COMPOSITION OF ORDINARY GENERATING FUNCTIONS

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Abstract

A solution is proposed for the problem of composition of ordinary generating functions. A new class of functions that provides a composition of ordinary generating functions is introduced; main theorems are presented; compositae are written for polynomials, trigonometric and hyperbolic functions, exponential and log functions. It is shown that the composition holds true for many integer sequences.

Abstract A solution is proposed for the problem of composition of ordinary generating functions. A new class of functions that provides a composition of ordinary generating functions is introduced; main theorems are presented; compositae are written for polynomials, trigonometric and hyperbolic functions, exponential and log functions. It is shown that the composition holds true for many integer sequences

1 Introduction

Generating functions are an efficient tool of solving mathematical problems. Given the ordinary generating functions $F(x) = \sum\limits_{n\geqslant 1} f(n)x^n$ and $R(x) = \sum\limits_{n\geqslant 0} r(n)x^n$, the operation of composition of generating functions $A(x) = R(F(x))$ is defined correctly. [3, 1, 6, 2]. However, coefficients of the composition of generating functions are difficult to find. Stanley [3] came close to the solution of the problem and proposed a formula for the composition of exponential generating functions based on ordered partitions of a finite set. Let us show that the basis for the composition of ordinary generating functions is ordered partitions of a positive integer n and put forward basic formulae for the coefficients of the composition of ordinary generating functions. For this purpose, we introduce several definitions.

Definition 1. *An ordinary generating function $F(x)$ is a series that belongs to the ring of formal power series in one variable $K[[x]]$:*

$$F(x) = \sum_{n \geq 0} f(n)x^n,$$

where $f(n) : P \to K$, P is a set of nonnegative numbers; K is a commutative field.

Further we consider only ordinary generating functions. The known generating functions are denoted as $F(x)$, $R(x)$, $G(x)$, and the desired generating function as $A(x)$.

Definition 2. *An ordered partition (composition) of a positive integer n is an ordered sequence of positive integers λ_i such that*

$$\sum_{i=1}^{k} \lambda_i = n,$$

where $\lambda_i \geq 1$ and $k = \overline{1, n}$ are parts of the ordered partition.

C_n is a set of all ordered partitions of n.
$\pi_k \in C_n$ is an ordered partition of C_n with k parts.
The ordered partitions of n have been much studied [4, 5].

2 Compositae and their properties

Let there be functions $f(n)$ and $r(n)$ and their generating functions $F(x) = \sum_{n \geqslant 1} f(n)x^n$, $R(x) = \sum_{n \geqslant 0} r(n)x^n$. Then, calculating the composition of the generating functions $A(x) = R(F(x))$ requires [2]

$$[F(x)]^k = \sum_{n \geq k} \sum_{\substack{\lambda_i > 0 \\ \lambda_1 + \lambda_2 + \ldots + \lambda_k = n}} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k) x^n. \tag{1}$$

Hence it follows that for the function $a(n)$ of the composition of generating functions with $n > 0$, the formula

$$a(0) = r(0),$$

$$a(n) = \sum_{k=1}^{n} \left[\sum_{\substack{\lambda_i > 0 \\ \lambda_1 + \lambda_2 + \ldots + \lambda_k = n}} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k) \right] r(k) \tag{2}$$

holds true. Further the composition of generating functions is written implying that $a(0) = r(0)$.

Remark 3. It should be noted that the summation in formulae (1),(2) is over all ordered partitions of n that have exactly k parts, because $\{\lambda_1 + \lambda_2 + \ldots + \lambda_k = n\}$, $\lambda_i > 0$, $i = \overline{1, k}$ (further we use the reduction $\pi_k \in C_n$).

Thus, the ordered partitions of n are the basis for calculation of the composition of generating functions.

Let us consider the following example. Assume that $f(0) = 0$, $f(n) = 1$ for all $n > 0$. This function is defined by the generating function $F(x) = \frac{x}{1-x}$. Then, the expression

$$\sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k)$$

gives the number of ordered partitions of n with exactly k parts; this number is equal to $\binom{n-1}{k-1}$ [4]. Thus,

$$\sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k) = \binom{n-1}{k-1}.$$

Hence it follows that the formula valid for any generating function $R(x) = \sum_{n \geqslant 0} r(n) x^n$ and $A(x) = R\left(\frac{x}{1-x}\right)$ is

$$a(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} r(k).$$

Example 4. *For $R(x) = \frac{x}{1-x}$, we have the composition $A(x) = \frac{x}{1-2x}$ and*

$$a(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.$$

Thus, we calculate the total number of ordered partitions of n.

Example 5. *We have $R(x) = e^x$, then for the composition $A(x) = e^{\frac{x}{1-x}}$ we can write*

$$a(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{1}{k!}$$

(see A000262 formula Herbert S. Wilf).

Example 6. *We have $R(x) = \frac{x}{1-x-x^2}$, then for the composition $A(x) = R(\frac{x}{1-x})$ we can write*

$$a(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} F(k),$$

where $F(k)$ is the Fibonacci numbers (see A001519, formula Benoit Cloitre).

Definition 7. *A composita of the generating function $F(x) = \sum_{n>0} f(n) x^n$ is the function*

$$F^{\Delta}(n, k) = \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k). \tag{3}$$

Calculation of $F^{\Delta}(n,k)$ is of prime importance to obtain a composition of generating functions, because from formula (2) it follows that the formula valid for the composition $A(x) = R(F(x))$ is

$$a(n) = \sum_{k=1}^{n} F^{\Delta}(n,k)r(k). \tag{4}$$

The basis for the derivation of a composita is calculation of the ordered partition π_k of C_n. From formula (1) it follows that the generating function of the composita is equal to

$$[F(x)]^k = \sum_{n\geq k} F^{\Delta}(n,k)x^n.$$

For $F(x)$, the condition $f(0) = 0$ holds true, and hence numbering for the composita begins with $k = 1$, $n = 1$. For $k = 1$, $F^{\Delta}(n,k) = f(n)$. For $k > n$, $F^{\Delta}(n,k)$ is equal to zero. This statement stems from the fact that there is no ordered partition of n in which the number of parts is larger than n.

The above example demonstrates that the Pascal triangle is a composita for the generating function $\frac{x}{1-x}$ and deriving the composition $A(x) = R\left(\frac{x}{1-x}\right)$ requires the use of

$$F^{\Delta}(n,k) = \binom{n-1}{k-1}.$$

Let us derive a recurrence formula for the composita of a generating function.

Theorem 8. *For the composita $F^{\Delta}(n,k)$ of the generating function $F(x) = \sum_{n>0} f(n)x^n$, the following relation holds true:*

$$F^{\Delta}(n,k) = \begin{cases} f(n), & k = 1, \\ [f(1)]^n, & k = n, \\ \sum_{i=0}^{n-k} f(i+1)F^{\Delta}(n-i-1,k-1) & k < n. \end{cases} \tag{5}$$

Proof. Let us derive a recurrence formula for the $c_{n,k}$ number of ordered partitions of n that have exactly k parts. Let us introduce the operation $pos[\lambda^*, \pi_k]$ of adjunction of the new part λ^* on the left to a certain ordered partition $\pi_k \in C_n$ providing that $\lambda^* > 0$. From the ordered partition $\pi_k \in C_n$ this operation obtains an ordered partition $\pi_{k+1} \in C_{\lambda^*+n}$. Let us extend this operation to sets. Assume that $C_{n,k} = \{\pi_k | \pi_k \in C_n\}$, then the set $\hat{C} = pos[\lambda^*, C_{n,k}]$ is a subset $C_{\lambda^*+n,k+1}$. Thus, we can write

$$C_{n,k} = pos[1, C_{n-1,k-1}] \cup pos[2, C_{n-1,k-1}] \cup \ldots \cup pos[n-k-1, C_{k-1,k-1}].$$

In this case, the condition

$$pos[i, C_{n-i,k-1}] \cap pos[j, C_{n-j,k-1}] = \oslash$$

is fulfilled for all $i \neq j$, because the first parts of the ordered partitions π_k do not coincide. Hence,

$$c_{n,k} = \sum_{i=0}^{n-k} c_{n-i-1,k-1}, \tag{6}$$

and $c_{k,k} = 1$ because we have the only ordered partition $\pi_k = \{1+1+\ldots+1 = n\}$, and $c_{n,1} = 1$ because $\pi_1 = \{n = n\}$.

Let us now consider expression (3). Using expression (6), we can write

$$\begin{aligned}&F^{\Delta}(n,k) = \\ &= f(1)F^{\Delta}(n-1,k-1) + f(2)F^{\Delta}(n-2,k-1) + \ldots + \\ &+ f(n-k+1)F^{\Delta}(k-1,k-1).\end{aligned}$$

The set $C_{n,n}$ consists of the only ordered partition $\{1+1+\ldots+1\}$, and then $F^{\Delta}_{n,n} = [f(1)]^n$; the set $C_{n,1}$ consists of $\{n\}$, and then $F^{\Delta}_{n,1} = f(n)$. Thus, the theorem is proved. $\square$

Consideration of formula (4) allows the conclusion that the composita does not depend on $R(x)$ and characterizes the generating function $F(x)$. In tabular form, the composita is represented as

$$\begin{array}{ccccccccccc} & & & & & F^{\Delta}_{1,1} & & & & & \\ & & & & F^{\Delta}_{2,1} & & F^{\Delta}_{2,2} & & & & \\ & & & F^{\Delta}_{3,1} & & F^{\Delta}_{3,2} & & F^{\Delta}_{3,3} & & & \\ & & F^{\Delta}_{4,1} & & F^{\Delta}_{4,2} & & F^{\Delta}_{4,3} & & F^{\Delta}_{4,4} & & \\ & \iddots & & \vdots & & \vdots & & \vdots & & \ddots & \\ F^{\Delta}_{n,1} & & F^{\Delta}_{n,2} & & \ldots & & \ldots & & F^{\Delta}_{n,n-1} & & F^{\Delta}_{n,n} \end{array}$$

or, knowing that $F^{\Delta}_{1,n} = f(n)$, $F^{\Delta}_{n,n} = [f(1)]^n$, as

$$\begin{array}{ccccccccccc} & & & & & f(1) & & & & & \\ & & & & f(2) & & f^2(1) & & & & \\ & & & f(3) & & F^{\Delta}_{3,2} & & f^3(1) & & & \\ & & f(4) & & F^{\Delta}_{4,2} & & F^{\Delta}_{4,3} & & f^4(1) & & \\ & \iddots & & \vdots & & \vdots & & \vdots & & \ddots & \\ f(n) & & F^{\Delta}_{n,2} & & \ldots & & \ldots & & F^{\Delta}_{n,n-1} & & f^n(1) \end{array}$$

Below are the terms of the composite of the generating function $F(x) = \frac{x}{1-x}$:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Theorem 9. *For a given ordinary generating function $F(x) = \sum_{n\geq 1} f(n)x^n$, the composita $F^{\Delta}(n,k)$ always exists and is unique.*

Proof. Without proof. $\square$

3 Calculation of compositae

Calculation of compositae is based on derivation of the generating function of a composita

$$[A(x)]^k = \sum_{n \geqslant k} A^{\Delta}(n,k) x^n$$

and operation on them.

Theorem 10. *Let there be a generating function* $F(x) = \sum_{n>0} f(n)x^n$*, its composita* $F^{\Delta}(n,k)$*, and a constant* α*. Then, the generating function* $A(x) = \alpha F(x)$ *has the composita*

$$A^{\Delta}(n,k) = \alpha^k F^{\Delta}(n,k).$$

Proof.

$$[A(x)]^k = [\alpha F(x)]^k = \alpha^k [F(x)]^k.$$

□

Theorem 11. *Let there be a generating function* $F(x) = \sum_{n>0} f(n)x^n$*, its composita* $F^{\Delta}(n,k)$*, and a constant* α*. Then, the generating function* $A(x) = F(\alpha x)$ *has the composita*

$$A^{\Delta}(n,k) = \alpha^n F^{\Delta}(n,k).$$

Proof. By definition, we have

$$A^{\Delta}(n,k) = \sum_{\pi_k \in C_n} \alpha^{\lambda_1} f(\lambda_1) \alpha^{\lambda_2} f(\lambda_2) \ldots \alpha^{\lambda_k} f(\lambda_k) =$$

$$= \alpha^n \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k) = \alpha^n F^{\Delta}(n,k).$$

□

Theorem 12. *Let there be a generating function* $F(x) = \sum_{n>0} f(n)x^n$*, its composita* $F^{\Delta}(n,k)$*, a generating function* $B(x) = \sum_{n \geqslant 0} b(n)x^n$ *and* $[B(x)^k] = \sum_{n \geqslant 0} B(n,k)x^n$*. Then, the generating function* $A(x) = F(x)B(x)$ *has the composita*

$$A^{\Delta}(n,k) = \sum_{i=k}^{n} F^{\Delta}(i,k) B(n-i,k).$$

Proof. Because $a(0) = f(0)b(0) = 0$, $A(x)$ has the composita $A^{\Delta}(n,k)$. On the other hand,

$$[A(x)]^k = [F(x)]^k [B(x)]^k.$$

This, reasoning from the rule of product of generating functions, gives

$$A^{\Delta}(n,k) = \sum_{i=k}^{n} F^{\Delta}(i,k) B(n-i,k).$$

□

For $B(x)$ $b(0) = 0$, the formula has the form:

$$A^{\Delta}(n,k) = \sum_{i=k}^{n-k} F^{\Delta}(i,k) B^{\Delta}(n-i,k).$$

Theorem 13. *Let there be generating functions $F(x) = \sum_{n>0} f(n)x^n$, $G(x) = \sum_{n>0} g(n)x^n$ and their compositae $F^{\Delta}(n,k)$, $G^{\Delta}(n,k)$. Then, the generating function $A(x) = F(x)+G(x)$ has the composita*

$$A^{\Delta}(n,k) = F^{\Delta}(n,k) + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^{\Delta}(i,j) G^{\Delta}(n-i,k-j) + G^{\Delta}(n,k).$$

Proof. According to the binomial theorem, we have

$$[A(x)]^k = \sum_{j=0}^{k} \binom{k}{j} [F(x)]^j [G(x)]^{k-j},$$

$$[F(x)]^j = \sum_{n \geqslant j} F^{\Delta}(n,j),$$

and

$$[G(x)]^{k-j} = \sum_{n \geqslant k-j} G^{\Delta}(n,k-j).$$

According to the rule of multiplication of series, we obtain

$$A^{\Delta}(n,k) = F^{\Delta}(n,k) + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^{\Delta}(i,j) G^{\Delta}(n-i,k-j) + G^{\Delta}(n,k).$$

□

Definition 14. *Let there be a composition of generating functions $A(x) = R(F(x))$. Then, the product of two compositae will be termed a composite of the composition $A(x)$ and denoted as $A^{\Delta}(n,k) = F^{\Delta}(n,k) \circ R^{\Delta}(n,k)$.*

Theorem 15. *Let there be two generating functions $F(x) = \sum_{n>0} f(n)x^n$ and $R(x) = \sum_{n>0} r(n)x^n$, and their compositae $F^{\Delta}(n,k)$ and $R^{\Delta}(n,k)$. Then, the expression valid for the product of the compositae $A^{\Delta} = F^{\Delta} \circ R^{\Delta}$ is*

$$A^{\Delta}(n,m) = \sum_{k=m}^{n} F^{\Delta}(n,k) R^{\Delta}(k,m). \tag{7}$$

Proof.

$$[A(x)]^m = [G(F(x)]^m = G^m(F(x)).$$

Hence, according to the composition rule and taking into account that the nonzero terms $G^{\Delta}(n,m)$ begin with $n \geqslant m$, we have

$$A^{\Delta}(n,m) = \sum_{k=m}^{n} F^{\Delta}(n,k) G^{\Delta}(k,m).$$

□

Corollary . Because the composition of generating functions is an associative operation and

$$F(x) \circ (R(x) \circ G(x)) = (F(x) \circ R(x)) \circ G(x),$$

the product of compositae is also an associative operation and

$$\sum_{k=m}^{n} \sum_{i=k}^{n} F^{\Delta}(n,i) R^{\Delta}(i,k) G^{\Delta}(k,m) = \sum_{k=m}^{n} \sum_{i=k}^{n} R^{\Delta}(n,i) G^{\Delta}(i,k) F^{\Delta}(k,m).$$

4 Compositae of generating functions

4.1 Identical composita

Definition 16. *An identical composita $Id^{\Delta}(n,k)$ is a composita of the generating function $F(x) = x$.*

By definition, $[F(x)]^k = x^k$. Then

$$F^{\Delta}(n,k) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases} \tag{8}$$

Thus, $F^{\Delta}(n,k) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta. It is easily seen that for any generating function $A(x)$, we have the identity

$$a(n) = \sum_{k=1}^{n} Id^{\Delta}(n,k) a(k).$$

The composita of the function $F(x) = x^m$ is

$$F^{\Delta}(n,k,m) = \delta_{\frac{n}{m},k}, \quad \bmod (n,m) = 0 \text{ or } n = km. \tag{9}$$

4.2 Compositae of polynomials

4.2.1 Composita for $P_2(x) = (ax + bx^2)$

Let us consider $P_2(x) = (ax + bx^2)$. Then, $p_2(0) = 0$, $p_2(1) = a$ and $p_2(2) = b$, and the rest are $p_2(n) = 0,\ n > 2$. The composita of the function $F(x) = ax$ is equal to $a^k \delta_{n,k}$, and the composita of the function $G(x) = bx^2$ is equal to $b^k \delta_{\frac{n}{2},k}$. Using sum theorem (13), we obtain

$$P_2^{\Delta}(n,k) = \sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} a^j \delta_{i,j} b^{k-j} \delta_{\frac{n-i}{2},k-j},$$

$\delta_{\frac{n-i}{2},k-j} = 1$ for $\frac{n-i}{2} = k - j$, whence $i = n - 2k + 2j$. So we have

$$P_2^{\Delta}(n,k) = \sum_{j=0}^{k} \binom{k}{j} a^j \delta_{n-2k+2j,j} b^{k-j}.$$

Now $\delta_{n-2k+2j,j} = 1$ for $n - 2k + 2j = j$, whence $j = 2k - n$. So we obtain

$$P_2^{\Delta}(n,k,a,b) = \binom{k}{n-k} a^{2k-n} b^{n-k} \tag{10}$$

for $\lceil \frac{n}{2} \rceil \le k \le n$.

Thus, the composition $A(x) = R(ax + bx^2)$ can be found using the expression:

$$a(n) = \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \binom{k}{n-k} a^{2k-n} b^{n-k} r(k).$$

For example, let us derive an expression for the coefficients of the generating function $A(x) = e^{x+\frac{1}{2}x^2}$ (see A000085). Taking into account that this function is an exponential generating function, we obtain

$$a(n) = n! \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \binom{k}{n-k} \frac{1}{2^{n-k}} \frac{1}{k!}.$$

Another example is $A(x) = R(F(x))$, where $R(x) = \frac{x}{1-x}$ and $F(x) = x + x^2$, $A(x) = \frac{x+x^2}{1-x-x^2}$. Hence

$$a(n) = \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \binom{k}{n-k}$$

(see A000045).

4.2.2 Composita for $P_3(x) = ax + bx^2 + cx^3$

The polynomial $P_3(x) = ax + bx^2 + cx^3$ can be expressed as

$$P_3(x) = ax + xP_2(x, b, c).$$

The composita ax is equal to $\delta(n,k)a^k$, and the composita $xP_2(x)$ to $A_2\Delta(n-k,k)$. Then, on the strength of the theorem on the composita of the sum of generating functions, we have

$$A_3^{\Delta}(n,k,a,b,c) = \sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} A_2\Delta(i-j,j,b,c)\delta(n-i,k-j)a^{k-j}.$$

Simplification gives $\delta(n-i,k-j) = 1$ for $n - i = k - j$, whence we have $i = n - k + j$ and

$$A_3^{\Delta}(n,k,a,b,c) = \sum_{j=0}^{k} \binom{k}{j} A_2(n-k,j,b,c)a^{k-j},$$

where $A_2^{\Delta}(n-k,j,b,c) = \binom{j}{n-k-j} b^{2j+k-n} b^{n-k-j}$. Hence,

$$A_3^{\Delta}(n,k,a,b,c) = \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} b^{n-k-j}.$$

Then, for the generating function $A(x)=\frac{1}{1-ax-bx^2-cx^3}$, the following expression holds true:

$$a(n)=\sum_{k=1}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{j}{n-k-j}a^{k-j}b^{2j+k-n}b^{n-k-j}.$$

4.2.3 Composita for $P(x)=ax+cx^3$

An important condition in the foregoing examples is that $a,b,c\neq 0$. Therefore, if $b=0$ the formula for the composita should be sought for over again. For example,

$$P(x)=ax+cx^3.$$

In this case, the expression for the composita is

$$P^{\Delta}(n,k)=\binom{k}{\frac{3k-n}{2}}a^{\frac{3k-n}{2}}c^{\frac{n-k}{2}},$$

where $(n-k)$ is exactly divisible by 2. For example, for the generating function $A(x)=\frac{1}{1-x-x^3}$ the following expression holds true:

$$a(n)=\sum_{k=1}^{n}\binom{k}{\frac{3k-n}{2}}$$

(see A000930).

4.2.4 Composita for $P_4(x)=ax+bx^2+cx^3+dx^4$

At $n=4$, the polynomial $P_4(x)=ax+bx^2+cx^3+dx^4$ can be expressed as

$$P_4(x)=P_2(x)+x^2P_2(x).$$

The generating function of the composita for $x^2P_2(x)$ is equal to

$$x^{2k}\binom{k}{n-k}c^{2k-n}b^{n-k}x^n=\binom{k}{n-k}c^{2k-n}b^{n-k}x^{n+2k},$$

and hence the expression for the composita is

$$\binom{k}{n-3k}c^{4k-n}b^{n-3k}.$$

Then the composita $P_4(x)$ has the following expression:

$$\sum_{j=0}^{k}\binom{k}{j}\sum_{i=j}^{n-k+j}\binom{j}{i-j}a^{2j-i}b^{i-j}\binom{k-j}{n-i-3(k-j)}c^{4(k-j)-(n-i)}d^{n-i-k+j}.$$

For example, for the generating function $A(x)=\frac{1}{1-ax-bx^2-cx^3-dx^4}$ the following expression holds true:

$$a(n)=\sum_{k=1}^{n}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=j}^{n-k+j}\binom{j}{i-j}a^{2j-i}b^{i-j}\binom{k-j}{n-i-3(k-j)}c^{4(k-j)-(n-i)}d^{n-i-k+j}.$$

At $a=b=c=d=1$, we obtain the generating function $A(x)=\frac{1}{1-x-x^2-x^3-x^4}$. Hence

$$a(n)=\sum_{k=1}^{n}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=j}^{n-k+j}\binom{j}{i-j}\binom{k-j}{n-i-3(k-j)}.$$

4.2.5 Composita for $P_5(x)=ax+bx^2+cx^3+dx^4+ex^5$

For finding the composita of the mth power polynomial, we can propose the recurrent algorithm

$$A_m^{\Delta}(n,k)=\sum_{j=0}^{k}A_{m-1}^{\Delta}(n-k,j)a^{k-j},$$

providing that $A_{m-1}(n,0)=1$. Using this recurrent algorithm, we obtain the composita of the 5th power polynomial:

$$\sum_{r=0}^{k}a^{k-r}\binom{k}{r}\sum_{m=0}^{r}b^{r-m}\left(\sum_{j=0}^{m}c^{m-j}\,d^{r-n+m+k+2j}\,e^{v}\binom{j}{v}\binom{m}{j}\right)\binom{r}{m}$$

, where $v=-r+n-m-k-j$.

4.3 Composita for $A(x)=(\frac{ax}{1-bx})$

For the generating function $F(x)=\frac{x}{(1-x)}$, $F^{\Delta}(n,k)=\binom{n-1}{k-1}$, and

$$A(x)=(ab^{-1}\frac{bx}{1-bx}).$$

Using theorems (10,11), we obtain

$$A^{\Delta}(n,k)=\binom{n-1}{k-1}a^kb^{n-k}.$$

4.4 Compositae of the exponent

Let us find the expression for the coefficients of the generating function $[B(x)]^k=e^{kx}$:

$$B(x)^k=e^{xk}=\sum_{n\geqslant 0}\frac{k^n}{n!},$$

whence it follows that

$$B(n,k) = \frac{k^n}{n!}.$$

Now, for $A(x) = xe^x$ the composita is equal to

$$A^{\Delta}(n,k) = B(n-k,k) = \frac{k^{n-k}}{(n-k)!}. \tag{11}$$

Let us write the composita for the generating function $A(x) = e^x - 1$:

$$A(x)^k = \sum_{m=0}^{k} \binom{k}{m} e^{mx} (-1)^{k-m},$$

whence it follows that the composita is

$$A^{\Delta}(n,k) = \sum_{m=0}^{k} \binom{k}{m} \frac{m^n}{n!} (-1)^{k-m} = \frac{k!}{n!} S_2(n,k), \tag{12}$$

where $S_2(n,k)$ is the Stirling numbers of the second kind. For the generating functions of the Bell numbers $A(x) = e^{e^x-1}$, we have

$$a(n) = n! \sum_{k=1}^{n} S_2(n,k) \frac{k!}{n!} \frac{1}{k!} = \sum_{k=1}^{n} S_2(n,k)$$

(see A000110).

4.5 Composita for $\ln(1+x)$

Let $F(x) = \ln(x+1)$. Then, knowing the relation [6]

$$\sum_{n=k}^{\infty} S_1(n,k) \frac{x^n}{n!} = \frac{[\ln(1+x)]^k}{k!},$$

where $S_1(n,k)$ is the Stirling numbers of the first kind, and using formula (1), we obtain the expression for the composita of the generating function $\ln(1+x)$:

$$F^{\Delta}(n,k) = \frac{k!}{n!} S_1(n,k). \tag{13}$$

4.6 Composita for the generating function of the Bernoulli numbers

The generating function of the Bernoulli numbers is

$$A(x) = \frac{x}{e^x - 1}.$$

This function can be represented as the composition $B(F(x))$, where $B(x) = \frac{\ln x}{x}$, $F(x) = e^x - 1$. Let us find the expression for the coefficients of the generating function $[B(x)]^k$:

$$[B(x)]^k = \sum_{n \geqslant 0} S_1(n,k) \frac{k!}{n!} x^{n-k},$$

whence

$$B(n,k) = S_1(n+k,k) \frac{k!}{(n+k)!}.$$

Knowing the composita of the function $F(x)$ (see 12),

$$F^{\Delta}(n,k) = \frac{k!}{n!} S_2(n,k).$$

Let us write the composition of the generating functions $A(x)^k = [B(e^x - 1)]^k$:

$$A(n,m) = \begin{cases} 1, & n = 0, \\ \sum_{k=1}^{n} S_2(n,k) \frac{k!}{n!} S_1(k+m,m) \frac{m!}{(k+m)!}, & n > 0. \end{cases}$$

Then the composita of $xA(x)$ is

$$A^{\Delta}(n,m) = \begin{cases} 1, & n = m, \\ \frac{m!}{(n-m)!} \sum_{k=1}^{n-m} \frac{k!}{(k+m)!} S_1(k+m,m) S_2(n-m,k), & n > m. \end{cases}$$

4.7 Composita for the generating function of the Fibonacci numbers

Let us find the composita for the generating function of the Fibonacci numbers:

$$A(x) = \frac{x}{1 - x - x^2}.$$

The function can be represented as the composition of the generating functions $A(x) = R(F(x))$, where $R(x) = \frac{x}{1-x}$, $F(x) = \frac{x}{1-x^2}$. Let us find the composita for $F(x)$:

$$F^{\Delta}(n,k) = \begin{cases} \binom{\frac{n+k}{2}-1}{k-1}, & \text{at } n+k - \text{even}, \\ 0, & \text{at } n+k - \text{odd}. \end{cases}$$

Now, using the operation of product of compositae, we find the composita of the generating function $A(x)$:

$$A^{\Delta}(n,m) = \sum_{k=m}^{n} \binom{\frac{n+k}{2}-1}{k-1} \binom{k-1}{m-1}, \quad \text{at } n+k - \text{even}.$$

Below are the first terms of the composita for the generating function of the Fibonacci numbers:

1
1 1
2 2 1
3 5 3 1
5 10 9 4 1
8 20 22 14 5 1
13 38 51 40 20 6 1

4.8 Composita for the generalized Fibonacci numbers

Let us find the composita of the generating function:

$$F(x) = x + x^2 + \ldots + x^m = \frac{x - x^{m+1}}{1 - x}.$$

Let us write $F(x)$ as the product of the functions $G(x) = x - x^{m+1}$ and $R(x) = \frac{1}{1-x}$. Let us find the composita for $G(x)$. For this purpose, we consider the compositae of the functions $y(x) = x$ and $z(x) = -x^m$. For $y(x)$, the composita is equal to $Id(n,k) = \delta_{n,k}$. For $z(x) = -x^m$, the composita is

$$Z^{\Delta}(n,k) = (-1)^k \delta_{\frac{n}{m},k}.$$

Then, on the strength of the theorem on the composite of sum of generating functions $y(x) + z(x)$, we have

$$G^{\Delta}(n,k) = \sum_{j=0} \binom{k}{j} \sum_{i=j}^{n-k+j} Id(i,j) Z^{\Delta}(n-i, k-j) =$$

$$= \sum_{j=0} \binom{k}{j} \sum_{i=j}^{n-k+j} \delta_{i,j} \delta_{\frac{n-i}{m},k-j} (-1)^{k-j}.$$

The function $\delta_{i,j} = 1$ is only for $i = j$, and hence

$$G^{\Delta}(n,k) = \sum_{j=0} \binom{k}{j} \delta_{\frac{n-j}{m},k-j} (-1)^{k-j}.$$

The function $\delta_{\frac{n-j}{m},k-j} = 1$ is only for $\frac{n-j}{m} = k - j$, and hence

$$G^{\Delta}(n,k) = \binom{k}{\frac{(m+1)k-n}{m}} (-1)^{\frac{n-k}{m}}.$$

It is known that $R(n,k) = \binom{n+k-1}{k-1}$. Then, with regard to the rule of finding the composita of the product of generating functions (case 2), we obtain

$$F^{\Delta}(n,k) = \sum_{i=k}^{n} \binom{k}{\frac{(m+1)k-i}{m}} (-1)^{\frac{i-k}{m}} \binom{n-i+k-1}{k-1}.$$

Let us consider the composita of the generating functions:

$$A(x) = \frac{F(x)}{1 - F(x)} = \frac{x - x^{m+1}}{1 - 2x - x^{m+1}}.$$

Hence, using the theorem on the product of compositae, we obtain the composita of the generating function $A(x)$:

$$A^{\Delta}(n,l)=\sum_{k=l}^{n}F^{\Delta}(n,k)\binom{k-1}{m-1}=$$

$$=\sum_{k=m}^{n}\sum_{i=k}^{n}\binom{k}{\frac{(m+1)k-i}{m}}(-1)^{\frac{i-k}{m}}\binom{n-i+k-1}{k-1}\binom{k-1}{l-1}.$$

For $l=1$, we derive the formula for the generalized Fibonacci numbers:

$$F_n^{(m)}=\sum_{k=1}^{n}\sum_{i=k}^{n}\binom{k}{\frac{(m+1)k-i}{m}}(-1)^{\frac{i-k}{m}}\binom{n-i+k-1}{k-1}. \tag{14}$$

4.9 Composita of the generating function for the Catalan numbers

Let $F(x)=x\frac{1-\sqrt{1-4x}}{2x}$, then the composita has the form

$$F^{\Delta}(n,k)=\sum_{i=0}^{n-k}C(i)F^{\Delta}_{n-i-1,k-1},$$

where $C(i)$ is the Catalan numbers. The composita $F^{\Delta}(n,k)$ has the following triangular form:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&2&&1&&1&& \\ &5&&5&&3&&1& \\ 14&&14&&9&&4&&1 \end{array}$$

Let us consider the sequence A009766 called the Catalan triangle. This triangle is given by the formula

$$a(n,m)=\binom{n+m}{n}\frac{n-k+1}{n+1}.$$

Below are the initial values of the triangle, and n and m begin with zero.

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&2&&1&&2&& \\ &1&&3&&5&&5& \\ 1&&4&&9&&14&&14 \end{array}$$

Comparison of two triangles suggests that $a(n,k)=F^{\Delta}(n+1,n-k+1)$. Hence, the composita for the Catalan generating function is equal to

$$F^{\Delta}(n,k)=\binom{2n-k-1}{n-1}\frac{k}{n}.$$

Thus, the expression valid for the coefficients of the composition $A(x)=R(\frac{1-\sqrt{1-4x}}{2})$ is

$$a(n)=\binom{2n-k-1}{n-1}\frac{k}{n}\cdot r(k).$$

4.10 Composita of the generating function $\frac{x}{\sqrt{1-x}}$

This generating function can be represented as the composition of the functions:

$$\frac{x}{\sqrt{1-x}}=x\frac{1}{1-\left(2\frac{\sqrt{1-\frac{4x}{4}}}{2}-1\right)}=x\frac{1}{1-2C(\frac{1}{4}x)},$$

where $C(x)=\frac{1-\sqrt{1-4x}}{2}$.

Using the formula of composition, we finally obtain

$$A^{\Delta}(n,m)=\begin{cases}1, & n=m,\\ \sum_{k=1}^{n-m}\binom{2n-2m-k-1}{n-m-1}\frac{k}{n-m}2^{k-2n+2m}\binom{k+m-1}{m-1}, & n>m.\end{cases}$$

4.11 Compositae of trigonometric functions

4.11.1 Composita of the sine

Using the expression

$$\sin(x)=\frac{e^{ix}-e^{-ix}}{2i},$$

we obtain $\sin(x)^k$:

$$\sin(x)^k=\frac{1}{2^k i^k}\sum_{m=0}^{k}\binom{k}{m}e^{imx}e^{-i(k-m)x}(-1)^{k-m}=\frac{1}{2^k i^k}\sum_{m=0}^{k}\binom{k}{m}e^{i(2m-k)x}(-1)^{k-m}.$$

Hence the composita is

$$\frac{1}{2^k}i^{n-k}\sum_{m=0}^{k}\binom{k}{m}\frac{(2m-k)^n}{n!}(-1)^{k-m}.$$

Taking into account that $n-k$ is an even number and the function is symmetric about k, we obtain the composita of the generating function $\sin(x)$:

$$A^{\Delta}(n,k)=\begin{cases}\frac{1}{2^{k-1}n!}\sum_{m=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{m}(2m-k)^n(-1)^{\frac{n-k}{2}-m}, & (n-k)-\text{even}\\ 0, & (n-k)-\text{odd}\end{cases}$$

Example 17. *For the Euler numbers we know the exponential generating function $\frac{1}{1-sin(x)}$. Hence,*

$$E_{n+1}=\sum_{\substack{k=1\\ n+k\ even}}^{n}\frac{1}{2^{k-1}}\sum_{m=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{m}(2m-k)^n(-1)^{\frac{n+k}{2}-m}$$

(see A000111).

Example 18. *For the generating function $A(x) = e^{\sin(x)}$, the valid expression is*

$$a_n = \sum_{\substack{k=1\\ n+k\ even}}^{n} \frac{1}{2^{k-1}k!} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{m} (2m-k)^n (-1)^{\frac{n+k}{2}-m}$$

(see A002017).

4.11.2 Compositae of the cosine

Knowing that

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2},$$

We have

$$[\cos x]^k = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} e^{(2j-k)ix} =$$

$$= \frac{1}{2^k} \sum_{n \geqslant 0} \sum_{j=0}^{k} \binom{k}{j} (2j-k)^n i^n \frac{x^n}{n!}.$$

Hence

$$B(n,k) = \frac{1}{2^k n!} (-1)^{\frac{n}{2}} \sum_{j=0}^{k} \binom{k}{j} (2j-k)^n.$$

Then, the composita of the generating function $xcos(x)$ is

$$A^{\Delta}(n,k) = \begin{cases} \frac{1}{2^{k-1}(n-k)!} (-1)^{\frac{n-k}{2}} \sum_{j=0}^{k} \binom{k}{j} (2j-k)^{n-k}, & n-k \ -\text{even} \\ 0, & n-k \ - \ \text{odd}. \end{cases}$$

The composita of the function $cos(x) - 1$ is equal to

$$A^{\Delta}(n,k) = \sum_{i=0}^{k} B(n,i)(-1)^{k-i}.$$

Let us consider the following example. Let there be a generating function $A(x) = sec(x) = \frac{1}{cos(x)} = \frac{1}{1+(\cos(x)-1)}$. Hence, on the strength of the formula of composition and composita $(\cos(x) - 1)$, we obtain

$$a(n) = \sum_{k=1}^{2n} \sum_{m=0}^{k} \binom{k}{m} 2^{1-m} \left(\sum_{j=0}^{\frac{m}{2}} (2j-m)^{2n} \binom{m}{j} \right) (-1)^{n+m}$$

(see A000364).

4.11.3 Composita for $\tan(x)$

For the tangent, we know the identity

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} - e^{-ix})}.$$

Division of the numerator and denominator by e^{ix} gives

$$\tan(x) = \frac{1 - e^{-2ix}}{i(1 - e^{-2ix})}.$$

Multiplication of the numerator and denominator by i, and addition and then subtraction of unity gives

$$\tan(x) = i\frac{e^{-2ix} - 1}{2 - (e^{-2ix} - 1)}.$$

Whence it follows that

$$\tan(x) = \frac{i}{2}\frac{e^{-2ix} - 1}{1 - \frac{1}{2}(e^{-2ix} - 1)}.$$

Thus, the function $\tan(x)$ is expressed as the composition of the functions

$$F(x) = \frac{i}{2}\frac{x}{1 + \frac{1}{2}x}$$

and functions $R(x) = e^{-2ix} - 1$. The composita for $F(x)$ is equal to

$$F^{\Delta}(n,k) = \frac{1}{2^n}(-1)^{n-k}\binom{n-1}{k-1}i^k.$$

The composita for $R(x)$ is equal to

$$R^{\Delta}(n,k) = (-2i)^n\frac{k!}{n!}S_2(n,k),$$

where $S_2(n,k)$ is the Stirling numbers of the second kind. Then, on the strength of the theorem on the product of compositae, we obtain the composita of the function $\tan(x)$:

$$A^{\Delta}(n,m) = \sum_{k=m}^{n}(-2i)^n S_2(n,k)\frac{k!}{n!}\frac{1}{2^k}(-1)^{k-m}\binom{k-1}{m-1}i^m.$$

After transformation, we obtain

$$A^{\Delta}(n,m) = (-1)^{\frac{n+m}{2}}\sum_{k=m}^{n}(2)^{n-k}S_2(n,k)\frac{k!}{n!}(-1)^{n+k-m}\binom{k-1}{m-1}.$$

Then at $k = 1$, the expression for the tangential numbers is

$$a(n) = (-1)^{n+1}\sum_{j=1}^{2\,n+1}(-1)^j\; j!\,2^{2\,n-j+1}\,S_2(2\,n+1,j)$$

(see A000182)

Let us consider the example $A(x)=e^{\tan(x)}$:

$$a(n)=\sum_{k=1}^{n}\frac{(-1)^{\frac{n+k}{2}}\sum_{j=k}^{n}\binom{j-1}{k-1}j!\,2^{n-j}\,(-1)^{n-k+j}\,S_2\,(n,j)}{k!}$$

(see A006229). For more examples, see A000828,A000831,A003707

4.11.4 Composita for $x^2\cot(x)$

It is known that

$$x^2\cot(x)=ix\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}=ix^2+\frac{2ix^2}{e^{2ix}-1}.$$

The composita ix^2 is equal to $\delta(\frac{n}{2},k)i^k$, and the composita for $\frac{2ix^2}{e^{2ix}-1}$ is equal to

$$(2i)^{n-k}B^{\Delta}(n,k),$$

where $B^{\Delta}(n,k)$ is the composita for the generating function of the Bernoulli numbers. Using the theorem on the composita of the sum of generating functions, we obtain the composita of the function $x^2\cot(x)$:

$$A^{\Delta}(n,k)=\delta(\frac{n}{2},k)i^k+\sum_{j=1}^{k}B^{\Delta}(n-2k+2j,j)(2i)^{n-2k+j}i^{k-j}=$$

$$=\delta(\frac{n}{2},k)i^k+i^{n-k}\sum_{j=1}^{k}B^{\Delta}(n-2k+2j,j)2^{n-2k+j}$$

4.11.5 Composita of the arc tangent $F(x)=\arctan(x)$

Let us consider the generating function of the arc tangent:

$$\arctan(x)=\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)}x^{2n+1}.$$

Let us find an expression for the composita of the arc tangent from the operation of product of compositae. For this purpose, the expression

$$\arctan(x)=\frac{i}{2}(\ln(1-ix))-\ln(1+ix))$$

is written as follows:

$$\arctan(x)=\frac{i}{2}\ln(1-\frac{2ix}{1+ix}).$$

The composita of the function $f(x)=\frac{2ix}{1+ix}$ is equal to

$$F^{\Delta}(n,k)=2^k\binom{n-1}{k-1}i^n,$$

whence it follows that

$$A_z^{\Delta}(n,m) = \frac{i^m}{2^m}\sum_{k=m}^{n} 2^k \binom{n-1}{k-1} i^n \frac{m!}{k!} S_1(k,m). \tag{15}$$

$$A_z^{\Delta}(n,m) = \frac{(-1)^{\frac{m+n}{2}}}{2^m}\sum_{k=m}^{n} 2^k \binom{n-1}{k-1} \frac{m!}{k!} S_1(k,m). \tag{16}$$

Below are the first terms of the composita of the arc tangent $A^{\Delta}(n,k)$ in the triangular form:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 1 & & & & & \\
 & & & & -\frac{1}{3} & & 0 & & 1 & & & & \\
 & & & 0 & & -\frac{2}{3} & & 0 & & 1 & & & \\
 & & \frac{1}{5} & & 0 & & -1 & & 0 & & 1 & & \\
 & 0 & & \frac{23}{45} & & 0 & & -\frac{4}{3} & & 0 & & 1 & \\
 -\frac{1}{7} & & 0 & & \frac{14}{15} & & 0 & & -\frac{5}{3} & & 0 & & 1
\end{array}$$

Example 19. *Let there be $R(x) = \frac{1}{1-x}$, then the coefficients of the generating function*

$$A(x) = \frac{1}{1-\arctan(x)}$$

are expressed by the formula:

$$a(n) = \sum_{m=1}^{n} \frac{(-1)^{\frac{m+n}{2}}}{2^m}\sum_{k=m}^{n} 2^k \binom{n-1}{k-1} \frac{m!}{k!} S_1(k,m).$$

Hence, summation of rows of the composita of the arc tangent gives the following series:

$$A(x) = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{5}x^5 + \frac{8}{45}x^6 + \dots.$$

Example 20. *Let $A(x) = e^{\arctan(x)}$, then the valid expression is*

$$a(n) = n!\sum_{m=1}^{n} \frac{(-1)^{\frac{3\,n+m}{2}}\ \sum_{i=m}^{n} \frac{2^i\, S_1(i,m)\,\binom{n-1}{i-1}}{i!}}{2^m}$$

(see A002019).

4.12 Compositae of hyperbolic functions

For the hyperbolic sine, we have the known expression:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Let us find the composita of this generating function. For this purpose, we write

$$\left(\frac{e^x - e^{-x}}{2}\right)^k = \frac{1}{2^k}\left(e^x + e^{-x}\right)^k = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}e^{(k-i)x}(-1)^i e^{-ix} =$$

$$= \frac{1}{2^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}e^{(k-2i)x}.$$

Let us write e^x as a series, then we obtain

$$\frac{1}{2^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\sum_{n\geqslant 0}\frac{(k-2i)^n}{n!}x^n.$$

Hence, the composita is

$$F^{\Delta}(n,k) = \frac{1}{2^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(k-2i)^n}{n!}.$$

For example, for $A(x) = e^{\sinh x}$ the valid expression is

$$a(n) = \sum_{k=1}^{n}\frac{\sum_{i=0}^{k}(-1)^i\,(k-2\,i)^n\,\binom{k}{i}}{2^k\,k!}$$

(see A002724).

For the hyperbolic cosine, we have

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Then,

$$\cosh^k(x) = \left(\frac{e^x + e^{-x}}{2}\right)^k = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}e^{(k-2i)x} =$$

$$= \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\sum_{n\geqslant 0}\frac{(k-2i)^n}{n!}x^n,$$

and hence the composita of the generating function $x\cosh(x)$ is

$$F^{\Delta}(n,k) = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\frac{(k-2i)^{n-k}}{(n-k)!}.$$

For example, for $A(x) = e^{\cosh x}$ the valid expression is

$$\sum_{k=1}^{n}\frac{\left(\sum_{i=0}^{k}(k-2\,i)^{n-k}\,\binom{k}{i}\right)\,\binom{n}{k}}{2^k}$$

see A003727).

5 Conclusion

The operation of the composition $A(x) = R(F(x))$ of ordinary generating functions requires:

1. Finding the composita $F^{\Delta}(n,k)$ of the generating function $F(x)$ with the use of theorems (10,11,12,13,15)

2. Writing the composition in the form

$$a(n) = \sum_{k=1}^{n} F^{\Delta}(n,k) r(n).$$

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