\gamma}$, inductively it is easily shown by Lemma 2.2 that $\alpha \leq u_{n+1} \leq u_n \leq U$ for every $n \geq 0$. Since

$$(1-u_n)^{-p} \leq (1-U)^{-p} \quad \in L^1(B),$$

by Lebesgue Theorem the function $u = \lim_{n\to\infty} u_n$ is a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$ so that $\alpha \leq u \leq U$.

In particular, for every $\lambda \in (0, \lambda_*)$, we can find a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$. In the same range of $\lambda' s$, this is still true for regular weak solutions as shown in the following lemma.

Lemma 2.4. Let (α, γ) be an admissible pair and u be a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$. Then, there exists a regular solution for every $0 < \mu < \lambda$.

Proof. Let $\epsilon \in (0,1)$ be given and let $\bar{u} = (1-\epsilon)u + \epsilon\Phi$, where Φ is given in (2.2). by lemma 2.2 $\sup_B \Phi < \sup_B u \leq 1$. Hence

$$\sup_B \bar{u} \leq (1-\epsilon) + \epsilon \sup_B \Phi < 1, \quad \inf \bar{u} \geq (1-\epsilon)\alpha + \epsilon \inf_B \Phi = \alpha$$

$$\begin{aligned}
\int_B (\bar{u} - \Phi)(\beta\Delta^2\varphi - \tau\Delta\varphi) &= (1-\epsilon)\int_B (u-\Phi)(\beta\Delta^2\varphi - \tau\Delta\varphi) = (1-\epsilon)\lambda \int_B \frac{\varphi}{(1-u)^p} \\
&= (1-\epsilon)^{p+1}\lambda \int_B \frac{\varphi}{(1-\bar{u}+\epsilon(\Phi-1))^p} \geq (1-\epsilon)^{p+1}\lambda \int_B \frac{\varphi}{(1-\bar{u})^p}.
\end{aligned}$$

Note that $0 \leq (1-\epsilon)(1-u) = 1 - \bar{u} + \epsilon(\Phi - 1) < 1 - \bar{u}$. So $\bar{u}$ is a weak super-solution of $(2.1)_{(1-\epsilon)^{p+1}\lambda,\alpha,\gamma}$ such that $\sup_B < 1$. By Lemma 2.2 we get the existence of a weak solution B of $(2.1)_{(1-\epsilon)^{p+1}\lambda,\alpha,\gamma}$ so that $\alpha \leq \omega \leq \bar{u}$. In particular, $\sup_B \bar{u} < 1$ and ω is a regular weak solution. Since $\epsilon \in (0,1)$ is arbitrarily chosen, the proof is done.

Now we recall some basic facts about the minimal branch

Theorem 2.2. $\lambda^* \in (0, +\infty)$ and the following holds:

1. For each $0 < \lambda < \lambda^*$ there exists a regular and minimal solution u_λ of $(2.1)_{\lambda,\alpha,\gamma}$;
2. For each $x \in B$ the map $\lambda \to u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$;
3. For $\lambda > \lambda^*$ there are no weak solutions of $(2.1)_{\lambda,\alpha,\gamma}$.

The proof is standard, see [8], here we omit it.

3. Stability of the minimal solutions

In this section we shall show that the extremal solution is regular in small dimensions. Let us begin with the following priori estimates along the minimal branch u_λ. In order to achieve this, we shall need yet another notion of H^2(B)- weak solutions, which is an intermediate class between classical and weak solutions.

Definition 3.1. We say that u is a $H^2(\mathrm{B})$- weak solution of $(2.1)_{\lambda,\alpha,\beta}$ if $u - \Phi \in H_0^2(\mathrm{B})$, $\alpha \leq u \leq 1 \in B$, $\frac{1}{(1-u)^p} \in L^1(B)$ and if

$$\int_B \Delta u \Delta \phi = \lambda \int_B \frac{\phi}{(1-u)^p}, \quad \forall \phi \in C^4(\bar{B}) \bigcap H_0^2(B),$$

where Φ is given in (2.2). We say that u is a $H^2(\mathrm{B})$- weak super-solution (resp. $H^2(\mathrm{B})$- weak sub-solution) of $(2.1)_{\lambda,\alpha,\beta}$ if for $\phi \geq 0$ the equality is replaced with $\geq$ (resp.$\leq$) and $u \geq \alpha$ (resp. $\leq$), $\partial_v u \leq \beta$ (resp. $\geq$) on ∂B.

Theorem 3.1. Suppose that (α, γ) is an admissible pair.

1. The minimal solution u_λ is stable, and is the unique semi-stable $H-$ weak solution of $(2.1)_{\lambda,\alpha,\gamma}$;
2. The function $u^* := \lim_{\lambda \to \lambda^*} u_\lambda$ is a well-defined semi-stable $H-$ weak solution of $(2.1)_{\lambda^*,\alpha,\gamma}$;
3. u^* is the unique $H-$ weak solution of $(2.1)_{\lambda^*,\alpha,\gamma}$, and when u^* is classical solution, then $\mu_1(u^*) = 0$;
4. If v is a singular, semi-stable $H-$ weak solution of $(2.1)_{\lambda,\alpha,\gamma}$, then $v = u^*$ and $\lambda = \lambda^*$.

The main tool which we use to prove the theorem 3.1 is the following comparison lemma which is valid exactly in the class H.

Lemma 3.2. Let (α, γ) is an admissible pair and u be a semi-stable $H-$ weak solution of $(P_{\lambda,\alpha,\gamma})$. Assume U is a $H-$ weak super-solution of $(2.1)_{\lambda,\alpha,\gamma}$. Then

1. $u \leq U$ a.e. in Ω;
2. If u is a classical solution and $\mu_1(u) = 0$ then $U = u$.

A more general version of Lemma 3.2 is available in the following.

Lemma 3.3. Let (α, γ) is an admissible pair and $\gamma' \leq 0$. Let u be a semi-stable $H-$ weak sub- solution of $(2.1)_{\lambda,\alpha,\gamma}$ with $u = \alpha' \leq \alpha, \Delta u = \beta' \geq \beta$ on $\partial\Omega$. Assume that U is a $H-$ weak super-solution of $(2.1)_{\lambda,\alpha,\gamma}$ with $U = \alpha, \Delta U = \beta$ on $\partial\Omega$. Then $U \geq u$ a.e. in Ω.

The proof of Lemma 3.2 and Lemma 3.1 are same as [8, 22], we omit it here.

We need also some a priori estimates along the minimal branch u_λ.

Lemma 3.4. Let (α, γ) be an admissible pair. Then for every $\lambda \in (0, \lambda^*)$, we have

$$p \int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} \leq \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^p},$$

where Φ is given by (2.1). In particular, there is a constant C independent of λ so that

$$\int_B |\Delta u_\lambda|^2 dx + \int_B \frac{1}{(1-u_\lambda)^{p+1}} \leq C. \tag{3.1}$$

Proof. Testing $(2.1)_{\lambda,\alpha,\gamma}$ on $u_\lambda - \Phi \in W^{4,2}(B) \cap H_0^2(B)$. We see that

$$\lambda \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^p} = \int_B (\Delta(u_\lambda - \Phi))^2 dx \geq p\lambda \int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} dx$$

in the view of $\beta\Delta^2\Phi - \tau\Delta\Phi = 0$. In particular, for $\delta > 0$ small we have that

$$\int_{|u_\lambda-\Phi|\geq\delta} \frac{1}{(1-u_\lambda)^{p+1}} \leq \frac{1}{\delta^2}\int_{|u_\lambda-\Phi|\geq\delta} \frac{(u_\lambda-\Phi)^2}{(1-u_\lambda)^{p+1}} \leq \frac{1}{\delta^2}\int_B \frac{1}{(1-u_\lambda)^p}$$
$$\leq \delta^{p-1}\int_B \frac{1}{(1-u_\lambda)^{p+1}} + C_\delta$$

by means of Young's inequality. Since for δ small

$$\int_{|u_\lambda-\Phi|\leq\delta} \frac{1}{(1-u_\lambda)^{p+1}} \leq C$$

for some $C > 0$, we get that

$$\int_B \frac{1}{(1-u_\lambda)^{p+1}} \leq C$$

for some $C > 0$ and for every $\lambda \in (0, \lambda^*)$. By Young' s and Hölder's inequalities, we have

$$\begin{aligned}\int_B |\Delta u_\lambda|^2 dx &= \int_B \Delta u_\lambda \Delta\Phi dx + \lambda\int_B \frac{u_\lambda-\Phi}{(1-u_\lambda)^p}dx \\ &\leq \delta\int_B |\Delta u_\lambda|^2 dx + C_\delta + C(\int_B \frac{1}{(1-u_\lambda)^{p+1}})^{\frac{p}{p+1}}.\end{aligned}$$

So we have

$$\int_B |\Delta u_\lambda|^2 dx + \int_B \frac{1}{(1-u_\lambda)^{p+1}} \leq C$$

where C is absolute constant.

Proof of the Theorem 3.1. (1) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0, \lambda^*)$. since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in B$, it is easily seen that $\lambda \to \mu_1(u_\lambda)$ is a decreasing and continuous function on $(0, \lambda^*)$. Define

$$\lambda_{**} := \sup\{0 < \lambda < \lambda^* : \mu_1(u_\lambda) > 0\}.$$

We have that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have $\mu_1(u_{\lambda_{**}}) = 0$, and for every $\mu \in (\lambda_{**}, \lambda^*)$, u_μ would be a classical super-solution of $(P_{\lambda_{**},\alpha,\gamma})$. A contradiction arises since Lemma 3.2 implies $u_\mu = u_{\lambda_{**}}$. Finally, Lemma 3.2 guarantees the uniqueness in the class of semi-stable $H-$ weak solutions.

(2) It follows from (3.1) that $u_\lambda \to u^*$ in a pointwise sense and weakly in $H^2(B)$, and $\frac{1}{1-u^*} \in L^{p+1}$. In particular, u^* is a $H-$ weak solution of $(P_{\lambda^*,\alpha,\gamma})$ which is also semi-stable as the limiting function of the semi-stable solutions $\{u_\lambda\}$.

(3) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the Implicit Function Theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By Lemma 3.2, u^* is then the unique $H-$ weak solution of $(P_{\lambda^*,\alpha,\gamma})$.

(4) If $\lambda < \lambda^*$, we get by uniqueness that $v = u_\lambda$. So v is not singular and a contradiction arises. Now by theorem 2.5 (3) we have that $\lambda = \lambda^*$. Since v is a semi-stable $H-$ weak solution of $(P_{\lambda^*,\alpha,\gamma})$ and u^* is a $H-$ weak super-solution of $(P_{\lambda^*,\alpha,\gamma})$, we can apply Lemma 3.2 to get $v \leq u^*$ a.e. in Ω. Since u^* is also a semi-stable solution, we can reverse the roles of v and u^* in Lemma 3.2 to see that $v \geq u^*$ a.e. in Ω. So equality

4. Regularity of the extremal solutions and the Proof of the Theorem 1.1 (i)

In this section we first show that the extremal solution is regular in small dimensions by the uniformly bounded of u_λ in $H_0^2(B)$. Second, using the refined version of Hardy-Rellich inequality, we prove the extremal solution is singular for $n \geq 13$ and p large enough. Now we give the proof of Theorem 1.1 (ii).

Proof of Theorem 1.1 (i). As already observed, estimate (3.1) implies that $f(u^*) = (1-u^*)^{-p} \in L^{\frac{p+1}{p}}(B)$. Since u^* is radial and radially decreasing. We need to show that $u^*(0) < 1$ to get the regularity of u^*. In fact, on the contrary suppose that $u^*(0) = 1$. By the standard elliptic regularity theory shows that $u^* \in W^{4,\frac{p+1}{p}}$. By the Soblev imbedding theorem, i.e. $W^{4,\frac{p+1}{p}} \hookrightarrow C^m(0 < m \leq 4 - \frac{pn}{p+1}, 1 \leq n \leq 4)$. We have u^* is a Lipschitz function in B for $1 \leq n \leq 3$.

Now suppose $u^*(0) = 1$ and $1 \leq n \leq 2$. Since

$$\frac{1}{1-u^*} \geq \frac{C}{|x|} \quad \text{in} \quad B$$

for some $C > 0$. One see that

$$+\infty = C^{p+1} \int_B \frac{1}{|x|^{p+1}} \leq \int_B \frac{1}{(1-u^*)^{p+1}} < +\infty.$$

A contradiction arises and hence u^* is regular for $1 \leq n \leq 2$.

For $n = 3$, by the Sobolev imbedding theorem, we have $u^* \in C^{\frac{p+4}{p+1}}(\bar{B})$, if $\frac{p+4}{p+1} \geq 2$, then $u^*(0) = 1, Du^*(0) = 0$ and

$$|Du^*(\varepsilon) - Du^*(0)| \leq M|\varepsilon| \leq M|x|$$

where $0 < |\varepsilon| < |x|$. Thus

$$|u(x) - u(0)| \leq |Du(\varepsilon)||x| \leq M|x|^2.$$

This inequality shows that

$$+\infty = C^{p+1} \int_B \frac{1}{|x|^{2(p+1)}} \leq \int_B \frac{1}{(1-u^*)^{p+1}} < +\infty.$$

A contradiction arises and hence u^* is regular for $n = 3$; if $\frac{p+4}{p+1} < 2$, then

$$|Du(\varepsilon) - Du(0)| \leq M|\varepsilon|^{\frac{4}{p-1}-1} \leq M|x|^{\frac{3}{p+1}}$$

where $0 < |\varepsilon| < |x|$. Thus

$$|u(x) - u(0)| \leq |Du(\varepsilon)||x| \leq M|x|^{\frac{4+p}{p+1}},$$

and a contradiction is obtained as above.

For $n = 4$, by the Sobolev imbedding theorem, we have $u^* \in C^{\frac{4}{p+1}}(\bar{B})$. If $1 < \frac{4}{p+1} < 2$, then $u^*(0) = 1, Du^*(0) = 0$ and where $0 < |\varepsilon| < |x|$. Thus

$$|u(x) - u(0)| \leq |Du(\varepsilon)||x| \leq M|x|^{\frac{4}{p+1}}.$$

If $\frac{4}{p+1} \leq 1$, then u^* is a Hölder's continues and

$$1 - u^*(x) \leq M|x|^{\frac{4}{p+1}},$$

and we obtain a contradiction as above.

Now we give the point estimate of singular extremal solution for dimensions $n \geq 5$.

Theorem 4.1. Let $n \geq 5$ and (u^*, λ^*) be the extremal pair of $(1.1)_\lambda$, when u^* is singular, then

$$1 - u^* \leq C_0|x|^{\frac{4}{p+1}},$$

where $C_0 := (\lambda^*/\bar{\lambda})^{\frac{1}{p+1}}$ and $\bar{\lambda} := \frac{8(p-1)}{(p+1)^2}[n - \frac{2(p-1)}{p+1}][n - \frac{4p}{p+1}]$.

In order to prove the Theorem 4.1, we need the lower bounds of λ^* and state as follows

Lemma 4.1. λ^* satisfies the following lower bounds for $n \geq 4$

$$\lambda^* \geq \bar{\lambda}$$

where $\bar{\lambda} = \frac{8(p-1)}{(p+1)^2}[n - \frac{2(p-1)}{p+1}][n - \frac{4p}{p+1}]$.

Proof. the proof is standard, here we include the proof for the sake of completeness. Notice that for $n \geq 4$ the function $\bar{u} = 1 - |x|^{\frac{4}{p+1}}$ satisfies

$$\frac{1}{(1 - \bar{u})^p} \in L^1(B)$$

and $\bar{u}$ is a weak solution of

$$\Delta^2 \bar{u} = \frac{\bar{\lambda}}{(1 - \bar{u})^p},$$

and $\bar{u}(1) = 0 = u_\lambda(1)$; $\frac{\partial u_\lambda}{\partial n}(1) \geq \frac{\partial \bar{u}_\lambda}{\partial n}(1)$. Therefore, $\bar{u}$ turns out to be a weak super-solution of $(1.1)_\lambda$ provided $\lambda < \bar{\lambda}$. Thus necessarily, we have

$$\lambda^* = \lambda_* \geq \bar{\lambda}.$$

The proof is done.

Proof of Theorem 4.1. First note that Lemma 4.1 gives the lower bound:

$$\lambda^* \geq \bar{\lambda}.$$

For $\delta > 0$, we define $u_\delta(x) := 1 - C_\delta|x|^{\frac{4}{p+1}}$ with $C_\delta := (\frac{\lambda^*}{\bar{\lambda}} + \delta)^{\frac{1}{p+}} > 1$. Since $n \geq 5$. we have that $u_\delta \in H^2_{loc}(R^n)$, $\frac{1}{1-u_\delta} \in L^3_{loc}(R^n)$ and u_δ is a H^2-weak solution of

$$\Delta^2 u_\delta = \frac{\lambda^* + \delta\bar{\lambda}}{(1 - u_\delta)^p} \quad \text{in} \quad R^n$$

We claim that $u_\delta \leq u^*$ in B, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set

is non-empty, and let $r_1 = \sup \Gamma$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$ and one infers that

$$\alpha := u^*(r_1) = u_\delta(r_1), \quad \beta = (u^*)'(r_1) \geq u'_\delta(r_1).$$

Setting $u_{\delta,r_1}(r) = r_1^{-\frac{4}{p+1}}(u_\delta(r_1 r) - 1) + 1$, we easily see that the function $u_{\delta,r_1}(r)$ is a $H^2(B)$-weak super-solution of $(2.1)_{\lambda^*+\delta\bar{\lambda},\alpha',\beta'}$, where

$$\alpha' := r_1^{-\frac{4}{p+1}}(u^*(r_1 r) - 1) + 1, \quad \beta' := r_1^{\frac{p-3}{p+1}}\beta.$$

Similarly, define $u^*_{r_1} = r_1^{-\frac{4}{p+1}}(u^*(r_1 r) - 1) + 1$, we have $u^*_{r_1}$ is singular semi-stable $H^2(B)$-weak solution of $(2.1)_{\lambda^*,\alpha',\beta'}$.

Now we claim that (α', β') is an admissible pair. Since u^* is radially decreasing, we have that $\beta' \leq 0$. Define the function

$$\omega(r) := (\alpha' - \frac{\beta'}{2}) + \frac{\beta'}{2}|x|^2 + \gamma(x),$$

where $\gamma(x)$ is a solution of $\Delta^2\gamma = \lambda^*$ in B with $\gamma = \partial_v \gamma = 0$ on ∂B. Then ω is a classical solution of

$$\begin{cases} \Delta^2 \omega = \lambda^* & \text{in } \ B \\ \omega = \alpha', \partial_v \omega = \beta' & \text{on } \ \partial B. \end{cases}$$

Since $\frac{\lambda^*}{(1-u^*_{r_1})^p} \geq \lambda^*$, by Lemma 2.1 we have

$$u^*_{r_1} \geq \omega \ \text{ a.e. } \ \text{in} B$$

Since $\omega(0) = \alpha' - \frac{\beta'}{2} + \gamma(0)$ and $\gamma(0) > 0$, we have

$$\alpha' - \frac{\beta'}{2} < 1$$

So (α', β') is an admissible pair and by Theorem 3.1 (4) we get that $(u^*_{r_1}, \lambda^*)$ coincides with the extremal pair of $(2.1)_{\lambda,\alpha',\beta'}$ in B.

Since (α', β') is an admissible pair and u_{δ,r_1} is a $H^2(B)$-weak super-solution of $(2.1)_{\lambda^*+\delta\bar{\lambda},\alpha',\beta'}$. We get from Proposition 2.1, the existence of a weak solution of $(2.1)_{\lambda^*+\delta\bar{\lambda},\alpha',\beta'}$. Since

$$\lambda^* + \delta\bar{\lambda} > \lambda^*,$$

we contradict the fact that λ^* is the extremal parameter of $(2.1)_{\lambda,\alpha',\beta'}$.

Thanks to the lower estimate on u^*, we get the following result.

Corollary 4.1. In any dimension $n \geq 1$, we have

$$\lambda^* > \bar{\lambda} = \frac{8(p-1)}{(p+1)^2}[n - \frac{2(p-1)}{p+1}][n - \frac{4p}{p+1}].$$

Proof. The function $\bar{u} := 1 - |x|^{\frac{4}{p+1}}$ is a $H^2(B)$-weak solution of $(2.1)_{\bar{\lambda},0,-\frac{4}{p+1}}$. If by

If $1 \le n \le 4$, u^* is then regular by Theorem (i). By Theorem 3.1 (3) there holds $\mu_1(u^*) = 0$. By Lemma 3.2 then yields that $u^* = \bar{u}$, which is a contradiction since then u^* will not satisfy the boundary conditions.

If now $n \ge 5$ and $\lambda^* = \bar{\lambda}$, then $C_0 = 1$ in Theorem 4.1, and we then have $u^* \ge \bar{u}$. It means again that $u^* = \bar{u}$, a contradiction that completes the proof.

In what follows, we will show that the extremal solution u^* of $(1.1)_\lambda$ in dimensions $n \ge 13$ is singular.

5. The extremal solution is singular for $n \ge 13$

We prove in this section that the extremal solution is singular for $n \ge 13$ and p large enough. For that we will need a a suitable Hardy-Rellich type inequality which was established by Ghoussoub-Moradifam in [14]. As in the previous section (u^*, λ^*) denotes the extremal pair of $(2.1)_\lambda$

Lemma 5.1. Let $n \ge 5$ and B be the unit ball in R^n. Then there exists $C > 0$, such that the following improved Hardy-Rellich inequality holds for all $\varphi \in H_0^2(B)$:

$$\int_B (\Delta\varphi)^2 dx \ge \frac{n^2(n-4)^2}{16} \int_B \frac{\varphi^2}{|x|^4} dx + C \int_B \varphi^2 dx$$

Lemma 5.2. Let $n \ge 5$ and B be the unit ball in R^n. Then the following improved Hardy-Rellich inequality holds for all $\varphi \in H_0^2(B)$:

$$\begin{aligned} \int_B (\Delta\varphi)^2 dx \quad \ge \quad & \frac{(n-2)^2(n-4)^2}{16} \int_B \frac{\varphi^2 dx}{(|x|^2 - 0.9|x|^{\frac{n}{2}+1})(|x|^2 - |x|^{\frac{n}{2}})} \\ + \quad & \frac{(n-1)(n-4)^2}{4} \int_B \frac{\varphi^2 dx}{|x|^2(|x|^2 - |x|^{\frac{n}{2}})}. \end{aligned} \tag{5.0}$$

As a consequence, the following improvement of the classical Hardy-Rellich inequality holds:

$$\int_B (\Delta\varphi)^2 dx \ge \frac{n^2(n-4)^2}{16} \int_B \frac{\varphi^2}{|x|^2(|x|^2 - |x|^{\frac{n}{2}})}.$$

Lemma 5.3. If $n \ge 13$, then $u^* \le 1 - |x|^{\frac{4}{p+1}}$.

Proof. Recall from Corollary 4.1 that $\bar{\lambda} < \lambda^*$. Let $\bar{u} = 1 - |x|^{\frac{4}{p+1}}$, we now claim that $u_\lambda \le \bar{u}$ for all $\lambda \in (\bar{\lambda}, \lambda^*)$. Indeed, fix such a λ and assume by contradiction that

$$R_1 := \inf\{0 \le R \le 1 : u_\lambda < \bar{u} \ \text{ in the interval } (R, 1)\} > 0.$$

From the boundary condition, one has that $u_\lambda < \bar{u}(r)$ as $r \to 1^-$. Hence, $0 < R_1 < 1, \alpha := u_\lambda(R_1) = \bar{u}(R_1)$ and $\beta := u'_\lambda(R_1) \le \bar{u}'(R_1)$. The same as the proof of Lemma 4.1, we have $u_\lambda \ge \bar{u}$ in B_{R_1} and a contradiction arises in view of the fact that $\lim_{x \to 0} \bar{u}(x) = 1$ and $\|u_\lambda\|_\infty < 1$. It follows that $u_\lambda \le \bar{u}$ in B for every $\lambda \in (\bar{\lambda}, \lambda^*)$ and in particular $u^* \le \bar{u}$ in B.

Lemma 5.4. Let $n \ge n(p)$. Suppose there exists $\lambda' > 0$ and a singular radial function $\omega(r) \in H^2(B)$ with $\frac{1}{1-\omega} \in L^\infty_{loc}(\bar{B} \setminus \{0\})$ such that and

$$p\beta\int_B \frac{\varphi^2}{(1-\omega)^{p+1}} \leq \int_B (\Delta\varphi)^2 \quad \text{for all } \varphi \in H_0^2(B) \tag{5.2}$$

1. If $\beta \geq \lambda'$, then $\lambda^* \leq \lambda'$.
2. If either $\beta > \lambda'$ or $\beta = \lambda' = \frac{H_n}{p}$, then the extremal solution u^* is necessarily singular.

Proof. (1). First, note that (5.2) and $\frac{1}{1-\omega} \in L^\infty_{loc}(\bar{B}\setminus\{0\})$ yield to

$$\frac{1}{1-\omega} \in L^1(B).$$

At the same time, (5.1) implies that $\omega(r)$ is a $H^2(B)$- weak stable sub-solution of $(1.1)_{\lambda'}$. If now $\lambda' < \lambda^*$, then by Lemma 3.3, we have

$$\omega(r) < u_{\lambda'},$$

which ia a contradiction since ω is singular while $u_{\lambda'}$ is regular.

(2) Suppose first that $\beta = \lambda' = \frac{H_n}{p}$ and that $n \geq 13$. Since by part (1) we have $\lambda^* \leq \frac{H_n}{p}$, we get from Lemma 5.3 and improved Hardy-Rellich inequality that there exists $C > 0$ so that for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2 - p\lambda^*\int_B \frac{\phi^2}{(1-u^*)^{p+1}} \geq \int_B (\Delta\phi)^2 - H_n\int_B \frac{\phi^2}{|x|^4} \geq C\int_B \phi^2.$$

It follows that $\mu_1(u^*) > 0$ and u^* must therefore be singular since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*.

Suppose now that $\beta > \lambda'$, and let $\frac{\lambda'}{\beta} < \gamma < 1$ in such a way that

$$\alpha := (\frac{\gamma\lambda^*}{\lambda'})^{\frac{1}{p+1}} < 1.$$

Setting $\bar{\omega} := 1 - \alpha(1-\omega)$, we claim that

$$u^* \leq \bar{\omega} \quad \text{in} \quad B. \tag{5.3}$$

Note that by the choice of α we have $\alpha^{p+1}\lambda' < \lambda^*$, and therefore to prove (3.4) it suffices to show that for $\alpha^{p+1}\lambda' \leq \lambda < \lambda^*$, we have $u_\lambda \leq \bar{\omega}$ in B. Indeed, fix such λ and note that

$$\Delta^2\bar{\omega} = \alpha\Delta^2\omega \leq \frac{\alpha\lambda'}{(1-\omega)^p} = \frac{\alpha^{p+1}\lambda'}{(1-\bar{\omega})^p} \leq \frac{\lambda}{(1-\bar{\omega})^p}.$$

Assume that $u_\lambda \leq \bar{\omega}$ dose not hold in B, and consider

$$R_1 := \sup\{0 \geq R \leq 1 | u_\lambda(R) > \bar{\omega}(R)\} > 0$$

Since $\bar{\omega}(1) = 1 - \alpha > 0 = u_\lambda(1)$, we then have

$$R_1 < 1, u_\lambda(R_1) = \bar{\omega}(R_1) \text{ and } u'_\lambda(R_1) \leq \bar{\omega}'(R_1).$$

Introduce, as in the proof of the Lemma 4.1, the functions u_{λ,R_1} and $\bar{\omega}_{R_1}$. We have that u_{λ,R_1} is a classical solution of $(2.1)_{\lambda,\alpha',\beta'}$, where

Since $\lambda < \lambda^*$ and then

$$\frac{p\lambda}{(1-\bar{\omega})^p} \leq \frac{p\lambda^*}{\alpha^{p+1}(1-\omega)^{p+1}} < \frac{p\beta}{(1-\omega)^{p+1}},$$

by (3.3) $\bar{\omega}_{R_1}$ is a stable $H^2(B)$-weak sub-solution of $(2.1)_{\lambda,\alpha',\beta'}$. By Lemma 3.3, we deduce that $u_\lambda \geq \bar{\omega}$ in B_{R_1} which is impossible, since $\bar{\omega}$ is singular while u_λ is regular. This establishes claim (3.4) which, combined with the above inequality, yields

$$\frac{p\lambda^*}{(1-u^*)^{p+1}} \leq \frac{p\lambda^*}{\alpha^{p+1}(1-\omega)^{p+1}} < \frac{p\beta}{(1-\omega)^p},$$

and Thus

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \frac{p\lambda^*\varphi^2}{(1-u^*)^{p+1}}}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth function, since otherwise, one could use the Implicit function Theorem to continue the minimal branch beyond λ^*.

Proof Theorem 1.1 (ii).

We have proven that the u^* is regular for $n \leq 12$. Now we only prove that u^* is a singular solution of $(1.1)_{\lambda^*}$ for $n \geq 13$, in order to achieve this, we shall find a singular $H-$weak sub-solution of $(1.1)_{\lambda'}$, denote by $\omega_m(r)$, which is stable, according to the Lemma 5.4.

Choosing

$$\omega_m(r) = 1 - a_1 r^{\frac{4}{p+1}} + a_2 r^m, \quad \bar{\lambda} = \frac{8(p-1)}{(p+1)^2}[n - \frac{2(p-1)}{p+1}][n - \frac{4p}{p+1}],$$

since $\omega(1) = \omega'(1) = 0$, we have

$$a_1 = \frac{m}{m - \frac{4}{p+1}}; \quad a_2 = \frac{\frac{4}{p+1}}{m - \frac{4}{p+1}}.$$

For any m fixed, when $p \to \infty$, we have

$$a_1 = 1 + \frac{4}{(p+1)m} + o(p^{-1}) \quad \text{and} \quad a_2 = a_1 - 1 = \frac{4}{(p+1)m} + o(p^{-1})$$

and

$$\bar{\lambda} = \frac{8(n-2)(n-4)}{p} + o(p^{-1}).$$

Note that

$$\begin{aligned}
\frac{\lambda_n'\bar{\lambda}}{(1-\omega_m(r))^p} &- \Delta^2\omega_m(r) = \frac{\lambda_n'\bar{\lambda}}{(1-\omega_m(r))^p} \\
&- a_1\bar{\lambda}r^{-\frac{4p}{p+1}} - a_2\frac{m(m-2)(m+n-2)(m+n-4)}{r^{4-m}} \\
&= \frac{\lambda_n'\bar{\lambda}}{(a_1r^{\frac{4}{p+1}}-a_2r^m)^p} - a_1\bar{\lambda}r^{-\frac{4p}{p+1}} \\
&- a_2\frac{m(m-2)(m+n-2)(m+n-4)}{r^{4-m}} \\
&= \bar{\lambda}r^{-\frac{4p}{p+1}}\Big[\frac{\lambda_n'}{(a_1-a_2r^{m-\frac{4}{p+1}})^p} - a_1 \\
&- \frac{a_2m(m-2)(m+n-2)(m+n-4)}{\bar{\lambda}}r^{\frac{4p}{p+1}+m-4}\Big] \\
&= \bar{\lambda}r^{-\frac{4p}{p+1}}\Big[\frac{\lambda_n'}{(a_1-a_2r^{m-\frac{4}{p+1}})^p} - a_1 \\
&- \frac{a_2m(m-2)(m+n-2)(m+n-4)}{\bar{\lambda}}r^{m-\frac{4}{p+1}}\Big] \\
&= \frac{\bar{\lambda}r^{-\frac{4p}{p+1}}}{(a_1-a_2r^{m-\frac{4}{p+1}})^p}\left[\lambda_n' - H(r^{m-\frac{4}{p+1}})\right] \qquad (5.4)
\end{aligned}$$

with

$$H(x) = (a_1-a_2x)^p\left[a_1 + \frac{a_2m(m-2)(m+n-2)(m+n-4)}{\bar{\lambda}}x\right]. \qquad (5.5)$$

(1) Let $m = 2$, then we can prove that

$$\sup_{[0,1]} H(x) = H(0) = a_1^{p+1} \longrightarrow e^2 \quad \text{as} \quad p \longrightarrow +\infty.$$

So (5.4) ≥ 0 is valid as long as

$$\lambda_n' = e^2.$$

At the same time, we have (since $a_1 - a_2r^{2-\frac{4}{p+1}} \geq a_1 - a_2 \geq 1$ in $[0,1]$)

$$\frac{n^2(n-4)^2}{16}\frac{1}{r^4} - \frac{p\beta_n}{r^4(a_1-a_2r^{2-\frac{4}{p+1}})^{p+1}} \geq r^{-4}\left[\frac{n^2(n-4)^2}{16} - p\beta_n\right]. \qquad (5.6)$$

Let $\beta_n = (\lambda_n' + \varepsilon)\bar{\lambda}$, where ε is arbitrary sufficient small, we need finally here

$$\frac{n^2(n-4)^2}{16} - p\beta_n = \frac{n^2(n-4)^2}{16} - p\lambda_n'\bar{\lambda} > 0.$$

For that, it is sufficient to have for $p \longrightarrow +\infty$

$$\frac{n^2(n-4)^2}{16} - 8(e^2+\varepsilon)(n-2)(n-4) + o(\frac{1}{p}) > 0.$$

So (5.6) ≥ 0 holds only for $n \geq 32$ when $p \longrightarrow +\infty$. Moreover, for p large enough

Thus it follows from Lemma 5.4 that u^* is singular and $\lambda^* \le e^2\bar{\lambda}$.

(2) Assume $13 \le n \le 31$. We shall show that $u = \omega_{3.5}$ satisfies the assumptions of Lemma 5.4 for each dimension $13 \le n \le 31$. Using Maple, for each dimension $13 \le n \le 31$ one can verify that inequality (5.4) ≥ 0 holds for the λ'_n given by Table 1. Then, by using Maple again, we show that there exists $\beta_n > \lambda'_n$ such that

$$\frac{(n-2)^2(n-4)^2}{16}\frac{1}{(|x|^2-0.9|x|^{\frac{n}{2}+1})(|x|^2-|x|^{\frac{n}{2}})} + \frac{(n-1)(n-4)^2}{4}\frac{1}{|x|^2(|x|^2-|x|^{\frac{n}{2}})} \ge \frac{p\beta_n}{(1-w_{3.5})^p}.$$

The above inequality and and improved Hardy-Rellich inequality (5.0) guarantee that the stability condition (5.2) holds for $\beta_n > \lambda'_n$. Hence by Lemma 5.4 the extremal solution is singular for $13 \le n \le 31$ the value of λ'_n and β_n are shown in Table 1.

Remark 1 *The values of λ'_n and β_n in Table 1 are not optimal.*

Table1

n	λ'_n	β_n
31	$3.15\bar{\lambda}$	$4\bar{\lambda}$
30-19	$4\bar{\lambda}$	$10\bar{\lambda}$
18	$3.19\bar{\lambda}$	$3.22\bar{\lambda}$
17	$3.15\bar{\lambda}$	$3.18\bar{\lambda}$
16	$3.13\bar{\lambda}$	$3.14\bar{\lambda}$
15	$2.76\bar{\lambda}$	$3.12\bar{\lambda}$
14	$2.34\bar{\lambda}$	$2.96\bar{\lambda}$
13	$2.03\bar{\lambda}$	$2.15\bar{\lambda}$

Remark 2 *The improved Hardy-Rellich inequality (5.0) is crucial to prove that u^* is singular in dimensions $n \ge 13$. Indeed by the classical Hardy-Rellich inequality and $u := w_2$, Lemma 5.4 only implies that u^* is singular n dimensions $n \ge 32$.*

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